Introduction to Plasma Physics
CERN School on Plasma Wave Acceleration

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Outline

- Lecture 1: Introduction – Definitions and Concepts
- Lecture 2: Wave Propagation in Plasmas
What is a plasma?

Simple definition: a *quasi-neutral* gas of charged particles showing *collective behaviour*.

**Quasi-neutrality:** number densities of electrons, \( n_e \), and ions, \( n_i \), with charge state \( Z \) are *locally balanced*:

\[
  n_e \simeq Z n_i. \tag{1}
\]

**Collective behaviour:** long range of Coulomb potential \( (1/r) \) leads to nonlocal influence of disturbances in equilibrium.

Macroscopic fields usually dominate over microscopic fluctuations, e.g.:

\[
  \rho = e(Z n_i - n_e) \Rightarrow \nabla \cdot \mathbf{E} = \rho/\varepsilon_0
\]
Where are plasmas found?

1. cosmos (99% of visible universe):
   - interstellar medium (ISM)
   - stars
   - jets

2. ionosphere:
   - \( \leq 50 \text{ km} = 10 \text{ Earth-radii} \)
   - long-wave radio

3. Earth:
   - fusion devices
   - street lighting
   - plasma torches
   - discharges - lightning
   - plasma accelerators!

Plasma properties

<table>
<thead>
<tr>
<th>Type</th>
<th>Electron density ( n_e (\text{ cm}^{-3}) )</th>
<th>Temperature ( T_e (\text{eV}^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stars</td>
<td>( 10^{26} )</td>
<td>( 2 \times 10^3 )</td>
</tr>
<tr>
<td>Laser fusion</td>
<td>( 10^{25} )</td>
<td>( 3 \times 10^3 )</td>
</tr>
<tr>
<td>Magnetic fusion</td>
<td>( 10^{15} )</td>
<td>( 10^3 )</td>
</tr>
<tr>
<td>Laser-produced</td>
<td>( 10^{18} - 10^{24} )</td>
<td>( 10^2 - 10^3 )</td>
</tr>
<tr>
<td>Discharges</td>
<td>( 10^{12} )</td>
<td>1-10</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>( 10^6 )</td>
<td>0.1</td>
</tr>
<tr>
<td>ISM</td>
<td>1</td>
<td>( 10^{-2} )</td>
</tr>
</tbody>
</table>

*1eV \equiv 11600K*

Table 1: Densities and temperatures of various plasma types
Debye shielding

What is the potential $\phi(r)$ of an ion (or positively charged sphere) immersed in a plasma?

Debye shielding (2): ions vs electrons

For equal ion and electron temperatures ($T_e = T_i$), we have:

$$\frac{1}{2} m_e v_e^2 = \frac{1}{2} m_i v_i^2 = \frac{3}{2} k_B T_e$$  \hspace{1cm} (2)

Therefore,

$$\frac{v_i}{v_e} = \left( \frac{m_e}{m_i} \right)^{1/2} = \left( \frac{m_e}{A m_p} \right)^{1/2} = \frac{1}{43}$$  \hspace{1cm} (hydrogen, $Z=A=1$)

Ions are almost stationary on electron timescale!
To a good approximation, we can often write:

$$n_i \approx n_0,$$

where the material (eg gas) number density, $n_0 = N_A \rho_m / A$. 
Debye shielding (3)

In thermal equilibrium, the electron density follows a Boltzmann distribution:

\[ n_e = n_i \exp\left(\frac{e\phi}{k_B T_e}\right) \quad (3) \]

where \( n_i \) is the ion density and \( k_B \) is the Boltzmann constant.

From Gauss’ law (Poisson’s equation):

\[ \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0} (n_i - n_e) \quad (4) \]

* See, eg: F. F. Chen, p. 9

Debye shielding (4)

Combining (4) with (3) in spherical geometry and requiring \( \phi \to 0 \) at \( r = \infty \), get solution:

\[ \phi_D = \frac{1}{4\pi \varepsilon_0} \frac{e^{-r/\lambda_D}}{r}. \quad (5) \]

Exercise

Debye length

\[ \lambda_D = \left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e}\right)^{1/2} = 743 \left(\frac{T_e}{\text{eV}}\right)^{1/2} \left(\frac{n_e}{\text{cm}^{-3}}\right)^{-1/2} \text{ cm} \quad (6) \]
Debye sphere

An *ideal* plasma has many particles per Debye sphere:

\[ N_D \equiv n_e \frac{4\pi}{3} \lambda_D^3 \gg 1. \]  

\[ \Rightarrow \text{Prerequisite for collective behaviour.} \]

Alternatively, can define *plasma parameter*:

\[ g \equiv \frac{1}{n_e \lambda_D^3} \]

Classical plasma theory based on assumption that \( g \ll 1 \), which also implies dominance of collective effects over collisions between particles.

