LOW FREQUENCY INSTABILITIES IN CONFINED PLASMAS: CONCEPTS & THEORETICAL FRAMEWORK

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ABSTRACT

Most experts consider that anomalous energy and particle transport in fusion devices are due to low frequency waves whose free energy sources are the equilibrium gradients and the associated drifts across the confining magnetic field (*drift waves*). We consider successively the cases where $k_{||}qR >> 1$ and $k_{||}qR \sim 1$ where $k_{||}$ is the parallel wave number, qR being the connection length. The first limit is particularly adequate if the gradient of the parallel flow velocity is significant; exact stability criteria are then obtained with the help of the Nyquist diagram in the framework of the local dispersion relation which applies. That is not the case if $k_{||}qR \sim 1$: here, the stability analysis leads to second order differential equations whose complex eigenvalues provide the wave frequencies and the growth/decay rates. The theoretical concepts are developed successively for cylindrical and axisymmetric toroidal geometries. Electrons are considered to be adiabatic.

I. INTRODUCTION

In a liquid in contact with a heat source, the temperature gradient generates ascending Benard cells which carry the energy to the surface in contact with the cooling atmosphere. Similarly, in a magnetically confined plasma, the inherent temperature, density and parallel flow velocity gradients are free energy sources which can sustain micro-instabilities and turbulence leading to anomalous transport. There is therefore a permanent interplay between macroscopic and microscopic processes. In axisymetric toroidal devices (tokamaks), the macroscopic length scales are the temperature, density and parallel flow velocity gradient scales [L, e.g., $L_N = (\partial_r \ln N)^{-1}$], the minor and major radii [r, respectively R] and the connection length qR (q being the safety factor). Microscopic scales are the ion and electron gyro-radii (a_i and a_e) and the ion and electron Debye lengths ($\lambda_{D,i}$ and $\lambda_{D,e}$). The stable operating regime of tokamaks is such that $\lambda_{D,i} \approx \lambda_{D,e} \approx a_e << a_i$. In this lecture, we consider oscillations with perpendicular length scales of the order of, or larger than the ion gyro-radius: $\vec{k}_\perp a_i \leq \vec{1}$.

We define the unit vectors parallel to the magnetic field lines, the pressure gradient, and the binormal: $\hat{n} = \vec{B}/B$, $\hat{p} = \vec{\nabla}P/|\vec{\nabla}P|$ ($\hat{p}$ is by definition orthogonal to the nested toroidal flux surfaces [M.D. Kruskal and R.M. Kulsrud, 1958]) and $\hat{b} = \hat{n}x\hat{p}$. The phase velocity of drift waves in the direction of the binormal, $v_{ph,\beta}$, is a linear combination of the temperature and density gradient drift velocities $\hat{p}\cdot\vec{\nabla}T_J / e_J B$ and $T_J \hat{p}\cdot\vec{\nabla}N_J / e_J N_J B$ (J is the species index), of the $\vec{E}x\vec{B}$ drift velocity $-\hat{p}.\vec{E}/B$, and of the flow parallel velocity $U_{||,i}$. Their angular frequency, which is the product of the wave vector component k_β ($\equiv \vec{k}.\hat{b}$) with the phase velocity $v_{ph,\beta}$, is of the order of the diamagnetic frequencies

$$\omega_j^* = k_\beta \partial_r \ln N_j / e_j B \qquad (1)$$

∂_r is a short-hand notation for $\hat{p}.\vec{\nabla}$ (∂_r and $\hat{p}.\vec{\nabla}$ are rigorously identical in cylindrical geometry; in more complex systems, general coordinates and Christoffel symbols must be introduced [A. Rogister & D. Li, 1992]). It follows that drift frequencies are small compared to the ion cyclotron frequency (thus the label *low frequency waves*):

$$\omega / \Omega_i \sim k_\beta a_i^2 / L << 1$$

It is generally assumed that the most readily unstable modes are those with the longest parallel wavelengths, on the ground that the time required for electrons to neutralize any charge separation which might arise is then largest. The argument is founded if the instability drive is related to electron inertia or to collisions. Its validity is less obvious otherwise; if the driving mechanism is a shear parallel flow, in particular, the unstable spectrum is not expected to be symmetric with respect to the parallel component of the wave vector $k_{||} = \vec{k}.\hat{n}$ and the most unstable modes are expected to have finite $k_{||}$'s.

