Drifts of electron orbits induced by toroidal electric field in tokamaks
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The drifts of electron orbits induced by the toroidal electric field in tokamaks are analyzed. Based on the relativistic Hamiltonian equations for guiding centre motion, the formula for the drift velocity $v_{dr}$ is derived. It describes the outward drift of passing particles as well as the inward drift (the Ware pinch) of trapped particles. Unlike the approximate formula for $v_{dr}$ given in Guan et al. [Phys. Plasmas 17, 092502 (2010)] for circular electron orbits, it describes qualitatively new features of the outward drift of electron orbits. Particularly, the new formula describes the evolution of the orbit’s shape, the formation of X-point and the associated separatrix. It is shown that the outward drift velocity is proportional to the inverse aspect ratio of tokamaks. [http://dx.doi.org/10.1063/1.4914935]
where \( u_\phi(x, z, \varphi, t, p_\varphi) = [\gamma_t^2 - 1] - 2 \omega_c I_s^{1/2}, \gamma_t = (-p_t - Z_q \phi)/\omega_0. \) In this case the Hamiltonian equations (1) can be rewritten as

\[
\frac{dz}{d\varphi} = -x \frac{\partial \psi}{\partial x} + \frac{\sigma \gamma_t x}{Z_q} \left( u_\phi + \frac{\omega_c I_s}{u_\phi} - \frac{x r_q Z_q \partial \psi}{u_\phi} \right),
\]

\[
\frac{dp_z}{d\varphi} = -Z_q \left( \frac{\partial \psi}{\partial z} \right) u_\phi - \frac{\sigma \gamma_t x}{Z_q} \left( \frac{\partial \psi}{\partial x} \right) \frac{dt}{d\varphi} \frac{\sigma \gamma_t x}{u_\phi},
\]

\[
\frac{dp_t}{d\varphi} = -Z_q \left( \frac{\partial \psi}{\partial t} \right) u_\phi - \frac{\sigma \gamma_t x}{Z_q} \left( \frac{\partial \psi}{\partial x} \right) \frac{dt}{d\varphi} \frac{\sigma \gamma_t x}{u_\phi}.
\]

Here, the parameter \( \sigma = \pm 1 \) stands for the direction of motion along the toroidal angle \( \varphi. \)

The system of Eq. (4) for the GC motion can be presented in a form similar to the equations for the magnetic field lines in the cylindrical coordinate system \((R, Z, \varphi).\) We consider the case when the electric field potential to vanish: \( \Phi \equiv 0.\) Using the definitions, \( B_R = B_\varphi \partial \psi/\partial z \) and \( B_Z = -B_\varphi x \partial \psi/\partial x \) for the poloidal components of the magnetic field \((B_\varphi = R_0 B_0/R \text{ is the toroidal magnetic field}),\) one can reduce the system of Eq. (4) to the form

\[
\frac{dZ}{d\varphi} = \frac{R B_Z^2}{B_\varphi}, \quad \frac{dR}{d\varphi} = \frac{R B_R}{B_\varphi}, \quad \frac{dt}{d\varphi} = \frac{\sigma R}{u_\phi}, \quad \frac{dH}{d\varphi} = -Z_q \frac{\partial (R A_{\phi})}{\partial t},
\]

where \( u_\phi = u_\phi R_0 / \gamma_t \), is the toroidal velocity, \( B_Z^2 = B_Z + \sigma B_\varphi / Z_q \left( u_\phi + \omega_c I_s / u_\phi \right).\)

Consider the dynamics of electrons in the presence of the toroidal electric field. The latter can be represented by means of the toroidal component of the vector potential \( A_{\phi}^{(ind)}(R, Z, \varphi) \)

\[
E_{\phi}(R, Z, \varphi) = \frac{1}{\omega_0} A_{\phi}^{(ind)}(R, Z, \varphi).
\]

For simplicity, we suppose that the electric field is determined by the loop voltage \( V: E_{\phi} = V/2\pi R_0.\)