Collisions in plasmas

At the other extreme, where \( N_D \leq 1 \), screening effects are reduced and collisions will dominate the particle dynamics. A good measure of this is the *electron-ion collision rate*, given by:

\[ \nu_{ei} = \frac{\pi^{\frac{3}{2}} n_e Z e^4 \ln \Lambda}{2^\frac{1}{2} (4\pi \varepsilon_0)^2 m_e^2 v_{te}^3} \text{ s}^{-1} \]

\[ v_{te} \equiv \sqrt{k_B T_e / m_e} \] is the electron thermal velocity and \( \ln \Lambda \) is a slowly varying term (Coulomb logarithm) \( O(10 \sim 20) \).

Can show that

\[ \frac{\nu_{ei}}{\omega_p} \simeq \frac{Z \ln \Lambda}{10 N_D} ; \quad \text{with} \quad \ln \Lambda \simeq 9 N_D / Z \]
Plasma classification

Model hierarchy

1. First principles N-body molecular dynamics
2. Phase-space methods – Vlasov-Boltzmann
3. 2-fluid equations
4. Magnetohydrodynamics (single, magnetised fluid)

- Time-scales: $10^{-15} - 10^3$ s
- Length-scales: $10^{-9} - 10$ m
- Number of particles needed for first-principles modelling (1): $10^{21}$ (tokamak), $10^{20}$ (laser-heated solid)
Consider electron layer displaced from plasma slab by length $\delta$. This creates two 'capacitor' plates with surface charge $\sigma = \pm en_e\delta$, resulting in an electric field:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{en_e\delta}{\varepsilon_0}$$

The electron layer is accelerated back towards the slab by this restoring force according to:

$$m_e \frac{dv}{dt} = -m_e \frac{d^2\delta}{dt^2} = -eE = \frac{e^2n_e\delta}{\varepsilon_0}$$

Or:

$$\frac{d^2\delta}{dt^2} + \omega_p^2\delta = 0,$$

where

$$\omega_p \equiv \left(\frac{e^2n_e}{\varepsilon_0 m_e}\right)^{1/2} \approx 5.6 \times 10^4 \left(\frac{n_e}{\text{cm}^{-3}}\right)^{1/2} \text{s}^{-1}. \quad (8)$$
Response time to create Debye sheath

For a plasma with temperature $T_e$ (and thermal velocity $v_{te} \equiv \sqrt{k_B T_e / m_e}$), one can also define a characteristic response time to recover quasi-neutrality:

$$t_D \simeq \frac{\lambda_D}{v_{te}} = \left( \frac{\varepsilon_0 k_B T_e}{e^2 n_e} \cdot \frac{m}{k_B T_e} \right)^{1/2} = \omega_p^{-1}.$$ 

External fields: underdense vs. overdense

If the plasma response time is shorter than the period of a external electromagnetic field (such as a laser), then this radiation will be *shielded out*.

**Figure 1:** Underdense, $\omega > \omega_p$: plasma acts as nonlinear refractive medium

**Figure 2:** Overdense, $\omega < \omega_p$: plasma acts like mirror
The critical density

To make this more quantitative, consider ratio:

$$\frac{\omega_p^2}{\omega^2} = \frac{e^2 n_e}{\varepsilon_0 m_e} \cdot \frac{\lambda^2}{4\pi^2 c^2}.$$  

Setting this to unity defines the wavelength for which $n_e = n_c$, or the critical density

$$n_c \simeq 10^{21} \lambda^{-2} \text{ cm}^{-3} \quad (9)$$

above which radiation with wavelengths $\lambda > \lambda_\mu$ will be reflected. cf: radio waves from ionosphere.

