Pion–nucleon scattering in an effective chiral field theory with explicit spin–3/2 fields[#1]

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Abstract

We analyze elastic–pion nucleon scattering to third order in the so–called small scale expansion. It is based on an effective Lagrangian including pions, nucleons and deltas as active degrees of freedom and counting external momenta, the pion mass and the nucleon–delta mass splitting as small parameters. The fermion fields are considered as very heavy. We present results for phase shifts, threshold parameters and the sigma term. We discuss the convergence of the approach. A detailed comparison with results obtained in heavy baryon chiral perturbation theory to third and fourth order is also given.

Keywords: *Pion-nucleon scattering, effective field theory, delta resonance*

1 Introduction and summary

Pion–nucleon scattering is an important testing ground for our understanding of the chiral dynamics of QuantumChromoDynamics (QCD). It has been investigated to third and fourth order in the chiral expansion, leading e.g. to precise predictions for the threshold parameters [1, 2]. This approach is based on an effective field theory with the active degrees of freedom being the asymptotically observable pion and nucleon fields. When going to higher energies, the usefulness of the chiral expansion is limited by the appearance of the nucleon resonances, the most prominent and important of these being the $\Delta(1232)$ with spin and isospin $3/2$. Its implications for hadronic and nuclear physics are well established. Consequently, one would like to have a consistent and systematic framework to include this important degree of freedom in baryon chiral perturbation theory, as first stressed by Jenkins and Manohar [3] and only recently formalized by Hemmert, Holstein and Kambor [4]. Counting the nucleon–delta mass splitting as an additional small parameter, one arrives at the so–called small scale expansion (which differs from the chiral expansion because the $N\Delta$ splitting does not vanish in the chiral limit). It has already been established that most of the low–energy constants appearing in the effective chiral pion–nucleon Lagrangian are saturated by the delta [5] and thus the resummation of such terms underlying the small scale expansion (SSE) lets one expect a better convergence as compared to the chiral expansion. In addition, the radius of convergence is clearly enlarged when including the delta as an explicit degree of freedom. This of course increases the complexity of the approach since the pertinent effective Lagrangian contains more structures consistent with all symmetries. The purpose of this paper is to analyze pion–nucleon scattering to third order in this framework. In particular, we want to address two questions. First, it has to be demonstrated that for a given energy range, the third order SSE calculation leads to improved results as compared to the ones obtained in the third order chiral expansion. Second, a precise description of the resonant phase, i.e. of the P_{33} partial wave, should be obtained. As we will demonstrate, the SSE will pass both these tests. Furthermore, our investigation can be used to test the convergence of the small scale expansion. Indeed, from many models and more phenomenological approaches it is believed that the most important delta contributions stem from the Born graphs with intermediate resonance fields. We will address this issue in what follows. Also, from the technical point of view, we provide for the first time a systematic evaluation of the many $1/m$ corrections (using the machinery spelled out in Ref.[4]). Finally, we wish to point out that a systematic Lorentz invariant formulation as it is available for the pion–nucleon effective field theory [6] is not yet available, but could be built based on the pioneering investigation in Ref. [7]. The heavy fermion approach used here suffers from the same deficiencies as the standard heavy baryon chiral perturbation theory, these problems are, however, not relevant to the main issue we are going to address, namely the extension to higher energies.

The pertinent results of the present investigation can be summarized as follows:

(i) We have constructed the one–loop amplitude for elastic pion–nucleon scattering based on an effective field theory including pions, nucleons and deltas to third order in the small scale expansion, $\mathcal{O}(\varepsilon^3)$, where ε collects all small parameters (external momenta, the pion mass and the nucleon–delta mass splitting). We have constructed the pertinent terms of the effective Lagrangian including the $1/m$ corrections. The amplitude contains altogether 14 low–energy constants from the nucleon, the nucleon–delta and the delta sector (if we count the leading $\pi N\Delta$ coupling constant $g_{\pi N\Delta}$ as a LEC).

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(ii) The values of the LECs can be determined by fitting to the two S– and four P–wave amplitudes for different sets of available pion–nucleon phase shifts in the physical region at low energies. We have performed two types of fits. In the first one, we fit to the Matsinos phase shifts in the range of 40 to 100 MeV pion momentum in the laboratory frame. This allows for a direct comparison with the results based on the chiral expansion. We find that the third order SSE results are clearly better than the ones of the third order chiral expansion and only slightly worse than the ones obtained at fourth order in the pion–nucleon EFT. Second, we have fitted to the Karlsruhe phase shifts for pion lab momenta between 40 and 200 MeV. This allows for a better study of the resonance region. In both cases, most fitted LECs are of "natural" size.

(iii) We have studied the convergence of the small scale expansion by comparing the best fits based on the first (leaving the coupling constant $g_{\pi N\Delta}$ free), second and third order representation of the scattering amplitudes. The third order corrections are in general not large, but they improve the description of most partial waves. This indicates convergence of the small scale expansion for this process. As anticipated, the most important contributions come from the tree (Born) graphs with intermediate delta states. This allows to pin down the coupling constant $g_{\pi N\Delta}$ fairly precisely.

(iv) We can predict the phases at *lower* and at *higher* energies, in particular the threshold parameters (scattering lengths and effective ranges). The results are not very different from the third and fourth order studies based on the chiral expansion, but the description of the scattering length and the energy dependence in the delta channel are clearly improved. While the convergence of the isovector S–wave scattering length is satisfactory to third order in the small scale expansion, for drawing a conclusion on the isoscalar S–wave scattering length a fourth order calculation is mandatory.

(iv) We have considered the pion–nucleon sigma term. To third order in the small scale expansion, it depends on the low energy constant c_1 and the coupling $g_{\pi N\Delta}$. For the KA85 phases, we get a value consistent with previous determinations. We have also performed fits using the sigma term extracted from a family of sum rules as input and shown that this procedure leads to more reliable predictions for some LECs.

The manuscript is organized as follows. In section 2, we discuss the effective Lagrangian underlying our calculation. It decomposes into separate nucleon, delta–nucleon and delta contributions. We spell out all details pertinent to our calculation including also the $1/m$ corrections. Many of these terms have so far not been available in the literature. Section 3 contains the results for the pion–nucleon scattering amplitudes $g^{\pm}, h^{\pm}$ to third order in the small scale expansion. In particular, we discuss the interplay between the low–energy constants from the various sectors and the role of the so–called off–shell parameters. The fitting procedure together with the results for the phase shifts and threshold parameters are presented in section 4. In particular, we give a detailed comparison with the results obtained in the chiral expansion and discuss the issue of convergence. We also discuss the sigma term and show how it can be used to further constrain the fits. The pertinent Feynman rules, explicit expressions for the various $1/m$ corrections, for the tree and loop contributions to the scattering amplitude and the analytical expressions for the various threshold parameters are given in the appendix.

2 Effective Lagrangian

In this section, we briefly discuss the effective Lagrangian underlying our calculation. We borrow heavily from the work of Hemmert et al. [4] and refer the interested reader to that paper to fill in the details omitted here. The explicit inclusion of the delta into an effective pion–nucleon field theory is motivated by the fact that in certain observables this resonance plays a prominent role already at low energies. The reason for this is twofold. First, the delta–nucleon mass splitting [#4] is small,

$$\Delta \equiv m_\Delta - m_N = 294 \text{ MeV} \simeq 3F_\pi \ , \tag{2.1}$$

with $F_\pi = 92.4 \text{ MeV}$ the weak pion decay constant. In the chiral limit of vanishing quark masses, neither Δ nor F_π vanish. Therefore, such an extended EFT does not have the same chiral limit as QCD, as it is well–known since long [8]. Second, the delta couples very strongly to the $\pi N \gamma$ system, e.g. the strong $\Delta N \gamma$ M1 transition plays a prominent role in charged pion photoproduction. One can set up a consistent power counting by using the well–known heavy baryon techniques [9, 10] and by treating the mass splitting Δ as an additional small parameter besides the external momenta and quark (meson) masses. Therefore, any matrix element or transition current has a low energy expansion of the form

$$\mathcal{M} = \varepsilon^n \, \mathcal{M}_1 + \varepsilon^{n+1} \, \mathcal{M}_2 + \varepsilon^{n+2} \, \mathcal{M}_3 + \mathcal{O}(\varepsilon^{n+3}) \ , \tag{2.2}$$

where the power n depends on the process under consideration (for pion–nucleon scattering, n equals one) and ε collects the three different small parameters,

$$\varepsilon \in \left\{ \frac{q}{\Lambda_\chi}, \frac{M_\pi}{\Lambda_\chi}, \frac{\Delta}{\Lambda_\chi} \right\} \ , \tag{2.3}$$

with $\Lambda_\chi \simeq 1 \text{ GeV}$ the scale of chiral symmetry breaking, M_π the pion mass, and q some external momentum. This power counting scheme is often called ε– or small scale expansion (SSE). As it is common in heavy baryon approaches, the expansion in the inverse of the heavy mass and with respect to the chiral symmetry breaking scale are treated simultaneously (because $m_N \sim m_\Delta \sim \Lambda_\chi$). The effective Lagrangian has the following low–energy expansion