In summary, we shall consider waves with frequencies and wave number components in the following ranges:

$$\omega \sim \omega_j^*, \tag{2}$$

$$\vec{k}_\perp a_i \le \vec{1}, \tag{3}$$

and

$$k_{||} qR >> 1, \tag{4a}$$

respectively

$$k_{||} qR \sim 1. \tag{4b}$$

(It will indeed be shown that the lowest characteristic value of $k_{||}$ within a radial eigenmode is $\sim 1/qR$).

II. $k_{||} qR >> 1$: PARALLEL VELOCITY SHEAR/ ION TEMPERTURE GRADIENT INSTABILITY

II.A. LOCAL DISPERSION RELATION

The electron and ion distribution functions can be split into their equilibrium and perturbed components. As the collision time scales are much shorter than the transport time scales, the equilibrium distributions are Maxwellian in leading order:

$$F_j^{(0)} = \frac{N_j}{(2\pi c_j^2)^{3/2}} exp[-\frac{v_\perp^{\ 2} + (v_{//} - U_{//,j})^2}{2c_j^2}] \tag{5}$$

where $v_{||} = \vec{v}.\hat{n}$ is the parallel component of the particle velocity and $v_\perp^2 = v^2 - v_{||}^2$; the $c_j = \sqrt{T_j / m_j}$ are the thermal velocities. There is no a priori constraint on $T_e - T_i$ if the temperature relaxation time scale ($\tau_{equi} \sim m_i / m_e \nu_e$) is larger than, or of the order of the transport time scale. The difference $U_{//,e} - U_{//,i}$ is proportional to the plasma current density along the magnetic field lines; it is usually smaller than the ion thermal speed.

Assuming $\vec{k}_\perp a_i < \vec{1}$ for simplicity, it is convenient to split the perturbed distribution functions into $f_j = \bar{f}_j + \tilde{f}_j$, where $\bar{f}_j$ is the gyro-phase average of f_j; $\vec{k}_\perp a_i < \vec{1}$ leads to $\tilde{f}_j < \bar{f}_j$. A straightforward expansion of Vlasov's equation shows that

$$\bar{f}_i^{(0)} = \frac{k_{//}(v_{//} - U_{//,i})F_i^{(0)} - \frac{k_\beta T_i}{eB} \partial_r F_i^{(0)}}{(\omega - \omega_E) - k_{//} v_{//}} \frac{e\phi}{T_i} \tag{6}$$

where ϕ is the electrostatic potential of the wave and

$$\omega_E = -k_\beta E_r / B \tag{7}$$

is the $\vec{E} \times \vec{B}$ Doppler frequency (ω_j^* and ω_E are usually of comparable magnitudes). The most general ion response corresponds to $\omega' \equiv \omega - \omega_E - k_{||} U_{||,i} \sim k_{||} c_i$ and, therefore, $k_{||} L \sim k_\beta a_i$. Moreover, $\omega - \omega_E - k_{||} U_{||,e} << k_{||} c_e$ which immediately leads to

$$f_e^{(0)} = \frac{e\phi}{T_e} F_e^{(0)} \tag{8}$$

(adiabatic electron response). We note that

$$\begin{aligned} &\frac{k_\beta T_i}{eB} \partial_r F_i^{(0)} \\ &= \omega_i^* \{1 + \eta_i [\frac{v_\perp^{\ 2} + (v_{//} - U_{//,i})^2}{2c_i^{\ 2}} - \frac{3}{2}] \\ &\quad + \frac{v_{//} - U_{//,i}}{c_i} \frac{\partial_r U_{//,i}}{c_i \partial_r \ln N_i} \} F_i^{(0)} \end{aligned} \tag{9}$$

where $\eta_i = \partial_r \ln T_i / \partial_r \ln N_i = L_N / L_T$. The expressions of the electron and ion densities at leading order are readily obtained. Inserting those into the charge neutrality condition provides the lowest order dispersion relation

