Numerical and Analytical Interpretation of Rotation and Radial Electric Fields in Collision Dominated Edge Plasmas

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Abstract

The ambipolarity constraint and the parallel momentum balance equation of neoclassical theory, accounting for finite Larmor radius effects and inertia, allow to describe the radial electric field and the related spin up in collision dominated edge plasmas with steep gradients. Thus they may contribute significantly to the understanding of the L-H transition.

The variation of the toroidal velocity from the last closed magnetic surface up to a position r within the plasma is predicted to be proportional to the integral of the product \( u_\theta \frac{\partial n_T}{\partial r} \) i.e., to \( \sum \int \frac{T^2}{T_1} \) Ld, if the interaction with the neutral gas can be neglected. The summation is over different radial domains, such as the edge pedestal. \( L_d \) is the radial extension of the respective domain. The dimensionless parameter \( \Lambda = \frac{\nu_1^2 c^2}{\Omega_r L_{T_1}} = \frac{\nu_1 a_{\theta p}}{\epsilon c} \) [where \( \nu_1 = \frac{q R_0}{ci} > 0.22 \) is the relevant collision parameter and \( a_{\theta p} \) the poloidal ion Larmor radius] characterizes the ratio of the diamagnetic rotation frequency to the heat diffusion rate along magnetic field lines. Conventional neoclassical theory assumes \( \Lambda \to 0 \). However, e.g. in ALCATOR C-MOD ohmic H-mode pedestals, \( \Lambda \) is sufficiently large that conventional neoclassical results are invalid: it follows from the neoclassical theory that the poloidal velocity decreases below the standard prediction \( v_{\text{neo}} = \frac{1.83 T_1}{eB} \) as \( \Lambda^2 \) increases and changes sign for \( \Lambda^2 = \Lambda_0^2 \) (typically \( \approx 1-2 \)). The equations are treated analytically using a linear interpolation for the poloidal velocity, \( v_\theta (\Lambda^2) \), based on \( v_\theta (\Lambda_0^2) = 0 \) and on the neoclassical value \( v_{\text{neo}} \) for small \( \Lambda \). This allows to account for finite \( \Lambda \) effects in the just mentioned integration.
The equations are also solved numerically (1) to benchmark with a simplified analytical theory with $\Lambda=0$ and vanishing neutral gas density; (2) to compare with the analytical theory accounting for finite $\Lambda$ effects and (3) to explore the parameter space in regions where the analytical theory is not valid, in particular in the cases where the neutral gas density is larger than $10^{14} \text{m}^{-3}$.

The method resorts to an ODE - solver for the classical momentum balance which is combined with a solver for transcendental equations yielding $v_\Theta$.

The results concern the comparison with the analytical solution and the experimental results of the ohmically heated ALCATOR plasma. For $\Lambda = 0$ the numerical solution and the analytical one agree exactly. For finite $\Lambda \approx 1$ the deviations are surprisingly small. The toroidal spin up of the ALCATOR plasma, characterized by a very short decay length $L_{\psi} = 0.76 \text{ cm}$, is $\approx 40 \text{ km sec}^{-1}$. This compares well the measured value of $35 \text{ km sec}^{-1}$. The radial electric field profile assumes the characteristic shape and absolute values reported by the DIII-D Group.
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1 Introduction

Anomalous plasma transport and the concomitant deterioration of the confinement times of tokamaks much below the neoclassical prediction, in particular if auxiliary heating is applied, are key issues in fusion research.

Therefore the surprising experimental discovery of the transition to a high confinement mode [1] above a certain operating power has evoked considerable interest in improved confinement regimes.

Thus e. g. the H (high) ([1] - [5]), the VH (very high) [6], the RI (radiation improved) [7] and the ERS (enhanced reversed shear) ([8] - [11]) modes were investigated. The importance of these modes is underlined by the fact that ignition of the International Tokamak Experimental Reactor (ITER) [12] requires an improved confinement regime.

The theoretical models for these transitions and in particular for the L (low) to H transition, ([13] - [17]) are generally speaking not fully selfconsistent. In fact, a model accounting for the interaction of turbulence, MHD - activity and neoclassical effects selfconsistently is out of range.

Here we concentrate on the revisited neoclassical theory ([22] - [24]) which, in contrast to the conventional neoclassical theory, allows within the framework of a rigorous analytical approach to calculate the velocity field parallel to the flux surfaces and the ambipolar electric field. This theory is valid in the collision dominated regime which is sometimes, e.g. in [4], encountered in the edge region.

In standard neoclassical theory it is assumed that all macro-
scopic lengths (connection length qR, minor radius a, and the absolute values of the decay lengths of the ion Temperature T and ion density n, \( L_T = \left\{ \frac{\partial n}{\partial r} T \right\}^{-1} = T \left( \frac{\partial T}{\partial r} \right)^{-1} \) and \( L_n = \left\{ \frac{\partial n}{\partial r} \right\}^{-1} = n \left( \frac{\partial n}{\partial r} \right)^{-1} \), respectively, are much larger than the ion Larmor radius.

If, however, the decay lengths of the plasma parameters are intermediate between the ion Larmor radius and the connection length, finite Larmor radius (F. L. R.) effects and inertia become important. Therefore standard neoclassical transport theory (predicting e. g. 'automatic' ambipolarity) does not apply to the collision dominated edge region.

Within the framework of the revisited neoclassical theory ([22]-[24],[27]), the analysis of poloidal or toroidal rotation in a collision dominated toroidal plasma with steep gradients is based on the fluid equations with mass and momentum sources. The revised theory has introduced important modifications into the parallel momentum equation, when the parameter

\[
\Lambda_1 = \frac{\nu_{ii} q^2 R^2}{\Omega_i L_T 4r}
\]

exceeds \( \frac{1}{3} \). (\( \Omega_i, \nu_{ii} \) are the ion gyro- and the ion - ion collision frequency, respectively.)

Since the poloidal velocity is determined in this theory by means of a cubic equation, it has also been speculated that the poloidal velocity as derived by the revisited neoclassical theory would not be unique and under certain conditions allow for bifurcated equilibria.

Poloidal plasma rotation in toroidal systems is related to various instability mechanisms [32]. For example, plasma rotation
is always accompanied by a radial electric field, whose origin appears to be complicated due to various competing effects, e. g. the friction with the neutral gas and steepening of the temperature profile. It is usually believed that the neoclassical transport should be ambipolar and independent of the radial electric field. However, this requirement is strictly valid only in a plasma without steep gradients as source of toroidal momentum.

Stringer [19] was the first to notice that the resistive diffusion rate in a toroidal plasma can not only be non-ambipolar, but can also be negative for some values of the poloidal rotation velocity.

Rosenbluth and Taylor [20], considered the stability of toroidal diffusion using a fluid model and proved that if the resistivity were the only dissipative mechanism, then even if all plasma deformations were excluded, there could be no stable poloidal rotation velocity.

The rapidly growing poloidal or toroidal rotations, can also be observed as spin-up phenomena, sometimes of unknown origin, in tokamaks.

An interaction between a poloidal or toroidal spin-up and the turbulence driven anomalous transport is also believed to be a likely reason for the aforementioned L-H mode transition in tokamaks.

A further consequence of an unstable rotation is that a poloidally asymmetric particle transport may also render the radial electric field unstable [21].

In the here envisaged high collisionality regime with steep gradients only stationary solutions are investigated. The F. L. R. effects enter the ion fluid equations via the gyro-stress ten-
Since ambipolarity is no longer automatically ensured in these equations, an ambipolarity constraint can be derived allowing to determine the selfconsistent radial electric field $E_r$. This constraint together with the momentum balance provide the possibility to compute the parallel ion speed $v_\parallel$ and the radial electric field $E_r$ or, equivalently, the poloidal velocity component $u_\Phi = v_E + \frac{B_0}{B} v_\parallel + v_n + \tilde{v}_T$ selfconsistently. As mentioned, both equations are nonlinear in $u_\Phi$ and $v_\parallel$. ([22]-[24],[27]). (The electric drift speed and the density or temperature related velocities $v_E$, $v_n$, $\tilde{v}_T$ are given by $v_E = -\frac{E_r}{B_0}$, $v_n = \frac{T}{eB_0} \frac{\partial n}{\partial r}$, $\tilde{v}_T = \frac{T}{eB_0} \frac{\partial n}{\partial T}$)

The momentum source or sinks due to the recycled neutrals and NBI are included in these equations.

The crucial parameter $\Lambda_1$ measures the ratio of the contributions arising from perpendicular viscosity to those from the parallel viscosity. To compare with the results of the standard neoclassical theory $|\Lambda_1|$ is chosen to approach zero.

By increasing $|\Lambda_1|$ until 0.5 (as indicated by ASDEX - parameters) the modifications due to the full theory can be investigated.
2 Ambipolarity and momentum balance in the revisited neoclassical theory

The revisited neoclassical theory provides an ambipolarity constraint, which together with the parallel and radial momentum equations allows to account e.g. for the radial electric field, the plasma rotation and the formation of transport barriers [24]. Equations for a two-component plasma, describing the continuity of species $j$ with sources $S_j(x,t)$, the momentum balance with friction $R_j(x,t)$ and momentum input $S_{JM}^j(x,t)$, and the energy balance with analogous terms and energy input are

$$\frac{\partial n_j}{\partial t} + \nabla(n_j \vec{u}_j) = S_j$$

(1)

$$m_j n_j \frac{d \vec{u}_j}{dt} = -\nabla \vec{P}_j - \nabla \vec{\Pi}_j - eZ_j n_j (\vec{E}_j + \frac{1}{c} \vec{u}_j \times \vec{B}) - \vec{R}_j + S_{JM}^j$$

(2)

and

$$\frac{3}{2} n_j \frac{d T_j}{dt} + P_j \nabla u_j = -\nabla \vec{q}_j - \vec{\Pi}_j : \nabla \vec{u}_j + R_j^E + S_j^E$$

(3)

The viscous tensor is given by

$$\vec{\Pi} = -\eta_0 \vec{W}_0 - \eta_1 (\vec{W}_1 + 4 \vec{W}_2) + \eta_3 (\vec{W}_3 + 2 \vec{W}_4)$$

(4)

The tensors $\vec{W}_0$, $\vec{W}_1$, $\vec{W}_2$, $\vec{W}_3$, $\vec{W}_4$, accounting for the parallel, perpendicular and gyro stress are given in [18] (recently extended and completed by Mikhailovsky and Tsypin [26]). The terms $S_j^E$ on the R.H.S. of equation (3) denotes energy sources and losses, whereas $R_j^E$ denotes terms relating to collisional energy transfer and frictional heating. The correct forms of these
terms require a kinetic and atomic approach. However, here we shall assume them, as given functions.

The radial electric field satisfies the radial momentum balance

\[ E_r + (\vec{U} \times \vec{B}) = \frac{1}{eN} \frac{\partial P_i}{\partial r} \]

In the revisited neoclassical theory [1], a plausible ordering inside the separatrix is introduced by means of a small parameter \( \mu \approx 0.1 \) as

\[ \frac{c_j}{qRv_j} \approx \frac{L_{\psi}}{r} \approx \frac{r}{qR} \approx \frac{B_\theta}{B_\phi} \approx \mu \]

and

\[ \frac{e_j V}{T_j} \approx \sqrt{\frac{m_e}{m_i}} \approx \frac{a_i}{L_{\psi}} \approx \mu^2 \]

where \( L_{\psi} \) is the radial scalelength of the temperature, \( c_j \) and \( a_j \) are the thermal speed and the gyro radius of the species \( j \), respectively; \( V \) is the loop voltage and \( \nu_j \) the collision frequency between like particles. Using the magnetic field aligned orthonormal unit vectors \((\hat{p}, \hat{b}, \hat{n})\) in radial, binormal and parallel directions, and the small parameter \( \mu \), the velocity of species \( j \) can be assumed as,

\[ \hat{p} \vec{U}_j = \mu^6 U_{\psi,j}^{(6)}(\psi, \chi) + ... \]

\[ \hat{b} \vec{U}_j = \mu^2 U_{\beta,j}^{(2)}(\psi) + \mu^3 U_{\beta,j}^{(3)}(\psi, \chi) ... \]

\[ \hat{n} \vec{U}_j = \mu U_{||,j}^{(1)}(\psi) + \mu^2 U_{||,j}^{(2)}(\psi, \chi) ... \]

Assuming that the magnetic field, density, temperature, potential, etc., are independent of the poloidal angle in dominant
order, these are also expanded in perturbation series. For example, the density and the magnetic field are written as,

\[ N(\psi, \chi) = N^{(0)}(\psi)[1 + \mu N^{(1)}(\psi, \chi)\ldots] \]

and

\[ B(\psi, \chi) = B^{(0)}(\psi)[1 + \mu B^{(1)}(\psi, \chi)\ldots] \]

respectively. For a tokamak plasma with circular cross section, also use is made of the toroidal unit vectors \((\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)\). Taking toroidal and parallel projections of the momentum equation and averaging them over the magnetic flux surfaces, and imposing the ambipolarity condition, one obtains a pair of coupled nonlinear equations for the toroidal and poloidal ion velocities in terms of other plasma variables, such as temperature, density, and electric field \([1]\).
3 Classical Toroidal Momentum Transport

The main result of the revisited neoclassical theory [27] is an equation describing the radial transport of toroidal momentum in a collisional subsonic plasma with steep gradients. For large aspect ratio and circular cross-section we get the time-dependent $u_\phi$ equation

$$\frac{\partial}{\partial r} \left[ \eta_2 \left( \frac{\partial u_\phi}{\partial r} - \frac{0.107q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial r} \frac{B_\phi}{B_\theta} u_\theta \right) \right] = m_i n_i \frac{\partial}{\partial t} + \nu_{\text{ex}} u_\phi - m_i n_i (\dot{M}_{\phi,i} + j_r B_\theta)$$

(5)

Q and S are given in the appendices F and E. $m_i, n_i$ are the ion mass and ion density, respectively.