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \ldots \tag{2.4}$$

where each of the terms $\mathcal{L}^{(n)}$ decomposes into a pure nucleon (πN), a nucleon–delta ($\pi N \Delta$) and a pure delta part ($\pi \Delta$),

$$\mathcal{L}^{(n)} = \mathcal{L}^{(n)}_{\pi N} + \mathcal{L}^{(n)}_{\pi N \Delta} + \mathcal{L}^{(n)}_{\pi \Delta} \ , \quad n = 1, 2, 3, \ldots \ . \tag{2.5}$$

We will now discuss these in succession. We note that the $N\Delta$ terms have to be understood symmetrically, i.e. we can have an in–going nucleon and an out–going delta or the other way around. The coupling of external fields is done by standard methods, for our purpose we only have to consider a scalar source to deal with the explicit chiral symmetry breaking due to the quark masses.

[#4]We use the same symbol for the delta resonance as well as the ΔN mass splitting. This cannot lead to confusion since from the context it is always obvious what is meant.

2.1 Single nucleon sector

This part is fairly standard, for completeness we collect the terms necessary for the following discussion. To lowest (first) order, the relativistic pion–nucleon Lagrangian takes the form

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}_N (i \not{D} - m + \frac{g_A}{2} \not{u}\gamma_5)\Psi_N \; , \tag{2.6}$$

where the bi–spinor Ψ_N collects the proton and neutron fields, $g_A = 1.26$ is the axial–vector coupling constant and $m = m_N$ the nucleon mass. All parameters in the Lagrangian should be taken at their chiral limit values. We do not exhibit this by special symbols but it should be kept in mind. The heavy baryon projection is most economically done using path integral methods as outlined in Ref.[10]. This leads to a set of matrices, which organize the transitions between the light–light $(N - N)$, light–heavy $(N - h)$ and heavy–heavy $(h - h)$ components of the nucleon fields, denoted by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, in order,

$$\mathcal{L}_{\pi N} = \bar{N}\mathcal{A}_N N + \bar{h}\mathcal{B}_N N + \bar{N}\gamma_0 \mathcal{B}_N^\dagger \gamma_0 h - \bar{h}\mathcal{C}_N h \; . \tag{2.7}$$

The inverse of $\mathcal{C}$ is then further expanded in inverse powers of the nucleon mass, leading to the decoupling of the positive and negative velocity sectors. These matrices have a chiral expansion, i.e. $\mathcal{A} = \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \ldots$. From the lowest order, we only need

$$\mathcal{C}_N^{(0)} = 2m \; , \tag{2.8}$$
$$\mathcal{A}_N^{(1)} = iv \cdot D + g_A S \cdot u \; , \tag{2.9}$$
$$\mathcal{B}_N^{(1)} = -\gamma_5 (2iS \cdot D + \frac{g_A}{2} v \cdot u) \; , \tag{2.10}$$
$$\mathcal{C}_N^{(1)} = iv \cdot D + g_A S \cdot u \; , \tag{2.11}$$

in terms of the nucleon four–velocity v_μ and the Pauli–Lubanski spin–vector S_μ (for more details, see e.g. the review [11]). We now turn to the second order terms. In the isospin limit of equal up and down quark masses, one has to deal with 4 operators,

$$\mathcal{L}_{\pi N}^{(2)} = \bar{\Psi}_N \left[c_1 \langle \chi_+ \rangle - \frac{c_2}{8m^2} \left(\langle u_\mu u_\nu \rangle \{D_\mu, D_\nu\} + \text{h.c.} \right) + \frac{c_3}{2} \langle u^2 \rangle + \frac{ic_4}{4} \sigma_{\mu\nu} [u_\mu, u_\nu] \right] \Psi_N \; , \tag{2.12}$$

where χ_+ includes the explicit chiral symmetry breaking and traces in flavor space are denoted by $\langle \ldots \rangle$. More precisely, $\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$, $u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger)$ and $U(x) = u^2(x)$ collects the pion fields (for details, see e.g. [11]). Note, however, that due to the presence of the delta degrees of freedom the numerical values of these LECs are different from the ones obtained in the pure pion–nucleon EFT. This is discussed in more detail in Ref.[12]. From the second order transition matrices we need the following terms:

$$\mathcal{A}_N^{(2)} = c_1 \langle \chi_+ \rangle + c_2 (v \cdot u)^2 + c_3 u^2 + c_4 [S^\mu, S^\nu] u_\mu u_\nu \; , \tag{2.13}$$
$$\mathcal{B}_N^{(2)} = -c_4 \gamma_5 [v \cdot u, S \cdot u] \; , \tag{2.14}$$
$$\mathcal{C}_N^{(2)} = - \left[c_1 \langle \chi_+ \rangle + c_2 (v \cdot u)^2 + c_3 u^2 + c_4 [S^\mu, S^\nu] u_\mu u_\nu \right] \; . \tag{2.15}$$

Finally, from the third order Lagrangian, we only need one $1/m$ correction to the dimension two operator $\sim c_2$ and the following dimension three operators

$$
\begin{aligned}
\mathcal{A}_N^{(3)} &= \left(i\, \frac{c_2}{2m} \langle (v \cdot u) u_\mu \rangle D^\mu + \text{h.c.} \right) + i d_1 [u_\mu, [v \cdot D, u^\mu]] + i d_2 [u_\mu, [D^\mu, v \cdot u]] \\
&\quad + i d_3 [v \cdot u, [v \cdot D, v \cdot u]] + d_5 [\chi_-, v \cdot u] - i\,[S^\mu, S^\nu] d_{14} \langle [v \cdot D, u_\mu] u_\nu \rangle \\
&\quad - i\,[S^\mu, S^\nu] d_{15} \langle u_\mu [D_\nu, v \cdot u] \rangle + d_{16} \langle \chi_+ \rangle S \cdot u + i\, d_{18} [S \cdot D, \chi_-] \;.
\end{aligned}
\tag{2.16}
$$

Note that in πN scattering, some of the LECs only appear in certain combinations, here $d_1 + d_2$ and $d_{14} - d_{15}$ are of relevance. Some of these LECs are needed for the renormalization and their finite parts depend on the regularization scale. We do not further specify this but it should be kept in mind. Furthermore, there are some additional LECs just needed for the renormalization, as spelled out e.g. in Ref.[1]. We refrain from writing down such terms here. As stressed before, the numerical values of the finite parts of these LECs are influenced by the presence of the delta and thus can not be taken over from the pure πN EFT. The remaining $1/m$ corrections from the elimination of the small components are standard and can be found in Ref.[1].