$$1 + \tau_i = D(\omega') \tag{10}$$

where $\tau_i = T_i / T_e$ and

$$D(\omega') = N_i^{-1} \int_{-\infty}^{\infty} dv'_{\|} (\omega' - k_{\|} v'_{\|})^{-1} \left\{ \omega' - \omega_i^* \left[1 + \eta_i \left(\frac{v_{\|}'^2}{2c_i^2} - \frac{1}{2} \right) + \frac{v'_{\|}}{c_i} \frac{\partial_r U_{\|,i}}{c_i \partial_r \ln N_i} \right] \right\} F_i^{(0)}$$

$$= \frac{1}{\sqrt{2\pi\tau_i}} \int_{-\infty}^{\infty} du (z - \xi u)^{-1} \left[z + \tau_i \left(1 - \frac{\eta_i}{2} \right) + \beta u + \frac{\eta_i}{2} u^2 \right] exp(-u^2 / 2\tau_i) \tag{11}$$

We have now defined $z = \omega' / \omega_e^*$, $\xi = k_{\|} c_s / \omega_e^*$, $\beta = \partial_r U_{\|,i} / c_s \partial_r \ln N_i$ and $v'_{\|} = v_{\|} - U_{\|,i}$ [so that $F_i^{(0)} \propto exp(v_\perp^2 + v_{\|}'^2) / 2c_i^2$]. If $|Im\,\omega'| << |Re\,\omega'|$, the function $D(\omega')$ can be rewritten as

$$D(\omega') = (2\pi\tau_i)^{-1/2} P \int_{-\infty}^{\infty} du (z - \xi u)^{-1} \left[z + \tau_i \left(1 - \frac{\eta_i}{2} \right) + \beta u + \frac{\eta_i}{2} u^2 \right] exp(-u^2 / 2\tau_i) - i \sqrt{\frac{\pi}{2\tau_i}} \frac{1}{|\xi|} \left[z + \tau_i \left(1 - \frac{\eta_i}{2} \right) + \beta \frac{z}{\xi} + \frac{\eta_i}{2} \frac{z^2}{\xi^2} \right] exp(-z^2 / 2\tau_i \xi^2) \tag{12}$$

where P is the Cauchy principal part integral and the imaginary part corresponds to the residue following the prescription of Landau (The latter follows from causality [L.D. Landau, 1946]). Worth noting is that

(i) the dispersion relation involves only two wave related dimensionless parameters, namely z and ξ; since $\omega_e^* \propto k_\beta$, those are proportional to ω' / k_β and $\omega' / k_{\|}$, respectively;

(ii) the radial derivative of the fluctuation does not appear, even if we order $\partial_r \ln(n_j, p_j, \phi) \sim L^{-1}$; in other words, the dispersion relation is scalar (local);

(iii) unlike the derivative of the equilibrium toroidal velocity, the derivative of the radial electric field does not enter the dispersion relation;

(iv) finally, the unstable oscillations will slowly disintegrate owing to $\partial_r \omega \neq 0$; their radial extent is limited by the equilibrium inhomogeneities; moreover, the value of $\partial_r \omega$ may be controlled by the fluxes to carry away.

II.B. NYQUIST STABILITY ANALYSIS

The dispersion function $D(\omega')$ can be written in a closed analytical form only if $\tau_i \to 0$. Exact instability criteria must otherwise be obtained by means of the Nyquist diagram technique [T.H. Stix, 1992, R.J. Goldsto & P.H- Rutherford, 1995]. Instability occurs if $\Im m\omega' > 0$ [as $\phi \propto exp(-i\omega t)$]; assuming $\omega_e^* > 0$ without loss of generality, that requests $\Im mz > 0$. Let z trace out a closed contour in the complex plane, going from $-\infty$ to $+\infty$ on the real axis and closing anti-clockwise on a semicircle at infinity in the upper half-plane. The function $D(z)$ will correspondingly trace out some closed contour in the complex D-plane (the Nyquist contour). If the point $D(z) = 1 + \tau_i$ [cf. Eq. (10)] falls in a region encircled by, and lying to the left of this contour, then the dispersion relation admits a root with $\Im mz > 0$, i.e. the plasma is unstable.