Plasma creation: field ionization

At the Bohr radius

$$a_B = \frac{\hbar^2}{me^2} = 5.3 \times 10^{-9} \text{ cm},$$

the electric field strength is:

$$E_a = \frac{e}{4\pi \varepsilon_0 a_B^2} \simeq 5.1 \times 10^9 \text{ Vm}^{-1}. \quad (10)$$

This leads to the atomic intensity:

$$I_a = \frac{\varepsilon_0 c E_a^2}{2} \simeq 3.51 \times 10^{16} \text{ Wcm}^{-2}. \quad (11)$$

A laser intensity of $I_L > I_a$ will guarantee ionization for any target material, though in fact this can occur well below this threshold value (eg: $\sim 10^{14}$ Wcm$^{-2}$ for hydrogen) via multiphoton effects.
Ionized gases: when is plasma response important?

Simultaneous field ionization of many atoms produces a plasma with electron density $n_e$, temperature $T_e \sim 1 - 10$ eV. Collective effects important if

$$\omega_p \tau_{\text{interaction}} > 1$$

Example (Gas jet)

$\tau_{\text{int}} = 100$ fs, $n_e = 10^{17}$ cm$^{-3}$ $\rightarrow \omega_p \tau_{\text{int}} = 1.8$

Typical gas jets: $P \sim 1$ bar; $n_e = 10^{18} - 10^{19}$ cm$^{-3}$

Recall that from Eq.9, critical density for glass laser $n_c(1 \mu) = 10^{21}$ cm$^{-3}$. Gas-jet plasmas are therefore underdense, since $\omega^2 / \omega_p^2 = n_e / n_c \ll 1$.

Exploit plasma effects for: short-wavelength radiation; nonlinear refractive properties; high electric/magnetic fields; particle acceleration!

Relativistic field strengths

Classical equation of motion for an electron exposed to a linearly polarized laser field $E = \hat{y}E_0 \sin \omega t$:

$$\frac{dv}{dt} \approx -\frac{eE_0}{m_e} \sin \omega t$$

$$\rightarrow v = \frac{eE_0}{m_e \omega} \cos \omega t = v_{os} \cos \omega t$$

Dimensionless oscillation amplitude, or 'quiver' velocity:

$$a_0 \equiv \frac{v_{os}}{c} \equiv \frac{p_{os}}{m_e c} \equiv \frac{eE_0}{m_e \omega c}$$
Relativistic intensity

The laser intensity $I_L$ and wavelength $\lambda_L$ are related to $E_0$ and $\omega$ by:

\[ I_L = \frac{1}{2} \varepsilon_0 c E_0^2, \quad \lambda_L = \frac{2\pi c}{\omega} \]

Substituting these into (13) we find:

\[ a_0 \simeq 0.85 (I_{18} \lambda_{\mu}^2)^{1/2}, \quad (14) \]

where

\[ I_{18} = \frac{I_L}{10^{18} \text{ Wcm}^{-2}}; \quad \lambda_{\mu} = \frac{\lambda_L}{\mu \text{m}}. \]

Implies that for $I_L \geq 10^{18}$ Wcm$^{-2}$, $\lambda_L \simeq 1$ $\mu$m, we will have relativistic electron velocities, or $a_0 \sim 1$. 

Further reading

Lecture 2: Wave propagation in plasmas

Plasma oscillations

Transverse waves

Nonlinear wave propagation

Further reading

Formulary

Model hierarchy

1. First principles N-body molecular dynamics
2. Phase-space methods – Vlasov-Boltzmann
3. 2-fluid equations
4. Magnetohydrodynamics (single, magnetised fluid)
The 2-fluid model

Many plasma phenomena can be analysed by assuming that each charged particle component with density \( n_s \) and velocity \( u_s \) behaves in a fluid-like manner, interacting with other species (s) via the electric and magnetic fields. The rigorous way to derive the governing equations in this approximation is via kinetic theory, which is beyond the scope of this lecture.

We therefore begin with the 2-fluid equations for a plasma assumed to be:
- thermal: \( T_e > 0 \)
- collisionless: \( \nu_{ie} \approx 0 \)
- and non-relativistic: velocities \( u \ll c \).

\[ \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s u_s) = 0 \]  
\[ n_s m_s \frac{d u_s}{d t} = n_s q_s (E + u_s \times B) - \nabla P_s \]  
\[ \frac{d}{d t}(P_s n_s^{-\gamma_s}) = 0 \]

\( P_s \) is the thermal pressure of species \( s \); \( \gamma_s \) the specific heat ratio, or \((2 + N)/N\), where \( N \) is the number of degrees of freedom.
Continuity equation

The continuity equation (Eq. 15) tells us that (in the absence of ionization or recombination) the number of particles of each species is conserved.

