On collisional diffusion in a stochastic magnetic field

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The effect of particle collisions on the transport in a stochastic magnetic field in tokamaks is investigated. The model of resonant magnetic perturbations generated by external coils at the plasma edge is used for the stochastic magnetic field. The particle collisions are simulated by a random walk process along the magnetic field lines and the jumps across the field lines at the collision instants. The dependencies of the local diffusion coefficients on the mean free path λ_{mfp}, the diffusion coefficients of field lines D_{FL}, and the collisional diffusion coefficients, $\chi_\perp$, are studied. Based on these numerical data and the heuristic arguments the empirical formula,$D_r = \chi_\perp + v_{||} D_{FL}/\left(1 + L_c/\lambda_{mfp}\right)$, for the local diffusion coefficient is proposed, where L_c is the characteristic length of order of the connection length $l_c = \pi q R_0$, q is the safety factor, R_0 is the major radius. The formula quite well describes the results of numerical simulations. In the limiting cases the formula describes the Rechester-Rosenbluth and Laval scalings.

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I. INTRODUCTION

The charged particle transport in a stochastic magnetic field has been a subject of many studies starting from the middle of 1970's and it remains still one of the important issues of the laboratory and space plasmas (see, for example, Refs. [1–18] and the references therein). At the first glance one may suppose that the transport is caused mainly due to the particles along stochastic magnetic field lines. However, the particle collisions, fluctuations of the plasma density and temperature (or a plasma turbulence) may strongly modify this simple picture of the transport making the problem more complex.

In the last decade this problem has become very important in magnetically confined fusion research. It was found that the stochastic magnetic fields created at the plasma edge by the externally applied resonant magnetic perturbations (RMPs) may mitigate the so–called the edge localized modes (ELMs) which may cause a damaging effect on the wall material due to release the huge repetitive energy and particles fluxes (see, e.g., [13, 19, 20] and the references therein). The mechanism of this effect is still not well understood. Therefore the study of particle transport in such a deterministic chaotic magnetic field is important to understand this problem.

The effect of particle collisions on the particle transport in a stochastic magnetic field has been considered in the numerous analytical and numerical works [1–8, 10, 14–16, 21–23]. Since it is difficult to perform the analytical calculations in dynamically chaotic systems the analytical theories are based on the heuristic arguments rather than on the rigorous mathematical treatments. In such theories a stochastic magnetic field is introduced by its property of the exponential divergency of magnetic field lines with close initial coordinates and it is characterized by the characteristic (Kolmogorov) length L_K. The predictions of these theories for the radial diffusion coefficient D_r have only a qualitative nature. Particularly, the different regimes of the transport have been identified. Among them one can note the followings: the collisionless regime where D_r is determined only by the diffusion coefficient of magnetic field lines, D_{FL}, $D_r = v_\| D_{FL}$, where $v_\|$ is the parallel thermal parallel velocity of particles; the so-called Rechester-Rosenbluth regime [1] with $D_{RR} = v_T D_{FL} \lambda_{mfp}/L_{K\delta} = 2D_{FL}\chi_\|/L_{K\delta}$ for the moderately collisional plasmas, and the Kadomtsev-Pogutse regime [3] with $D_{KP} = D_{FL} 2 l_{cr}^{-1}\sqrt{\chi_\| \chi_\perp}$ for extremely collisional plasmas. Another regime called a long-mean–free path has been identified in [7] with the diffusion coefficient $D_L = v_T D_{FL}/(1 + l_{c\delta}/\lambda_{mfp})$ with $l_{c\delta} = (1/2) L_K \ln |\chi_\| \chi_\perp v_T^2/l_{cr}^2|$ (see [4, 5, 7] for the detailed discussions). Here λ_{mfp} is the mean free path, $\chi_\|$ and $\chi_\perp$ are the parallel and the perpendicular diffusion coefficients, $L_{K\delta} = L_K = \ln\left[l_{cr}/L_K(\chi_\|/\chi_\perp)^{1/2}\right]$, and l_{cr} is the correlation length of the stochastic magnetic field.

These collisional regimes have been verified in a number of numerical simulations (see, e.g.,[6, 16, 23]). Among them one can note the work by Rax and White [6] and its generalization [16] where the standard mapping is used as a model of a stochastic magnetic field. This model, however, does not quite well represent a stochastic magnetic field in real tokamak experiments, for example, like ergodic divertors in the Tore Supra tokamak [9] and in the TEXTOR tokamak [24], and a stochastic plasma in the DIII–D tokamak [19]. In these experiments the stochastic magnetic field is created by the externally RMPs with strongly radially dependent amplitudes and covers a finite region at the plasma edge with open magnetic field lines. These features of the perturbation magnetic field in real tokamak plasmas are not captured by the simple mapping models for particle transport in a unlimited domain as in Refs. [6, 16]. The determination of diffusion coefficients of field lines and particles in a finite domain is different than in a unlimited one (see, e.g., [25]).

In the present work we reexamine a collisional particle transport in a realistic model of a stochastic magnetic field. To simplify the problem we will not take into account the trapped particle transport. The problem is studied using the direct numerical simulations by introducing particle collisions as a random walk process along the field lines and the random jumps across magnetic surfaces.

Specifically we have used the model of RMPs created at the plasma edge by the external set of helical coils like in the ergodic divertor of the Tore Supra [9, 26] and the dynamic ergodic divertor of the TEXTOR tokamak [24, 27]. The statistical properties of chaotic magnetic field lines in a stochastic zone are similar to the ones in the divertor tokamaks with the magnetic separatrix [28].