Equations (11) and (12) yield $\lim_{|z| \to \infty} D(z) = 1$ on the semicircle $|z| \to \infty$ and, respectively,

$$\lim_{z \to \pm\infty} D = 1 + \frac{\tau_i}{z} - i \sqrt{\frac{\pi}{2\tau_i}} \frac{\eta_i}{2} \frac{z^2}{|\xi|^3} \exp(-\frac{z^2}{2\tau_i \xi^2})$$

for large positive and negative *real* values of z. Since, on the one hand, $\Re eD > 1$ for $z \to +\infty$ and $\Re eD < 1$ for $z \to -\infty$ and, on the other hand, $\Im mD < 0$ (we assume $\eta_i > 0$), D moves in an anti-clockwise fashion along a (infinitely small) trajectory which is tangent to the point $D = 1$ and otherwise below the real axis as $\Re ez$ passes from large and positive values to large and negative values along the semicircle at infinity. In view of the preceding paragraph, a necessary condition for instability is thus that the Nyquist (D) contour crosses the real axis as z goes from $-\infty$ to $+\infty$ along the real axis, i.e., that the residue in Eq. (12) vanishes for some values of $z = z_i$. The z_i's are thus solutions of

$$\frac{\eta_i}{2\xi^2} z^2 + \left(\frac{\beta}{\xi} + 1 \right) z + \tau_i \left(1 - \frac{\eta_i}{2} \right) = 0 :$$

$$z_i = -\frac{\xi}{\eta_i}(\beta+\xi)\left[1 \pm \sqrt{1-\frac{\eta_i\tau_i(2-\eta_i)}{(\beta+\xi)^2}}\right] \qquad (13)$$

We note that the residue is proportional to the numerator $N(u,z)$ of the integrand in (12) evaluated at $u = z/\xi$. $N(u,z_i)$ therefore divides by $(z_i - \xi u)$ and $D(z_i)$ can be obtained exactly:

$$D(z_i) =$$

$$-\frac{1}{\xi\sqrt{2\pi\tau_i}}\int_{-\infty}^{\infty} du\left(\frac{\eta_i}{2}u + \beta + \frac{\eta_i z_i}{2\xi}\right) exp\left(-\frac{u^2}{2\tau_i}\right) \qquad (14)$$

$$= -\left(\frac{\beta}{\xi} + \frac{\eta_i z_i}{2\xi^2}\right)$$

We note that $D(z_1) > D(z_2)$ if $z_1 < z_2$. The point $D(z) = 1+\tau_i$ will therefore lay to the left of, and be encircled by the Nyquist contour if

$$D(z_2) < 1+\tau_i < D(z_1). \qquad (15)$$

(15) is the sufficient condition for instability. Introducing (13) into (14) yields

$$D(z_2) - 1 = -\frac{\beta+\xi}{2\xi}\left[1 \mp \sqrt{1-\frac{\eta_i\tau_i(2-\eta_i)}{(\beta+\xi)^2}}\right] \qquad (14')$$

and

$$D(z_1) - 1 = -\frac{\beta+\xi}{2\xi}\left[1 \pm \sqrt{1-\frac{\eta_i\tau_i(2-\eta_i)}{(\beta+\xi)^2}}\right] \qquad (14'')$$

where the upper sign corresponds to $(\beta+\xi)/\xi < 0$ and the lower sign to $(\beta+\xi)/\xi > 0$. It can be verified that either relation $D(z_i) = 1+\tau_i$ leads to

$$(1+\tau_i)\xi^2 + \beta\xi + \eta_i(2-\eta_i)/4 = 0$$

which has the solutions

$$\xi_i = \frac{-\beta \pm \sqrt{\beta^2 - \eta_i(2-\eta_i)(1+\tau_i)}}{2(1+\tau_i)} \qquad (16)$$

Consideration of Inequalities (15) under the conditions $(\beta+\xi)/\xi < 0$, $(\beta+\xi)/\xi > 0$, $0 < \eta_i < 2$ and $\eta_i > 2$ shows that the perturbations are unstable if

$$|\beta| \geq |\beta_{thr}| = \sqrt{\eta_i(2-\eta_i)(1+\tau_i)} \qquad (17)$$

At threshold, all values of ξ correspond to damping, except

$$\xi_i = \xi_{thr} = -\beta_{thr}/2(1+\tau_i) \qquad (17')$$

which is marginally stable. Introducing (17) and (17') into (15) reveals that equality is satisfied on the left-hand side. The frequency of the marginally unstable mode thus corresponds to the largest of the two values of z_i, namely