Noting that the charge and current densities can be written \( \rho_s = q_s n_s \) and \( \mathbf{J}_{s} = q_s n_s \mathbf{u}_s \) respectively, Eq. (15) can also be written:

\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot \mathbf{J}_s = 0,
\]  

(18)

which expresses the conservation of charge.

Momentum equation

(Eq. 16) governs the motion of a fluid element of species \( s \) in the presence of electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \).

Remark: In the absence of fields, and assuming strict quasineutrality (\( n_e = Z n_i = n; \mathbf{u}_e = \mathbf{u}_i = \mathbf{u} \)), we recover the Navier-Stokes equations.

Exercise

In the plasma accelerator context we will usually deal with unmagnetised plasmas, and stationary ions \( \mathbf{u}_i = 0 \), in which case the momentum equation reads:

\[
n_e m_e \frac{d \mathbf{u}_e}{dt} = -n_e e \mathbf{E} - \nabla P_e
\]  

(19)

Note that \( \mathbf{E} \) can include both external and internal field components (via charge-separation).
Longitudinal plasma waves

A characteristic property of plasmas is their ability to transfer momentum and energy via collective motion. One of the most important examples of this is the oscillation of the electrons against a stationary ion background, or *Langmuir wave*. Returning to the 2-fluid model, we can simplify Eqs.(15-17) by setting $u_i = 0$, restricting the electron motion to one dimension ($x$) and taking $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$:

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0 \\
n_e \left( \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} \right) = -\frac{e}{m} n_e E - \frac{1}{m} \frac{\partial P_e}{\partial x} \\
\frac{d}{dt} \left( \frac{P_e}{n_e^{\gamma_e}} \right) = 0
\]  

(20)

Poisson's equation

The above system (20) has 3 equations and 4 unknowns.

To close it we need an expression for the electric field, which, since $B = 0$, can be found from Gauss’ law (Poisson’s equation) with $Zn_i = n_0 = \text{const}$:

\[
\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0} (n_0 - n_e)
\]  

(21)
Longitudinal plasma waves (3)

1D electron fluid equations

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} (n_e u_e) = 0
\]

\[
n_e \left( \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} \right) = -\frac{e}{m} n_e E - \frac{1}{m} \frac{\partial P_e}{\partial x}
\]  \hspace{1cm} (22)

\[
\frac{d}{dt} \left( \frac{P_e}{n_e^{1/2}} \right) = 0
\]

\[
\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0} (n_0 - n_e)
\]

Wave propagation in plasmas

Longitudinal plasma waves (4)

Linearization

This system is nonlinear, and apart from a few special cases, cannot be solved exactly. A common technique for analyzing waves in plasmas therefore is to \textit{linearize} the equations, assuming the perturbed amplitudes are small compared to the equilibrium values:

\[
n_e = n_0 + n_1,
\]

\[
u_e = u_1,
\]

\[
P_e = P_0 + P_1,
\]

\[
E = E_1,
\]

where \( n_1 \ll n_0, P_1 \ll P_0 \). These expressions are substituted into (22) and all products \( n_1 \partial_t u_1, u_1 \partial_x u_1 \) etc. are neglected to get a set of linear equations for the perturbed quantities...
Linearized equations

\[ \frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_1}{\partial x} = 0, \]

\[ n_0 \frac{\partial u_1}{\partial t} = -\frac{e}{m} n_0 E_1 - \frac{1}{m} \frac{\partial P_1}{\partial x}, \quad (23) \]

\[ \frac{\partial E_1}{\partial x} = -\frac{e}{\varepsilon_0} n_1, \]

\[ P_1 = 3k_B T_e n_1. \]

N.B. Expression for \( P_1 \) results from specific heat ratio \( \gamma_e = 3 \) and assuming isothermal background electrons, \( P_0 = k_B T_e n_0 \) (ideal gas) – see Kruer (1988).