The magnetic field line equations in the Hamiltonian representation are integrated using the canonical mapping procedure [29]. This mapping has advantages over the standard mapping used in Refs. [6, 16]: (i) the integration step can be chosen to have a desired accuracy; (ii) the mapping is an invariant with respect to the change of motion direction.

The work consists of six sections. The basic equations of magnetic field lines, the model of RMPs and the mapping procedure to integrate the Hamiltonian field line equations are recalled in Sec. II. There we have also given the brief description of the diffusive and the correlation properties of field lines and particles in the collisionless plasmas. The random walk model describing the diffusion due to particle collisions is introduced in Sec. II E. The numerical simulations of the particle transport in the stochastic magnetic field with collisions are described in Sec. III. In Sec. IV the empirical formula for the radial diffusion coefficient of particles is proposed and its limiting cases are discussed. In the final Sec. V we have summarized the obtained results and made some conclusive remarks.

II. BASIC EQUATIONS OF MAGNETIC FIELD LINES

A. Hamiltonian equations of magnetic field lines

We employ the equations for field lines in the magnetic flux coordinates $(\vartheta, \varphi, \psi_t, \psi_p)$ with (ϑ, φ) being the poloidal and the toroidal angles and $(\psi \equiv \psi_t, \psi_p)$ being the toroidal and poloidal fluxes, respectively. The field lines satisfy the Hamiltonian equation,

$$\frac{d\vartheta}{d\varphi} = \frac{\partial H}{\partial \psi}, \qquad \frac{d\psi}{d\varphi} = -\frac{\partial H}{\partial \vartheta}, \tag{1}$$

with the Hamiltonian function $H \equiv \psi_p(\psi, \vartheta, \varphi)$, the time-like variable φ, the canonical variables (ϑ, ψ). In the equilibrium plasma the Hamiltonian H is determined by the safety factor $q(\psi)$, i.e., $H_0(\psi) = \int d\psi/q(\psi)$.

In the presence of non-axisymmetric magnetic perturbations the Hamiltonian H can be presented as the sum,

$$H = H_0(\psi) + \epsilon H_1(\psi, \vartheta, \varphi), \tag{2}$$

of the poloidal fluxes corresponding the equilibrium plasma $H_0(\psi)$ and the magnetic perturbations $\epsilon H_1(\psi, \vartheta, \varphi)$. The latter is a $2\pi-$periodic function of the poloidal ϑ and the toroidal angles φ, and therefore can be presented by a Fourier series,

$$\epsilon H_1(\psi, \vartheta, \varphi) = \epsilon \sum_{m,n} H_{mn}(\psi) \cos\left(m\vartheta - n\varphi + \chi_{mn}\right), \tag{3}$$

where $H_{mn}(\psi)$ and χ_{mn} are the Fourier amplitudes and phases of the perturbation. The parameter ϵ stands for the dimensionless perturbation amplitude, $\epsilon = B_r/B_0$, defined as the ratio of the magnetic perturbation amplitude B_r to the toroidal field B_0 of the equilibrium field at the major radius R_0.

B. Model of magnetic perturbations

To be specific we consider the magnetic perturbations created at the plasma edge by a set helical coils. Such coils have been used in the ergodic divertors of the Tore Supra and the TEXTOR tokamaks [9], [27]. The magnetic perturbations of such coils can be described by the analytical model. Such a model would be useful to study generic features of magnetic field lines at the plasma edge, particularly, the statistical properties, transport of field lines and particles in the stochastic magnetic fields.

Based on the general features of magnetic perturbations created by a helical coils, studied in Refs. [9, 27, 30, 31] the spectra of magnetic perturbations H_{mn} can be presented by the following analytic formula,

$$\begin{aligned} H_{mn}(\psi) &= C_m \left(\frac{r}{r_0}\right)^{m/\gamma} \frac{\sin\left([m/\gamma - m_0]\theta_c\right)}{m[m/\gamma - m_0]\theta_c}, \\ r &= r(\psi) = R_0\sqrt{2\psi}, \end{aligned} \tag{4}$$

where r_0, γ, θ_c, m_0 are parameters of the model: r_0 is the minor radius of helical coils, γ is the parameter equal to the derivative of the poloidal angle ϑ with respect to the geometrical poloidal angle θ, i.e., $d\vartheta/d\theta$ taken at the high $\theta = \pi$ (or low $\theta = 0$) field side. The quantity θ_c is a half of the poloidal extension of coils $\Delta\theta$. One can assume that γ and θ_c do not depend on the toroidal flux ψ. The parameter $C_m = 1$ if the perturbation is localized on the low field side as in the TORE SUPRA-ED. In this case $\gamma > 1$. If the perturbation is localized on the high field side like in the TEXTOR-DED one has $C_m = (-1)^m$ and $\gamma < 1$ and

The poloidal spectrum $H_{mn}(\psi)$ is localized near the central mode $m_0^* = m_0/\gamma$. By the appropriate choice of γ one can localize the central resonant magnetic surface ψ_{mn}, $q(\psi_{mn}) \approx m_0^*/n = m_0/n\gamma$. The width of the spectrum Δm is determined by θ_c, i.e., $\Delta m \approx \pi\gamma/\theta_c$.

One can model the safety factor profile as

$$q(\psi) = q_a \frac{r^2/a^2}{1 - (1 - r^2/a^2)^{\mu}}, \tag{5}$$

where q_a and $\psi_a = a^2/2R_0^2$ are a safety factor and the toroidal flux at the plasma boundary $r = a$, respectively,

μ is a parameter, related to the value of $q(\psi)$ at the magnetic axis $r=0$, i.e., $q(0)=q_a/\mu$.