2.2 $N\Delta$–sector

We now turn to the sector of the nucleons coupled to deltas and pions as well as external sources. Here, we only consider external scalar sources related to the explicit chiral symmetry breaking. The first order relativistic Lagrangian consistent with the requirement of point transformation invariance takes the form

$$
\mathcal{L}_{\pi N\Delta}^{(1)} = g_{\pi N\Delta} \left[\bar{\Psi}_\mu^i \Theta_{\mu\alpha}(z_0) w_\alpha^i \Psi_N + \bar{\Psi}_N w_\alpha^i \Theta_{\alpha\mu}(z_0) \Psi_\mu^i \right] \;,
\tag{2.17}
$$

with $\Theta_{\alpha\mu}(z_0) = g_{\mu\nu} + z_0 \gamma_\mu \gamma_\nu$, Ψ_μ^i is a conventional Rarita–Schwinger spinor and $w_\alpha^i = \frac{1}{2} \langle \tau^i u_\alpha \rangle$. We have to deal with two parameters, the leading pion–nucleon–delta coupling constant $g_{\pi N\Delta}$ and the so–called off–shell parameter z_0. The latter will be discussed in more detail below. The heavy baryon projection is more tedious since the delta field has a large and a small spin–3/2 as well as four (off–shell) spin–1/2 components. To keep only the large (light) spin–3/2 part, one has to insert appropriate spin–isospin projection operators. The technology to do that is spelled out in Ref.[4]. The effective Lagrangian has the genuine form

$$
\mathcal{L}_{\pi N\Delta} = \bar{T} \mathcal{A}_{N\Delta} N + \bar{G} \mathcal{B}_{N\Delta} N + \bar{T} \gamma_0 \mathcal{D}_{N\Delta}^\dagger \gamma_0 h + \bar{G} \gamma_0 \mathcal{C}_{N\Delta}^\dagger \gamma_0 h + \text{h.c.} \;,
\tag{2.18}
$$

i.e. one has to deal with four types of transition matrices. These are denoted by $\mathcal{A}_{N\Delta}$ (light–nucleon (N) to light–delta transitions (T)), $\mathcal{B}_{N\Delta}$ (light–nucleon (N) to heavy–delta transitions (G)), $\mathcal{C}_{N\Delta}$ (heavy–delta transitions (G) to heavy–nucleon (h)), and $\mathcal{D}_{N\Delta}$ (light–delta transitions (T) to heavy–nucleon (h)). Note that the field G has 5 components. To order ε, these transition matrices read

$$
\mathcal{A}_{N\Delta}^{(1)} = P_+ g_{\pi N\Delta}\, {}_3 P_{\mu\alpha} w_\alpha^i P_+ \;,
\tag{2.19}
$$

$$
\mathcal{B}_{N\Delta}^{(1)} = g_{\pi N\Delta}
\begin{pmatrix}
0 \\
-\frac{4(1+3z_0)}{3} P_+ S_\mu S \cdot w^i\, P_+ \\
2 z_0 P_- \gamma_5 v_\mu S \cdot w^i\, P_+ \\
-2 z_0 P_- \gamma_5 S_\mu v \cdot w^i\, P_+ \\
(1 + z_0) P_+ v_\mu v \cdot w^i\, P_+
\end{pmatrix} \;,
\tag{2.20}
$$

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$$\mathcal{D}_{N\Delta}^{(1)} = 0 , \tag{2.21}$$

$$\mathcal{C}_{N\Delta}^{T\,(1)} = g_{\pi N\Delta} \begin{pmatrix} P_- \, w_\alpha^i \, {}_3P_{\alpha\mu}P_- \\ 2z_0 P_- v \cdot w^i S_\mu \gamma_5 \, P_+ \\ (1+z_0)P_- v \cdot w^i v_\mu P_- \\ \frac{-4(1+3z_0)}{3} P_- S \cdot w^i S_\mu P_- \\ -2z_0 P_- S \cdot w^i v_\mu \gamma_5 \, P_+ \end{pmatrix} , \tag{2.22}$$

with

$$_3P_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu - \frac{4}{1-d} S_\mu S_\nu , \tag{2.23}$$

in d space–time dimensions and the $P_\pm$ are the usual velocity projection operators. The second order relativistic $\pi N\Delta$ Lagrangian has two terms of relevance to our study, these read

$$\mathcal{L}_{\pi N\Delta}^{(2)} = \bar{\Psi}_\mu^i \Theta_{\mu\alpha}(z) \left[ib_3 w_{\alpha\beta}^i \gamma^\beta + i\frac{b_8}{m} w_{\alpha\beta}^i iD^\beta \right] \Psi_N + \text{h.c.} , \tag{2.24}$$

with z another off–shell parameter also discussed below. To be more precise, any new structure which appears with a low energy constant has a separate off–shell parameter. However, these can be absorbed in the corresponding LECs (as discussed in more detail below) and we thus collectively call these new off–shell parameters z. The LECs b_i ($i = 3, 8$) are finite and will appear in the tree contribution to the πN scattering amplitude. The heavy baryon projection is done in terms of the appropriate second order transition matrices

$$\mathcal{A}_{N\Delta}^{(2)} = P_+ \, {}_3P_{\mu\alpha} i(b_3 + b_8) w_{\alpha\beta}^i v_\beta , \tag{2.25}$$

$$\mathcal{B}_{N\Delta,1}^{(2)} = -P_- \, {}_3P_{\mu\alpha} \gamma_5 2ib_3 w_{\alpha\beta}^i S_\beta P_+ , \tag{2.26}$$

$$B_{N\Delta,2}^{(2)} = -\frac{4}{3} P_+ S_\mu \left[(1+3z)i(b_3 + b_8) w_{\alpha\beta}^i S^\alpha v^\beta - 3zib_3 w_{\alpha\beta}^i v_\alpha S_\beta \right] , \tag{2.27}$$

$$\mathcal{B}_{N\Delta,3}^{(2)} = P_- v_\mu \gamma_5 \left[-(1+z)2ib_3 w_{\alpha\beta}^i v^\alpha S^\beta + 2zi(b_3 + b_8) w_{\alpha\beta}^i S_\alpha v_\beta \right] , \tag{2.28}$$

$$\mathcal{B}_{N\Delta,4}^{(2)} = \frac{4}{3} P_- S_\mu \gamma_5 \left[(1+3z)2ib_3 w_{\alpha\beta}^i S_\alpha S_\beta - \frac{3}{2} zi(b_3 + b_8) w_{\alpha\beta}^i v_\alpha v_\beta \right] , \tag{2.29}$$

$$\mathcal{B}_{N\Delta,5}^{(2)} = P_+ v_\mu \left[(1+z)i(b_3 + b_8) w_{\alpha\beta}^i v_\alpha v_\beta - 2zib_3 w_{\alpha\beta}^i S_\alpha S_\beta \right] , \tag{2.30}$$

$$\mathcal{D}_{N\Delta}^{(2)} = P_- \gamma_5 2ib_3 S_\beta w_{\beta\alpha}^i \, {}_3P_{\alpha\mu} P_+ , \tag{2.31}$$

$$\mathcal{C}_{N\Delta}^{(3)} = P_+ \, {}_3P_{\mu\alpha} \frac{ib_8}{m} w_{\alpha\beta}^i iD_\beta . \tag{2.32}$$

Note that at this order the first non–vanishing contribution to $\mathcal{D}_{N\Delta}$ appears. Finally, at third order, we only need the relativistic effective Lagrangian and the corresponding transition matrix $\mathcal{A}_{N\Delta}^{(3)}$. These read:

$$\mathcal{L}_{\pi N\Delta}^{(3)} = \bar{\Psi}_{\mu'}^i \Theta_{\mu'\mu}(z) \left[\frac{f_1}{m} [D_\mu, w_{\alpha\beta}^i] \gamma_\alpha iD_\beta - \frac{f_2}{2m^2} [D_\mu, w_{\alpha\beta}^i] \{D_\alpha, D_\beta\} \right.$$
$$\left. + f_4 w_\mu^i \langle \chi_+ \rangle + f_5 [D_\mu, i\chi_-^i] \right] \Psi_N + \text{h.c.} , \tag{2.33}$$

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$$\mathcal{A}_{N\Delta}^{(3)} = i\frac{b_8}{m}w_{\mu\nu}^i iD_\nu + (f_1 + f_2)[D_\mu, w_{\alpha\beta}^i]v_\alpha v_\beta + f_4 w_\mu^i\langle\chi_+\rangle + f_5[D_\mu, i\chi_-^i] \ . \tag{2.34}$$

Four new LECs, which we call f_i, appear. However, only the combinations $f_1 + f_2$ and $2f_4 - f_5$ are of relevance in case of pion–nucleon scattering. Altogether, in the $N\Delta$ sector we have 5 LECs and the same number of (unobservable) off–shell parameters. We have not counted b_3 and b_8 separately, since the latter can be absorbed in other LECs as detailed below. The Feynman rules for the resulting $\pi N\Delta$ vertex to first, second and third order are collected in appendix A. The $1/m$ corrections originating from the elimination of the various "small" components are discussed below.