Exercise

Wave equation

We can now eliminate \( E_1, P_1 \) and \( u_1 \) from (23) to get:

\[ \left( \frac{\partial^2}{\partial t^2} - 3v_{te}^2 \frac{\partial^2}{\partial x^2} + \omega_p^2 \right) n_1 = 0, \quad (24) \]

with \( v_{te}^2 = k_B T_e / m_e \) and \( \omega_p \) given by (8) as before.

Finally, we look for plane wave solutions of the form

\[ A = A_0 e^{i(\omega t - kx)}, \]

so that our derivative operators become:

\[ \frac{\partial}{\partial t} \rightarrow i\omega; \quad \frac{\partial}{\partial x} \rightarrow -ik. \]

Substitution into (24) yields finally:

Bohm-Gross dispersion relation for electron plasma waves

\[ \omega^2 = \omega_p^2 + 3k^2 v_{te}^2 \quad (25) \]
Electromagnetic waves

To describe transverse electromagnetic (EM) waves, we need two more of Maxwell's equations: Faraday's law (35) and Ampère's law (36), which we come to in their usual form later. To simplify things, taking our cue from the previous analysis of small-amplitude, longitudinal waves, we linearize and again apply the harmonic approximation $\frac{\partial}{\partial t} \rightarrow i\omega$:

$$\nabla \times E_1 = -i\omega B_1,$$  \hspace{1cm} (26)

$$\nabla \times B_1 = \mu_0 J_1 + i\varepsilon_0 \mu_0 \omega E_1,$$  \hspace{1cm} (27)

where the transverse current density is given by:

$$J_1 = -n_0 e u_1.$$ \hspace{1cm} (28)

Wave propagation in plasmas

Electromagnetic waves (2)

Ohm's law

We now look for pure EM plane-wave solutions with $E_1 \perp k$. Also note that the group and phase velocities $v_p, v_g \gg v_{te}$, so that we can assume a cold plasma with $P_e = n_0 k_B T_e = 0$.

The linearized electron fluid velocity and corresponding current are then:

$$u_1 = -\frac{e}{i\omega m_e}E_1,$$

$$J_1 = \frac{n_0 e^2}{i\omega m_e}E_1 \equiv \sigma E_1,$$ \hspace{1cm} (29)

where $\sigma$ is the AC electrical conductivity.
Electromagnetic waves (3)

Dielectric function

By analogy with dielectric media (see eg: Jackson), in which Ampere’s law is usually written \( \nabla \times \mathbf{B}_1 = \mu_0 \partial_t \mathbf{D}_1 \), by substituting (29) into (36), can show that

\[
\mathbf{D}_1 = \varepsilon_0 \varepsilon \mathbf{E}_1
\]

with

\[
\varepsilon = 1 + \frac{\sigma}{i\omega\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}.
\]  

(30)

Electromagnetic waves (4)

Refractive index

From (30) it follows immediately that:

**Refractive index**

\[
\eta \equiv \sqrt{\varepsilon} = \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}
\]

(31)

with

**Dispersion relation**

\[
\omega^2 = \omega_p^2 + c^2 k^2
\]

(32)

**Exercise**

The above expression can also be found directly by elimination of \( \mathbf{J}_1 \) and \( \mathbf{B}_1 \) from Eqs. (26)-(29).
Propagation characteristics
Underdense plasmas

From the dispersion relation (32) a number of important features of EM wave propagation in plasmas can be deduced.

For underdense plasmas \((n_e \ll n_c)\):

- Phase velocity: 
  \[ v_p = \frac{\omega}{k} \simeq c \left( 1 + \frac{\omega^2_p}{2\omega^2} \right) > c \]

- Group velocity: 
  \[ v_g = \frac{\partial \omega}{\partial k} \simeq c \left( 1 - \frac{\omega^2_p}{2\omega^2} \right) < c \]

Propagation characteristics (2)
Overdense plasmas

In the opposite case, \(n_e > n_c\), the refractive index \(\eta\) becomes imaginary, and the wave can no longer propagate, becoming evanescent instead, with a decay length determined by the collisionless skin depth \(c/\omega_p\).
Nonlinear wave propagation

The starting point for most analyses of nonlinear wave propagation phenomena is the Lorentz equation of motion for the electrons in a cold \( T_e = 0 \), unmagnetized plasma, together with Maxwell’s equations.