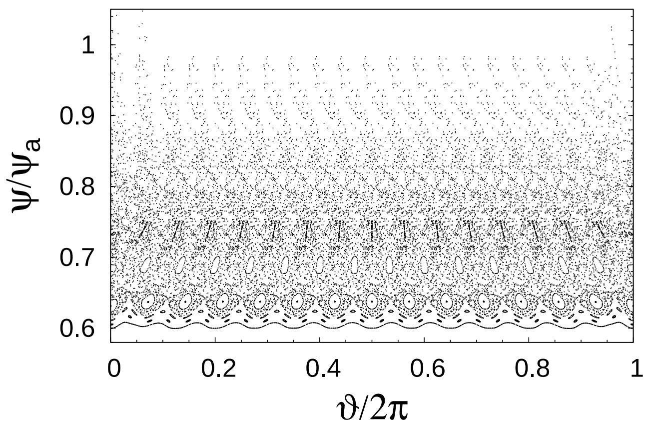

FIG. 1: Poincaré sections of field lines in the (ϑ,ψ)-plane: (a) for the model of Tore Supra ED with the toroidal mode $n=6$, perturbation parameter $\epsilon=10^{-3}$, $R_0=2.38$ m, $a=$0.8 m, $r_0=$0.85 m, $\gamma_1=1.81$, $\theta_c=\pi/7$, $m_0=10$, $q_0=0.9$, $q_a=3.6$.

C. Mapping of field lines

The Hamiltonian equation (1) is integrated using the symplectic mapping procedure [29, 32, 33]. Let (ϑ_k,ψ_k) be the values of the poloidal angle and the toroidal flux (ϑ,ψ) at the sections $\varphi=\varphi_k=(2\pi/N)\,k$, $k=0,\pm1,\pm2,\cdots$. The integer number N, $(N\geq 1)$, determines the integration step of the mapping, $\Delta\varphi=2\pi/N$. The forward mapping $(\vartheta_k,\psi_k)\to(\vartheta_{k+1},\psi_{k+1})$ is given by

$$\Psi_k=\psi_k-\epsilon\frac{\partial S^{(k)}}{\partial\vartheta_k},\qquad \Theta_k=\vartheta_k+\epsilon\frac{\partial S^{(k)}}{\partial\Psi_k},\tag{6}$$

$$\Psi_{k+1}=\Psi_k,\qquad \bar{\Theta}_k=\Theta_k+\frac{\varphi_{k+1}-\varphi_k}{q(\Psi_k)},\tag{7}$$

$$\psi_{k+1}=\Psi_{k+1}+\epsilon\frac{\partial S^{(k+1)}}{\partial\vartheta_{k+1}},\ \vartheta_{k+1}=\bar{\Theta}_k-\epsilon\frac{\partial S^{(k+1)}}{\partial\Psi_{k+1}},\tag{8}$$

where $S^{(k)}\equiv S(\vartheta_k,\Psi_k)$ is the value of the generating function $S(\vartheta,\Psi,\varphi;\epsilon)$ at the section φ_k. In the first order of ϵ we have

$$\begin{aligned}S(\vartheta,\Psi,\varphi,\varphi_0)=&-(\varphi-\varphi_0)\sum_{m,n}H_{mn}(\Psi)\\&\big[a(x_{mn})\sin\left(m\vartheta-n\varphi+\chi_{mn}\right)\\&+b(x_{mn})\cos\left(m\vartheta-n\varphi+\chi_{mn}\right)\big],\end{aligned}\tag{9}$$

where $\varphi_0=(\varphi_{k+1}+\varphi_k)/2$, and

$$a(x)=\frac{1-\cos x}{x},\qquad b(x)=\frac{\sin x}{x}.\tag{10}$$

The backward mapping $(\vartheta_k,\psi_k)\to(\vartheta_{k-1},\psi_{k-1})$ can be obtained from Eqs.(6)–(8) by simple changing the sign of $\Delta\varphi$.

The typical Poincaré section of magnetic field lines at the plasma edge is shown in Fig. 1. The parameters of the model correspond to the ergodic divertor of the Tore Supra tokamak.

D. Particle diffusion in collisionless plasmas

The charged particles predominantly follow magnetic field lines and the radial transport of electrons across magnetic surfaces is determined by the radial deviation of field lines from the magnetic surfaces $r=$const (or $\psi=$const). The diffusion coefficient of particles is defined as $D_\perp=\langle(\Delta r)^2\rangle/2\Delta t$, where Δr is a random radial advance of particle during time period Δt, and $\langle(\cdots)\rangle$ stands for averaging over magnetic surface. Suppose that v_T is the thermal velocity of particles. Particles make a full toroidal turn in time $\Delta t=\Delta l/v_T$ where $\Delta l=2\pi R_0$ is the connection length of field line. Therefore we have

$$D_r=\frac{\langle(\Delta r)^2\rangle}{2\Delta t}=\frac{\Delta l}{\Delta t}\frac{\langle(\Delta r)^2\rangle}{2\Delta l}=D_{FL}v_T,\tag{11}$$

where

$$D_{FL}=\frac{\langle(\Delta r)^2\rangle}{2\Delta l},\tag{12}$$

is the diffusion coefficients of magnetic field lines. It means that in the collisionless plasma the diffusion of charged particles are mainly determined by the radial diffusion of field lines.