The poloidal rotation driven by the temperature gradient seen inside the parenthesis on the right-hand-side of (5) results from the gyro stress tensor and acts like another source term, i.e., as a toroidal momentum source or sink, depending on the sign of its radial gradient. External momentum sources can be direct, such as fast ions provided by the neutral beam injection (NBI), collisions by alpha particles, or indirect and due to particle sources such as aforementioned charge exchange reactions with cold recycled neutrals. The poloidal rotation velocity $u_\theta$, the charge exchange reactions with the recycled cold neutrals, the momentum injection $\dot{M}_{\phi,i}$ by NBI and by a probe generating the radial current density $j_r$ determine the toroidal flow velocity $u_\phi$ [25]. The charge exchange reactions are characterized by the charge exchange frequency

$$\nu_{\text{ex}} = < \sigma v >_{\text{ex}} n_0$$

(6)
< \sigma v >_{cx} \text{ is the rate coefficient for charge exchange, and } n_0 \text{ the neutral gas density. We define}

\[ g = \frac{u_\phi}{v_T} \]
\[ h = \frac{u_\theta}{v_T} \]

and

\[ x = \frac{r - r_{inf}}{L_\psi} \]

Here we used the (constant, positive) velocity

\[ v_T = \frac{1}{eB_\phi} \frac{T_{inf}}{L_\psi} \]

\( T_{inf} \) is the temperature at the inflection point \( P_{inf} \) of the temperature profile (Fig. 1). The inflection point has the radius \( r_{inf} \) and is defined as the locus of vanishing curvature of the temperature profile. \( L_\psi \) is the temperature scale length at \( P_{inf} \) in absolute value (appendix A). We get

\[
\frac{v_T}{L_\psi} \frac{\partial}{\partial x} \left[ \eta_2 \left( \frac{\partial g}{\partial x} - \frac{0.107q^2}{1 + \frac{Q}{S_\perp}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} \right) \right]
\]

\[ = m_i n_{inf} \hat{n} v_T \left( \frac{\partial}{\partial t} + \nu_{cx} \right) g - m_i n_{inf} \hat{n} \left( \dot{M}_{\phi,i} + j_r B_\theta \right) \quad (7) \]

\( n_{inf} \) is the density at the inflection point and \( \hat{n} \) is defined by \( n_i = \hat{n} n_{inf} \). The characteristic time \( t_c \) is given by

\[
t_c^{-1} = \frac{1}{m_i n_{inf} L_\psi^2 \eta_{2,inf}} \quad (8)\]

13
We introduce this time by multiplying with $\frac{1}{m_in_{inf}L^2_\psi}$. With the decomposition
\[ \eta_2 = \eta_{2,inf} \hat{\eta}_2 \]
we can write
\[
\frac{\eta_{2,inf}}{m_in_{inf}L^2_\psi} \left( \frac{\partial}{\partial x} [\hat{\eta}_2 (\frac{\partial g}{\partial x} - \frac{0.107q^2}{1 + \frac{q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B^\phi}{B^\theta} h)] \right)
\]
\[ = \hat{n} (\frac{\partial}{\partial t} + \nu_{cx}) g - \frac{\hat{n} \dot{M}_{\phi,i}}{v_T} + \frac{J_r B^\theta}{m_i v_T n_{inf}} \]  
(9)
We define
\[ G = \frac{\partial g}{\partial x} - \frac{0.107q^2}{1 + \frac{q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B^\phi}{B^\theta} h \]  
(10)
and get
\[
\frac{\eta_{2,inf}}{m_in_{inf}L^2_\psi} \frac{\partial}{\partial x} (\hat{\eta}_2 G)
\]
\[ = \hat{n} (\frac{\partial}{\partial t} + \nu_{cx}) g - \frac{\hat{n} \dot{M}_{\phi,i}}{v_T} + \frac{j_r B^\theta}{m_i v_T n_{inf}} \]  
(11)
By differentiating we obtain
\[
\frac{\eta_{2,inf}}{m_in_{inf}L^2_\psi} (G \frac{\partial}{\partial x} \hat{\eta}_2 + \hat{\eta}_2 \frac{\partial}{\partial x} G)
\]
\[ = \hat{n} (\frac{\partial}{\partial t} + \nu_{cx}) g - \frac{\hat{n} \dot{M}_{\phi,i}}{v_T} + \frac{J_r B^\theta}{m_i v_T n_{inf}} \]  
(12)
or
\[
\frac{\eta_{2,inf}}{m_in_{inf}L^2_\psi} \hat{\eta}_2 (G \frac{\partial}{\partial x} \ln(\hat{\eta}_2) + \frac{\partial}{\partial x} G)
\]
\[ = \hat{n} (\frac{\partial}{\partial t} + \nu_{cx}) g - \frac{\hat{n} \dot{M}_{\phi,i}}{v_T} + \frac{J_r B^\theta}{m_i v_T n_{inf}} \]  
(13)
Furthermore we get

$$\frac{\eta_{2,\inf}}{m_i n_{\inf} L^2} \frac{\partial}{\partial x} G = \hat{n} \left( \frac{\partial}{\partial t} + \nu_{ex} \right) g - \frac{\hat{n} M_{\phi,i}}{v_T} +$$

$$\frac{J_r B_0}{m_i v_T n_{\inf}} - \frac{1}{m_i n_{\inf} L^2} \eta_2 G \frac{\partial}{\partial x} \ln(\eta_2)$$

(14)

and

$$\frac{\partial}{\partial x} G = \frac{m_i n_{\inf} L^2}{\eta_{2,\inf} \hat{n}_2} = \left[ \hat{n} \left( \frac{\partial}{\partial t} + \nu_{ex} \right) g - \frac{\hat{n} M_{\phi,i}}{v_T} +$$

$$\frac{J_r B_0}{m_i v_T n_{\inf}} - \frac{\eta_{2,\inf}}{m_i n_{\inf} L^2} \hat{n}_2 G \frac{\partial}{\partial x} \ln(\eta_2) \right]$$

(15)

The coefficient

$$\frac{1}{m_i n_{\inf} L^2} \eta_{2,\inf} = \eta'$$

containing the viscosity can be rewritten as

$$\eta' = \frac{\frac{6}{5} n_{\inf} k T_{\inf} \tau_{ii}}{\Omega_i^2 \tau_{ii}^2 m_i n_{\inf} L^2}$$

(16)

We use

$$k T_{\inf} = m_i c_{\inf}^2 \hat{T},$$

where

$$c_{\inf}$$

is the thermal speed at the inflection point and get

$$\eta' = \frac{\frac{6}{5} \hat{T} c_{\inf}^2}{\Omega_i^2 \tau_{ii} L^2}$$

(17)

or (appendix D)

$$\eta' = 8.57 \times 10^{-7} \frac{a_{i,\inf}^2 n_{\inf} Z_{eff,\inf} T_{\inf}^2}{L^2} \frac{1}{A}$$

(18)

$$a_{i,\inf}$$

is the ion Larmor radius at the inflection point.

We assume

$$Z_{eff,i} = Z_{eff,\inf} \hat{T}^2$$
where $Z$ is a constant. Then the radial dependence of $\eta_2$ is given by (one $\hat{T}$ is cancelled)

$$\hat{\eta}_2 = \hat{n} \hat{T}^{(-\frac{1}{2}+Z)}$$

(19)

We get the derivative

$$\frac{\partial}{\partial x}(\hat{\eta}_2) = \hat{T}^{-\frac{1}{2}+Z} \frac{\partial \hat{n}}{\partial x} + n\left(-\frac{1}{2} + Z\right) \hat{T}^{-\frac{3}{2}+Z} \frac{\partial \hat{T}}{\partial x}$$

(20)

The logarithmic derivative is

$$\frac{\partial}{\partial x} \ln(\eta_2) = \frac{\partial}{\partial x} \ln(\hat{\eta}_2) = \frac{\hat{T}^{-\frac{1}{2}+Z} \frac{\partial \hat{n}}{\partial x} + \hat{n}\left(-\frac{1}{2} + Z\right) \hat{T}^{-\frac{3}{2}+Z} \frac{\partial \hat{T}}{\partial x}}{\hat{n} \hat{T}^{-\frac{1}{2}+Z}}$$

(21)

or

$$\frac{\partial}{\partial x} \ln(\eta_2) = \frac{1}{\hat{n}} \frac{\partial \hat{n}}{\partial x} + \left(-\frac{1}{2} + Z\right) \frac{\partial \hat{T}}{\hat{T} \partial x}$$

(22)
4 Poloidal rotation

Using the ambipolarity condition and the extended forms of the stress tensors in the parallel momentum equation, one can cancel the time derivative and the source terms. The result - in the lowest order - is a nonlinear equation between the radial derivatives of the poloidal and toroidal plasma velocities. For large aspect ratio and circular cross-section it can be written as [25]

\[
\begin{align*}
\frac{u_\theta}{eB_\phi} + \frac{1.833}{eB_\phi} \frac{\partial T}{\partial r} &= 0.36 \frac{\eta_1}{\eta_0 (1 + \frac{\omega^2}{27})} q^2 R^2 eB_\phi \frac{\partial \ln T}{\partial r} \\
\{ T \frac{\partial u_\phi}{\partial r} + \frac{1}{2} u_\phi^2 \\n- u_\phi \frac{B_\phi}{B_\theta} (u_\theta - \frac{T}{eB_\phi} \frac{\partial \ln \left( T n_i^2 \right)}{\partial r} ) + \\
1.9 \frac{B_\phi^2}{B_\theta^2} [u_\theta - 0.8 \frac{T}{eB_\phi} \frac{\partial \ln \left( T n_i^{1.6} \right)}{\partial r} ]^2 \}
\end{align*}
\]

This may be rewritten as

\[
\begin{align*}
\frac{u_\theta}{eB_\phi} + \frac{1.833}{eB_\phi} \frac{\partial T}{\partial r} &= \\
T_{36} \left\{ T_i \frac{\partial u_\phi}{\partial r} + \frac{1}{2} u_\phi^2 \\n- u_\phi \frac{B_\phi}{B_\theta} (u_\theta - \frac{T_i}{eB_\phi} \frac{\partial \ln \left( T n_i^2 \right)}{\partial r} ) + \\
1.9 \frac{B_\phi^2}{B_\theta^2} [u_\theta - 0.8 \frac{T_i}{eB_\phi} \frac{\partial \ln \left( T n_i^{1.6} \right)}{\partial r} ]^2 \} \end{align*}
\]
$T_{36}$ is given by

$$T_{36} = 0.36 \frac{\eta_1}{\eta_0 (1 + \frac{Q^2}{S^2})} q^2 R^2 eB_\phi \frac{\partial \ln T}{\partial r}$$

We use $\nu_T = \frac{1}{eB_\phi} \frac{T_{\nu T}}{L_\psi}$ and get the following expression for the LHS of the preceding equation

$$u_\theta + \frac{1.833 \partial T}{eB_\phi} = \nu_T (h + 1.833 \frac{\partial \hat{T}}{\partial x})$$

(25)

$T_{36}$ may be rewritten as

$$T_{36} = 0.36 \frac{\eta_1}{\eta_0 (1 + \frac{Q^2}{S^2})} q^2 R^2 eB_\phi \frac{\partial \ln T}{\partial r} =$$

$$0.36 \frac{\eta_1}{\eta_0 (1 + \frac{Q^2}{S^2})} q^2 eB_\phi L_\psi \frac{R}{T} \frac{R}{L_\psi L_\psi} \frac{\partial \ln T}{\partial r}$$

$$= 0.36 \frac{\eta_1}{\eta_0 (1 + \frac{Q^2}{S^2})} q^2 \frac{1}{\nu_T T L_\psi^2} R^2 \frac{\partial \ln T}{\partial x}$$

(26)

The viscosities $\eta_1, \eta_0$ can be written as [18]

$$\eta_1 = \frac{6}{5} n_i T \tau_{ii}$$

(27)

$$\eta_0 = \frac{0.96 n_i T \nu_{ii}}{\nu_{ii}^2}$$

(28)

For the ratio $\frac{\eta_1}{\eta_0}$ we get

$$\frac{\eta_1}{\eta_0} = \frac{\frac{6}{5} n_i T \nu_{ii}}{\frac{0.96 n_i T \nu_{ii}}{\nu_{ii}^2}} = \frac{6 \nu_{ii}^2}{5 \cdot 0.96 \Omega_i^2}$$