2.3 Single Δ sector

As in the previous paragraphs, we start with the dimension one effective Lagrangian coupling the massive spin-3/2 fields to pions,

$$\begin{aligned}
\mathcal{L}_{\pi\Delta}^{(1)} = -\bar{\Psi}_\mu^i \ &[(i\,\slashed{D}^{ij} - m_\Delta\delta^{ij})g_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma^\lambda(i\,\slashed{D}^{ij} - m_\Delta\delta^{ij})\gamma_\lambda\gamma_\nu \\
&+ \frac{g_1}{2}g_{\mu\nu}\,\slashed{u}^{ij}\gamma_5 + \frac{g_2}{2}(\gamma_\mu u_\nu^{ij} + u_\mu^{ij}\gamma_\nu)\gamma_5 + \frac{g_3}{2}\gamma_\mu\,\slashed{u}^{ij}\gamma_5\gamma_\nu]\Psi_\nu^j \ ,
\end{aligned} \tag{2.35}$$

with $u_\mu^{ij} = u_\mu\,\delta^{ij}$ and D_μ^{ij} the appropriate chiral covariant derivative. If one now performs simultaneously the $1/m$ expansion of the nucleon and delta fields, one can only rotate away the nucleon mass, so that the $N\Delta$ mass splitting remains in the delta propagator. Since this is, however, a quantity of order ε, it can be expanded systematically. In the heavy fermion approach, the Lagrangian takes the form

$$\mathcal{L}_{\pi\Delta} = \bar{T}\mathcal{A}_\Delta T + \bar{G}\mathcal{B}_\Delta T + \bar{T}\gamma_0\mathcal{B}_\Delta^\dagger\gamma_0 G - \bar{G}\mathcal{C}_\Delta G \ . \tag{2.36}$$

The corresponding contributions from the three types of transition matrices $\mathcal{A}_\Delta$, $\mathcal{B}_\Delta$, and $\mathcal{C}_\Delta$ are given by

$$\mathcal{A}_\Delta^{(1)} = -(iv \cdot D^{ij} - \Delta\delta^{ij} + g_1 S \cdot u^{ij})g_{\mu\nu} \ , \tag{2.37}$$

$$\mathcal{B}_{\Delta,1}^{(1)} = P_-\,{}_3P_{\mu\nu}(2iS \cdot D^{ij} + \frac{g_1}{2}v \cdot u^{ij})\gamma_5 P_+ \ , \quad \mathcal{B}_{\Delta,2}^{(1)} = -P_+(\frac{2}{3}g_1 + g_2)S_\mu u_\nu^{ij} P_+$$

$$\mathcal{B}_{\Delta,3}^{(1)} = P_-\frac{g_2}{2}v_\mu u_\nu^{ij}\gamma_5 P_+ \ , \quad \mathcal{B}_{\Delta,4}^{(1)} = \frac{4}{3}P_- S_\mu iD_\nu^{ij}\gamma_5 P_+ \ , \quad \mathcal{B}_{\Delta,5}^{(1)} = 0 \ , \tag{2.38}$$

$$\mathcal{C}_{\Delta,11}^{-1\,(0)} = -\frac{1}{2m}P_-\,{}_3P_{\mu\nu}P_- \ , \quad \mathcal{C}_{\Delta,22}^{-1\,(0)} = \frac{1}{2m}P_+\,{}_1P_{\mu\nu}P_+ \ , \quad \mathcal{C}_{\Delta,23}^{-1\,(0)} = -\frac{1}{2m}\frac{2}{3}P_+ S_\mu v_\nu\gamma_5 P_- \ ,$$

$$\mathcal{C}_{\Delta,32}^{-1\,(0)} = \frac{2}{3}P_- v_\mu S_\nu\gamma_5 P_+ \ , \quad \mathcal{C}_{\Delta,33}^{-1\,(0)} = -\frac{1}{2m}P_-\,{}_2P_{\mu\nu}P_- \ , \quad \mathcal{C}_{\Delta,44}^{-1\,(0)} = -\frac{1}{2m}P_-\,{}_1P_{\mu\nu}P_- \ , \tag{2.39}$$

$$\mathcal{C}_{\Delta,45}^{-1\,(0)} = -\frac{1}{2m}2P_- S_\mu v_\nu\gamma_5 P_+ \ , \quad \mathcal{C}_{\Delta,45}^{-1\,(0)} = \frac{1}{2m}2P_+ v_\mu S_\nu\gamma_5 P_- \ , \quad \mathcal{C}_{\Delta,55}^{-1\,(0)} = -\frac{1}{2m}\frac{5}{3}P_+\,{}_2P_{\mu\nu}P_+ \ ,$$

$$\mathcal{C}_{\Delta,11}^{-1\,(1)} = \left(\frac{1}{2m}\right)^2 P_-\,{}_3P_{\mu\alpha}(iv \cdot D^{ij} + \Delta\delta^{ij} + g_1 S \cdot u^{ij})\,{}_3P_{\alpha\nu}P_- \ ,$$

$$\mathcal{C}_{\Delta,12}^{-1\,(1)} = \left(\frac{1}{2m}\right)^2 2P_-\,{}_3P_{\mu\alpha}\,iS \cdot D^{ij}\gamma_5\,{}_1P_{\alpha\nu}P_+ \ ,$$

$$\mathcal{C}_{\Delta,13}^{-1\,(1)} = -\left(\frac{1}{2m}\right)^2\frac{2}{3}P_-\,{}_3P_{\mu\alpha}\,iD_\alpha^{ij}v_\nu P_- \ ,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,14} = \left(\frac{1}{2m}\right)^2 \frac{2g_1}{3} P_- \,_3P_{\mu\alpha} u^{ij}_\alpha S_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,15} = -\left(\frac{1}{2m}\right)^2 (g_1 + \frac{2}{3}g_2) P_- \,_3P_{\mu\alpha} u^{ij}_\alpha \gamma_5 v_\nu P_+ \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,22} = \left(\frac{1}{2m}\right)^2 \frac{8}{3} P_+ S_\mu [-iv \cdot D^{ij} + 2\Delta\delta^{ij} - \frac{1}{3}S \cdot u^{ij}(g_1 - 4g_2 + 8g_3)] S_\nu P_+ \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,23} = \left(\frac{1}{2m}\right)^2 \frac{4}{3} P_+ S_\mu [iv \cdot D^{ij} + \frac{1}{3}S \cdot u^{ij}(g_1 + 2g_2)] \gamma_5 v_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,24} = \left(\frac{1}{2m}\right)^2 P_+ S_\mu [-\frac{16}{9}iS \cdot D^{ij} - \frac{4}{3}v \cdot u^{ij}(g_1 + 2g_2)] \gamma_5 S_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,25} = \left(\frac{1}{2m}\right)^2 \frac{1}{9} P_+ S_\mu [40iS \cdot D^{ij} + v \cdot u^{ij}(14g_1 + 16g_2 + 16g_3)] v_\nu P_+ \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,33} = \left(\frac{1}{2m}\right)^2 P_- v_\mu [\frac{2}{3}iv \cdot D^{ij} + \frac{4}{3}\Delta\delta^{ij} + \frac{10}{9}g_1 S \cdot u^{ij}] v_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,34} = \left(\frac{1}{2m}\right)^2 P_- v_\mu [-\frac{40}{9}iS \cdot D^{ij} - \frac{2}{3}g_1 v \cdot u^{ij}] S_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,35} = \left(\frac{1}{2m}\right)^2 P_- v_\mu [4iS \cdot D^{ij} + \frac{1}{3}v \cdot u^{ij}(g_1 - 2g_2)] \gamma_5 v_\nu P_+ \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,44} = \left(\frac{1}{2m}\right)^2 \frac{8}{3} P_- S_\mu [iv \cdot D^{ij} - 2\Delta\delta^{ij} + \frac{5}{3}g_1 S \cdot u^{ij}] S_\nu P_- \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,45} = \left(\frac{1}{2m}\right)^2 P_- S_\mu [-\frac{4}{3}iv \cdot D^{ij} + \frac{16}{3}\Delta\delta^{ij} + S \cdot u^{ij}(-4g_1 + \frac{8}{9}g_2)] \gamma_5 v_\nu P_+ \,,$$

$$\mathcal{C}^{-1\,(1)}_{\Delta,55} = \left(\frac{1}{2m}\right)^2 P_+ v_\mu [-\frac{2}{3}iv \cdot D^{ij} + 4\Delta\delta^{ij} + \frac{2}{9}S \cdot u^{ij}(-17g_1 + 12g_2 - 8g_3)] v_\nu P_+ \,. \tag{2.40}$$