The diffusion coefficient D_{FL} can be expressed through the corresponding diffusion coefficients $D(\psi)$ in the magnetic flux coordinate ψ,

$$D(\psi)=\langle(\Delta\psi)^2\rangle/2\Delta\varphi.\tag{13}$$

To do this one should use the relation between the radial coordinate r and the toroidal flux ψ. Then

$$D_{FL}=\left(R_0^3/r^2\right)D(\psi).\tag{14}$$

In the quasilinear approximation one can obtain,

$$D^{(Q)}(\psi)=\frac{\epsilon^2n^2q(\psi)}{8\pi}|R_n^{(reg)}(\psi)|^2.\tag{15}$$

Here $R_n(\psi)$ is the Poincaré integrals $R_n(\psi)$ defined as $R_n(\psi)=2\pi q(\psi)H_{mn}(\psi)\Big|_{m=nq(\psi)}$, is given by

$$\begin{aligned}R_n^{(reg)}(\psi)&=\frac{2\pi C_m}{n}\left(\frac{r}{r_0}\right)^{nq(\psi)/\gamma}\frac{\sin y}{y},\\ y&=[nq(\psi)/\gamma-m_0]\theta_c.\end{aligned}\tag{16}$$

Figure 2 shows the typical dependence of the second moments of the radial displacement, $\langle\Delta r\rangle$, $\langle(\Delta r)^2\rangle$ on the toroidal angle φ. At the same figure the dependence of the phase correlation function $G_m(\varphi)$ on the toroidal angle φ for the central mode $m = m_0^* = 18$ is shown [right–hand axis]. The latter is defined as

$$G_m(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} e^{im(\vartheta(\vartheta_0,\varphi)-\vartheta_0)}d\vartheta_0, \qquad (17)$$

where $\vartheta(\vartheta_0, \varphi)$ is the angle variable of the field line with the initial condition $\vartheta(\vartheta_0, \varphi = 0) = \vartheta_0$.

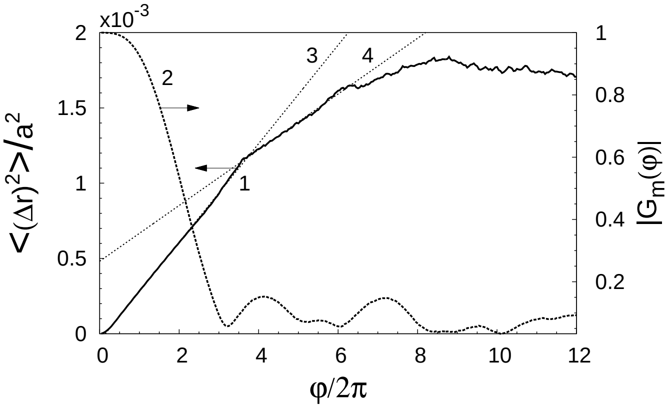

FIG. 2: Typical dependence of the second moment $\langle(\Delta r)^2\rangle$ of the radial displacement Δr and the correlation function $G_m(\varphi)$ (17) [right–hand axis] along the toroidal angle φ. Field lines are launched from the magnetic surface $\psi = 0.85\psi_a$. The toroidal mode $n = 6$, the perturbation parameter $\epsilon = 10^{-3}$, and other parameters are the same as Fig. 1. Curve 1 corresponds to the numerical calculations, curve 2 describes the correlation function $G_m(\varphi)$ for the mode $m = 18$, and curve 3 describes the linear function $f(\varphi) = d_0 + 2D\varphi$ with the coefficients d_0, D, obtained by fitting to the second moment $\langle(\Delta r)^2\rangle$ at the initial linearly growing range of φ, and curve 4 is a linear function $f_c(\varphi) = d_1 + 2D_c\varphi$ with the coefficients d_1, D_c obtained by fitting in the second linearly growing range of $\langle(\Delta r)^2\rangle$.

As seen from Fig. 2 the moment $\langle\Delta r\rangle$, $\langle(\Delta r)^2\rangle$ grow linearly at the initial interval of φ, $0 < \varphi < \varphi_1$. The upper limit φ_1 slightly larger than the decay length φ_c of the correlation function $G_m(\varphi)$. There is the second interval of φ, $\varphi_1 < \varphi < \varphi_2$, with the linear dependencies of the moments on φ. For the values $\varphi > \varphi_2$ the second moment $\langle(\Delta r)^2\rangle$ reaches its saturated value and does not grow anymore.

The diffusion coefficients of field lines D_{FL} are determined at the first interval $0 < \varphi < \varphi_1$. Figure 3 shows the dependence of these coefficients on the normalized toroidal flux coordinate ψ/ψ_a. The dotted curves 1−3 present the numerical calculations for the three different perturbation amplitudes $\epsilon = 5\times10^{-4}, 10^{-3}, 2\times10^{-3}$, respectively. The solid curves 4−6 present the corresponding dependencies given by the quasilinear formula (14), (15).

As seen from Fig. 3 the numerically calculated diffusion coefficients closely follow the quasilinear approximation (15) at the certain radial intervals $\psi_1 < \psi_t < \psi_{lam}$.