(29)
Therefore the factor \( T_{36} \) in the preceding term becomes

\[
T_{36} = 0.36 \frac{\eta_1}{\eta_0 (1 + \frac{\Omega_i^2}{\sigma^2})} q^2 \frac{1}{v_T \hat{T} \frac{L_v^2}{\partial x}} \frac{R^2 \partial ln T}{\partial x}
\]

\[
= 0.36 \frac{6}{5} \frac{\nu_i^2}{0.96 \Omega_i^2 (1 + \frac{\Omega_i^2}{\sigma^2})} q^2 \frac{1}{v_T \hat{T} \frac{L_v^2}{\partial x}} \frac{R^2 \partial ln T}{\partial x}
\]

\[
= 0.45 \frac{\nu_i^2}{\Omega_i^2 (1 + \frac{\Omega_i^2}{\sigma^2})} q^2 \frac{1}{v_T \hat{T} \frac{L_v^2}{\partial x}} \frac{R^2 \partial ln T}{\partial x}
\]  

(30)

The other terms of equation (24) are

\[
\frac{T}{eB_\theta} \frac{\partial u_\phi}{\partial r} = \frac{B_\phi}{B_\theta eB_\psi L_v} \frac{T}{\partial u_\phi} \frac{\partial u_\phi}{\partial x} = \frac{B_\phi}{B_\theta} v_T^2 \hat{T} \frac{\partial g}{\partial x}
\]

\[
-u_\phi \frac{B_\phi}{B_\theta} (u_\theta - \frac{T}{eB_\phi} \frac{\partial ln \left( Tn_i^2 \right)}{\partial r}) = -v_T^2 \hat{T} \frac{g_\phi}{B_\theta} (h - \hat{T} \frac{\partial ln \left( Tn_i^2 \right)}{\partial x})
\]

(31)

and

\[
1.9 \frac{B_\phi^2}{B_\theta^2} \left[ u_\phi - 0.8 \frac{T_i}{eB_\phi} \frac{\partial ln \left( Tn_i^{1.6} \right)}{\partial r} \right]^2 = 1.9 \frac{B_\phi^2}{B_\theta^2} v_T^2 \left[ h - 0.8 \hat{T} \frac{\partial ln \left( Tn_i^{1.6} \right)}{\partial x} \right]^2
\]  

(32)

Thus equation (24) can be written as

\[
v_T (h + 1.833 \frac{\partial \hat{T}}{\partial x}) =
\]

\[
T_{36} \left\{ \frac{B_\phi}{B_\theta} v_T^2 \frac{\hat{T}}{\partial x} \frac{\partial g}{\partial x} + \frac{1}{2} v_T^2 \frac{\partial \hat{T}}{\partial x} \right\}
\]

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\[ -v_T^2 \frac{B_\phi}{B_\theta} (h - \hat{T} \frac{\partial \ln (Tn_i^2)}{\partial x}) \]

\[ +1.9 \frac{B_\phi^2}{B_\theta^2} v_T^2 [h - 0.8 \hat{T} \frac{\partial \ln (Tn_i^{1.6})}{\partial x}]^2 \]  

(33)

We multiply with \( \hat{T} v_T \) and cancel \( v_T^2 \)

\[ (h + 1.833 \frac{\partial \hat{T}}{\partial x}) \hat{T} = \]

\[ C_\eta \left\{ \frac{B_\phi}{B_\theta} \frac{\partial g}{\partial x} + \frac{1}{2} g^2 \right\} \]

\[ -g \frac{B_\phi}{B_\theta} (h - \hat{T} \frac{\partial \ln (Tn_i^2)}{\partial x}) \]

\[ +1.9 \frac{B_\phi^2}{B_\theta^2} [h - 0.8 \hat{T} \frac{\partial \ln (Tn_i^{1.6})}{\partial x}]^2 \]  

(34)

\( C_\eta \) is defined by

\[ C_\eta = \nu_T \hat{T} \hat{T}_{36} = \]

\[ 0.45g^2 \frac{R^2}{L_{\psi}^2} \frac{\nu_{ii}(n_{inf}, T_{inf})^2}{\Omega_i^2} (\hat{T}^{-\frac{3}{2}} T^2 \hat{n})^2 \frac{1}{(1 + \frac{q^2}{3})} \frac{\partial \ln T}{\partial x} \]  

(35)
5 Description of the plasma rotation by a 2nd order equation and a transcendental equation

We start with equation (7) for $g$

\[
\frac{v_T}{L^2_\psi} \frac{\partial}{\partial x} \left[ \eta_2 \left( \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{q^2}{S_x^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} \right) \right] = m_i n_{\text{inf}} \hat{n} v_T \left( \frac{\partial}{\partial t} + \nuex \right) g - m_i n_{\text{inf}} \hat{n} \dot{M}_{\phi,i} + J_r B_\theta
\]  

(36)

$n_{\text{inf}}$ is the density at the inflection point and $\hat{n}$ is defined by $n_i = \hat{n} n_{\text{inf}}$. To get the characteristic time we multiply - as before - by $\frac{1}{m_i n_{\text{inf}} v_T}$ and get using $\eta_2 = \eta_{2,\text{inf}} \hat{n}_2$

\[
\frac{\eta_{2,\text{inf}}}{m_i n_{\text{inf}} L^2_\psi} \left( \frac{\partial}{\partial x} \left[ \hat{n}_2 \left( \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{q^2}{S_x^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} \right) \right] \right) = \hat{n} \left( \frac{\partial}{\partial t} + \nuex \right) g - \frac{\hat{n} \dot{M}_{\phi,i}}{v_T} + \frac{J_r B_\theta}{m_i v_T n_{\text{inf}}}
\]  

(37)

The characteristic time $t_c$ is given by

\[
t_c^{-1} = \frac{1}{m_i n_{\text{inf}} L^2_\psi \eta_{2,\text{inf}}}
\]

We define

\[
G = \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{q^2}{S_x^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h
\]  

(38)

and get the first order equation

\[
\frac{\eta_{2,\text{inf}}}{m_i n_{\text{inf}} L^2_\psi} \frac{\partial}{\partial x} \left( \hat{n}_2 G \right)
\]
\[ \frac{\partial}{\partial t} + \nu_{ex} g - \frac{\hat{n}_i \dot{M}_{\phi,i}}{v_T} + \frac{J_r B_0}{m_i v_T \eta_{inf}} \]

By differentiating we obtain
\[
\frac{\eta_{2,inf}}{m_i \eta_{inf} L_{\psi}^2} \left( G \frac{\partial}{\partial x} \eta_2 + \hat{n}_2 \frac{\partial}{\partial x} G \right) \]
\[ = \hat{n}_i \frac{\partial}{\partial t} + \nu_{ex} g - \frac{\hat{n}_i \dot{M}_{\phi,i}}{v_T} + \frac{J_r B_0}{m_i v_T \eta_{inf}} \]

or, resolving with respect to \( \frac{\partial}{\partial x} G \) we get
\[
\frac{\partial}{\partial x} G = \frac{m_i \eta_{inf} L_{\psi}^2}{\eta_{2,inf} \hat{n}_2} \left[ \hat{n}_i \left( \frac{\partial}{\partial t} + \nu_{ex} g - \frac{\hat{n}_i \dot{M}_{\phi,i}}{v_T} + \frac{J_r B_0}{m_i v_T \eta_{inf}} \right) \right] - \frac{\eta_{2,inf}}{m_i \eta_{inf} L_{\psi}^2} \hat{n}_2 G \frac{\partial}{\partial x} \ln(\eta_2) \]

In the last term the factor \( \hat{n}_2 \) appears because the 'ln' was introduced. The equation can be written in the form (omitting \( \frac{\partial}{\partial t} \))
\[ \frac{\partial}{\partial x} G = T_{CX} + T_{NB} + T_{Pr} - G \frac{\partial}{\partial x} \ln(\eta_2) \]

Here
\[ T_{CX} = t_c \frac{\hat{n}}{\eta_2} \nu_{ex} g = t_c \hat{T}^{0.5 - z} \nu_{ex} g \]
\[ T_{NB} = t_c \frac{\hat{n}}{\eta_2 v_T} \dot{M}_{\phi,i} \]

and
\[ T_{Pr} = t_c \frac{J_r B_0}{m_i v_T \eta_{inf} \eta_2} \]
account for the friction caused by the neutral gas, the momentum source due to neutral beam injection and the momentum source due to a probe, respectively.
6 Boundary values

The solution of the preceding equations is strongly influenced by the boundary values which are therefore discussed in the following.

Firstly we assume a symmetric streaming of the scrape-off plasma into the divertor. This yields as boundary value \( h(r = r_s) = 0 \).

We note that this assumption determines the value of \( \Lambda(r = r_s) \) at the boundary. Via the temperature profile we then get the radius \( r_s \). Thus the condition \( h(r = r_s) = 0 \) is not a boundary condition for the differential equations, but determines the radial extension.

Secondly we assume \( g(r = r_s) = 0 \) because of the absence of momentum sources such as neutral beam injection (NBI) and because neutrals may be reducing \( g(r = r_s) \) to a low value. (We note, however, that at JET [33] and JT60, i.e. devices with strong NBI, a considerable spin up at the separatrix had been observed.)

The analytical or numerical integration can start at the inside \((r=0)\) or at the outside \((r = r_s)\).

For integrating from the outside to the inside - what may be considered as the 'natural' procedure - \( g \) and \( \frac{\partial n}{\partial x} \) are prescribed and \( G \) follows from equation (38). If the neutral gas can be neglected, NBI and the probe are switched off, the boundary condition is \( G = 0 \).

If the neutral gas density is finite as in chapter 13, the derivative \( \frac{\partial n}{\partial x} \big|_{r=r_s} \) must be adjusted such that \( \frac{\partial n}{\partial x} \big|_{r=0} = 0 \) ('shooting method').

For integrating from the inside to the outside \( g \) and \( G \) are pre-
scribed. From equation (38) follows $G(x = -\infty) = 0$. Now $g(r = 0)$ must be adjusted to hit a prescribed boundary value $g(r = r_s)$ at the outside.
7 Halfanalytical solution

We envisage the equation

\[ \frac{v_T}{L_\psi} \frac{\partial}{\partial x} \left[ \eta_2 \left( \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h \right) \right] \]

\[ = m_i N_{inj} v_T \left( \frac{\partial}{\partial t} + \nu_{ex} \right) g - m_i N_{inj} \hat{M}_{\phi,i} \quad (43) \]

Considering stationarity and neglecting injection, probe current and the neutral gas yields

\[ \frac{\partial}{\partial x} \left[ \eta_2 \left( \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h \right) \right] = 0 \quad (44) \]

Integrating radially we get

\[ \eta_2 \left( \frac{\partial g}{\partial x} - \frac{0.107 q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h \right) = Const \quad (45) \]

In the interior the term \( \frac{0.107 q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h \) vanishes, therefore

\[ \eta_2 \left( \frac{\partial g}{\partial x} \right) \bigg|_{x=-\infty} = Const \quad (46) \]

In an infinitesimally small volume around the magnetic axis the thermodynamic quantities are constant because of the nesting of the flux surfaces around this axis. Therefore we get for the preceding equation Const=0. We integrate radially again and get the general solution

\[ g - g_0 = \int_{x=-\infty}^{x} dx' \left[ \frac{0.107 q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} h \right] \quad (47) \]
\( g_0 \) is the boundary value of \( g \) at \( r=0 \) or \( x = -\infty \) (concerning the boundary values also see chapter 6).