The other elements of $\mathcal{C}^{-1}_\Delta$ are given by

$$\mathcal{C}^{-1}_{\Delta,ji} = \gamma_0 \mathcal{C}^{-1\,\dagger}_{\Delta,ij} \gamma_0 \,. \tag{2.41}$$

From the second order relativistic Lagrangian, we only need one term, which is the mass insertion on the Δ propagator, i.e. the analog to the dimension two nucleon term $\sim c_1$,

$$\mathcal{L}^{(2)}_{\pi\Delta} = \bar{\Psi}^i_\mu \Theta_{\mu\mu'}(z_1) \, a_1 \langle \chi_+ \rangle \delta^{ij} \, g_{\mu'\nu'} \Theta_{\nu'\nu}(z'_1) \Psi^j_{\nu'} \tag{2.42}$$

which translates into

$$\mathcal{A}^{(2)}_\Delta = P_+ \,_3P_{\mu\mu'} a_1 \langle \chi_+ \rangle g_{\mu'\nu'} \delta^{ij} \,_3P_{\nu'\nu} P_+ \,, \tag{2.43}$$

$$\mathcal{B}^{(2)}_\Delta = 0 \,. \tag{2.44}$$

For our purpose, no dimension three operator of the $\pi\Delta$ sector is needed, thus

$$\mathcal{A}^{(3)}_\Delta = 0 \,. \tag{2.45}$$

The genuine $1/m$ corrections will be considered in the next section.

2.4 1/m corrections

To calculate the $1/m$ corrections related to the elimination of the "small" components, it is most economical to use the path integral formalism outlined in Ref.[10]. Here, an algebraic complication due to the multiplication of degrees of freedom to be eliminated and also due to the transition between the nucleon and the delta sectors appears. The machinery to do that is spelled out in great detail in Ref.[4], here we only collect the pertinent results and basic definitions. We note however that we give the most detailed list of such terms so far worked out, only a subset of these is found in Ref.[4]. After the canonical change of variables to make the action quadratic in the fields, the various pieces of the effective Lagrangian take the form:

$$\tilde{\mathcal{L}}_{\pi N} = \bar{N}\mathcal{A}_N N + \bar{N}\Big[\gamma_0 \tilde{\mathcal{B}}_N^\dagger \gamma_0 \tilde{\mathcal{C}}_N^{-1}\tilde{\mathcal{B}}_N + \gamma_0 \mathcal{B}_{N\Delta}^\dagger \gamma_0 \mathcal{C}_\Delta^{-1}\mathcal{B}_{N\Delta}\Big] N \ , \tag{2.46}$$

$$\tilde{\mathcal{L}}_{\pi\Delta} = \bar{T}\mathcal{A}_\Delta T + \bar{T}\Big[\gamma_0 \mathcal{B}_\Delta^\dagger \gamma_0 \mathcal{C}_\Delta^{-1}\mathcal{B}_\Delta + \gamma_0 \tilde{\mathcal{D}}_{N\Delta}^\dagger \gamma_0 \tilde{\mathcal{C}}_N^{-1}\tilde{\mathcal{D}}_{N\Delta}\Big] T \ , \tag{2.47}$$

$$\tilde{\mathcal{L}}_{\pi N\Delta} = \bar{T}\mathcal{A}_{N\Delta} N + \bar{T}\Big[\gamma_0 \tilde{\mathcal{D}}_{N\Delta}^\dagger \gamma_0 \tilde{\mathcal{C}}_N^{-1}\tilde{\mathcal{B}}_N + \gamma_0 \mathcal{B}_\Delta^\dagger \gamma_0 \mathcal{C}_\Delta^{-1}\mathcal{B}_{N\Delta}\Big] N + \text{h.c.} \ , \tag{2.48}$$

employing the definitions:

$$\tilde{\mathcal{B}}_N = \mathcal{B}_N + \mathcal{C}_{N\Delta}\mathcal{C}_\Delta^{-1}\mathcal{B}_{N\Delta} \ , \tag{2.49}$$

$$\tilde{\mathcal{C}}_N = \mathcal{C}_N - \mathcal{C}_{N\Delta}\mathcal{C}_\Delta^{-1}\mathcal{C}_{N\Delta}^\dagger \ , \tag{2.50}$$

$$\tilde{\mathcal{D}}_{N\Delta} = \mathcal{D}_{N\Delta} + \mathcal{C}_{N\Delta}\mathcal{C}_\Delta^{-1}\mathcal{B}_\Delta \ . \tag{2.51}$$

We note that the only $1/m$ corrections for $\mathcal{L}_{\pi\Delta}$ we need are propagator insertions. From these equations our previously made remark that the low energy constants of the single nucleon sector get modified due to the presence of the delta becomes quite obvious since the light nucleon to light nucleon Lagrangian is modified by the appearance of the nucleon to delta transition operators, e.g. the last term in Eq.(2.46). We come back to this point below. The algebra to work out these terms is somewhat tedious and we collect the final results in appendix B.

3 Pion–nucleon scattering

3.1 Basic definitions

In this section, we only give a few basic definitions pertinent to elastic pion–nucleon scattering. For a more detailed discussion, we refer to Ref.[1]. In the center-of-mass system (cms), the amplitude for the process $\pi^a(q_1) + N(p_1) \to \pi^b(q_2) + N(p_2)$ takes the following form (in the isospin basis, with a, b denoting cartesian isospin indices of the pions):

$$T_{\pi N}^{ba} = \left(\frac{E+m}{2m}\right)\Big\{\delta^{ba}\Big[g^+(\omega,t) + i\vec{\sigma}\cdot(\vec{q}_2 \times \vec{q}_1)\,h^+(\omega,t)\Big]$$

$$+ i\,\epsilon^{bac}\tau^c\Big[g^-(\omega,t) + i\vec{\sigma}\cdot(\vec{q}_2 \times \vec{q}_1)\,h^-(\omega,t)\Big]\Big\} \tag{3.1}$$

with $\omega = v \cdot q_1 = v \cdot q_2$ the pion cms energy, $E_1 = E_2 \equiv E = (\vec{q}^{\,2} + m^2)^{1/2}$ the nucleon energy and $\vec{q}_1^{\,2} = \vec{q}_2^{\,2} \equiv \vec{q}^{\,2} = [(s - M_\pi^2 - m^2)^2 - 4m^2 M_\pi^2]/(4s)$. $t = (q_1 - q_2)^2$ is the invariant momentum transfer squared and s denotes the total cms energy squared. Furthermore, $g^\pm(\omega, t)$ refers to the

isoscalar/isovector non-spin-flip amplitude and $h^\pm(\omega, t)$ to the isoscalar/isovector spin-flip amplitude. This form is most suitable for a heavy fermion calculation (as done here) since it is already defined in a two–component framework.