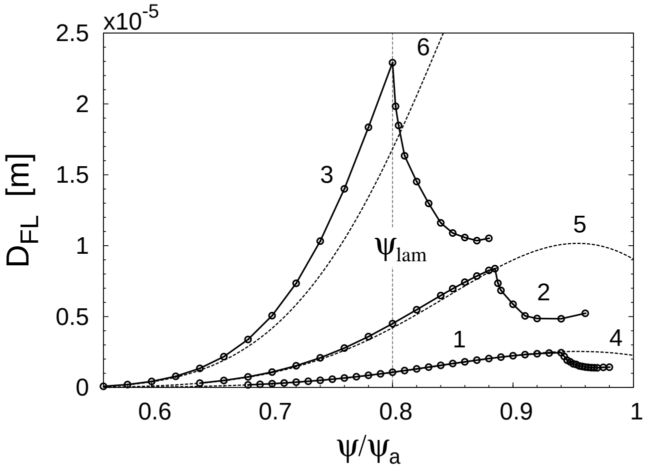

FIG. 3: Radial profiles of local field line diffusion coefficients D_{FL}. The toroidal mode $n = 6$, perturbation parameter $\epsilon = 5\times10^{-4}$ (curves 1 and 4), $\epsilon = 10^{-3}$ (curves 2 and 5), $\epsilon = 2\times10^{-3}$ (curves 3 and 6). Other parameters are the same as in Fig. 1. Curves 1-3 correspond to the numerical calculations, curves 4-6 − to the quasilinear values calculated by Eqs. (14), (15).

For the large perturbation ϵ the numerical values of D_{FL} start to deviate from the quasilinear formula (see curve 3 and 6 in Fig. 3 (b)). However, for $\psi_t > \psi_{lam}$ the numerical coefficients abruptly decay and completely deviate from the quasilinear formula. The area of the stochastic layer $\psi_t > \psi_{lam}$ can define the quantitative measure of the laminar zone where field lines leave the stochastic layer after a small number of poloidal turns (see, e.g., [31]). The lower boundary of this zone ψ_{lam} depends on the perturbation amplitude ϵ: it decreases with increasing ϵ.

One should note that the diffusion coefficient determined in the second linear interval $\varphi_1 < \varphi < \varphi_2$ changes irregular for larger ϵ although it coincides with D_{FL} at small ϵ.

In Table I we have listed the values of the connection length l_c, the Kolmogorov length L_K, and the field line diffusion coefficient D_{FL} at the three different magnetic surfaces. The Kolmogorov length L_K is found through the averaged Lyapunov exponents calculated directly using the procedure described in [27].

ψ/ψ_a	$l_c = \pi q(\psi)R_0$, m	L_K, m	D_{FL}, m
0.72	19.5002	78.4031	1.62534e-06
0.79	21.3059	73.2367	3.04457e-06
0.85	22.8911	68.3215	7.02784e-06

TABLE I: The values of the connection length l_c, the Kolmogorov length L_K, and the field line diffusion coefficient D_{FL} at the three different magnetic surfaces. The perturbation parameter $\epsilon = 10^{-3}$.

E. Random walk model for particle collisions

In order to introduce particle collisions we consider the following simple model of random walk of test particles. Let y be a coordinate along the magnetic field line, while x, ($x \approx r$) be a coordinate perpendicular to the field lines. We assume that a particle motion is a random walk process: it moves freely along the field line with a step l_s. After each step it collides with other particle with the probability p, $0 < p < 1$. After the collision the particle changes its direction to the opposite one simultaneously being displaced on the distance $\pm\rho$ along the $x-$ axis, i.e., across the field lines.

This random walk model describes the diffusion process with the following parallel $\chi_\parallel$ and the perpendicular $\chi_\perp$ diffusion coefficients [see, e.g., [6]]

$$\chi_\parallel = l_s v_\parallel \frac{1-p}{2p}, \qquad \chi_\perp = \frac{\rho^2 v_\parallel}{2l_s} p, \tag{18}$$

where $v_\parallel$ is a parallel velocity of a particle.

III. SIMULATION OF A PARTICLE DIFFUSION

For the numerical simulations of particle transport with collisions we need the two parameters, the probability of collision p and the perpendicular displacement ρ. They should be determined by the parallel and the perpendicular diffusion coefficients $\chi_\parallel$ and $\chi_\perp$. The first one $\chi_\parallel$ is given by the mean free path of particles, electrons and ions $\lambda_e \approx \lambda_{mfp}$, i.e., $\chi_\parallel = \lambda_{mfp} v_\parallel/2$. From Eq. (18) it follows that the probability p should be taken equal to $l_s/(l_s + \lambda_{mfp})$ and the perpendicular displacement ρ is given by $\rho = \sqrt{2\chi_\perp/pv_\parallel}$.

To be specific we consider the transport of H^+ ion (protons) in the stochastic magnetic field. We calculate the local diffusion coefficient D of particles near the given magnetic surface as functions of the two parameters, the mean–free path length λ_{mfp} and the perpendicular diffusion coefficient $\chi_\perp$. Below we use the following formula $\lambda_{mfp} = v_{T_i}\tau_i = 8.5 \times 10^{21}(T_i^2/n)$ m, where T_i is the ion temperature in keV, n is a plasma density in the unit m^{-3}, and the Coulomb logarithm is taken equal to $\ln\Lambda = 17$ (see [34, 35]). For simplicity the parallel velocity $v_\parallel$ is taken equal to the thermal velocity $v_T = (kT_i/m_i)^{1/2} = 3.09 \times 10^5 T_i^{1/2}$ m/s, where k is the Boltzmann's constant.

The calculations are made by running the randomly walking particles with collisions along the field lines using the forward and backward mappings (6)- (8). Figure 4 (a) and (b) show the typical dependencies of the second moment of radial displacement $\langle(\Delta r)^2\rangle$ the phase correlation function $G_m(t) \equiv G_m(\varphi(t))$ (17) on the time t for the two different ion temperatures: $T_i = 50$ eV (curve 1) and 700 eV (curve 2). The corresponding mean–free paths are λ_{mfp} =1.06 m and 208.25 m, respectively. The particles are launched from the magnetic surface $\psi = 0.72\psi_a$.