We also make two assumptions:

1. The ions are initially assumed to be singly charged \( Z = 1 \) and are treated as a immobile \( v_i = 0 \), homogeneous background with \( n_0 = Z n_i \).

2. Thermal motion is neglected – justified for underdense plasmas because the temperature remains small compared to the typical oscillation energy in the laser field \( v_{os} \gg v_{te} \).


**Lorentz-Maxwell equations**

Starting equations (SI units) are as follows

\[
\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p = -e(E + \mathbf{v} \times \mathbf{B}), \quad (33)
\]

\[
\nabla \cdot \mathbf{E} = \frac{e}{\varepsilon_0}(n_0 - n_e), \quad (34)
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (35)
\]

\[
c^2 \nabla \times \mathbf{B} = -\frac{e}{\varepsilon_0} n_e \mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}, \quad (36)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (37)
\]

where \( p = \gamma m_e \mathbf{v} \) and \( \gamma = (1 + p^2/m_e^2c^2)^{1/2} \).

---

**Electromagnetic waves**

To simplify matters we first assume a plane-wave geometry like that above, with the transverse electromagnetic fields given by \( \mathbf{E}_L = (0, E_y, 0); \mathbf{B}_L = (0, 0, B_z) \).

From Eq. (33) the transverse electron momentum is then simply given by:

\[
p_y = eA_y, \quad (38)
\]

where \( E_y = \partial A_y/\partial t \).

This relation expresses conservation of canonical momentum.
**The EM wave equation I**

Substitute \( E = -\nabla \varphi - \partial A / \partial t \); \( B = \nabla \times A \) into Ampère Eq.(36):

\[
c^2 \nabla \times (\nabla \times A) + \frac{\partial^2 A}{\partial t^2} = \frac{J}{\varepsilon_0} - \nabla \frac{\partial \varphi}{\partial t},
\]

where the current \( J = -en_e v \).

Now we use a bit of vectorial magic, splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts:

\[
J = J_\perp + J_\parallel = \nabla \times \Pi + \nabla \Psi
\]

from which we can deduce (see Jackson!):

\[
J_\parallel - \frac{1}{c^2} \nabla \frac{\partial \varphi}{\partial t} = 0.
\]

**The EM wave equation II**

Now apply Coulomb gauge \( \nabla \cdot A = 0 \) and \( v_y = eA_y / \gamma \) from (38), to finally get:

\[
\frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y. \tag{39}
\]

The nonlinear source term on the RHS contains two important bits of physics:

\[
n_e = n_0 + \delta n \rightarrow \text{Coupling to plasma waves}
\]

\[
\gamma = \sqrt{1 + p^2 / m_e c^2} \rightarrow \text{Relativistic effects}
\]
Electrostatic (Langmuir) waves I

Taking the longitudinal (x)-component of the momentum equation (33) gives:

\[
\frac{d}{dt}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e\gamma} \frac{\partial A_y^2}{\partial x}
\]

We can eliminate \(v_x\) using Ampère’s law (36)_x:

\[0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t},\]

while the electron density can be determined via Poisson’s equation (34):

\[n_e = n_0 - \frac{\varepsilon_0}{e} \frac{\partial E_x}{\partial x}.\]

Electrostatic (Langmuir) waves II

The above (closed) set of equations can in principle be solved numerically for arbitrary pump strengths. For the moment, we simplify things by linearizing the plasma fluid quantities:

\[n_e \approx n_0 + n_1 + ...\]
\[v_x \approx v_1 + v_2 + ...\]

and neglect products like \(n_1 v_1\) etc. This finally leads to:

\[
\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right)E_x = -\frac{\omega_p^2 e}{2m_e\gamma_0^2} \frac{\partial}{\partial x} A_y^2 \tag{40}
\]

The driving term on the RHS is the relativistic ponderomotive force, with \(\gamma_0 = (1 + a_0^2/2)^{1/2}\).
Cold plasma fluid equations: summary

**Electromagnetic wave**

\[ \frac{\partial^2 A_y}{\partial t^2} - c^2 \nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y \]

**Electrostatic (Langmuir) wave**

\[ \left( \frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0} \right) E_x = -\frac{\omega_p^2 e}{2m_e \gamma_0^2} \frac{\partial}{\partial x} A_y^2 \]

Cold plasma fluid equations: outlook

These coupled fluid equations and their fully non-linear variations describe a vast range of nonlinear laser-plasma interaction phenomena:

- plasma wake generation: Bingham, Assmann
- blow-out regime: Silva
- laser self-focussing and channelling: Najmudin, Cros
- parametric instabilities
- harmonic generation, ...

