Coarsening of Cracks in a Uniaxially Strained Solid

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We discuss the stress relaxation in a uniaxially strained solid due to the coarsening of a system of parallel cracks. We emphasize similarities and differences of this process to Ostwald ripening in a first order phase transition. A conventional mean-field approximation breaks down and several independent length scales have to be taken into account. Strong elastic interactions between the cracks determine the growth behavior. We derive scaling laws for the coarsening of the different length scales involved and the time evolution of stress relaxation, finally leading to the equilibrium state of a fractured body. The characteristic size of the cracks grows linearly in time which is much faster than in usual Ostwald ripening.

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In the early theory of fracture, Griffith [1] developed a criterion which predicts the irreversible growth of a crack, provided a critical length is exceeded, and conversely its disappearance if it is shorter. The critical length depends on the external tension P in the sense that a higher tension leads to a lower threshold of instability. The origin of this instability is founded on a competition between a release of elastic energy, caused by the opening of the crack, and an increase of surface energy. The critical Griffith crack is analogous to the critical nucleus in a first order phase transition.

Statistical mechanics of cracks in the nucleation stage has been discussed in [2] using this analogy between the theory of fracture and first order phase transitions. A solid is unstable against an external tension $P < 0$ because the true ground state of the system is fractured into pieces. In a sense, a solid under stretching is similar to a supercooled gas: the point of zero external stress plays the role of the liquid-gas condensation point. During this nucleation stage all cracks are well separated and hence a dilute gas approximation of straight cracks in an infinite medium is legitimate.

However, the late stage of phase separation is characterized by Ostwald ripening when a new phase coarsens in order to lower the interfacial free energy. Larger droplets are energetically more favorable because of their smaller interface curvature. Thus they grow at the expense of smaller inclusions which resolve again and finally disappear. This collective behavior leads to a growth of the average droplet size and simultaneously to a decrease of their number. Finally the system reaches the full thermodynamic equilibrium. The same coarsening process should take place in a system of cracks according to the analogy mentioned above. However, to our best knowledge, this problem has not been addressed in the literature so far and it is the subject of this Letter.

The coarsening of a new phase is generally well understood for vanishingly small initial supersaturations (characterized by a dimensionless parameter $\Delta_0 \ll 1$ which has to be defined appropriately) as long as only one length scale, the size of the particles R, is relevant in the system. In this case the distance between the different precipitates is large and a mean-field theory in the sense of Lifshitz, Slyozov, and Wagner [3] is applicable.

It is well known that long-range interactions mediated through the elasticity of the medium cause a significant change in the scaling properties of Ostwald ripening (see, e.g., [4], and references therein). In our earlier publication [5] we studied the coarsening of melt inclusions inside an overheated crystal. The scenario depicted there behaves crucially different from the classical coarsening theory. We pointed out that in this situation the assumption of a single relevant length scale existing in the system cannot hold. The density difference of the two phases causes noticeable elastic deformations of the surrounding crystal. As a consequence the precipitates are not spherical but cracklike inclusions with a height $h \ll R$. It turns out that this is incompatible with the mean-field requirement that different inclusions stay remote from each other, even in the limit of a vanishingly small overheating Δ_0. Strong interactions between adjacent inclusions alter the growth laws and lead to nontrivial predictions. The Griffith condition plays a central role and gives a strong hint towards a closer connection between the theory of first order phase transitions and the theory of fracture.

The main aim of this Letter is to analyze the coarsening of cracks in uniaxially strained solids. We emphasize that this process is not accompanied by a real phase transition and it is very much different from the one discussed in [5]. However, the strong analogy between cracklike inclusions in melting processes and real cracks in a stressed solid still holds. Since the lateral extent of the cracks is much bigger than their opening, they come close to each other during the coarsening stage. This limit has so far been almost inaccessible to analytic and numerical calculations, and therefore it is quite remarkable that a derivation of the scaling laws is possible.

If a uniform isotropic tension is applied at infinity, all crack orientations are present with the same probability and the approach between several of them finally leads

to the formation of a crack network [6]. If, on the other hand, we stretch a block of a perfect solid in only one direction (which we call the z direction) and then fix it in this position that a uniform strain $u_{zz}^{(0)}$ is maintained in the solid (see Fig. 1), then the situation is different. Our goal is to describe the time evolution in this arrangement.

Obviously the initial uncracked configuration is far from equilibrium: a high elastic energy density $w_{el}^{(0)} \sim E u_{zz}^{(0)2}$ could be dramatically reduced through the appearance of cracks in the system (E is the Young modulus). Finally, if only one long crack traverses the whole block, the system is completely unstretched and only a small amount of surface energy remains due to the two new crack surfaces. Refraining from these finite size effects we assume that a statistical ensemble of cracks appears in the solid with a growing mean size similar to the coarsening during a phase transition.