$$z_{thr} = 1 - 0.5\eta_i \qquad (17'')$$

We note that $\beta/\xi \equiv -k_\beta \partial_r U_{||,,i}/k_{||}\Omega_i$; therefore

$$(k_\beta \partial_r U_{||,i}/k_{||}\Omega_i)_{thr} = 2(1+\tau_i) \qquad (18)$$

Moreover, Inequality (17) requests that either

$$\eta_i \leq 1 - \sqrt{1-\beta^2/(1+\tau_i)} \qquad (19)$$

or

$$\eta_i \geq 1 + \sqrt{1-\beta^2/(1+\tau_i)} \qquad (20)$$

(20) reduces to the standard criterion $\eta_i > 2$ for the ion temperature gradient instability if $\beta = 0$.

II.C. DISCUSSION

i) We note that

$$\left(\frac{\omega'^2}{k_{||}^2 c_i^2}\right)_{thr} = \left(\frac{z^2}{\xi^2\tau_i}\right)_{thr} = \frac{(1+\tau_i)|2-\eta_i|}{\tau_i\eta_i}. \qquad (21)$$

Near marginal instability, the velocity integral in (11) may therefore be expanded for $\omega'/k_{||}v'_{||} >> 1$ if $\tau_i\eta_i \to 0$ and for $\omega'/k_{||}v'_{||} << 1$ if $\eta_i \to 2$. For $\beta = 0$, to the two stability limits $\eta_i = 2$ (hence $\omega' = 0$) and $\eta_i = 0$

(hence $\omega' = \omega_e^*$) correspond two different asymptotic expansions. For finite values of β, the stability limits (19) and (20) can be obtained by expansion only if $\tau_i \to 0$.

If $\tau_i \to 0$, the dispersion function $D(z)$ can readily be evaluated; that leads to the dispersion relation

$$z^2 - z - \xi(\beta + \xi) = 0 \qquad (22)$$

The corresponding marginal instability relations are $|\beta_{thr}| = 1$, $\xi_{thr} = -\beta_{thr}/2$ and $z_{thr} = 1/2$. Those agree with (17), (17') and (17'') if $\eta_i = 1$; it is interesting to note that η_i does not appear in (22) and that the right-hand side of (17) is maximum for $\eta_i = 1$. Restricting the calculation to the case $\eta_i = 0$, N. D' Angelo [1965] and P.J. Catto *et al.*[1973] obtained (from the two fluids and Vlasov equations, respectively) marginal instability relations which differ qualitatively from (17), (17') and (17'').

ii) The threshold condition (18) can be rewritten as

$$k_{||} qR = (k_\beta a_i)(qR\partial_r U_{||,i}/c_i)/2(1+\tau_i) \qquad (18')$$

If $(qR\partial_r U_{||,i}/c_i)/2(1+\tau_i)$ is large compared to unity (as may be the case in the pedestal of high confinement discharges), there are therefore marginally unstable modes with finite values of $k_\beta a_i$ for which $k_{||} qR >> 1$. Those are not linked to any particular rational surface.

III. $k_{||} qR \sim 1$: ELECTRON AND ION DRIFT WAVES

Under most experimental conditions, neither the product $(qR\partial_r U_{||,i}/c_i)/2(1+\tau_i)$, nor the value of $k_{||} qR$ appearing in (18') are large numbers. In such cases, rational surfaces play a leading role. To simplify the discussion, we ignore the torus curvature at first.

III.A. CYLINDRICAL GEOMETRY

The parallel wave vector can be written as

$$k_{||} = \vec{k}.\hat{n} = k_\varphi \frac{B_\varphi}{B} + k_\theta \frac{B_\theta}{B} = [m + nq(r)]\frac{B_\theta}{rB} \qquad (23)$$

where $m = k_\theta r$ and $n = k_\varphi R$ are the poloidal and toroidal mode numbers. In a plasma with finite magnetic shear, i.e. $\hat{s} \equiv r\partial_r \ln q \neq 0$, $k_{||}$ is a function of position which vanishes at the *rational* surface $r = r_{m,n}$ defined by

$$q(r_{m,n}) = -m/n \qquad (24)$$

Expanding the right-hand side of (23) in the neighborhood of $r = r_{m,n}$ yields

$$k_{||} = n(r - r_{m,n})(\partial_r q)(B_\theta / rB) = -\frac{k_\theta \hat{s}}{qR}(r - r_{m,n}) \qquad (25)$$