The quantities of interest are the partial wave amplitudes $f_{l\pm}^\pm(s)$, where l refers to the orbital angular momentum, the superscript '$\pm$' to the isospin (even or odd) and the subscript '$\pm$' to the total angular momentum $(j = l \pm s)$, are given in terms of the invariant amplitudes via

$$f_{l\pm}^\pm(s) = \frac{E + m}{16\pi\sqrt{s}} \int_{-1}^{+1} dz \left[g^\pm P_l(z) + \vec{q}^{\,2} h^\pm \left(P_{l\pm1}(z) - zP_l(z) \right) \right] , \tag{3.2}$$

where $z = \cos(\theta)$ is related to the scattering angle. The $P_l(z)$ are the conventional Legendre polynoms. For a given isospin I, the phase shifts $\delta_{l\pm}^I(s)$ can be extracted from the partial waves via

$$f_{l\pm}^I(s) = \frac{1}{2i|\vec{q}|} \left[\exp(2i\delta_{l\pm}^I(s)) - 1 \right] . \tag{3.3}$$

For vanishing inelasticity, which is the case for the energy range considered in this work ($\sqrt{s} \leq 1.3\,\mathrm{GeV}$), the phase shifts are real. They are given by

$$\delta_{l\pm}^I(s) = \arctan(|\vec{q}|\,\mathrm{Re}\,f_{l\pm}^I(s)) . \tag{3.4}$$

In the low energy region, one could equally well use the definition without the arctan, the difference being of higher order. For the phase shifts in the kinematical region considered here, this difference is not negligible. In fact, this arctan prescription is nothing but a unitarization procedure which is mandated by the appearance of the poles in the delta propagator at $\omega = \Delta$. Although there is some arbitrariness in this unitarization procedure, it has already been shown in Ref.[13] that it leads to the proper resonance width (for sharp resonances like the delta or the ρ in pion–pion scattering, for more details see Ref.[13]). Consequently, the phase shifts presented in what follows are based on Eq.(3.4).

3.2 Small scale expansion of the amplitudes

In this section we discuss the small scale expansion of the non-spin-flip and spin-flip amplitudes $g^\pm, h^\pm$. These consist of essentially three pieces, which are the tree contributions with intermediate nucleons and deltas, counterterm parts of polynomial type as well as the unitarity corrections due to the pion loops (again with intermediate nucleons and deltas). The tree plus counterterm and loop graphs with intermediate deltas are shown in Fig.1 and Fig.2, respectively. We remark that the $\pi\pi N\Delta$ vertices only start at order $\mathcal{O}(\varepsilon^2)$ and thus all loop graphs of relevance here have $N\Delta$ couplings linear in the pion field. Formally, the small scale expansion of the various amplitudes has the form

$$X = X^{\mathrm{tree}} + X^{\mathrm{ct}} + X^{\mathrm{loop}} , \quad X = g^\pm, h^\pm , \tag{3.5}$$

where the tree contribution subsumes all Born terms with fixed coefficients, and the counterterm amplitude the ones proportional to the dimension two and three LECs. The last term in Eq.(3.5) is the leading one–loop amplitude consisting of terms of order ε^3. The latter is a complex–valued function and restores unitarity in the perturbative sense. Its various terms are all proportional to $1/F_\pi^4$. Note that we treat the chiral symmetry breaking scale $\Lambda_\chi \simeq 1\,\mathrm{GeV}$ on the same footing as the nucleon and delta mass. These amplitudes are functions of two kinematical variables, which we

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choose to be the pion energy and the invariant momentum transfer squared, i.e. $X = X(\omega, t)$. In what follows, we mostly suppress these arguments. It is also instructive to compare these amplitudes with the ones of the pure pion–nucleon EFT, based on the *chiral expansion*. These terms are, of course, contained in the SSE to third order and their explicit expressions are given in Ref.[1] (we remind the reader that the numerical values of the LECs are different, for the reasons discussed above).[#5] The novel tree and loop terms, shown in Fig.1 and Fig.2, are of fourth order (or higher) in the chiral expansion, since the delta is frozen out in that approach and generates second (and higher) order contact interactions (as discussed in more detail in Ref.[5]). Therefore, the amplitudes based on the third order small scale expansion can be considered as partial fourth order chiral calculations. Of course, they also contain higher order terms in the chiral expansion, since the delta is not frozen out. One can therefore expect that in most channels the third order SSE calculation should lead to a better description of the phase shifts than given by the third order chiral expansion. Such an expectation is indeed borne out by the actual results to be presented below.

The full one–loop amplitude to order ε^3 is obtained after mass and coupling constant renormalization,

$$(\mathring{g}_A, \mathring{m}, \mathring{\Delta}, F, M) \rightarrow (g_A, m, \Delta, F_\pi, M_\pi) \ . \tag{3.6}$$

The pion mass, decay constant and Z–factor are standard,

$$M_\pi^2 = M^2 \left\{ 1 + \frac{2M^2}{F^2} \ell_3 + \frac{\Delta_\pi}{2F^2} \right\} , \tag{3.7}$$

$$Z_\pi = 1 - \frac{2M^2}{F^2} \ell_4 - \frac{\Delta_\pi}{F^2} , \tag{3.8}$$

$$F_\pi = F \left\{ 1 + \frac{M^2}{F^2} \ell_4 - \frac{\Delta_\pi}{F^2} \right\} , \tag{3.9}$$

in terms of the LECs $\ell_{3,4}$ and the divergent pion self–energy tadpole Δ_π given e.g. in Ref.[11]. Similarly, for the nucleon mass shift, the nucleon Z–factor and the pion–nucleon coupling, we use (for more details, see Refs.[2, 12]):

$$
\begin{aligned}
m = {}& \mathring{m} - 4M^2 c_1 + \frac{9}{4} \left(\frac{g_A}{F} \right)^2 J_2(0) \\
& + \frac{4}{3} \left(\frac{g_{\pi N\Delta}}{F} \right)^2 \left[(M^2 - \Delta^2) \left(J_0(-\Delta) - \frac{2\Delta}{M^2} \Delta_\pi \right) + \frac{\Delta}{48\pi^2} (3M^2 - 2\Delta^2) \right] , \tag{3.10}
\end{aligned}
$$

$$
\begin{aligned}
Z_N = {}& 1 - 8M^2 \tilde{d}_{28}(\lambda) + \frac{9}{4} \left(\frac{g_A}{F} \right)^2 J_2'(0) - \frac{3M^2}{32\pi^2} \left(\frac{g_A}{F} \right)^2 \\
& + \frac{4}{3} \left(\frac{g_{\pi N\Delta}}{F} \right)^2 \Big[(M^2 - \Delta^2) J_0'(-\Delta) + 2\Delta J_0(-\Delta) \\
& \qquad\qquad + \frac{2}{M^2} (M^2 - 3\Delta^2) \Delta_\pi + \frac{1}{16\pi^2} (-M^2 + 2\Delta^2) \Big] , \tag{3.11}
\end{aligned}
$$

$$
\frac{g_A}{F_\pi} = \frac{\mathring{g}_A}{F} \left\{ 1 - \frac{M^2}{F^2} \ell_4 + \frac{4M^2}{g_A} d_{16}(\lambda) + \frac{g_A^2}{4F^2} \left(\Delta_\pi - \frac{M^2}{4\pi^2} \right) \right.
$$

[#5]In this paper, we only display the novel delta contributions to the amplitudes, threshold parameters and so on. The purely nucleonic terms can be found in Ref.[1].

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$$+ \left(\frac{g_{\pi N\Delta}}{F} \right)^2 \left(\frac{1}{3} \left(4 - \frac{100}{81} \frac{g_1}{g_A} \right) \left[(M^2 - \Delta^2) J_0'(-\Delta) + 2\Delta J_0(-\Delta) \right] \right.$$

$$- \frac{32}{27\Delta} \left[M^2 J_0(0) - (M^2 - \Delta^2) J_0(-\Delta) \right] + \frac{8}{27M^2} \Delta_\pi \left[(M^2 - 19\Delta^2) - \frac{25}{9} \frac{g_1}{g_A} (M^2 - 3\Delta^2) \right]$$

$$\left. + \frac{1}{24\pi^2} (-M^2 + 2\Delta^2)(2 - \frac{70}{27} \frac{g_1}{g_A}) + \frac{2}{81\pi^2} (-3M^2 + 2\Delta^2) \right) \bigg\} \,, \tag{3.12}$$

in terms of the standard loop functions $J_{0,2}(\omega)$ [11]. Finally, for the delta, we only need the mass shift,

$$m_\Delta = \overset{\circ}{m}_\Delta - 4M^2 a_1 - \frac{25}{108} \left(\frac{g_1}{F} \right)^2 M^2 J_0(0) - \frac{1}{3} \left(\frac{g_{\pi N\Delta}}{F} \right)^2 (M^2 - \Delta^2) \left(J_0(\Delta) + \frac{2\Delta}{M^2} \Delta_\pi \right) . \tag{3.13}$$

We refrain from giving the much more complicated expressions for the delta Z–factor and the renormalization of $g_{\pi N\Delta}$. Also, we always work with renormalized LECs, all infinities appearing in the loop diagrams are accounted for by the corresponding infinite parts of the pertinent counterterms. What is still missing in the SSE is a systematic investigation of renormalization as it is the case in (heavy) baryon chiral perturbation theory.