The poloidal flux \( \psi \) in the Hamiltonian function (3) is given by

\[
\psi = \psi^{(0)}(x, z) + \psi^{(ind)}(x, z, \varphi, t, p_\varphi),
\]

\[
\psi^{(ind)}(x, z, \varphi, t, p_\varphi) = \frac{1}{R_0 B_0^2} \int E_{\phi}(t') dt',
\]

where \( E_{\phi}(t) \) is the normalized toroidal electric field

\[
E_{\phi}(t) = \frac{R E_\varphi(R, Z, t)}{B_0 R_0^2} = \frac{V}{2\pi R_0 B_0^2}.
\]

The variation of energy with time is given by

\[
\frac{dh}{dt} = \frac{\partial h}{\partial \psi} \frac{d\psi}{dt} = -Z_q \frac{\partial h}{\partial \psi} \frac{d\psi}{dt} = Z_q \frac{d\psi}{dt} \frac{\sigma q_{eff} E_{\phi}(t)}{x \gamma_t}.
\]

The energy grows if \( Z_q u_\phi E_{\phi}(t) > 0.\) Furthermore, we assume that the loop voltage \( V \) and thus \( E_{\phi}(t) \) are constants in the poloidal cross section. Then the increment of the particle energy in one poloidal turn is given by

\[
\Delta E = E_{ref} \Delta h = E_{ref} \int_{t}^{t+T} \frac{dh}{dt} dt = E_{ref} \sigma q_{eff} Z_q E_{\phi}(t) = \Delta E_{\phi}(t) = \sigma q_{eff} V^2,
\]

where \( q_{eff} \) is the effective safety factor defined as \( q_{eff} = |\Delta \varphi|/2\pi, \Delta \varphi \) is the increment of the toroidal angle \( \varphi \) in one poloidal turn, and \( T \) is the normalized transition time.

We now estimate the drift velocity \( v_{dr} \) of the RE orbit induced by the toroidal electric field. We assume axisymmetry, i.e., \( \psi = \psi(x, z, t) \) and \( u_\phi = u_\phi(x, z, p_\phi). \) The toroidal momentum \( p_\phi \) is then a constant of motion. According to (3), the drift surface at time \( t \) is determined by

\[
p_\phi = -Z_q \psi(x, z, t) + \sigma R u_\phi(x, z, p_\phi) = \text{const},
\]

where the poloidal flux \( \psi(x, z, t) \) is given by (8). According to the latter and (12), the poloidal flux \( \psi \) and the energy \( h = -p_\phi \) get, respectively, increments \( \Delta \psi = T Z_q E_{\phi} \) and \( \Delta h \) in one poloidal turn. Since the increment \( \Delta p_\phi = 0, \) the drift surface is shifted along the radial direction by the distance \( \Delta \lambda, \) the expression of which can be obtained from (13)

\[
\Delta \lambda = \frac{Z_q E_{\phi} (x \sigma q_{eff} / \gamma_t, u_\phi - T)}{\left[ Z_q \frac{\partial h}{\partial \psi} - \sigma [u_\phi + \omega_c I_s / u_\phi] \right]}. \]

The expression of the orbit’s drift velocity is obtained from (14)

\[
v_{dr} = \omega_0 \frac{\Delta R}{T} = \frac{R_0 E_{\phi}}{R Z_q^2} \left( 1 - \frac{RT_{av}}{R_0 T} \right),
\]

where \( T_{av} = \omega_0^{-1} T \) is the actual transition time, and \( \Delta R = R_0 \Delta \lambda \) is the radial shift of the orbit. The quantity,

\[
T_{av} = 2 \pi q_{eff} / \omega_0 \beta_0 u_\phi = \frac{2 \pi q_{eff} R_0}{v_\phi},
\]

is the average transition time.

Expression (15) is obtained by expansion of (13) with respect to the radial shift \( \Delta x \) in one poloidal turn, keeping only the first term. For realistic plasma parameters the shift \( \Delta x \) is extremely small (~10^-7) and the procedure is well justified. (That has been confirmed by numerical calculations.)