We consider the neoclassical approximation of equation (24) for \( C_\eta = 0 \)

\[
h = -1.833 \frac{\partial \hat{T}}{\partial x}
\]  

(48)

With this equation we get

\[
g - g_0 = \int_{x=-\infty}^{x} dx' \left[ \frac{0.107q^2}{(1 + \frac{Q^2}{S^2})} \frac{\partial \ln T}{\partial x} \frac{B_\phi}{B_\theta} \left( \frac{\partial T}{\partial x} \right) \right]
\]  

(49)

or

\[
g - g_0 = \int_{x=-\infty}^{x} dx' \left[ \frac{0.107q^2}{(1 + \frac{Q^2}{S^2})} \frac{B_\phi}{B_\theta} \left( \frac{-1.833 \frac{\partial \hat{T}}{\partial x}^2}{\hat{T}} \right) \right]
\]  

(50)
8 Analytical integration resorting to a special temperature profile

We evaluate equation (50) in the case of the profile

\[ T = T_{ inf}(1 - \tanh \frac{T - T_{inf}}{\Delta T}) = T_{inf}(1 - \tanh(x)) \]  \hspace{1cm} (51)

which is normalized to \( T_{inf} \), the temperature at the inflection point \( r = r_{inf} \) (Fig. 1). The profile (51) stands for the pedestal at the boundary and thus contains the main property leading to the spin up of the plasma. With the derivatives

\[ \frac{\partial \ln T}{\partial x} = -\frac{1}{(1 - \tanh(x)) \cosh^2(x)} \] \hspace{1cm} (52)

and

\[ \frac{\partial T}{\partial x} = -\frac{1}{\cosh^2(x)} \] \hspace{1cm} (53)

we get

\[ g(x') - g_0 = \int_{-\infty}^{x'} dx \frac{0.107q^2}{(1 + \frac{q^2}{\alpha^2})(1 - \tanh(x)) \cosh^2(x)} \frac{1}{B_0} \times \]

\[ (1.833 \frac{1}{\cosh^2(x)}) \] \hspace{1cm} (54)

We use the identities

\[ \frac{1}{(1 - \tanh(x)) \cosh^4(x)} = \frac{(1 - \tanh(x))^2}{(1 - \tanh(x))} = \]

\[ \frac{(1 - \tanh(x))^2(1 + \tanh(x))^2}{(1 - \tanh(x))} = (1 - \tanh(x))(1 + \tanh(x))^2 \] \hspace{1cm} (55)
and
\[
\frac{1}{(1 - \tanh(x)) \cosh^4(x)} =
\]
\[(1 - \tanh^2(x))(1 + \tanh(x)) = \frac{1 + \tanh(x)}{\cosh^2(x)} \quad (56)
\]

This yields
\[
g(x') - g_0 = \frac{0.107 q^2}{(1 + \frac{Q^2}{S^2}) B_\theta} \times
\]
\[(1.833 (\tanh(x) + 0.5 \tanh^2(x))) \bigg|_{x' = -\infty} \quad (57)
\]
9 Analytical treatment in the case of finite $\Lambda_r$

The $u_\phi$ - equation is in the case of vanishing neutral gas density given by

$$\frac{\partial g}{\partial x} = -\frac{0.107q^2}{1 + \frac{q^2}{S^2}} \frac{\partial \ln T}{\partial x} B_\phi \frac{\partial T}{\partial x} h \quad (58)$$

With $h = h' \frac{\partial T}{\partial x}$ and $g = g' \frac{B_\phi}{B_\theta}$ we obtain

$$\frac{\partial g'}{\partial x} = -\frac{0.107q^2}{1 + \frac{q^2}{S^2}} \frac{\partial \ln T}{\partial x} h' \frac{\partial T}{\partial x} \quad (59)$$

The $u_\theta$ - equation can be written as

$$\frac{\tilde{\nu}_T}{\partial t} (h + 1.833 \frac{\partial T}{\partial x}) = \tilde{\nu}_T (h' + 1.833) =$$

$$T_{36} \left\{ \frac{B_\phi}{B_\theta} \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} \frac{\partial g}{\partial x} + \frac{1}{2} \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} B_\phi \right\}$$

$$\frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} B_\theta (h - T \frac{\partial \ln (Tn_i^2)}{\partial x})$$

$$+ 1.9 \frac{B_\phi^2}{B_\theta^2} \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} [h - 0.8 \frac{\partial \ln (Tn_i^{1.6})}{\partial x}]$$

or

$$\tilde{\nu}_T (h' + 1.833) =$$

$$T_{36} \left\{ \frac{B_\phi^2}{B_\theta^2} \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} \frac{\dot{B}_\theta}{B_\phi} \frac{\partial g}{\partial x} + \frac{1}{2} \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} \frac{B_\phi}{B_\theta} \right\}$$

$$- \frac{\tilde{\nu}_T^2}{[\frac{\partial T}{\partial x}]^2} \frac{B_\theta}{B_\phi} (h - T \frac{\partial \ln (Tn_i^2)}{\partial x})$$

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\[ +1.9 \frac{\ddot{v}_T^2}{[\frac{\partial T}{\partial x}]^2} \frac{h - 0.8 \hat{T} \frac{\partial \ln (T n_i^{1.6})}{\partial x}}{2} \]  

We replace \( \frac{\partial h}{\partial x} \) according to equation (58) cancel one \( \ddot{v}_T \), and employ \( h = h' \frac{\partial h}{\partial x} \): 

\[
(h' + 1.833) = 
T_{36} \ddot{v}_T \left[ - \frac{B_\phi}{B_0} \right]^2 \{ \frac{1}{[\frac{\partial T}{\partial x}]^2} \hat{T} \frac{0.107 q^2 \frac{\partial \ln T}{\partial x}}{1 + \frac{q^2}{S^2}} \frac{1}{2} + \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{B_\theta}{B_0} \} \frac{g^2}{B_\phi} 
- \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{B_\theta}{B_0} (h - \hat{T} \frac{\partial \ln (T n_i^2)}{\partial x}) 
+ 1.9 \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{h - 0.8 \hat{T} \frac{\partial \ln (T n_i^{1.6})}{\partial x}}{2} \} 
\]

(62)

Using \( v_T = \ddot{v}_T \frac{L_T}{L_\psi} \) and 

\[
\Lambda_r = \frac{\nu_{ii} q^2 R^2}{\Omega_i L_T r} 
\]

we get for \( T_{36} \left[ \frac{B_\phi}{B_0} \right]^2 \) successively the relations

\[
T_{36} \left[ \frac{B_\phi}{B_0} \right]^2 = 0.36 \frac{6}{5} \frac{\nu_{ii}}{0.96 \Omega_i} \left( 1 + \frac{q^2}{S^2} \right) q^2 \frac{R^2}{\ddot{v}_T L_T} \frac{L_T}{L_\psi} \frac{B_\phi}{B_0} 
\]

\[
T_{36} \left[ \frac{B_\phi}{B_0} \right]^2 = 0.36 \frac{6}{5} \frac{\nu_{ii}}{0.96 \Omega_i} \left( 1 + \frac{q^2}{S^2} \right) q^2 \frac{R^2}{\ddot{v}_T L_T} \frac{L_T}{L_\psi} \frac{B_\phi}{B_0} 
\]

\[
T_{36} \left[ \frac{B_\phi}{B_0} \right]^2 = 0.36 \frac{6}{5} \frac{\nu_{ii}}{0.96 \Omega_i} \left( 1 + \frac{q^2}{S^2} \right) q^2 \frac{R^2}{\ddot{v}_T L_T} \frac{L_T}{B_\phi} B_\phi 
\]

\[
T_{36} \left[ \frac{B_\phi}{B_0} \right]^2 = 0.45 \frac{\lambda_r^2 t^2}{q^2 R^2} \frac{1}{\ddot{v}_T} \frac{B_\phi}{B_0} \]

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and finally
\[ T_{36}\frac{B_{\phi}^2}{B_0} = 0.45 \frac{\Lambda_r^2}{(1 + 2\frac{Q^2}{S^2})} \]

With \( C_\eta = \nu T \hat{T} T_{36} \) we obtain

\[ T_{36}\frac{B_{\phi}^2}{B_0} \tilde{v}_T = \frac{C_\eta}{T} \frac{B_{\phi}^2}{B_0} \frac{\partial T}{\partial x} = 0.45 \frac{\Lambda_r^2}{(1 + 2\frac{Q^2}{S^2})} \]

For \( h' \) we get the equation

\[ (h' + 1.833) = \]

\[ 0.45 \frac{\Lambda_r^2}{(1 + 2\frac{Q^2}{S^2})} \left\{ \frac{1}{[\frac{\partial T}{\partial x}]^2} \hat{T} 0.107 q^2 \frac{\partial ln T}{\partial x} h' + \frac{1}{2} \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{B_{\phi}^2}{B_\phi} \frac{\partial^2 T}{\partial x^2} \
- \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{B_{\phi}}{B_\phi} \frac{\partial h}{\partial x} - \frac{1}{[\frac{\partial T}{\partial x}]^2} \frac{\partial ln (T n_{i1}^2)}{\partial x} \frac{\partial T}{\partial x} \right\} \]

\[ + 1.9 \frac{1}{[\frac{\partial T}{\partial x}]^2} \left\{ h - 0.8 \frac{\partial ln (T n_{i1}^{1.6})}{\partial x} \right\}^2 \] \hspace{1cm} (63)

We use (appendix C)

\[ \Lambda_r^2 = \Lambda_{r,in} \hat{T}^{-2c} \left[ \frac{\partial \hat{T}}{\partial x} \right]^2 \]

with

\[ 2c = 5 \frac{2}{\eta} - 2Z \]

or

\[ Z = \frac{5}{2} \frac{1}{\eta} - c \]

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E. g., the following cases are envisaged in chapter (13):

1. $c=1, Z=0 \rightarrow \eta = \frac{2}{3}$

2. $c=1, \eta = \frac{3}{2} \rightarrow Z=\frac{5}{6} = 0.833$

3. $c=1, \eta = 1.6 \rightarrow Z=0.875$

The choice $c=1$ is important for the analytical treatment. Using the model profile (51) we have

$$\frac{\partial \hat{T}}{\partial x} = \frac{\partial (1 - \tanh(x))}{\partial x} = \frac{1}{\cosh^2(x)} = -(1 - \tanh^2(x)) = -(1 - \tanh(x))(1 + \tanh(x)) = -\hat{T}(2 - \hat{T})$$

We rewrite equation (63) as

$$h' + 1.833 = \frac{0.45\Lambda_0^2}{1 + 0.26[0.625(1 + 2\eta_i^{-1})]^2}\left\{ \frac{0.107q^2}{1 + \frac{Q^2}{S^2}} h' + \frac{1}{2} \frac{1}{[\frac{\partial h'}{\partial x}]^2} (g')^2 \right\}$$

$$- \frac{1}{\partial \hat{T}} g'(h' - (1 + \frac{2}{\eta})) + 1.9[h' - 0.8(1 + \frac{1.6}{\eta}]^2 \right\}$$

Now we assume $h' = 0$ and get an equation for $\Lambda = \Lambda_0$ at the zero of $h'$

$$1.83 = \frac{0.45\Lambda_0^2}{1 + 0.26[0.625(1 + 2\eta_i^{-1})]^2}\left\{ \frac{0.5}{T_0^2(2 - \hat{T}_0)^2} (g_0')^2 \right\}$$

$$+ \frac{g'}{T_0(2 - \hat{T}_0)} (1 + 2\eta_i^{-1}) + 1.9[0.8(1 + 1.6\eta_i^{-1})]^2 \right\}$$

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$T_0$, $g'_0$ are temperature and the normalized velocity $g'$ at the zero of $h'$. We approximate the solution by

$$h' = h'_{-\infty}(1 - \frac{\Lambda^2}{\Lambda_0^2})$$

For small $\Lambda_r$ we get the solution at the magnetic axis.

We use the $u_\theta$ - equation and take the limit for $x=-\infty$. Only the term with $g'$ remains

$$(h')_{-\infty} = -1.83 + \frac{0.45}{T^2c} \Lambda_r(-\infty)[\hat{T}(2 - \hat{T})]^2 \frac{1}{2(g')_{-\infty}} \frac{1}{[\hat{T}(2 - \hat{T})]^2}$$

or

$$(h')_{-\infty} = -1.83 + \frac{0.225}{2c} \Lambda_r^2(-\infty)(g')_{-\infty}$$

We use the equation

$$\frac{Q}{S} = 0.5(h' - 0.625(1 + \frac{2}{\eta}))\Lambda_r$$

and combine it with the Ansatz for $h'$

$$\frac{Q}{S} = 0.5((h')_{-\infty}(1 - \frac{\Lambda^2}{\Lambda_0^2}) - 0.625(1 + \frac{2}{\eta}))\Lambda_r$$

From the relation

$$f^2 = \frac{\Lambda_r^2}{\Lambda_0^2}$$

we get

$$f = \frac{\Lambda_{r,inf}}{\Lambda_0} \frac{2 - \hat{T}}{\hat{T}c^{-1}}$$

At the boundary we obtain

$$1 = \frac{\Lambda_{r,inf}}{\Lambda_0} \frac{2 - \hat{T}_0}{\hat{T}_0c^{-1}}$$
We get
\[ \frac{Q}{S} = 0.5((h')_{-\infty} - 0.625(1 + \frac{2}{\eta}) - (h')_{-\infty}f^2)\Lambda_r \]
or if we replace \( \Lambda_r \) -
\[ \frac{Q}{S} = 0.5((h')_{-\infty} - 0.625(1 + \frac{2}{\eta}) - (h')_{-\infty}f^2)\Lambda_0 f \]
We discuss
\[ F = \frac{2 - \hat{T}}{\hat{T}^{c-1}} \]
as a function of \( \hat{T} \). \( F \) has an extremum for
\[ \hat{T} = 2\frac{c-1}{c-2} \]
We investigate \( \frac{Q^2}{S^2} \) as function of \( f^2 \). It has a maximum for
\[ f^2 = f_{max}^2 = \frac{1}{3}[1 - \frac{0.625(1 + \frac{2}{\eta})}{(h')_{-\infty}}] \]
and a minimum for
\[ f^2 = f_{min}^2 = 1 - \frac{0.625(1 + \frac{2}{\eta})}{(h')_{-\infty}} \]
This value is larger than unity and outside the range of interest.
The maximum value is
\[ \frac{Q^2}{S^2} = 0.26\Lambda^2_0[(h')_{-\infty}]^2 \frac{4}{27}[1 - \frac{0.625(1 + \frac{2}{\eta})}{(h')_{-\infty}}]^3 \]
We approximate \( \frac{Q^2}{S^2} \) by a parabola which interpolates between \( f=0, f=f_m \) and \( f=1 \). We get
\[ \frac{Q^2}{S^2} = 0.26\Lambda^2_0[(h')_{-\infty}]^2[f_m^2(9f_m^2 - 1) - (5f_m^2 - 1)f^2]f^2 \]
The toroidal velocity is then given by

\[ g'(\hat{T}) = g'(\hat{T}_0) + 0.107q^2(h')_{-\infty} \int_{\hat{T}_0}^{\hat{T}} \frac{(1 - f^2)(2 - \hat{T})d\hat{T}}{1 + \alpha f^2 + \beta f^4} \]