These cracks are mainly aligned perpendicular to the z axis because this maximizes the elastic energy release rate. Apart from the average size of the cracks R we introduce another length scale ℓ, which is the characteristic vertical distance between them. Only the assumption $\ell \ll R$ leads to consistent predictions. One can easily understand that a single (macroscopic) length scale L is not sufficient: In this case the spacing between neighboring cracks would be of the same order L as the cracks themselves, and thus they would be almost independent. From dimensional reasons the average crack opening would be of order $u \sim PL/E$ and the strain would be $u_{zz}^{(0)} \sim u/L$. Hence the average stress would remain almost constant $P \sim E u_{zz}^{(0)}$. Under these circumstances the system can never reach its equilibrium configuration with $P = 0$ and therefore the assumption of only one relevant length scale cannot be correct. In the following we will restrict our calculations to some simple order-of-magnitude estimates,

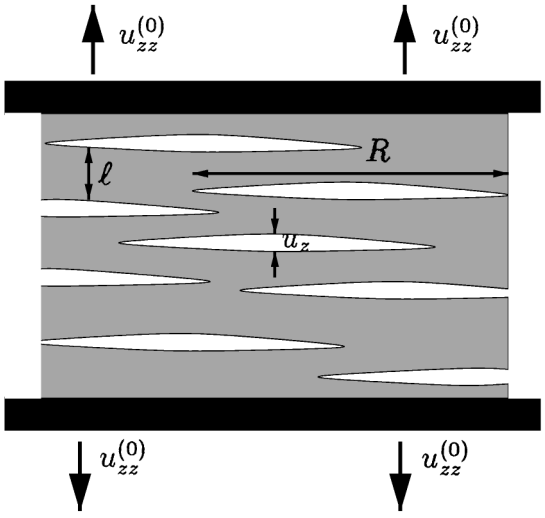

FIG. 1. An infinitely large block of solid is fixed between two pistons (black). Initially no cracks exist in the system and a homogeneous strain $u_{zz}^{(0)}$ is present everywhere.

assuming that the length scales of the system obey the order relation $R \gg \ell \gg u_z$; later we will see that these assumptions are self-consistent.

In the late stage of the coarsening process the main displacement comes from opening of the cracks and not from stretching. Thus

$$u_{zz}^{(0)} \sim u_z/\ell\,, \tag{1}$$

where we denote the average opening of the cracks by u_z. This equation corresponds to the global conservation law of conventional Ostwald ripening when almost all excess heat of the initially overheated system is converted to latent heat in order to perform the phase transition. The strain $u_{zz}^{(0)}$ in Eq. (1) corresponds to the supersaturation parameter Δ_0 mentioned above.

The distribution of cracks throughout the solid is irregular. It means that all of them are nearly perfectly aligned but the spatial distribution is random. The shape of the different cracks is not unique either. Because of these reasons an exact solution of the elastic problem is not possible. However, simple, basically dimensional arguments of the theory of thin plates allow us to derive the desired scaling behavior. First of all, let us present this scaling description, and then we will enlighten the elastic problem more carefully giving an exact two-dimensional calculation in the case of a regular arrangement of cracks.

The solid between two neighboring cracks in the vertical direction can be understood as a thin circular disk of lateral extent R and height ℓ. Its top and bottom are free surfaces with boundary conditions $\sigma_{zz} = \sigma_{rz} = 0$.

There are strong interactions between different plates via effective boundary conditions at the edges of the cracks where three (or more) of those plates come together and exert forces and torques on each other [7]. These forces produce an average stress $P \sim \sigma_{zz}$ in the plates, which causes a bending of them. Bending of thin plates subject to a stress P is described by the fourth order equation [7]

$$\frac{E\ell^3}{12(1-\nu^2)}\,\Delta^2 u_z = P\,. \tag{2}$$

Here $\Delta = \partial_x^2 + \partial_y^2$ is the two-dimensional Laplacian; ν and E are the Poisson and Young coefficients, respectively. From this equation it follows readily that an average stress P is related to the opening u_z by

$$u_z \sim P\,\frac{1-\nu^2}{E\ell^3}\,R^4. \tag{3}$$

The corresponding deformation volume (the volume of the void inside the crack) behaves as

$$V \sim P\,\frac{1-\nu^2}{E\ell^3}\,R^6 \tag{4}$$

and the related release of the elastic energy per plate is

$$W_{el} \sim -PV \sim -P^2\,\frac{1-\nu^2}{E\ell^3}\,R^6. \tag{5}$$

On the other hand the growth of the cracks is accompanied by an increase of surface energy

$$W_s \sim \alpha R^2 \tag{6}$$

with α being the surface tension.