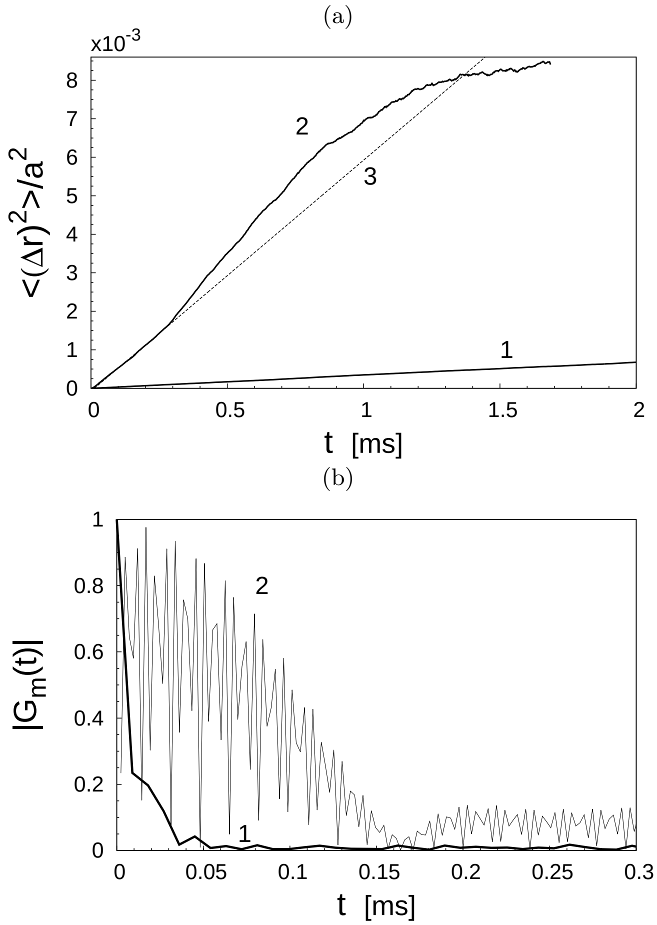

FIG. 4: (a) Typical time–dependencies of the second moments of radial displacements $\langle(\Delta r)^2\rangle$: curve 1 corresponds to the particles of thermal energy $T_i = 50$ eV, curve 2 corresponds – the one of $T_i = 700$ eV, curve 3 corresponds to the straight line obtained by fitting the curve 2 at the initial linear growth interval. The perpendicular diffusion $\chi_\perp = 0.1$ m^2/s and the perturbation amplitude $\epsilon = 10^{-3}$. Averaging is done over $N = 10^4$ orbits. (b) The same as in (a) but for the correlation function $G_m(t)$ with (17) with $m = 18$.

The perpendicular diffusion coefficient $\chi_\perp$ is taken equal to 0.1 m^2/s for the both cases.

The local diffusion coefficient D_r is found by fitting the dependence of the second moment $\langle(\Delta r)^2\rangle$ on the time t with the linear function $2D_r t + a$ at the initial linear growth range of t. At this range is taken as $t > t_c$ where t_c is the decorrelation time of phases. The latter is equal to the decay time of the the phase correlation function $G_m(t)$. As seen from Fig. 4 (b) $t_c \approx 0.04$ ms for $T_i = 50$ eV (curve 1) and $t_c \approx 0.15$ ms for $T_i = 700$ eV (curve 2). The example of a fitting is shown by curve 3 in Fig. 4. We have performed a large number of numerical calculations of the diffusion coefficient D for the different values of $\chi_\perp$, λ_e, and the perturbation amplitude ϵ. The typical dependencies of D_r on the mean free path λ_{mfp} are shown in Fig. 5. There the local diffusion coefficients D_r at the given magnetic surfaces ψ_t are plotted as functions of the mean free path λ_{mfp}: symbols

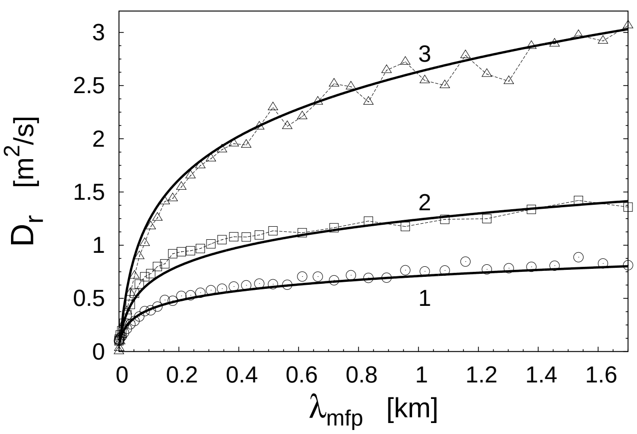

FIG. 5: Dependencies of the radial diffusion coefficients D_r on the mean free path length λ_{mfp}. Curve 1 and symbol $\odot$ correspond to $\chi_\perp = 0.1$ m^2/s at the magnetic surface $\psi_t = 0.72\psi_a$; curve 2 and $\Box$ correspond to $\chi_\perp = 10^{-3}$ m^2/s at $\psi_t = 0.79\psi_a$; curve 3 and $\triangle$ correspond to $\chi_\perp = 10^{-3}$ m^2/s at $\psi_t = 0.85\psi_a$. Solid curves 1, 2, and 3 describe to the empirical formula (20) with the corresponding diffusion coefficients of particles, $\chi_\perp$, and field lines, D_{FL}, and the characteristic lengths $L_c = l_c = \pi R_0 q(\psi_t)$, respectively. The perturbation amplitude $\epsilon = 10^{-3}$.