\(
\alpha = 0.26\Lambda_0^2(h')^2f_m(9f_m - 1)
\)

and

\(
\beta = -0.26\Lambda_0^2h'^2f_m(5f_m - 1)
\)

One can factorize \(1 + \alpha f^2 + \beta f^4\) and gets

\[ 1 + \alpha f^2 + \beta f^4 = \beta(f^2 - a)(f^2 - b) \]

where

\[
\begin{pmatrix}
  a \\
  b
\end{pmatrix} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2\beta}
\]

In the expression

\[ f = \frac{\Lambda_{r,inf}2 - \hat{T}}{\Lambda_0 \hat{T}^{c-1}} \]

we put \(\hat{T} = \hat{T}_0\) and get

\[ 1 = \frac{\Lambda_{r,inf}2 - \hat{T}_0}{\Lambda_0 \hat{T}_0^{c-1}} \]

or

\[
\frac{\Lambda_{r,inf}}{\Lambda_0} = \frac{\hat{T}_0^{c-1}}{2 - \hat{T}_0}
\]

\[
\Lambda_0 = \Lambda_{r,inf}2 - \hat{T}_0 \frac{\hat{T}_0^{c-1}}{\hat{T}_0^{c-1}}
\]

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We consider $c=1$ implying
\[ f = \frac{\Lambda_{r,\text{inf}}}{\Lambda_0} (2 - \hat{T}) \]
because in this case a simple analytical integration is possible. This integration yields the jump
\[ u_\phi(\hat{T} = 2) - u_\phi(\hat{T}_0) = -\frac{0.107q^2(h')-\infty}{2\sqrt{\alpha^2 - 4\beta}} \frac{\Lambda_0}{\Lambda_{r,\text{inf}}} \ln\left[ \left( \frac{a}{1-a} \right)^{1-a} \left( \frac{-b}{b-1} \right)^{b-1} \right] \]
The radial dependance is given by
\[ u_\phi(\hat{T}) - u_\phi(\hat{T}_0) = -\frac{0.107q^2(h')-\infty}{2\sqrt{\alpha^2 - 4\beta}} \frac{\Lambda_0}{\Lambda_{r,\text{inf}}} \ln\left[ f^2 - a \right]^{1-a} \left[ b - f^2 \right]^{b-1} \]
We assume
\[ \eta = 1.6 \]
and (neglecting the small term $\frac{0.225}{2\pi} \Lambda_r^2 (-\infty)(g')^2_{-\infty}$ approximately)
\[ (h')_{-\infty} = -1.83 \]
and use the boudary value
\[ (h')_{rs} = 0 \]
to compute $\Lambda_0^2$. We get the equation
\[ 1.83 = \frac{0.45\Lambda_0^2}{1 + 0.26[0.625(1 + 2\eta_i^{-1})]^2} \Lambda_0^2 \left\{ \frac{u_{\phi,0}^2}{\hat{T}_0^2(2 - \hat{T}_0)^2} + \frac{u_{\phi,0}}{\hat{T}_0(2 - \hat{T}_0)} (1 + 2\eta_i^{-1}) + 1.9[0.8(1 + 1.6\eta_i^{-1})]^2 \right\} \]
In addition we assume $u_{\phi,0} = 0$ (concerning the boundary values also see chapter 6) and get

\[
1.83 = \frac{0.45\Lambda_0^2}{1 + 0.26^2\mu_i^{-1}} \{ +1.9[0.8(1 + 1.6\eta_i^{-1})]^2 \}
\]

or

\[
1.83 + 1.83 \cdot 0.26[0.625(1 + 2\eta_i^{-1})]^2\Lambda_0^2 = 0.45\Lambda_0^2\{ +1.9[0.8(1 + 1.6\eta_i^{-1})]^2 \}
\]

Therefore

\[
\Lambda_0^2(1.83 \cdot 0.26[0.625(1 + 2\eta_i^{-1})]^2 - 0.45 \cdot 1.9[0.8(1 + 1.6\eta_i^{-1})]^2) = -1.83
\]

or

\[
\Lambda_0^2 = \frac{-1.83}{(1.83 \cdot 0.26[0.625(1 + 2\eta_i^{-1})]^2 - 0.45 \cdot 1.9[0.8(1 + 1.6\eta_i^{-1})]^2)}
\]

and finally

\[
\Lambda_0^2 = 1.47
\]

$g'$ and $h'$ allow to compute the radial electric field which is given by

\[
E_r = B_{\theta}u_{\phi} - B_{\phi}u_{\theta} + \frac{T_i \partial P_i}{e_i \partial r}
\]

This can be rewritten as

\[
\frac{E_r}{B_{\phi}} = \frac{B_{\theta}}{B_{\phi}} u_{\phi} - u_{\theta} + \frac{T}{e_i B_{\phi} L_{\psi}} \frac{1}{nT} \left[ n \frac{\partial T}{\partial x} + T \frac{\partial n}{\partial x} \right]
\]

Introducing the ratio $\eta$ entails

\[
\frac{E_r}{B_{\phi}} = \frac{B_{\theta}}{B_{\phi}} u_{\phi} - u_{\theta} + \frac{T_i}{e_i B_{\phi} L_{\psi}} \frac{1}{T} \frac{\partial \hat{T}}{\partial x} \left[ 1 + \frac{1}{\eta} \right]
\]
or with the help of the velocity $v_T$

\[
\frac{E_r}{B_\phi} = v_T \left\{ \frac{B_\theta}{B_\phi} g - h + \frac{\partial \hat{T}}{\partial x} \left[ 1 + \frac{1}{\eta} \right] \right\}
\]

We introduce $g'$ and $h'$ and get

\[
\frac{E_r}{B_\phi} = v_T \left\{ g' + \frac{\partial \hat{T}}{\partial x} \left[-h' + 1 + \frac{1}{\eta} \right] \right\}
\]

The radial electric field is then

\[
E_r = \frac{T_{inf}}{c_i L_\psi} \left[ g' + (-h' + 1 + \frac{1}{\eta}) \frac{\partial \hat{T}}{\partial x} \right]
\]
10 Numerical solution method

The second order equation is replaced by two first order equations. The 1st equation is that for $\frac{\partial g}{\partial x}$ which can be written as

$$\frac{\partial g}{\partial x} = g + \frac{0.107q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T B}{\partial x}$$

(65)

The 2nd equation is that for $G$ which is related to the first derivative of $g$ by

$$G = \frac{\partial g}{\partial x} - \frac{0.107q^2}{1 + \frac{Q^2}{S^2}} \frac{\partial \ln T B}{\partial x}$$

(66)

Thus the 2nd equation looks like

$$\frac{\partial}{\partial x} G = \frac{m_i n_{in,f} L_x^2}{\eta_2,inf \tilde{\eta}_2} \left[ \hat{n} v_{cx} g - \hat{n} \hat{M}_{\phi,i} \right]$$

$$+ \frac{J_r B_\theta}{m_i v_T n_{in,f}} - \frac{\eta_2,inf}{m_i n_{in,f} L_x^2} \hat{\eta}_2 G \frac{\partial}{\partial x} \ln(\eta_2)$$

(67)

or

$$\frac{\partial}{\partial x} G = \frac{1}{\eta' \tilde{\eta}_2} \left[ \hat{n} v_{cx} g - \hat{n} \hat{M}_{\phi,i} \right]$$

$$+ \frac{J_r B_\theta}{m_i v_T n_{in,f}} - \eta' \hat{\eta}_2 G \frac{\partial}{\partial x} \ln(\eta_2)$$

(68)

In addition we have the equation for the poloidal rotation which can be written as a nonlinear equation for $h$

$$(h + 1.833 \frac{\partial T}{\partial x}) \hat{T} = T_{45} \left[ \frac{B_{\phi}}{B_\theta} \hat{T} \frac{\partial g}{\partial x} + \frac{1}{2}g^2 \right.$$}

$$- \frac{B_{\phi}}{B_\theta} (h - \hat{T} \frac{\partial \ln (T n_i^2)}{\partial x})$$

39
\[ +1.9 \frac{B_0^2}{B_0} [h - 0.8 \hat{T} \frac{\partial \ln (T n_i^{1.6})}{\partial x}]^2 \] (69)

For a given temperature and density profile and the radial dependence of the normalized toroidal velocity \( g \) this equation is solved for \( h \) by means of the solver ZEROIN [28] for transcendental equations.
11 Friction due to recycled neutrals

The charge exchange frequency of a neutral gas with the density \( n_0 \frac{1}{m^3} \) is given by

\[
\nu_{cx} = 4.7 \times 10^{-14} \frac{m^3}{sec} n_0 \frac{1}{m^3}
\]

We multiply with the characteristic time (8) and get as charge exchange term in equation (67)

\[
T_{CX} = \frac{\hat{n}}{\hat{n}_2} t_c \nu_{cx} g = \hat{T}^{0.5 - Z} t_c \nu_{cx} g
\]

We assume that the neutral gas density decays as

\[
n_0 = N_0 \exp\left(\frac{L_{ps}}{L_{neu}} x\right)
\]

\( N_0 \) is the neutral gas density at the inflection point.

In the case of the Alcator [30] we have \( t_c = 3.3 \times 10^{-3} \) sec, \( N_0 = 5 \times 10^{14} \) and one gets

\[
T_{cx}(r = r_{inf}) = 4.7 \times 10^{-14} \times 5 \times 10^{14} \times 3.3 \times 10^{-3} = 8.08 \times 10^{-2}
\]

This has an influence on the step \( g - g_0 \) already.
12 Neutral injection

The injected current $I_b$ in Amp gives rise to an increase of the mean density by

$$\dot{N}_0 = \frac{I_b[A]}{e_0}$$

$e_0$ is the elementary charge. Resorting to the injected power $P$ and the beam energy $E_0$ we get

$$\dot{N}_0 = \frac{P_{MW}}{E_0,keV} \frac{10^3}{e_0}$$

or

$$\dot{N}_0 = 0.624 \ 10^{22} \frac{P_{MW}}{E_0,keV}$$

At large energies the rate coefficient $< \sigma v >_{cx}$ saturates at

$$< \sigma v >_{cx} = 1.5 \ 10^{-7} \frac{cm^3}{sec}$$

The charge exchange frequency is

$$< \sigma v >_{cx} \dot{N}_0 = 1.5 \ 10^{-13} \frac{m^3}{sec} \ 0.624 \ 10^{22} \frac{P_{MW}}{E_0,keV}$$

or

$$< \sigma v >_{cx} \dot{N}_0 = 0.94 \ 10^9 \frac{m^3}{sec} \frac{P_{MW}}{E_0,keV}$$

Accounting for the geometry of a pencil beam and of a circular tokamak plasma [24] we get for the deposited momentum

$$\dot{M}_{||,i} = \frac{\dot{N}_0 < \sigma v >_{cx}}{\sqrt{2\epsilon(2\pi)^2 r R}}$$
This can be rewritten as

\[ \dot{M}_{\|,i} = 0.94 \times 10^7 \frac{-1}{r^2 R^2} \frac{P_{MW}}{E_{0,keV}} \frac{100}{\sqrt{24\pi^2}} \]

or as

\[ \dot{M}_{\|,i} = 0.94 \times 10^7 \frac{-1}{r^2 R^2} \frac{P_{MW}}{E_{0,keV}} 1.79 \]

Finally we have

\[ \dot{M}_{\|,i} = 1.68 \times 10^7 \frac{-1}{r^2 R^2} \frac{P_{MW}}{E_{0,keV}} \]
13 Results

The equations for $u_\phi$ and $u_\theta$ are solved numerically (1) to benchmark with a simplified analytical theory for $\Lambda_1=0$ and vanishing neutral gas density and (2) to compare with the theory accounting for finite $\Lambda_1$ effects and (3) to explore the parameter space in regions where the analytical theory is not valid, in particular in the case where the neutral gas density larger than $10^{17} m^{-3}$ and/or neutral injection is important.

The method resort to an ODE - solver for the classical momentum balance which is combined with a solver for transcendental equations yielding $v_\theta$.

Several integration procedures such as the Adams - Pece -, Burlish - Stoer -, or the fifth order Cash - Carp - Runge - Kutta - method were applied showing that the results are almost independent from the special numerical method.

13.1 Comparison with the simplified analytical theory of section 8 and the halfanalytical theory of section 7

For the case $T_{inf}=165$ eV, $L_\phi = 0.76$ cm, $r_{inf}=20.8$ cm, $r_s = 21.5$ cm (Fig. 1), we get the jump $\Delta u_\phi = u_\phi - u_{\phi 0} = 140 \text{ km sec}^{-1}$, the analytical and the numerical solutions agree exactly (Fig. 2). This jump is much larger than the measured one ($\Delta u_\phi \approx 35 \text{ km sec}^{-1}$) because the approximation $Q_2^\phi = 0$ was made, i. e. in the numerical and the analytical approach $\Lambda_1 = C_\eta = 0$ was assumed.