Near the crack tips the theory of thin plates breaks down and stresses exhibit the typical $\sigma \sim r^{-1/2}$ behavior at distances $r \ll \ell$ from the tip. Local equilibrium requires an additional variation of the total energy $W = W_s + W_{el}$ with respect to R. This results in a relation between the length scales and the stress in the system:

$$R \sim \frac{E\alpha}{1 - \nu^2} \frac{1}{P^2} \left(\frac{\ell}{R}\right)^3. \tag{7}$$

It is a generalization of the Griffith condition for a single crack which can be retained by setting $\ell \sim R$ on the right-hand side. Equation (7) gives the critical size: bigger cracks grow and smaller ones shrink.

Another equation to determine the three unknowns R, ℓ, and P follows from the conservation law (1) in combination with (3):

$$P \sim u_{zz}^{(0)} \frac{E}{1 - \nu^2} \frac{\ell^4}{R^4}. \tag{8}$$

One of the most important differences to usual Ostwald ripening is the lack of a conserved order parameter. In the case of crack propagation, dissipation predominantly takes place in the close vicinity of the tips and diffusional transport of vacancies through the bulk is not required. Therefore kinetics of this mechanism is relatively fast and is commonly described by the phenomenological equation

$$R\dot{R} = -\kappa \frac{dW}{dR} \sim \kappa\left(P^2 \frac{1 - \nu^2}{E\ell^3} R^5 - \alpha R\right). \tag{9}$$

Here κ is the kinetic coefficient that relates the energy release rate to the tip velocity. It contains information about small scale physics in the vicinity of the tip of the propagating crack. The combination $\Gamma(\dot{R}) = \alpha + \dot{R}/\kappa$ is the so-called fracture energy that is different from the ordinary surface energy α since it contains also the dissipation term $\dot{R}/\kappa$. The additional factor R in Eq. (9) appears due to the use of polar coordinates to describe growth of an almost circular crack.

Because of the Griffith (7) condition, both energy contributions in Eq. (9) are of the same order and one immediately obtains

$$R \sim \alpha\kappa t. \tag{10}$$

Also, combining Eqs. (7) and (8), we find

$$\ell \sim \left(\frac{E}{1 - \nu^2}\right)^{-1/5} \alpha\, u_{zz}^{(0)-2/5} (\kappa t)^{4/5}. \tag{11}$$

Thus the average stress indeed decreases to zero as

$$P \sim \left(\frac{E}{1 - \nu^2}\right)^{1/5} u_{zz}^{(0)-3/5} (\kappa t)^{-4/5}. \tag{12}$$

The preceding three formulas are the main results of this Letter. They undoubtedly show the transition to equilibrium, i.e., the decrease of the stress and the growth of the length scales. From Eq. (1) it follows readily that the condition $u_z \ll \ell \ll R$ is clearly fulfilled, provided that $u_{zz}^{(0)} \ll 1$. This justifies the application of the theory of thin plates self-consistently.

We point out that the growth law (10) is quite remarkable in some ways. It describes the growth with a constant velocity $v \sim \alpha\kappa$. In comparison, in the usual cases of Ostwald ripening the transition to equilibrium happens with a decaying velocity, $R \sim t^\beta$ with $\beta < 1$ [3,5], and the process becomes slower and slower. The obtained scaling $R \sim t$ is probably the fastest growth law which is possible in dissipative systems. A similar linear law was predicted in [8] for the viscous hydrodynamic coarsening, but the physics involved is rather different. It is also noticeable that the velocity v is completely determined by material parameters and does not depend on the "supersaturation parameter" $u_{zz}^{(0)}$. Hence there is no reason for this velocity to be tunably small. If however the material excels by a high robustness against fracturing, characterized by a small κ, this velocity can be smaller than the speed of sound and below the branching threshold [9], since the stress relaxes with time and the characteristic size of cracks maintains near the Griffith size (7). In this case, first, the used static approximation of the theory of elasticity is legitimate; and second, it is even conceivable to observe the process described here in small-scale laboratory experiments, provided that still a statistical ensemble of cracks is present. Otherwise the system can break into pieces by the very fast growth of one single or only a few cracks. For higher velocities it should still be possible to observe the statistical behavior in geological systems which can extend over kilometers.