$\odot$, $\Box$, and $\triangle$ correspond to the diffusion coefficients at the magnetic surfaces $\psi_t = 0.72\psi_a$, $\psi_t = 0.79\psi_a$, and $\psi_t = 0.85\psi_a$, respectively. The corresponding values of the perpendicular diffusion coefficient $\chi_\perp$ are fixed and taken as 0.1m^2/s, 10^{-3} m^2/s, 0 m^2/s, respectively.

The numerical simulations reveal the following facts. For the value of the mean free paths λ_e shorter than the connection length $l_c = \pi R_0 q(\psi)$ the diffusion caused by the chaotic field lines is damped and the diffusion coefficient D tends to the perpendicular diffusion $\chi_\perp$ at $\lambda_e \ll l_c$. With the increasing λ_{mfp} the coefficient D_r monotonically grows and at $\lambda_{mfp} \gg l_c$ it approaches the value $v_T D_{FL}$ of the collisionless case. On the other hand D_r grows linearly with $\chi_\perp$ independently on the mean free path λ_{mfp}.

IV. EMPIRICAL FORMULA FOR THE DIFFUSION COEFFICIENT D_r IN A STOCHASTIC FIELD

Using the above facts from the numerical simulations and the general arguments one can propose an empirical formula for the local diffusion coefficient D of particles in a stochastic magnetic field accounting the collision of particles. We present the increment of a particle's radial coordinate, $\Delta r(t)$ in time as a sum of the two random processes,

$$\Delta r(t) = \Delta r_\perp(t) + \Delta r_\parallel(t),$$

corresponding to the radial increments due to the random kicks $\pm\rho$ at collisions, $\Delta r_\perp(t)$, and the motion along the chaotic field lines, $\Delta r_\parallel(t)$. The latter, however, is modulated by the random changes of the motion direction with the mean free path λ_{mfp}. At small $\rho \ll a$ the functions $\Delta r_\perp(t)$ and $\Delta r_\parallel(t)$ can be considered as independent random processes with $\langle \Delta r_\perp(t)\Delta r_\parallel(t)\rangle = 0$. Then we have

$$\begin{aligned}\langle(\Delta r(t))^2\rangle &= \langle(\Delta r_\perp(t))^2\rangle + \langle(\Delta r_\parallel(t))^2\rangle \\ &= 2\chi_\perp t + 2D_\parallel t = 2D_r t,\end{aligned} \tag{19}$$

where $D_\parallel$ is the diffusion coefficient of the modulated parallel motion along the chaotic field lines.

The dependence of the coefficient $D_\parallel$ on the mean free path λ_{mfp} can be established using the following arguments. At the values of λ_{mfp} much shorter than the characteristic length L_c which is of order of the connection length $l_c = \pi R_0 q(\psi)$, $\lambda_{mfp} \ll L_c$, the particle moves randomly forth and back along chaotic field lines so often that it is not affected strongly by the stochastic field. Therefore, one expects that $D_\parallel \to 0$ at the limit $\lambda_{mfp} \to 0$. On the other hand at the long mean free paths $\lambda_{mfp} \gg L_c$ the coefficient $D_\parallel$ should approach to $v_\parallel D_{FL}$ in the collisionless regime, i.e., $D_\parallel \to v_\parallel D_{FL}$ at the limit $\lambda_{mfp} \to \infty$.

The simplest formula satisfying these conditions can be written as

$$D_r = \chi_\perp + \frac{v_\parallel D_{FL}}{1 + L_c/\lambda_{mfp}}. \tag{20}$$

This empirical formula describes the local diffusion coefficient of particles D_r in a collisional plasma perturbed by the stochastic magnetic field. It is determined only by a few parameters: the local diffusion of field lines D_{FL}, the characteristic length L_c, ($L_c \sim l_c = \pi R_0 q(\psi)$), and the perpendicular diffusion coefficient $\chi_\perp$, and the local mean free path λ_{mfp}. The latter in turn depends on the local temperature T_i and the density n of the plasma, as $\lambda_{mfp} \propto T_i^2/n$.

The comparison of this formula with the numerically calculated coefficients is shown in Fig. 5 at the three different magnetic surfaces. Solid curves 1-3 correspond to the analytical formula (20) for D_r with the corresponding diffusion coefficients of field lines D_{FL} found numerically. The corresponding characteristic lengths L_c is taken equal to the connection lengths l_c. The values of L_c found by fitting the numerical data with the formula (20) are the following: $L_c = 19.8$ m at the magnetic surface $\psi = 0.72\psi_a$, $L_c = 15.0$ m at $\psi = 0.79\psi_a$, and $L_c = 33.2$ m at $\psi = 0.85\psi_a$. These values are of order of the connection length l_c but they are smaller that the Kolmogorov length L_K by a factor between 2.5 and 4 (see Table I). Overall the empirical formula (20) for the diffusion coefficient D_r satisfactorily describes the dependence of the numerically diffusion coefficients of the mean–free path λ_{mfp}.

A. Limiting cases

The empirical formula (20) formally coincides with the diffusion coefficient derived in [7] in the long-mean–free–path regime $D_L = v_T D_{FL}/(1 + l_{c\delta}/\lambda_{mfp})$ if L_c is replaced by the length $l_{c\delta} = (1/2)L_K \ln|\chi_\| \chi_\perp v_\|^2/l_{cr}^2|$. However, L_c practically does not depend on the perpendicular diffusion coefficient $\chi_\perp$, while $l_{c\delta}$ has a singularity at $\chi_\perp = 0$.