The halfanalytical treatment in chapter (7) resorts to the neo-classical solution (48). One gets exact agreement with the ODE
- integration (Fig. 3) if in the numerical method the solution of equation (24) is replaced by the neoclassical value \((C_\eta = 0)\). The corresponding jump of the approximate solution \(\Delta u_\phi = 43 \text{ km sec}^{-1}\) is now close to measured one.

Since the neoclassical approximation of \(u_\theta\) is rather accurate the halfanalytical solution (circles) agrees well with the exact solution (triangles). In Fig. 4 both solution are compared. We see that the 'neoclassical jump' is around 10\% larger than the corresponding jump of the exact numerical solution \(\Delta u_\phi = 35 \text{ km sec}^{-1}\) which is now close to measured one.

13.2 Comparison with the approximate analytical theory of section 9, accounting for finite \(\Lambda_1\)

The input data are those of ALCATOR C-MOD [30] (partly used already before): \(T_{\text{inf}} = 165 \text{ eV}, n_i = 1.87 \times 10^{20} \text{ m}^{-3}, L_\psi = 0.76 \text{ cm}, r_{\text{inf}} = 20.8 \text{ cm}, r_s = 21.5 \text{ cm}, \epsilon = 1.6, Z_{\text{eff inf}} = 1.57, Z = 0.875, \eta = 1.6, R = 67 \text{ cm}, B_\theta = 0.625 \text{ T} \text{ and } B_\phi = 5.2 \text{ T}.\) Here \(Z_{\text{eff inf}}\) and \(Z\) are defined by \(Z_{\text{eff}}(r) = Z_{\text{eff inf}} r^Z\), i. e. \(Z_{\text{eff inf}}\) is \(Z_{\text{eff}}(r)\) at the inflection point and \(Z\) is a constant. We proceed with the comparison of numerical and analytical solutions in the case of the exact expression for \(\Lambda\). The jumps \(\Delta u_\phi\) and the radial dependence of \(u_\phi\) according to the numerical integration (triangles) and according to the analytical theory (quadrangles) agree well (Fig. 5). The deviation of the maxima (jumps) is \(\approx 8\%\). Both jumps, \(\Delta u_\phi = 34 \text{ km sec}^{-1}\) (numerical) and \(\Delta u_\phi = 37 \text{ km sec}^{-1}\) (analytical), agree approximately with the experimental one (\(\approx 35 \text{ km sec}^{-1}\)) [30]. The deviations of the maxima of \(u_\theta\) and of the minima of \(E_r\) according to the numerical integration and
the analytical theory are less than \( \approx 8\% \) and \( 10\% \), respectively (Fig. 6 and Fig. 7). The radial dependence shown in Fig. 7 is similar to that reported at DIII - D.

13.3 General radial profiles of the plasma parameters

In Fig. 8 the profile

\[
\hat{T} = \hat{T}_0 + (1 - \hat{T}_0) \exp\left(-\frac{x}{1 - \hat{T}_0}\right)
\]

for \( x < 0 \) and

\[
\hat{T} = \exp[-x]
\]

for \( x > 0 \), discussed in appendix A in more detail, is shown. This profile is continuous, has a continuous derivative, but in contrast to the profile (51) the second derivative is discontinuous. In Fig. 9 the normalized velocity \( g \) is shown in the range \([x=-12,...,x=1.4]\). As in the solutions before the toroidal velocity increases within the distance \( L_\psi \) to roughly the same value as before in Fig. 5. The poloidal velocity (Fig. 10) has a sharp maximum of \( 10 \frac{\text{km}}{\text{sec}} \). It becomes zero at \( r=22 \text{ cm} \). Analogously the electric field (Fig. 11) has a sharp local minimum.

We now generalize the profile (51) to

\[
T = T_{inf}\{(1 - \tanh(x)) + (\frac{T_0}{T_{inf}} - 2)(1 - [\frac{r}{r_{inf}}]^2)^2}\}

\( T_0 \) is the temperature at the plasma center (Fig. 12). This profile has been measured at ALCATOR C - MOD [30]. In Fig. 13 \( g \) is shown in the same range as before in Fig. 5. The major part of the step in \( u_\phi \) is located in the boundary region. \( u_\phi \) increases there to around \( 32 \frac{\text{km}}{\text{sec}} \). In the central part of the plasma the
increase is around $10\ km/sec$. $(T_0 = 1020\ eV$ was assumed.) The poloidal velocity has a dominant peak in the boundary region (Fig. 14), and the electric field is somewhat modified in the central region (Fig. 15). However, one has to keep in mind the restricted validity of these calculation including the central core plasma, because there another collisionality regime may be entered where modifications of the present theory are needed.

13.4 Influence of the neutral gas

It turns out that the solution is very sensitive with respect to the neutral gas density. This may indicate that the neutral gas plays a decisive role in the L - H transition. In fact, the neutral gas physics in limiter and divertor devices are quite different.

In Fig. 16 the influence of the neutral gas is investigated. Even the very low density of $N_0 = 5 \cdot 10^{14}/m^3$ an effect on the step can be seen which is reduced in size by 20% and gets a pedestal at the outside. In the poloidal rotation velocity and the radial electric field the influence of such a small neutral gas density can be neglected.

If the neutral density $N_0$ is increased to $N_0 = 5 \cdot 10^{15}/m^3$ the step in $g$ is reduced to about 20%, i.e. almost removed (Fig. 16). Therefore the system may not be able to transit into the H - mode (Fig. 10).

Since not all charge exchange neutrals leave the plasma and contribute to the momentum loss (as it is assumed in equation [15]), the aforementioned neutral gas densities are probably underestimated by a factor 2 - 3. However, the inclusion of a more detailed neutral gas model would be beyond the scope of this
14 Conclusions

The good agreement between the numerical and the analytical model shows that both models are consistent and the comparison with the experimental data should be reasonable. In fact, this comparison shows that in (ohmic discharges of) the ALCA-TOR - device neoclassical physics prevail during the H - mode. Surprising is the strong influence of the neutral gas. A very small density \( n_0 = 5 \times 10^{15} \text{ m}^{-3} \) destroys the step already. Preliminary calculations accounting for neutral beam injection indicate that in this case the neoclassical viscosity must be corrected by an anomalous contribution.
Figure 1: Ion temperature $T[10\text{eV}]$. The inflection point of the temperature profile, $P_{inf}$, is the locus of vanishing curvature. The decay length there and the maximum at $r=0$ (165 eV) are adjusted to ALCATOR C-MOD data.
Figure 2: Toroidal velocity $U_\phi [km/sec]$. Due to the assumptions $\frac{Q}{S} = 0$ and $C_\phi = 0$ the analytical and the numerical solutions agree exactly. By the same reason the jump of the toroidal velocity $\Delta u_\phi = u_\phi - u_{\phi_0} = 140 \text{ km/sec}$ is much too large.
Figure 3: Toroidal velocity $U_\phi [km/sec]$. The half-analytical treatment in chapter (7) resorts to the neoclassical solution (48). One gets exact agreement with the ODE-integration if the solution of equation (24) is replaced by the neoclassical value ($C_\eta = 0$). The corresponding jump of the approximate solution $\Delta u_\phi = 43 \frac{km}{sec}$ is now close to measured one.
Figure 4: Toroidal velocity $U_\phi [km/sec]$. One gets approximate agreement of the halfanalytical solutions (triangles) with the exact ODE - integration. As in Fig. 3 the halfanalytical solution is obtained by replacing $u_\phi$ by the neoclassical value ($C_\eta = 0$). The corresponding jump of the approximate solution $\Delta u_\phi = 43 \frac{km}{sec}$ is now close to measured one.
Figure 5: Toroidal velocity $U_\phi [km/sec]$. The jumps $\Delta u_\phi$ and the radial dependence of $u_\phi$ according to the numerical integration (triangles) and according to the analytical theory (quadrangles) of chapter (9) agree well. The deviation of the maxima (jumps) is $\approx 8\%$. Both jumps agree approximately with the experimental one ($\approx 35 \text{ km/sec}$) [30].
Figure 6: Poloidal velocity $U_\theta [km/sec]$. The deviations of the maxima of $u_\theta$ according to the numerical integration and the analytical theory are less than $\approx 8\%$. 
Figure 7: Radial electric field $E_r [kV/m]$. The deviations of minima of $E_r$ according to the numerical integration and the analytical theory are less than $\approx 10\%$. The radial dependence is similar to that reported at DIII - D.
Figure 8: Ion temperature $T[10eV]$. The temperature profile in the appendix (A) was chosen to demonstrate the effect of different profile shapes. It has an inflection point at $r = r_{inf}$. As in Fig. 1 the central value is 2 times larger than the value at the inflection point, so that similar results can be expected as Figs. 6 and 7.
Figure 9: Toroidal velocity $U_\phi [\text{km/sec}]$. The total jump in $g$ is three times larger than in Fig. 5.
Figure 10: Poloidal velocity $U_\theta [km/sec]$. The maximum of $u_\theta$ is roughly the same as in Fig. 6 analytical theory are less than $\approx 8\%$. The different profile shape of Fig. 8 effects a rather sharply peaked maximum.
Figure 11: Radial electric field $E_r[kV/m]$. The minimum of $E_r$ is of the same order as in Fig. 7. Even in the case of a strongly changed temperature profile the electric field is positive at $r=0$. 
Figure 12: Ion temperature $T[10eV]$. The more realistic temperature profile has a pedestal like in Fig 1 and a parabolic part connecting the edge with the center.
Figure 13: Toroidal velocity $U_\phi [km/sec]$. The input data are those of AL-CATOR C-MOD [30]. The jump $\Delta u_\phi$ and the radial dependence of $u_\phi$ in the edge are the same as in the corresponding Fig. 5. In going to the center the spin-up increases further because of the temperature gradient.
Figure 14: Poloidal velocity $U_\theta [km/sec]$. The maximum of $u_\theta$ due to the pedestal is still pronounced. The poloidal rotation velocity at smaller radii is - according to the temperature gradient - finite but small.
Figure 15: Radial electric field $E_r[kV/m]$. The electric field deviates at the plasma interior from that of Fig. 7 by at most 20%.
Figure 16: Toroidal velocity $U_\phi [km/sec]$. Even at the very low neutral density of $N_0 = 5 \cdot 10^{14} \frac{1}{m^3}$ an effect on the step can be seen which is reduced in size by 25%.
Figure 17: Toroidal velocity $U_0[\text{km/sec}]$. If the neutral density $N_0$ is increased to $N_0 = 5 \cdot 10^{15} \frac{1}{m^3}$, the step in $g$ is reduced to about 16%, i.e. almost removed.
References


A Temperature and density profiles with $L_\psi$ as decay length

Inside the last closed flux surface with radius $r_s$ an exponential decay has been assumed for the temperature and density profile.

$$
\hat{T} = \hat{T}_0 + (1 - \hat{T}_0)\exp\left(-\frac{r - r_{inf}}{L_\psi(1 - \hat{T}_0)}\right)
$$

(73)

or

$$
\hat{T} = \hat{T}_0 + (1 - \hat{T}_0)\exp\left(-\frac{x}{1 - \hat{T}_0}\right)
$$

(74)

with $\hat{T}(r = 0) = \hat{T}_0$ and $\hat{T}(r = r_{inf}) = 1$. $L_\psi$ is related to the temperature scale-length $L_{Ts}$ (see below). Analogously we have for the density profile

$$
\hat{n} = \hat{n}_0 + (1 - \hat{n}_0)\exp\left(-\frac{r - r_{inf}}{L_\psi(1 - \hat{n}_0)}\right)
$$

(75)

or with $r - r_{inf} = L_\psi x$

$$
\hat{n} = \hat{n}_0 + (1 - \hat{n}_0)\exp\left(-\frac{L_\psi}{L_{\psi_n}(1 - \hat{n}_0)}x\right)
$$

(76)

with $\hat{n}(r = 0) = \hat{n}_0$.

$\hat{T}_0, \hat{n}_0$ are the central values and $r_{inf}$ is the radius of the inflection point. $L_{\psi_n}$ is related to the density scale length in an analogous way as is $L_\psi$ to the temperature scale length.