3.3 Counterterm combinations and off–shell parameters

First, we enumerate the novel low energy constants related to the $\pi\Delta$ and $\pi N\Delta$ sectors. Consider first the $\pi N\Delta$ couplings. At leading order, there is only $g_{\pi N\Delta} = 1.05$ from the width of the decay $\Delta \to N\pi$ [4]. Since the necessary resummation at the pole of the delta includes some higher order terms, we will also perform fits leaving this coupling free. We expect, however, that the so determined value is not very different from 1.05. In addition, from the dimension two and three Lagrangians, we have the LEC combinations $b_3 + b_8$, $f_1 + f_2$ and $2f_4 - f_5$. We note that there is a $1/m$ correction to b_8, nonetheless the LECs b_3 and b_8 need not be treated separately because the term b_8/m can be absorbed in the dimension three LECs from the nucleon sector, as detailed below. From the $\Delta\pi$ sector, we only have the coupling g_1, which we leave free. In the large N_c limit of QCD, one obtains the relation $g_1 = 9g_A/5$, with $g_A = 1.26$ the axial coupling constant measured in neutron β–decay. However, we will not use this relation but rather consider the value determined from the fit to the data as a check of the large N_c expansion of QCD. In addition, there is the dimension two LEC a_1. It only leads to a mass shift of the delta and can thus be absorbed completely in the physical nucleon–delta mass splitting. All these couplings are accompanied by separate off–shell parameters, which we (collectively) have denoted by z_0, z and z'.[#6] These off–shell parameters are not observable, so they can be chosen freely. According to the common practice, we set $z = z' = z_0 = \frac{1}{2}$, which is of course a little bit arbitrary. As stated before, due to the explicit appearance of the Δ in the theory, there are new $1/m$ corrections to $\mathcal{L}_{\pi N}$, which can be absorbed into the LECs c_i, d_i of the nucleon sector. This also means that a certain part of these LECs is explained by Δ properties (as it is well known, see Ref.[5]). In the following, we will not absorb these pieces in the c_i and d_i LECs (with one

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[#6]Note that the special treatment of the leading $\pi N\Delta$ coupling is done for historical reasons.

exception to be given below), because such a redefinition also depends on the off–shell parameters z, z' and z_0. If one were to perform this redefinition, it would take the form

$$c_2 \longrightarrow c_2 - 2\frac{g_{\pi N\Delta}^2}{9m}(3 + 4z_0^2) + 8\Delta\frac{g_{\pi N\Delta}^2}{9m^2}(1 + 2z_0^2) , \tag{3.14}$$

$$c_3 \longrightarrow c_3 + \frac{g_{\pi N\Delta}^2}{9m}(1 + 8z_0 + 12z_0^2) - 2\Delta\frac{g_{\pi N\Delta}^2}{9m^2}(1 + 6z_0 + 8z_0^2) , \tag{3.15}$$

$$c_4 \longrightarrow c_4 + \frac{g_{\pi N\Delta}^2}{9m}(1 + 8z_0 + 12z_0^2) - 2\Delta\frac{g_{\pi N\Delta}^2}{9m^2}(1 + 6z_0 + 8z_0^2) , \tag{3.16}$$

$$(\bar{d}_1 + \bar{d}_2) \longrightarrow (\bar{d}_1 + \bar{d}_2) + \frac{1}{4}\frac{g_{\pi N\Delta}^2}{9m^2}(5 + 4z_0^2) - 2\frac{g_{\pi N\Delta}(b_3 + b_8)}{9m}z_0 - \frac{1}{2}\frac{g_{\pi N\Delta}b_8}{9m}(1 + 4z + 12zz_0) , \tag{3.17}$$

$$\bar{d}_3 \longrightarrow \bar{d}_3 - 3\frac{g_{\pi N\Delta}^2}{9m^2} + \frac{1}{4}\frac{g_{\pi N\Delta}(b_3 + b_8)}{9m}(16 + 5z - 4zz_0) - \frac{1}{4}\frac{g_{\pi N\Delta}b_8}{9m}(4 + 5z - 20zz_0) , \tag{3.18}$$

$$\bar{d}_5 \longrightarrow \bar{d}_5 + \frac{1}{2}\frac{g_{\pi N\Delta}^2}{9m^2}(2 - z_0) - \frac{1}{8}\left(\frac{g_{\pi N\Delta}(b_3 + b_8)}{9m} - \frac{g_{\pi N\Delta}b_8}{9m}\right)(6 + 13z + 4zz_0) , \tag{3.19}$$

$$(\bar{d}_{14} - \bar{d}_{15}) \longrightarrow (\bar{d}_{14} - \bar{d}_{15}) - \frac{g_{\pi N\Delta}^2}{9m^2}(1 + 4z_0^2) - 8\frac{g_{\pi N\Delta}(b_3 + b_8)}{9m}z_0 - 2\frac{g_{\pi N\Delta}b_8}{9m}(1 + 4z + 12zz_0) . \tag{3.20}$$

In this case, the c_i now also have contributions proportional to Δ/m. Since c_2 and c_4 also appear in the third order amplitude, using these renormalized LECs would lead to contributions of fourth order, which is beyond the accuracy of our calculation. This we consider another reason not to make this redefinition. Similar problems related to the mixing of various chiral orders arise also in the fourth order analysis performed in the pure pion–nucleon EFT, see Ref.[2]. However, in one case, we have to perform this redefinition. The contribution proportional to b_8 has to be absorbed into $(\bar{d}_1 + \bar{d}_2)$, $\bar{d}_3$, $\bar{d}_5$ and $(\bar{d}_{14} - \bar{d}_{15})$. The resulting LECs and combinations thereof are labeled by a subscript "Δ". If that is not done, one introduces an additional redundant fit parameter. So this is the only redefinition we are making. The other effects from b_8 can then be absorbed into the following combinations

$$(f_1 + f_2) - \frac{b_8}{2m} \quad \text{and} \quad (2f_4 - f_5) - \frac{b_8}{4m} . \tag{3.21}$$

So we end up with the nine LECs from $\mathcal{L}_{\pi N}^{(2,3)}$ plus four LECs (or combinations thereof) from $\mathcal{L}_{\pi N\Delta}^{(2,3)}$ (counting the leading $\pi N\Delta$ coupling as a free parameter, although we also perform fits with its value fixed) and one LEC from the $\Delta\pi$ sector.

4 Results

4.1 The fitting procedure

There are various possibilities to fix the LECs. We proceed here along the similar lines as in Refs.[1, 2], namely we fit to the phase shifts given by different partial wave analyses in the low energy region.