Plasma-accelerated particle beams, on the other hand, cannot be treated with fluid theory and require a more sophisticated kinetic approach. – see Pukhov
Further reading

1. J. Boyd and J. J. Sanderson, *The Physics of Plasmas*

 Constants

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<th>Name</th>
<th>Symbol</th>
<th>Value (SI)</th>
<th>Value (cgs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boltzmann constant</td>
<td>$k_B$</td>
<td>$1.38 \times 10^{-23}$ J K$^{-1}$</td>
<td>$1.38 \times 10^{-16}$ erg K$^{-1}$</td>
</tr>
<tr>
<td>Electron charge</td>
<td>$e$</td>
<td>$1.6 \times 10^{-19}$ C</td>
<td>$4.8 \times 10^{-10}$ statcoul</td>
</tr>
<tr>
<td>Electron mass</td>
<td>$m_e$</td>
<td>$9.1 \times 10^{-31}$ kg</td>
<td>$9.1 \times 10^{-28}$ g</td>
</tr>
<tr>
<td>Proton mass</td>
<td>$m_p$</td>
<td>$1.67 \times 10^{-27}$ kg</td>
<td>$1.67 \times 10^{-24}$ g</td>
</tr>
<tr>
<td>Planck constant</td>
<td>$h$</td>
<td>$6.63 \times 10^{-34}$ Js</td>
<td>$6.63 \times 10^{-27}$ erg-s</td>
</tr>
<tr>
<td>Speed of light</td>
<td>$c$</td>
<td>$3 \times 10^8$ ms$^{-1}$</td>
<td>$3 \times 10^{10}$ cms$^{-1}$</td>
</tr>
<tr>
<td>Dielectric constant</td>
<td>$\varepsilon_0$</td>
<td>$8.85 \times 10^{-12}$ Fm$^{-1}$</td>
<td>—</td>
</tr>
<tr>
<td>Permeability constant</td>
<td>$\mu_0$</td>
<td>$4\pi \times 10^{-7}$</td>
<td>—</td>
</tr>
<tr>
<td>Proton/electron mass ratio</td>
<td>$m_p/m_e$</td>
<td>1836</td>
<td>1836</td>
</tr>
<tr>
<td>Temperature = 1eV</td>
<td>$e/k_B$</td>
<td>11604 K</td>
<td>11604 K</td>
</tr>
<tr>
<td>Avogadro number</td>
<td>$N_A$</td>
<td>$6.02 \times 10^{23}$ mol$^{-1}$</td>
<td>$6.02 \times 10^{23}$ mol$^{-1}$</td>
</tr>
<tr>
<td>Atmospheric pressure</td>
<td>1 atm</td>
<td>$1.013 \times 10^5$ Pa</td>
<td>$1.013 \times 10^6$ dyne cm$^{-2}$</td>
</tr>
</tbody>
</table>
## Formulae