The first and second derivative have the simple forms

$$
\frac{d\hat{T}}{dx} = -\exp\left(-\frac{x}{1 - \hat{n}_0}\right)
$$

(77)

$$
\frac{d\hat{n}}{dx} = -\frac{L_\psi}{L_{\psi_n}}\exp\left(-\frac{L_\psi}{L_{\psi_n}}\frac{x}{1 - \hat{n}_0}\right)
$$

(78)
and
\[ \frac{d^2 \hat{T}}{dx^2} = \frac{1}{1 - \hat{T}_0} \exp\left(\frac{-x}{1 - \hat{T}_0}\right) \] (79)
\[ \frac{d^2 \hat{n}}{dx^2} = \left[ \frac{L_\psi}{L_{\psi_n}} \right]^2 \frac{1}{1 - \hat{n}_0} \exp\left(\frac{-L_\psi}{L_{\psi_n}} \frac{x}{1 - \hat{n}_0}\right) \] (80)

At the inflection point we have
\[ \left(\frac{d\hat{T}}{dx}\right)_{x=0} = -1 \] (81)
\[ \left(\frac{d\hat{n}}{dx}\right)_{x=0} = -\frac{L_\psi}{L_{\psi_n}} \] (82)

The temperature and density scale lengths are
\[ L_T = \frac{T}{\frac{dT}{dr}} = \frac{L_\psi \hat{T}}{\frac{dT}{dx}} = L_\psi \left[ \frac{\hat{T}_0}{-\exp\left(\frac{-x}{1 - \hat{T}_0}\right)} - (1 - \hat{T}_0) \right] \] (83)
\[ L_n = \frac{n}{\frac{dn}{dr}} = \frac{L_\psi \hat{n}}{\frac{dn}{dx}} = \frac{L_\psi \left[ \hat{n}_0 + (1 - \hat{n}_0) \exp\left(-\frac{x - r_{inf}}{L_{\psi_n}(1 - \hat{n}_0)}\right) \right]}{-\frac{L_\psi}{L_{\psi_n}} \exp\left(-\frac{L_\psi}{L_{\psi_n}} \frac{x}{1 - \hat{n}_0}\right)} \]
\[ = L_{\psi_n} \left[ \frac{\hat{n}_0}{-\exp\left(-\frac{L_\psi}{L_{\psi_n}} \frac{x}{1 - \hat{n}_0}\right)} - (1 - \hat{n}_0) \right] \] (84)

respectively.

The temperature and density scale lengths at the inflection point are then
\[ L_{T_{inf}} = \left(\frac{T}{\frac{dT}{dr}}\right)_{inf} = -L_\psi \] (85)
\[ L_{n_{inf}} = \left(\frac{n}{\frac{dn}{dr}}\right)_{inf} = -L_{\psi_n} \] (86)
They are then independent from the maximum values $\hat{T}_0$ and $\hat{n}_0$.

Outside the last closed flux surface also an exponential decay has been assumed for the temperature and density profile. For the temperature we have

$$\hat{T} = \exp\left[\frac{r - r_{inf}}{L_\psi}\right]$$  \hspace{1cm} (87)$$

or

$$\hat{T} = \exp[-x]$$  \hspace{1cm} (88)$$

with $\hat{T}(x = 0) = 1$ and $\hat{T}(r = \infty) = 0$. The first and second derivative have the simple form

$$\frac{d\hat{T}}{dx} = -\exp[-x]$$  \hspace{1cm} (89)$$

$$\frac{d^2\hat{T}}{dx^2} = \exp[-x]$$  \hspace{1cm} (90)$$

The temperature scale length is

$$\frac{T}{\frac{dT}{dr}} = \frac{L_\psi}{\frac{dT}{dx}} = -L_\psi$$  \hspace{1cm} (91)$$

It is a constant and equal to $(\frac{T}{\frac{dT}{dr}})_{r=r_{inf}}$ so that the radial temperature dependance is continuous and 'smooth' at $r=r_{inf}$. (the curvature, however, is discontinuous.)

For the density analogous formulae are valid. In the calculations we assume $L_{\psi n} = L_\psi$. Thus we have utilized the same model behaviour for both T and n with the same scale length $L_\psi$. 

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B Ion - ion collisionality and collision frequency

In the following we collect some standard formulas for the collisionality and the collision frequency. For the collisionality \( \hat{\nu}_i \) we have \((N_i = \frac{n_i}{10^{13}}, N_{iof} = \frac{n_{iof}}{10^{13}})\)

\[
\hat{\nu}_i = \frac{qR}{c_i \tau_i} = 7.14 \times 10^2 \frac{qRN_iZ_{eff,i}}{T_i^2} \tag{92}
\]

With the thermal speed

\[
c_i = \sqrt{\frac{kT_i}{A}} = 0.99 \times 10^4 \sqrt{\frac{T_i[\text{eV}]}{m_i \text{ sec}}} \tag{93}
\]

we get the collision frequency

\[
\nu_i = \frac{1}{\tau_i} = 7.14 \times 10^2 \frac{N_iZ_{eff,i}}{T_i^2} 0.99 \times 10^4 \sqrt{\frac{T_i[\text{eV}]}{A}} \tag{94}
\]

or

\[
\nu_i = \frac{1}{\tau_i} = 7.14 \times 10^6 \frac{N_iZ_{eff,i}}{T_i^2 \sqrt{A}} \tag{95}
\]

The collision time is then

\[
\tau_i = \frac{1}{\nu_i} = \frac{1.414}{7.14} 10^{-6} \frac{T_i^{\frac{3}{2}} \sqrt{\frac{A}{2}}}{N_iZ_{eff,i}} \tag{96}
\]

or

\[
\tau_i = \frac{1}{\nu_i} = 1.98 \times 10^{-7} \frac{T_i^{\frac{3}{2}} \sqrt{\frac{A}{2}}}{N_iZ_{eff,i}} \tag{97}
\]

This formula compares well with that of Braginski if a Coulomb logarithm \( \lambda_{coul} = 15 \) is assumed. More exactly we have

\[
\tau_i = \frac{1}{\nu_i} = \frac{1.98 \times 10^6}{(\frac{\lambda_{coul}}{15})} T_i^{\frac{3}{2}} \sqrt{\frac{A}{2} n_iZ_{eff,i}} \tag{98}
\]
C  Parameter $\Lambda_1$

The connection of $\Lambda_1$ with the velocity $S$ is given by

$$\Lambda_1 = \frac{2\tilde{v}_T B_\phi}{S \ B}$$

$S$ is defined by

$$S = \frac{8rc_i}{\nu_i q R}$$

(99)

Therefore

$$\Lambda_1 = \frac{2\tilde{v}_T B_\phi}{\frac{8rc_i}{\nu_i q R} \ B}$$

where $\tilde{v}_T = \frac{1}{eB_\phi} T_{LT}$ with $L_T = \frac{T}{\frac{\partial \nu_i}{\partial r}}$. Therefore we get

$$\Lambda_1 = \frac{2\frac{1}{eB_\phi} T_{LT,inf} B_\phi}{\frac{8rc_i}{\nu_i q R} \ B} = \frac{\nu_i q^2 R^2}{\Omega_i L_T 4r}$$

At the inflection point we have ($L_T = L_{T,inf}$)

$$\Lambda_{1,inf} = \frac{2\frac{1}{eB_\phi} T_{inf} B_\phi}{\frac{8rc_i}{\nu_i q R} \ B}$$

or

$$\Lambda_{1,inf} = 2\frac{1}{eB_\phi} L_{T,inf} 8r_{inf} 4r_{inf} \nu_{ii,inf} c_{inf}^2$$

with $\Omega_i = \frac{eB}{m_i}$ we get

$$\Lambda_{1,inf} = \frac{\nu_{ii,inf}}{\Omega_i L_{T,inf} 4r_{inf}} q^2 R^2$$
The quantity $\Lambda_{1, inf}$ used in appendix G is defined by

$$
\Lambda'_{1, inf} = \frac{\nu_{ii, inf}}{\Omega_i} \frac{q^2 R^2}{L_\psi^4 r_{inf}} = \frac{L_{T, inf}}{L_\psi} \Lambda_{1, inf} = -\Lambda_{1, inf}
$$

for $L_{T, inf} = -L_\psi$ as in the case of the preceding temperature profiles.

The radial dependence due to the plasma parameters is given by

$$
\Lambda_1 = \Lambda_{1, inf} \frac{\hat{n}}{n} \frac{L_\psi}{L_T} \frac{r_{inf}}{r} \frac{L_{T, inf}}{L_\psi^4 r_{inf}} \frac{d\ln(T)}{dr}
$$

or

$$
\Lambda_1 = \Lambda_{1}' \frac{L_\psi}{L_T}
$$

$\Lambda_{1}'$ is given by

$$
\Lambda_{1}' = \Lambda_1 \frac{L_T}{L_\psi} = \frac{\nu_{ii} q^2 R^2}{\Omega_i \frac{L_\psi^4 r_{inf}}{L_T}}
$$

we assume $\hat{n} = \hat{T}^{\frac{\eta}{2}}$ and $\hat{Z}_{eff} = \hat{T}^Z$ where $\eta$ can be written as

$$
\eta = \frac{\frac{d\ln(\hat{T})}{dx}}{\frac{d\ln(\hat{n})}{dx}}
$$

We get (appendix B)

$$
\Lambda_1 = \Lambda_{1, inf} \frac{1}{\hat{T}^{\frac{5}{2} - \frac{1}{\eta} - Z}} \frac{r_{inf}}{r} \frac{L_{T, inf}}{L_\psi^4 r_{inf}} \frac{d\ln(T)}{dr}
$$

or

$$
\Lambda_1 = \Lambda_{1, inf} \frac{1}{\hat{T}^{\frac{5}{2} - \frac{1}{\eta} - Z}} \frac{r_{inf}}{r} \frac{d\hat{T}}{dx}
$$

If we assume that the exponent of $\hat{T}$, $\frac{5}{2} - \frac{1}{\eta} - Z$, is equal to zero, we get

$$
\Lambda_1 = \Lambda_{1, inf} \frac{r_{inf}}{r} \frac{L_{T, inf}}{L_\psi^4 r_{inf}} \frac{d\ln(T)}{dr}
$$
or
\[ \Lambda_1 = \Lambda_{1,inf} \frac{r_{inf} L_{T,inf}}{r L_T} \]

It is sometimes convenient to introduce the quantity \( \Lambda_r \). We use the expression
\[ \Lambda_1 = \frac{2 \dot{v}_T B_\phi}{S B} \]
and replace \( S \)
\[ \Lambda_1 = 2 \dot{v}_T \frac{\dot{v}_i q R B_\phi}{8 r c_i B} \]
Then we replace \( \dot{v}_T \) and get
\[ \Lambda_1 = 2 \frac{T}{e B_\phi L T} \frac{\dot{v}_i q R B_\phi}{8 r c_i B} \]
We use \( q = \frac{r B_\phi}{RB_\theta} \), \( T = m_i c_i^2 \), \( e B = m_i \Omega_i \) and obtain
\[ \Lambda_1 = 2 \frac{m_i c_i^2}{m_i \Omega_i L T R B_\theta 8 r c_i B} \frac{\dot{v}_i r R}{1} \]
With \( a_i = \frac{\dot{v}_i}{\Omega_i} \) we get
\[ \Lambda_1 = \frac{a_i}{L T B_\theta} \frac{\dot{v}_i}{B_\phi} \]
We define \( \Lambda_r = 4 \Lambda_1 \) and obtain
\[ \Lambda_r = \frac{a_i}{L T B_\theta} B_\phi \]
With \( a_{pi} \), the Larmor radius in the poloidal field we get the simple expression
\[ \Lambda_r = \frac{a_{pi}}{L T} \frac{\dot{v}_i}{B_\phi} \]
D  Reciprocal time $\eta'$

The quantity $\eta'$, is defined as

$$
\eta' = \frac{1}{m_i n_{inf} L_\psi^2} \eta_2 = \frac{6}{5} \frac{N_{inf} k T_{\tau_{ii}}}{\Omega_i^2 \tau_{ii}^2 m_i n_i L_\psi^2}
$$

(100)

With the expression for the thermal speed $c_i$ we get

$$
\eta' = \frac{6}{5} \frac{c_i^2}{\Omega_i^2 \tau_{ii} L_\psi^2}
$$

(101)

Replacing $\tau_{ii}$ by means of the collisionality yields

$$
\eta' = \frac{6}{5} \frac{c_i^2}{\Omega_i^2 L_\psi^2} \nu_i c_i
$$

(102)

With the expressions for the collisionality and the thermal speed we get

$$
\eta' = \frac{6}{5} \frac{c_i^2}{\Omega_i^2 L_\psi^2} \frac{7.14 10^2 q R N Z_{eff}}{T_{\tau_{ii}}^2} c_i
$$

(103)

Introducing the Larmor radius $a_i = \frac{c_i}{\Omega_i}$ gives

$$
\eta' = \frac{6}{5} \frac{a_i^2}{L_\psi^2} \frac{7.14 10^2 q R N Z_{eff i} T_{\tau_{ii}}^2}{T_i^2} 0.99 10^4 \sqrt{\frac{T_i eV}{A}}
$$

(104)

or

$$
\eta' = \frac{6}{5} \frac{a_i^2}{L_\psi^2} \frac{N_i Z_{eff i}}{T_i^2} 0.99 10^4 \sqrt{\frac{1}{A}}
$$

(105)

and finally

$$
\eta' = 8.57 10^{-7} \frac{a_i^2}{L_\psi^2} \frac{n_i Z_{eff i} T_{\tau_{ii}}}{T_i^{\frac{3}{2}}} \sqrt{\frac{1}{A}}
$$

(106)
Replacing \( \tau_{ii} \) by means of \( \Lambda_1 \) we get

\[
\eta' = \frac{6 c_i^2}{\Omega_i^2 L_\psi^2} |\Lambda_1| \frac{L_\psi A r_i}{q^2 R^2 \Omega_i} \tag{107}
\]

or, simplifying

\[
\eta' = \frac{6 a_i^2}{L_\psi^2} |\Lambda_1| \frac{L_\psi A r_i}{q^2 R^2 \Omega_i} \tag{108}
\]