It may be verified that $1/k_\theta \hat{s}$ is the distance between the neighboring rational surfaces $r_{m,n}$ and $r_{m\pm 1,n}$. Moderate values of $k_{||} qR$ therefore request that the mode (m,n) be localized in the neighborhood of the rational surface defined in (24) and overlaps a few neighboring rational surfaces at most. The short scale radial dependence introduced in the dispersion equation by the parallel wave vector under those conditions must be balanced by other short scale dependent processes; in cylindrical geometry, those are finite Larmor radius effects:

$$k_r a_i \to a_i \partial_r$$

The technique of the Nyquist diagram has not been extended to that situation. Expansions of the dispersion function are therefore being used; for that purpose, it is usually assumed that

$$\lambda \equiv L_{N(T)}/R << 1 \qquad (26)$$

and $L_{N(T)} \sim r$. The perturbations are then described by second order ordinary differential equations in which the complex frequency plays the role of eigenvalue:

$$[a_s^2(\partial_x^2 - k_\beta^2) + \frac{k_\theta^2 c_s^2}{\omega'^2}\frac{x^2}{L_s^2} + c(\omega')]f(x,\omega') = 0 \qquad (27)$$

where $x = r - r_{m,n}$ and $k_{||} = -k_\theta x / L_s$; $L_s = qR/\hat{s}$ is the magnetic shear length. The solutions of Eq. (27) (eigenmodes) are of the form

$$f(x,\omega') = H_l(\sqrt{i\sigma}x / a_s)exp(-i\sigma x^2 / 2a_s^2) \quad (28)$$

where the $H_l(\zeta)$'s are Hermite polynomials of order l and the parameters σ and ω' are solutions of

$$\sigma = \pm k_\theta c_s a_s / \omega' L_s \quad (29)$$

$$i\sigma = [c(\omega') - k_\theta^2 a_s^2] / (2l+1) \quad (30)$$

Two classes of oscillations can be identified according to the frequency range: the electron and the ion drift branches.

III.A.1 The electron drift branch

The frequencies of the electron drift modes are in the range

$$\omega - \omega_E = \omega' \sim \omega^*$$

and the parallel phase velocity is such that

$$c_i << \omega' / k_{||} \sim k_\beta a_s (qR/L) c_s << c_e .$$

The electron response is *adiabatic* under those conditions (moving much faster than the waves, they average out its structure); thus $f_e = -(e_e\phi / T_e)F_e$, $t_e = 0$ and $n_e = (e\phi / T_e)N_e$. Inspection of the fundamental equations shows that the complex frequency can be expanded as

$$\omega' = \omega_e^* + \omega^{(1)} \quad (31)$$

where $\omega^{(1)} \sim \lambda\omega_e^*$; moreover,

$$t_i / T_i \sim n_i / N_i = n_e / N_e \quad (32)$$

The coefficient $c(\omega')$ is

$$c(\omega') = -[1 + \tau_i(1+\eta_i))]^{-1} \omega^{(1)} / \omega_e^* .$$

The only acceptable solution of (29), (30) is a damped one, in which the wave energy is radiated away from the reference rational surface [L.D. Pearlstein and H.L. Berk, 1969]: $\omega^{(1)} = \Re e\omega^{(1)} + i\gamma$, where

$$\gamma = -[1 + \tau_i(1+\eta_i)](1+2l) \,|\, k_\theta \,|\, c_s a_s / \,|\, L_s \,| \quad (33)$$

The damping rate (33), also known as shear damping, vanishes in the limit $\hat{s} \to 0$. In a tokamak, the modes can be excited by wave particle resonant interactions or collisions between passing and trapped electrons (processes not included in the present analysis); they are then referred to as dissipative and collisionless trapped electron modes.

III.A.2 The ion drift branch

The frequencies of ion drift modes are in the range

$$\omega - \omega_E = \omega' \sim \lambda\omega^*$$

and their parallel phase velocity is such that

$$c_i \sim \omega' / k_{||} << c_e .$$

The electron response is again *adiabatic*. Inspection of the equations shows that

$$t_i / T_i \sim \lambda^{-1} n_i / N_i \quad (34)$$

There are an unstable and a stable bounded solutions if $\eta_i > 2/3$ whose growth/damping rates are

$$\gamma = \pm(\eta_i - 2/3)^{1/2} \,|\, \hat{s} \,|\, c_i / qR \quad (35)$$

(Slightly different results were obtained by B. Coppi et al. [1967]). It must be emphasized that the theory of the ion drift branch is on a much less firm ground than that of the electron branch, owing to the smaller value of $\omega' / k_{||} c_i$; that may partly explain the discrepancy between the threshold values given here and in Section II.B.