Local equilibrium requires the possibility of a retraction of subcritical cracks. Under some circumstances this process might be forbidden or hampered by oxidation of the surfaces. Then we expect small cracks to remain in the system, even if they are subcritical. Nevertheless they do not influence the scaling behavior. Indeed, inside a plate between two large cracks the horizontal stress is much bigger than its vertical component $\sigma_{zz} \sim P$. We estimate $\sigma_{rr} \sim E\ell\partial_R^2 u_z \sim PR^2/\ell^2$. If such a big plate is horizontally cut by much smaller cracks, the arising "microplates" are mainly stretched or compressed in the horizontal direction and consequently hardly bent. We conclude that almost the whole opening comes from the big growing cracks which therefore characterize the coarsening. Nevertheless the full hierarchy of length scales remains in the system, hence exhibiting a fractal behavior. It turns out that the surface energy of the smallest "frozen" cracks of length L, which are still remote from each other, contain nearly the whole surface energy density $w_s \sim W_s/L^3 \sim \alpha/L$ which is proportional to the initial elastic energy density $w_{el}^{(0)} \sim E u_{zz}^{(0)2}$.

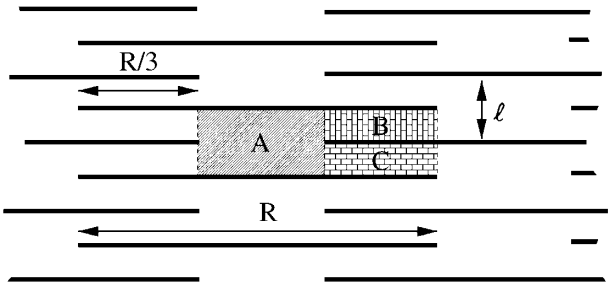

FIG. 2. Cross section through the solid with perfectly aligned cracks (solid lines). In this regular structure all cracks have exactly the same length R, their horizontal overlap is $R/3$, and their distance ℓ. The system consists (periodically continued) of three different plate types denoted by A, B, and C.

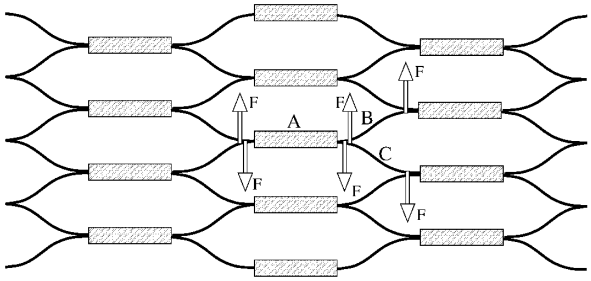

FIG. 3. Schematic picture of the elastic situation in the perfectly regular structure. The thicker plate A is unbent, whereas forces F deform the plates B and C (solid lines). At each triple junction of plates the total force and torque are zero. The vertical strings of B/C plates form a network of springs, horizontally connected through A plates.

The rest dissipates and leads to a small increase of the temperature.

Solution of the elastic problem for a regular arrangement of cracks.—In the remaining part of this paper we illuminate the elastic situation more carefully. In order to make analytical progress, we consider a two-dimensional solid with cuts inside as a basis. These cuts are chosen to be perfectly aligned as depicted in Fig. 2, all of them having the same length R.

If we neglect irregularities, this configuration of course does not lead to a coarsening, but since we are basically interested in the analysis of the elastic properties, we disregard this problem here. Since the horizontal distance is also of order R, we prescribe the horizontal overlap between the cracks to be $R/3$ without loss of generality. Obviously three different regions of the solid between neighboring cracks appear, denoted by letters A, B, and C. The array of B and C plates forms a network of springs in the vertical direction; horizontally they are connected through the thicker A plates. We already mentioned this interpretation in our previous publication [5]. Application of an external strain $u_{zz}^{(0)}$ bends the rods differently. From the symmetry present in this special arrangement it readily follows that B and C bend exactly the same but in opposite directions; A remains unbent (Fig. 3). Therefore it is sufficient to consider part B. At its bottom and top it is bounded by stress free surfaces, thus the right-hand side of Eq. (2) vanishes and it reads $u_z(x)'''' = 0$. Choosing the origin $x = 0$ to be in the middle of B, i.e., $-R/6 < x < R/6$, symmetry allows only the general solution $u_z = ax^3 + bx$. It also requires the conditions $u_z'(\pm R/6) = 0$ to allow the periodic continuation in the x direction. To produce the average strain $u_{zz}^{(0)}$, the opening of each crack is determined by the condition $u_{zz}^{(0)}\ell/2 = 2u_z(R/6)$. Therefore the missing coefficients are $a = -27u_{zz}^{(0)}\ell/R^3$ and $b = 9u_{zz}^{(0)}\ell/4R$.

To maintain this shape point forces and torques must appear at the edge of the plate. The sum of them is zero at each triple junction of plates as one can easily see from Fig. 3.

These forces are given by [7]

$$F = -\frac{E(\ell/2)^3}{12(1-\nu^2)}\, u_z'''(\pm R/6) = \frac{27E}{16(1-\nu^2)}\, u_{zz}^{(0)}\, \frac{\ell^4}{R^3}\,. \tag{13}$$

The average stress in the system is therefore $P \sim F/R$ which is exactly the relation (8).

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