\( a_i \) is the Larmor radius. Outside (inside) the inflection point we have

\[
\eta' = \eta'_{inf} \hat{T}^{-\frac{1}{2}} \hat{n} \tag{109}
\]

with

\[
\eta'_{inf} = \frac{6 a_i^2}{L_\psi^2} |\Lambda_{1,inf}| \frac{L_\psi A r_{inf}}{q^2 R^2} \Omega_i \tag{110}
\]
We use two expressions giving the exact result, the first resorts to $\chi||$ and the second to $\Lambda_1$. The first is

$$\frac{S}{\bar{v}_T} = \frac{2r\chi||}{4nq^2R^2\bar{v}_T}$$  \hspace{1cm} (111)$$

We use

$$\chi|| = 4c^2n\tau_{ii} = \frac{4Ti_n\tau_{ii}}{m_i}$$

and get

$$\frac{S}{\bar{v}_T} = \frac{2r\chi||}{nq^2R^2\bar{v}_T} = \frac{2r4Ti}{m_i\nu_{ii}q^2R^2\bar{v}_T}$$  \hspace{1cm} (112)$$

or with $\bar{v}_T = \frac{T}{eB_o} \frac{\partial n_T}{\partial r}$

$$\frac{S}{\bar{v}_T} = \frac{2r4Ti_n}{m_i\nu_{ii}nq^2R^2\bar{v}_T} = \frac{\Omega_iL_T8r}{\nu_{ii}q^2R^2} = \frac{2}{\Lambda_1}$$  \hspace{1cm} (113)$$

At the inflection point we have

$$\frac{S_{inf}}{\bar{v}_{T,inf}} = \frac{2r_{inf}4T_{inf}n}{m_i\nu_{ii,inf}nq^2R^2\bar{v}_{T,inf}} = \frac{\Omega_iL_{T,inf}8r_{inf}}{\nu_{ii,inf}q^2R^2} = \frac{2}{\Lambda_{1,inf}}$$  \hspace{1cm} (114)$$

or with $v_T = \frac{1}{eB_o} \frac{T_{inf}}{L_o}$

$$\frac{S_{inf}}{v_T} = \frac{2r_{inf}4T_{inf}n}{m_i\nu_{ii,inf}nq^2R^2v_T} = \frac{\Omega_iL_{\psi}8r_{inf}}{\nu_{ii,inf}q^2R^2} = \frac{2}{\Lambda_{1,inf}}$$  \hspace{1cm} (115)$$

Alternatively the formula

$$\frac{S_{inf}}{v_T} = \frac{2}{\Lambda'_{1,inf}}$$  \hspace{1cm} (116)$$
can be employed which is analytically identical with the aforementioned formula. The scaling with the plasmaparameters is given by

\[ \frac{S}{v_T} = \frac{S}{\bar{v}_T} \frac{\partial \hat{T}}{\partial x} = \frac{S}{\bar{v}_T} L_\psi \frac{\partial \hat{T}}{\partial r} = \frac{S}{\bar{v}_T} L_\psi \frac{\partial \hat{T}}{\bar{v}_T} = \]  

\[ \frac{S}{\bar{v}_T} L_\psi \hat{T} = \frac{S}{v_T} \frac{\hat{T}^3}{\hat{n} L_T} \]  

(117)  

\[ \frac{S}{v_T} = \frac{S_{inf}}{v_T} \frac{\hat{T}^3}{\hat{n}} \]  

(118)  

\[ \frac{S}{4v_T} = \frac{1}{2\Lambda_{1,inf}} \frac{\hat{T}^3}{\hat{n}} \]  

(119)  

We also have

\[ \frac{S}{\bar{v}_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} = \frac{8}{\Lambda_r \frac{B_\phi}{B}} \]  

(120)  

and

\[ \frac{S}{v_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \frac{\bar{v}_T}{v_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \frac{L_\psi}{L_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \]  

(121)  

\[ \frac{S}{v_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \frac{\bar{v}_T}{v_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \frac{L_\psi}{L_T} = \frac{2}{\Lambda_1 \frac{B_\phi}{B}} \]  

(122)  

**F Velocity Q**

Q is defined in [22] and is given by

\[ Q = (4u_\theta - 5v_n - 2.5\bar{v}_T) \frac{B_\phi}{B} \]  

\[ Q = (4u_\theta - 5v_n(1 + \frac{1}{2\eta})) \frac{B_\phi}{B} \]  

with the neoclassical value \( u_\theta = \kappa_0 \frac{\partial T}{\partial x} v_T \)

\[ Q = (4\kappa_0 \frac{\partial T}{\partial x} v_T - 5v_n(1 + \frac{1}{2\eta})) \frac{B_\phi}{B} \]  

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We get by normalizing
\[
\frac{Q}{4v_T} = h - \hat{T} \frac{\partial \ln(n^2 T)}{\partial x} \frac{2.5}{4}
\]
and
\[
\frac{Q}{\hat{v}_T} = (4h' - \frac{5}{\eta} - 2.5) \frac{B_\phi}{B}
\]

$h'$ is defined by
\[
h' = \frac{u_\theta}{\hat{v}_T}
\]

Using the neoclassical expression
\[
u_\theta = \frac{\kappa}{eB} \frac{\partial T}{\partial r}
\]
and
\[
h' = \frac{\kappa}{eB} \frac{\partial T}{\partial r} = \kappa
\]
we get
\[
\frac{Q}{\hat{v}_T} = (4\kappa - \frac{5}{\eta} - 2.5) \frac{B_\phi}{B}
\]

It follows the derivative \( \frac{\partial Q}{\partial h} = 4v_T \). We get the ratio
\[
\frac{Q}{S} = \frac{h - \hat{T} \frac{\partial \ln(n^2 T)}{\partial x} \frac{2.5}{4}}{\frac{1}{2\kappa} \frac{B_\phi}{B}}
\]
\[
\frac{Q}{S} = \frac{h - \hat{T} \frac{\partial \ln(n^2 T)}{\partial x} \frac{2.5}{4}}{\frac{S_{int} T^2}{4v_T} \frac{B_\phi}{n B}}
\]

We also get by using $Q$ and $S$ normalized to $\hat{v}_T$
\[
\frac{Q}{S} = \frac{(4h' - \frac{5}{\eta} - 2.5) \frac{B_\phi}{B}}{\frac{8}{\Lambda_r} \frac{B_\phi}{B}}
\]
or with $\frac{2\delta}{8} = 0.3125$

\[
\frac{Q}{S} = \frac{(0.5h' - 0.3125(1 + \frac{2}{\eta}))\frac{B_0}{B}}{\frac{1}{\Lambda_r} \frac{B_0}{B}}
\]

\[
\frac{Q}{S} = (0.5h' - 0.3125(1 + \frac{2}{\eta}))\Lambda_r
\]

\[
\frac{Q}{S} = 0.5(h' - 0.625(1 + \frac{2}{\eta}))\Lambda_r
\]
Evaluation of $T_{36}$

Two exact expressions are given, the first involves the collision-frequency and the second $\Lambda_1$.

G.1 1st expression

In the first case we have

$$T_{36} = 0.36 \frac{\eta_{1s}}{\eta_{0s}(1 + \frac{Q^2}{S^2})} q^2 \frac{1}{\tilde{T}v_T L_\psi^2} \left( \frac{\partial \ln T}{\partial x} \right)$$

or

$$T_{36} = 0.36 \frac{6}{5} \frac{\nu_{1s}^2}{0.96} \frac{1}{\Omega^2} \frac{1}{(1 + \frac{Q^2}{S^2})} q^2 \frac{1}{\tilde{T}v_T L_\psi^2} \left( \frac{\partial \ln T}{\partial x} \right)$$

G.2 2nd expression

We introduce

$$\Lambda'_1 = \frac{\nu_i q^2 R^2}{\Omega_i L_\psi 4 r_{inf}}$$

($\Lambda'_1$ does not depend on $L_T$) into $T_{36}$ and get

$$T_{36} = 0.36 \frac{6}{5} \frac{\nu_{1s}^2 L_\psi^2 16r^2}{0.96} \frac{1}{q^4 R^4} \frac{1}{(1 + \frac{Q^2}{S^2})} q^2 \frac{1}{\tilde{T}v_T L_\psi^2} \left( \frac{\partial \ln T}{\partial x} \right)$$

or

$$T_{36} = 0.36 \frac{6}{5} \frac{\nu_{1s}^2 16r^2}{0.96} \frac{1}{q^2 R^2 (1 + \frac{Q^2}{S^2})} \tilde{T} v_T \left( \frac{\partial \ln T}{\partial x} \right)$$

The dependence on the plasma parameters is given by

$$T_{36} = 0.45 q^2 \frac{R^2 \nu_{1s,inf}^2 (\tilde{T}^2 \frac{1}{2} \tilde{n} \tilde{T}^z)^2}{\Omega^2} \frac{1}{(1 + \frac{Q^2}{S^2})} \frac{1}{v_T} \left( \frac{\partial \tilde{T}}{\partial x} \right)$$

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H Electric drift velocity

The electric drift speed is given by

\[- \frac{E_r}{B_\phi} = v_E = v_\Theta - \frac{B_\theta}{B} v_{||,i} - v_N - \tilde{v}_T\]

We use

\[g = \frac{v_{||,i}}{v_T}\]

\[h = \frac{v_\Theta}{v_T}\]

and get the normalized drift speed

\[\hat{v}_E = \frac{v_E}{v_T} = \frac{v_\Theta - \frac{B_\theta}{B} v_{||,i} - v_N - \tilde{v}_T}{v_T}\]

\[\hat{v}_E = h - \frac{B_\theta}{B} g - \frac{v_N}{v_T} - \frac{\tilde{v}_T}{v_T}\]

The normalized speeds are given by

\[\frac{\tilde{v}_T}{v_T} = \frac{T}{eB} \frac{\partial \ln T}{\partial r} = T \frac{\partial \ln n}{\partial r} L_\psi = T \frac{\partial \ln n}{\partial x} = \frac{\partial T}{\partial x}\]

\[\frac{v_N}{v_T} = \frac{\frac{T}{eB} \frac{\partial \ln n}{\partial r}}{\frac{\tilde{T}}{eB} \frac{\partial \ln n}{\partial r}} = T \frac{\partial \ln n}{\partial r} L_\psi = \frac{\tilde{T}}{\tilde{n}} \frac{\partial \ln n}{\partial x} = \frac{\hat{T}}{\hat{n}} \frac{\partial \ln n}{\partial x}\]
I Temperature profile in Alcator C - MOD

\[ \frac{T_i(r)}{T_i(0)} = \frac{T_i(0) - 2T(r_{inf})}{T_i(0)} \left(1 - \left[\frac{r}{r_{inf}}\right]^2\right)^2 + \]

\[ \frac{T_i(r_{inf})}{T_i(0)} \left[1 - \tanh \frac{r - r_{inf}}{\Delta_T}\right] \quad (123) \]

or

\[ T_i(r) = (T_i(0) - 2T(r_{inf})) \left(1 - \left[\frac{r}{r_{inf}}\right]^2\right)^2 + \]

\[ \frac{T_i(r_{inf})}{T_i(0)} \left[1 - \tanh \frac{r - r_{inf}}{\Delta_T}\right] \quad (124) \]

The decaylength at the inversion point is \( \Delta_T \) again because

\[ \frac{dT_i(r)}{dr} = -(T_i(0) - 2T(r_{inf}))2 \left(1 - \left[\frac{r}{r_{inf}}\right]^2\right)^2 \frac{r}{r_{inf}^2} \]

\[ \frac{T_i(r_{inf})}{\Delta_T} \frac{1}{-\cosh^2 \frac{r - r_{inf}}{\Delta_T}} \quad (125) \]

at \( r = r_{inf} \) we have

\[ \frac{dT_i(r)}{dr} = -\frac{T_i(r_{inf})}{\Delta_T} \quad (126) \]

The derivative of the pedestal is given by

\[ \frac{d}{dr} \frac{T_i(r)}{T_i(0)} = \frac{d}{dr} \frac{T_i(r_{inf})}{T_i(0)} \left[1 - \tanh \frac{r - r_{inf}}{\Delta_T}\right] \quad (127) \]

and

\[ \frac{d}{dr} \frac{T_i(r)}{T_i(0)} = \frac{T_i(r_{inf})}{T_i(0)} \frac{1}{\Delta_T} \left[-\frac{1}{\cosh^2 \left[\frac{r - r_{inf}}{\Delta_T}\right]}\right] \quad (128) \]
The logarithmic derivative of the pedestal can be written as

\[
\frac{1}{T_i(r)} \frac{d}{dr} T_i(r) = T_i(r_{inf}) \frac{1}{\Delta_T} \left[ - \frac{1}{\cosh^2\left(\frac{r-r_{inf}}{\Delta_T}\right)} \right]
\]

or as

\[
\frac{1}{T_i(r_{inf})[1 - \tanh\frac{r-r_{inf}}{\Delta_T}]}
\]

or as

\[
\frac{1}{T_i(r)} \frac{d}{dr} T_i(r) = \frac{1}{\Delta_T} \left[ - \frac{1}{\cosh^2\left(\frac{r-r_{inf}}{\Delta_T}\right)} \right]
\]

\[
\frac{1}{[1 - \tanh\frac{r-r_{inf}}{\Delta_T}]}
\]