In particular, when \( T = T_{av} \) and for low-energy electrons, we have \( B_\varphi = B_Z \) and (15) can be reduced to the expression obtained in Refs. 2 and 3 for the circular GC orbits.
is much smaller than $R$ (see, also Ref. 6). The average transition time $T_{av}$ (16) and the transition time $T$. For a given orbit, the quantities $B_Z^2$ and $T_{av}$ depend on the radial position $R$ on the orbit, i.e., $v_{dr}$ is a local function of $R$. Furthermore, we consider the drift velocities $v_{dr}(R_i)$ and $v_{dr}(R_o)$ at the orbit’s two radial positions in the equatorial plane $Z = 0$, i.e., its innermost $R_i$ and outermost $R_o$ points. For REs, the outermost point drifts faster than the innermost point, i.e., $v_{dr}(R_o) > v_{dr}(R_i)$. This leads to an elongation of the orbit along the radial direction. In particular, an initially circular orbit evolves into an oval shaped one owing to the electron acceleration. Figures 1(a) and 1(b) illustrate the evolution of the GC orbit and the outward drift velocities $v_{dr}(R_i)$, $v_{dr}(R_o)$ in a tokamak plasma.8

At a certain critical energy $E_{cr}$, the GC orbit bifurcates by creating an unstable fixed point (or X-point) inside the plasma region. With the further increase of the RE energy, the orbit crosses the separatrix (a homoclinic orbit associated the X-point) and hits the wall.

Equation (15) describes not only the outward drift of passing particles but also the inward drift of trapped particles known as the Ware pinch7 (see, also Ref. 1). Indeed, for trapped particles the quantity $T_{av}$ (16) is much smaller than the transit time $T$ and the expression of the drift velocity $v_{dr}$ (15) reduces to

$$v_{dr} = \frac{R_0 E_a}{RB_Z^2} \approx \frac{R_0 E_a}{RB_Z}.$$

The average value of $v_{dr}$ (18) over the radial coordinate $R$ coincides the standard formula for the Ware pinch $v_p = -E_p/[B_Z]$. It is important to note that the drift of the electron GC orbit in the toroidal electric field is an adiabatic process. The area encircled by the GC orbit in the poloidal section is thus conserved, i.e., the integral $J = (2\pi)^{-1} \int_{R_0}^{R_1} R_{pol}dz$ is an adiabatic invariant (C is the closed contour along the GC orbit). The time-dependence of $J$ for passing electrons is shown by curve 3 in Fig. 1(b). Figure 2 illustrates the evolution of an initially trapped electron orbit into a passing one: at a certain time, due to its inward drift, the banana orbit turns into a

FIG. 1. (a) Evolution of the GC orbit in the presence of the toroidal electric field. Curve 1 corresponds to the separatrix of the GC orbit of energy $E = 27$ MeV. The plasma current $I_p = 150$ kA, the toroidal field $B_t = 2.5$ T, the major and minor radii $R_0 = 175$ cm and $a = 46$ cm, and the loop voltage $V = 40$ V. (b) Time evolution of the outward drift velocity $v_{dr}$ corresponding to the orbit in (a). Curve 1 corresponds to $v_{dr}(R_0)$, curve 2 to $v_{dr}(R_i)$, and curve 3 (the right hand axis) describes the energy growth. The average value of $v_{dr}$ (18) over the radial coordinate $R$ coincides the standard formula for the Ware pinch $v_p = -E_p/[B_Z]$.

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FIG. 2. (a) Evolution of the GC orbit of the initially trapped electron in the presence of the toroidal electric field. (b) Time dependence of the drift velocity $v_{dr}$ (15) of the orbit; curve 1 corresponds to the inward drift of trapped electrons, curves 2 and 2' correspond to the outermost and innermost points of the orbit, curve 3 (the right hand side axis) describes the energy growth. The plasma parameters are the same as in Fig. 1, but the plasma current $I_p = 100$ kA.
circular orbit and starts drifting outwards with the growth of the electron energy.