In the limit $\lambda_{mfp} \ll L_c$ the formula (20) can be approximated by $D_r \approx v_\| D_{FL}\lambda_{mfp}/L_c$ which coincides with the Rechester–Rosenbluth scaling $D_r \approx v_\| D_{FL}\lambda_{mfp}/L_{K\delta}$ [1].

Finally, the Kadomtsev–Pogutse regime was not recovered from the numerical simulations. Probably, this regime is not valid for the typical tokamak plasmas.

B. Radial profiles of diffusion coefficients

Below we present the radial profiles of the diffusion D_r at the plasma edge for the inhomogeneous temperature and density profiles. To take into account the ambipolarity effect we replace the parallel velocity $v_\|$ in Eq. (20) by the speed of sound $c_s = \sqrt{k(T_e + \gamma_i T_i)/m_k}$, where T_e is the electron temperature and γ_i is the adiabatic index. For simplicity we put $T_e = T_i$ and $\gamma_i = 3$. We choose the following radial profiles of the plasma temperature $T_i(r)$

$$T_i(r) = T_0 \begin{cases} \exp\left(-r^2/a_T^2\right), & \text{for } r < a_T, \\ e^{-1}(a-r)/\left(a - a_T\right), & \text{for } a_T < r < a_T, \end{cases} \tag{21}$$

and the plasma density $n(r)$

$$n(r) = \frac{n_0}{\tanh\left[\left(a_N - r\right)/\Delta_N\right]}. \tag{22}$$

Here a_T, a_N, Δ_N are the parameters.

The radial profiles of the diffusion coefficients D_r to these temperature and density profiles are plotted in Fig. 6 for a several values of the plasma density n_0 at the plasma center. As seen from Fig. 6 the collisions reduce the particle diffusion in a stochastic magnetic field. Particularly, the diffusion is decreased with increasing the plasma density, i.e., with the increase of the collisionality.

The decrease of the particle diffusion in a stochastic magnetic field with the increasing of the plasma density is in an agreement with the experimental observations in the Tore Supra and the TEXTOR tokamaks [12, 36] and in the DIII-D tokamak [37, 38]. In the latter experiments it has been found that the pump–out effect of the electron density at the plasma edge during the operation with the applied magnetic perturbations depends on the plasma collisionality. Namely, the outward particle flux is decreased with increasing the plasma density.

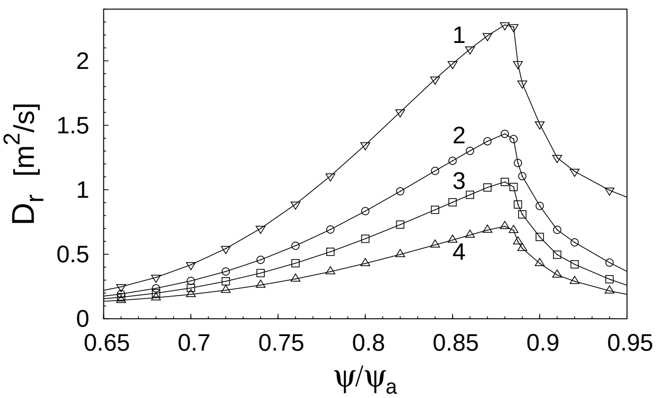

FIG. 6: Radial profiles of the radial diffusion coefficient D for the different density profiles of the plasma: curve 1 with a symbol $\triangledown$ corresponds to the collisionless case; curve 2 ($\odot$) corresponds to $n_0 = 2\times10^{19}$ m^{-3}, curve 3 ($\Box$) – $n_0 = 4\times10^{19}$ m^{-3}, curve 4 ($\triangle$) – $n_0 = 8 \times 10^{19}$ m^{-3}. The perturbation amplitude $\epsilon = 10^{-3}$ and $\chi_\perp = 0.01$ m^2/s, the temperature $T_0 = 0.5$ keV .

V. SUMMARY AND CONCLUSIONS

We have studied the collisional effects on the particle transport in a stochastic magnetic field. For this purpose the realistic model of magnetic field perturbations in tokamaks created by the external coils is proposed. The random walk model is used to simulate the particle collisions with the given parallel and the perpendicular to field lines diffusion coefficients. The forward and backward mappings of field lines constructed from the Hamiltonian equations are employed to run the test particles along the field lines and to calculate the second moments of radial displacements. The local radial diffusion coefficients of particles due to the chaotic field lines are calculated at the different magnetic surfaces. The dependencies of the diffusion coefficients on the mean–free path are obtained.

The empirical formula for the radial diffusion coefficients of particles in stochastic magnetic field is proposed. It depends on a few parameters of the plasma and the stochastic magnetic field: the local diffusion coefficient of magnetic field lines D_{FL}, the connection length of field lines $l_c = \pi R_0 q(\psi)$, the mean–free path λ_{mfp}, and the perpendicular diffusion coefficient $\chi_\perp$. The empirical formula describes well the numerically found dependencies of the diffusion coefficients on the mean–free path at the different magnetic surfaces.

The empirical formula coincides with the scaling obtained by [7] in the so–called long mean–free path regime. In the limit of the long mean–free paths, $\lambda_{mfp} \gg l_c$, it gives the diffusion coefficient in the collisionless regime. At the small values of the mean–free path, $\lambda_{mfp} \ll l_c$, the Rechester–Rosenbluth scaling [1] is recovered.

One can expect that this formula is generic for the collisional transport in a stochastic magnetic field created by typical RMPs in tokamaks. Particularly, it qualitatively explains the dependence of the pump–out effect on the plasma density experimentally observed in the Tore Supra, the TEXTOR and in the DIII-D tokamaks [12, 36, 37] during the operations with the externally applied RMPs.

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