III.B. TOROIDAL GEOMETRY

The description of drift waves in toroidal geometry differs from that in cylindrical geometry in many respects.

i) The frequency $\omega_{B,i}$ associated with the curvature and $\vec{\nabla} B$ drifts couple the Fourier components m,n and $m \pm 1, n$; it also introduces extra radial derivatives associated with the displacement of particles from the magnetic surfaces according to

$$\omega_{B,i} = -\Omega_i^{-1}(\frac{v_\perp^2}{2} + v_{||}^2)\frac{(\hat{n}x\vec{k}).\vec{\nabla}B}{B}$$
$$\rightarrow -\Omega_i^{-1}(\frac{v_\perp^2}{2} + v_{||}^2)\frac{B_\varphi}{B}R^{-1}\sin\theta\,\partial_r \quad (36)$$

That may lead, under certain conditions, to a reversal of the sign of the term $\partial_x^2 f(x,\omega')$ in Eq. (27).

ii) The requirement of long paparallel wavelengths cannot be fulfilled if a simple Fourier decomposition in θ is made (see Appendix A). To avoid that difficulty, J.W. Connor *et al.* [1978, 1979] introduced the *ballooning formalism*, a transformation of the original two-dimensional problem into a one-dimensional one in an infinite domain without periodicity constraint. Another, less abstract representation which leads to identical results, has been introduced by A. Rogister and G. Hasselberg [1985]. It relies on the fact that the distance between the rational surfaces ... $r_{m-2,n}$, $r_{m-1,n}$ $r_{m,n}$, $r_{m+1,n}$, $r_{m+2,n}$ ... being constant, the mode representation must satisfy certain invariance properties with respect to translations (There are here similarities with the theory of crystal lattices [Kittel, 1976]).

It is now well recognized that toroidal modes with a ballooning structure can, *under certain conditions*, be generated from the coupling of cylindrical modes. Two limiting cases must be envisaged: the cylindrical modes m,n and $m\pm1,n$ overlap either weakly or strongly.

III.B.1. Toroidal geometry without overlapping.

If the width of the cylindrical eigenmodes is small compared to the distance between the neighboring rational surfaces m,n and $m\pm1,n$, they can obviously not be coupled by the toroidal geometry. However, a *primary* oscillation with poloidal and toroidal mode numbers m and n drives *side bands* $m\pm1,n$ via the θ dependence of $\omega_{B,i}$. Those side bands introduce radial derivatives and modify the dynamics of the primary mode.

In the case of the electron branch, the coefficient of the second order derivative in Eq. (27) is, as a result, multiplied by $1+2q^2$ [A. Rogister, 1995]:

$$a_s^2(\partial_x^2 - k_\theta^2) \rightarrow (1+2q^2)a_s^2(\partial_x^2 - k_\theta^2) \quad (37)$$

It can be verified that this leads to an increase of the shear damping rate (33) by $(1+2q^2)^{1/2}$. We note that the factor $1+2q^2$ also appears in the neoclassical theory of heat transport.

In the case of the ion drift branch, the multiplication factor of $a_s^2(\partial_x^2 - k_\theta^2)$ is:

$$\left\{1+12.5(1+\tau_i)[L_N / R(1.5\eta_i - 1)k_\theta a_i]^2\right\} \quad (38)$$

The growth rate remains essentially unchanged. The results obtained in the framework of this model allow to explain the formation of internal transport barriers in negative central shear discharges [A. Rogister, 2000 and 2001]

III.B.2. Toroidal geometry with strong overlapping.