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Formula (SI)</th>
<th>Formula (cgs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debye length</td>
<td>$\lambda_D$</td>
<td>$\left( \frac{e^0 k_B T_e}{e^2 n_e} \right)^{\frac{1}{2}} \text{ m}$</td>
<td>$\left( \frac{k_B T_e}{4 \pi e^2 n_e} \right)^{\frac{1}{2}} \text{ cm}$</td>
</tr>
<tr>
<td>Particles in Debye sphere</td>
<td>$N_D$</td>
<td>$\frac{4 \pi}{3} \lambda_D^3$</td>
<td>$\frac{4 \pi}{3} \lambda_D^3$</td>
</tr>
<tr>
<td>Plasma frequency (electrons)</td>
<td>$\omega_{pe}$</td>
<td>$\left( \frac{e^2 n_e}{e^0 m_e} \right)^{\frac{1}{2}} \text{ s}^{-1}$</td>
<td>$\left( \frac{4 \pi e^2 n_e}{m_e} \right)^{\frac{1}{2}} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Plasma frequency (ions)</td>
<td>$\omega_{pi}$</td>
<td>$\left( \frac{2^2 e^2 n_i}{e^0 m_i} \right)^{\frac{1}{2}} \text{ s}^{-1}$</td>
<td>$\left( \frac{4 \pi Z^2 e^2 n_i}{m_i} \right)^{\frac{1}{2}} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Thermal velocity</td>
<td>$v_{te} = \omega_{pe} \lambda_D$</td>
<td>$\left( \frac{k_B T_e}{m_e} \right)^{\frac{1}{2}} \text{ ms}^{-1}$</td>
<td>$\left( \frac{k_B T_e}{m_e} \right)^{\frac{1}{2}} \text{ cms}^{-1}$</td>
</tr>
<tr>
<td>Electron gyrofrequency</td>
<td>$\omega_c$</td>
<td>$\frac{eB}{m_e} \text{ s}^{-1}$</td>
<td>$\frac{eB}{m_e} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Electron-ion collision frequency</td>
<td>$\nu_{ei}$</td>
<td>$\frac{\pi}{2} \left( \frac{m_p}{m_i} \right)^{\frac{1}{2}} n_e Z e^4 \ln \Lambda_{\text{cm}} \text{ s}^{-1}$</td>
<td>$\frac{4(\pi n_e)}{3 m_p^2 v_{te}^3} \ln \Lambda_{\text{cm}} \text{ s}^{-1}$</td>
</tr>
<tr>
<td>Coulomb-logarithm</td>
<td>$\ln \Lambda$</td>
<td>$\ln \left( \frac{9 N_D}{Z} \right)$</td>
<td>$\ln \left( \frac{9 N_D}{Z} \right)$</td>
</tr>
</tbody>
</table>

### Useful formulae

Plasma frequency

$$\omega_{pe} = 5.64 \times 10^4 n_e^{\frac{3}{2}} \text{ s}^{-1}$$

Critical density

$$n_c = 10^{21} \lambda_L^{-2} \text{ cm}^{-3}$$

Debye length

$$\lambda_D = 743 T_e^{\frac{1}{2}} n_e^{-\frac{1}{2}} \text{ cm}$$

Skin depth

$$\delta = \frac{c}{\omega_p} = 5.31 \times 10^5 n_e^{-\frac{1}{2}} \text{ cm}$$

Elektron-ion collision frequency

$$\nu_{ei} = 2.9 \times 10^{-6} n_e T_e^{-\frac{3}{2}} \ln \Lambda \text{ s}^{-1}$$

Ion-ion collision frequency

$$\nu_{ij} = 4.8 \times 10^{-8} Z^4 \left( \frac{m_p}{m_i} \right)^{\frac{1}{2}} n_i T_i^{-\frac{3}{2}} \ln \Lambda \text{ s}^{-1}$$

Quiver amplitude

$$a_0 = \frac{P_{osc}}{m_e c} = \left( \frac{1.37 \times 10^{18} \text{ W cm}^{-2} \mu \text{ m}^2}{I \lambda_L^2} \right)^{\frac{1}{2}}$$

Relativistic focussing threshold

$$P_c = 17 \left( \frac{n_c}{n_e} \right) \text{ GW}$$

$T_e$ in eV; \( n_e, n_i \) in $\text{cm}^{-3}$, \( \lambda_L \) in $\mu\text{m}$
# Maxwell’s Equations

<table>
<thead>
<tr>
<th>Name</th>
<th>(SI)</th>
<th>(cgs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’ law</td>
<td>$\nabla \cdot E = \rho / \varepsilon_0$</td>
<td>$\nabla \cdot E = 4\pi \rho$</td>
</tr>
<tr>
<td>Gauss’ magnetism law</td>
<td>$\nabla \cdot B = 0$</td>
<td>$\nabla \cdot B = 0$</td>
</tr>
<tr>
<td>Ampère</td>
<td>$\nabla \times B = \mu_0 J + \frac{1}{c^2} \frac{\partial E}{\partial t}$</td>
<td>$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t}$</td>
</tr>
<tr>
<td>Faraday</td>
<td>$\nabla \times E = -\frac{\partial B}{\partial t}$</td>
<td>$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$</td>
</tr>
<tr>
<td>Lorentz force</td>
<td>$E + v \times B$</td>
<td>$E + \frac{1}{c} v \times B$</td>
</tr>
<tr>
<td>per unit charge</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Wave propagation in plasmas

Formulary