The strong growth of the drift velocity at outermost point of a GC orbit shown by curve 2 in Fig. 1(b) is related to the creation of the X-point and the separatrix at a certain critical energy. This effect is described by the effective poloidal field $B_Z^e$ in Eq. (15): $B_Z^e$ decreases with the growth of the electron energy $E$ and at the critical value $E_{cr}$, the field $B_Z^e$ zeroes at the radial distance $R_s$ within the plasma region. The coordinates $(R_s, Z_s)$ of the X-point are determined by the equations, $dR/dr = 0, dZ/d\phi = 0$, or according to (5) by the zeros of the effective poloidal field,

$$B_Z^e(R_s, Z_s) = 0, \quad B_Z(R_s, Z_s) = 0. \quad (19)$$

The critical energy $E_{cr}$ for the creation of the X-point can be estimated using the definition of the effective magnetic field $B_Z^e$ (6). For sufficiently large energy with the relativistic factor $\gamma \gg 1$, one can assume the longitudinal velocity to be close to the speed of light, $v_{\parallel} \approx c$. Neglecting the last term in (6), Eq. (19) can be written as

$$B_Z^e = \frac{\sigma B_0 c}{Z_s R_0 \omega_0} \gamma \approx 0, \quad B_Z = B_Z(R_s, Z_s). \quad (20)$$

From (20) one obtains the critical energy $E_{cr} = m_0 c^2 \gamma_s$,

$$E_{cr} \approx m_0 c^2 \left| \frac{B_0 R_0 \omega_0}{B_0} \right| c = \frac{\sigma c k B_z}{R_s} \approx 2.998 \times 10^8 R_s |B_Z| \text{MeV}, \quad (21)$$

where we set $\sigma = 1, Z_s = -1$, and $B_Z^e$ is expressed in Tesla (T) and $R_s$ is given in meter (m). The critical energy $E_{cr}$ is therefore determined by the product of the poloidal field $B_Z$ at the X-point to it’s radial position $R_s$.

We should note that the formation of the separatrix of the GC orbit in the toroidal electric field has first been predicted three decades ago in Ref. 5. This phenomenon has been only recently confirmed by numerical simulations in realistic tokamak conditions during plasma disruptions, 6.

The outward drift velocity $v_{\parallel d}$ (15) is proportional to the factor $|1 - RT_{av}/R_0 \bar{T}|$. This factor is only weekly sensitive to the toroidal magnetic field $B_0$ and plasma current $I_p$. However, it strongly depends on the tokamak aspect ratio $R_0/a$: at the given energy $E$, it decreases as $R_0/a$ increases. For large aspect ratios $R_0/a \gg 1$, the transit time $T$ approaches $T_{av}$ so that $|1 - RT_{av}/R_0 \bar{T}| \rightarrow |R_0 - R|/R_0 \sim a/R_0$, i.e., the outward drift velocity $v_{\parallel d}$ is proportional to the inverse aspect ratio: $v_{\parallel d} \propto a/R_0$. Numerical calculations of $|1 - RT_{av}/R_0 \bar{T}|$ presented in Fig. 3 confirms such a dependence. Therefore, one expects the outward drift velocity in spherical tokamaks is larger than in standard tokamaks.

We have derived the formula for the radial drift velocity of electron orbits induced by the toroidal electric field in tokamaks. It describes the outward drift of passing electrons as well as the inward drift of trapped electrons. The outward drift of electrons may give a significant contribution to the decay of the relativistic electron current created during plasma disruptions in tokamaks.

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8. For calculations we have used the model of a tokamak magnetic field with a circular cross section described in Ref. 1 (Sec. 3.4) and in Ref. 4 with the toroidal corrections. We supposed that the directions of the plasma current $I_p$ and the toroidal electric field $E_i$ are opposite to the toroidal field $B_0$ as in the TEXTOR tokamak. Then $\sigma = 1$ for the REs. The evolution of the trapped electrons is obtained by the integration of Eq. (1) with Hamiltonian (2).