The last case to consider is when the distance between the neighboring rational surfaces m,n and $m\pm1,n$ is small compared to the characteristic radial scale of the cylindrical modes. Solutions satisfying the condition

$$\phi_{m+p,n}(x - x_{m+p,n}) = \exp(ip\theta_0)\phi_{m,n}(x - x_{m+p,n})$$

may be sought ($p << m$; we note that those form however a restricted set since the phase factor is assumed to be proportional to p). We may expand

$$\phi_{m\pm1,n}(x - x_{m,n})$$
$$= [1 \pm \Delta\partial_x + \frac{\Delta^2}{2}\partial_x^2]\phi_{m\pm1,n}(x - x_{m\pm,n})$$
$$= \exp(\pm i\theta_0)[1 \pm \Delta\partial_x + \frac{\Delta^2}{2}\partial_x^2]\phi_{m,n}(x - x_{m,n})$$

where $\Delta = x_{m+1,n} - x_{m,n}$. Two successive substitutions in $\omega_{B,i}\phi_{m\pm1,n}$ lead to the multiplication factor

$$1+(1+2\hat{s})L_N\Delta^2\cos\theta_0 / Ra_s^2. \quad (39)$$

of the second order derivative term in (27). L_N being generally negative, this coefficient may change sign if $\cos\theta_0 > 0$ ($\theta = 0$ refers to the outer equatorial plane). When this occurs, the electron drift branch is no longer radiating and magnetic shear damping disappears [J.B. Taylor, 1977]. It may moreover be shown (with the help of Poisson's summation formula [M.J. Lighthill, 1962]) that the superposition of the cylindrical modes peaks at $\theta = \theta_0$ (*ballooning mode*).

IV. SUMMARY

Oscillations are adequately described by a *local* dispersion relation if $k_{||}qR >> 1$; that is the case of the most readily unstable modes when $(qR\partial_r U_{||,i} / c_i) / 2(1+\tau_i)$ is a large number. Under those conditions, $k_{||}$ may be considered as nearly constant and the oscillations are localized to a region in which their distortion and disintegration resulting from the frequency gradient is balanced by the linear excitation. Exact stability criteria are obtained by means of the Nyquist diagram.

If, on the contrary, $k_{||}qR$ has to be finite for instability, then the mode must be localized near a *rational* surface. $k_{||}$ is then proportional to the distance from that surface and the strong space dependence which this introduces has to be balanced by other processes. In cylindrical geometry, localization of the mode is provided by finite gyro-radius effects; those lead to a second order differential eigenvalue problem whose solutions yield the local radial wave functions and the complex frequencies. In toroidal geometry, the departure of the particle orbits from rational surfaces which is associated with the $\vec{\nabla}B$ and curvature drifts combines with finite gyro-radius effects. The toroidal geometry may have a stabilizing or destabilizing role according to whether the width of the cylindrical modes is larger or smaller than the distance between the neighboring rational surfaces ($r_{m,n}$ and $r_{m\pm1,n}$). Superposition of cylindrical modes centered on those surfaces may lead to poloidally localized "ballooning" modes.

The derivation of the eigenvalue equation always requires certain working assumptions, the most crucial being that the parallel phase velocity is large compared to the ion thermal speed. Further investigations may be required as concern the ion drift branch, since the assumption $\omega >> k_{||}c_i$ leads to incorrect results in the limit $k_{||}qR >> 1$ where the Nyquist diagram technique must be used.

APPENDIX A

In large aspect ratio tokamaks, $B_\varphi \propto (1-\varepsilon\cos\theta)$, $R \propto 1+\varepsilon\cos\theta$, $B_\theta \propto (1-\Lambda\cos\theta)$ and $r \to r[1-\cos\theta\int_0^r \partial_{r'}\Delta dr'/r']$ [B.B. Kadomtsev and O.P. Pogutse, 1970]; here, $\varepsilon = r/R$, Δ is the Shafranov shift of the flux surface and $\Lambda = \varepsilon - \partial_r\Delta$; we note that $\partial_r\Delta$ is $O(\varepsilon)$. As a result, the term in the parenthesis on the right-hand side of Eq.(23) can be written as

$$m+n\nu(r,\theta) = [m+nq(r)] + n[\nu(r,\theta) - q(r)]$$

where

$$\nu(r,\theta) - q(r) = -[\varepsilon + \Delta' + \int_0^r \Delta' dr'/r']\cos\theta \,.$$

That difference would lead to $k_{||}qR \sim n\varepsilon\cos\theta$, which is usually a large number, if a simple Fourier representation in θ were used.

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