This allows for a better comparison with the results obtained in the chiral expansion. As input we use the phase shifts of the Karlsruhe (KA85) group [14] and from the analysis of Matsinos [15] (EM98). Of course, there is also the phase shift analysis of the VPI/GW group, which we do not use here for two reasons. First, the solution called SP98 from the VPI/GW group [16] is no longer available (it was used in Refs. [1, 2]) and, second, no threshold parameters for the newest solution SP00 have been published. Clearly, once these are available we can update our investigation, but we do not believe that it will lead to wildly different results and new insight compared to what is presented below. Since no uncertainties are available for the KA85 solution, we mimic the ones of the Matsinos analysis in that case, which is 1.5% for S_{31}, 0.5% for S_{11}, 1% for P_{33} and 3.5% for the other P–waves. This assignment gives more weight to the better determined larger partial waves and is more natural than one common global error. We remark that the Matsinos analysis only includes data up to 200 MeV pion laboratory momentum. For this analysis, we proceed as in Refs. [1, 2], namely fit to the S– and P–wave phase shifts for momenta between 40 and 100 MeV and predict the phases at lower and higher energies. For the KA85 solution, we extend the fit region up to 200 MeV and thus predict the phases up to about 300 MeV (and of course also in the threshold region). As in Refs. [1, 2] the LEC $\bar{d}_{18}$ is fixed by means of the Goldberger–Treiman discrepancy, i.e. by the value for the pion–nucleon coupling constant extracted in the various analyses. The actual values of $g_{\pi N}$ are $g_{\pi N} = 13.4 \pm 0.1$ and 13.18 ± 0.12 for KA85 and EM98, respectively. For the terms including the Δ, we perform fits with $g_{\pi N\Delta}$ fixed (at the value of 1.05) and letting it free (as discussed above). Throughout, we use $g_A = 1.26$, $F_\pi = 92.4\,\text{MeV}$, $m = 938.27\,\text{MeV}$, $\Delta = 294\,\text{MeV}$ and $M_\pi = 139.57\,\text{MeV}$.

4.2   Phase shifts and threshold parameters

LEC	Fit 1	Fit 1*	Fit 2	Fit 2*
c_1	0.77 ± 0.02	0.39 ± 0.02	-0.44 ± 0.01	-0.32 ± 0.01
c_2	-17.9 ± 0.03	-15.5 ± 0.03	-0.67 ± 0.03	-1.59 ± 0.03
c_3	20.2 ± 0.06	16.7 ± 0.04	-0.07 ± 0.03	1.15 ± 0.03
c_4	-15.6 ± 0.04	-12.5 ± 0.03	-2.51 ± 0.04	-3.44 ± 0.04
$(\bar{d}_1 + \bar{d}_2)_\Delta$	-5.91 ± 0.06	-5.81 ± 0.07	0.03 ± 0.04	-0.36 ± 0.04
$(\bar{d}_3)_\Delta$	7.68 ± 0.06	6.60 ± 0.07	-0.93 ± 0.04	-0.44 ± 0.04
$(\bar{d}_5)_\Delta$	-1.07 ± 0.03	-0.59 ± 0.04	0.75 ± 0.02	0.71 ± 0.02
$(\bar{d}_{14} - \bar{d}_{15})_\Delta$	-5.18 ± 0.20	-0.09 ± 0.21	-0.44 ± 0.15	-0.60 ± 0.15
$\bar{d}_{18}$	-0.98 ± 0.19	-0.97 ± 0.19	-1.35 ± 0.14	-1.38 ± 0.14
$g_{\pi N\Delta}$	1.32 ± 0.03	$1.05^\star$	0.98 ± 0.05	$1.05^\star$
$b_3 + b_8$	-12.0 ± 0.10	-11.1 ± 0.13	0.51 ± 0.09	-0.28 ± 0.08
$f_1 + f_2 - \frac{b_8}{2m}$	29.3 ± 0.30	32.4 ± 1.30	-19.1 ± 0.57	-18.6 ± 0.51
$2f_4 - f_5 - \frac{b_8}{4m}$	40.2 ± 0.73	53.5 ± 0.92	-30.3 ± 0.92	-27.0 ± 0.82
g_1	-1.42 ± 0.02	-2.65 ± 0.03	-1.10 ± 0.04	-0.94 ± 0.04

Table 1: Values of the LECs in appropriate units of inverse GeV for the various fits described in the text.

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We are now in the position to present results. For the analysis of Matsinos, we use 17 points for each partial wave in the range of $q_\pi = 41.4 - 96.3$ MeV. We have performed fits leaving the coupling $g_{\pi N\Delta}$ free or fixing its value at 1.05. We call these fits 1 and 1*, respectively. Therefore, we have to determine 14 and 13 parameters for these two fits. Since the data basis of Ref.[15] includes only data up to 200 MeV, we use these fits mostly for comparison with the results of the chiral expansion presented in Refs.[1, 2]. This is different for the Karlsruhe data basis, which extends up to very high energies. Consequently, for this case (KA85), we have fitted to the data up to 200 MeV pion lab momentum (i.e. 10 points per partial wave at $q_\pi = 40, 60, 79, 97, 112, 130, 153, 172, 185, 200$ MeV). We extend the fits to higher energies than it was done in Refs.[1, 2]; this is possible due to the explicit inclusion of delta and allows to study explicitly the resonance region. Again, we perform fits with varying and fixed $g_{\pi N\Delta}$, which are denoted by fits 2 and 2*, respectively. The resulting LECs are collected in table 1. We point out again that the values for the LECs from the nucleon sector (the c_i and d_i) can not be compared with the ones obtained in the chiral expansion for the various reasons discussed above. For the Matsinos data, the $\chi^2/$dof is very small and slightly better for fit 1 than for fit 1*. The reasons for this very small $\chi^2/$dof are discussed in some detail in Ref.[2]. For the KA85 phases, the $\chi^2/$dof is essentially the same for both cases (and larger than for fits 1,1*, see again Ref.[2].). We also note that while the value of $g_{\pi N\Delta}$ is somewhat larger than the canonical value of 1.05 for fit 1, it comes out slightly lower in the case of fit 2. In the following, we will mostly discuss the results of the fits 1*,2* with $g_{\pi N\Delta} = 1.05$. Most LECs come out of natural size, i.e. of order one, with the exception of the LEC combinations from $\mathcal{O}(\varepsilon^3)$. This can be traced back to the fact that there are large cancelations at this order between the loop graphs with intermediate deltas and the counterterms. Such a phenomenon was also observed in the third order SSE calculation of neutral pion photoproduction, see [17]. Also, in case of fits 1,1*, the values for the dimension two LECs are quite large, and, in particular, the value for c_1 is positive. Furthermore, it is important to stress that we find some sizeable correlations between the LECs $c_{1,2,3}$. This is not unexpected since the corresponding terms contribute only to the small isoscalar S–wave scattering amplitudes (in certain combinations, e.g. the S–wave isoscalar scattering length is only sensitive to the LEC combination $-2c_1 + c_2 + c_3$). This will be taken up in the next subsection. We note that the value for the $\pi\Delta$ coupling g_1 comes out very different from the large N_c prediction $g_1 = 9g_A/5 = 2.27$. Note, however, since g_1 only appears in the third order loop contribution, one can not expect to pin it down very precisely.

The resulting fits and predictions based on the EM98 and KA85 phases are shown in Figs.3 and 4, respectively, in comparison to the results based on the third and fourth order chiral amplitudes. The number of LECs to be fitted was 9 and 14 in these cases. The results of the third order SSE calculation are clearly better than the ones based on the third order chiral expansion and comparable to the fourth order results, although the overall description is still slightly better in the latter case. As expected, the most prominent improvement can be found in the P_{33} partial wave, which is well described up to the delta pole, see Fig.4. At lower energies, one can predict the threshold parameters, which are collected in table 2 for the fits 1* and 2* in comparison with the direct determination from the Matsinos and Karlsruhe phase shift analyses. The overall consistency is satisfactory but not as good as in the case of the fourth order chiral expansion. Again, we find a significant improvement of the scattering volume in the P_{33} channel, as expected due to the explicit inclusion of the delta.

It is also important to discuss the issue of convergence. For that, we redo the fits for the amplitudes at first and second order in the small scale expansion. To leading order, one only has the coupling $g_{\pi N\Delta}$, whereas at second order one has the additional four LECs from the nucleon sector and one LEC combination from $\mathcal{L}_{\pi N\Delta}^{(2)}$. The resulting phase shifts are shown in Fig.5 for the EM98 analysis and in

Obs.	Fit 1	Fit 2	EM98	KA85
a_{0+}^+	0.41	-0.94	0.41 ± 0.09	-0.83
b_{0+}^+	-4.22	-4.60	-4.46	-4.40
a_{0+}^-	7.74	8.95	7.73 ± 0.06	9.17
b_{0+}^-	1.42	1.63	1.56	0.77
a_{1-}^+	-5.49	-5.54	-5.46 ± 0.10	-5.53
a_{1+}^+	13.08	13.27	13.13 ± 0.13	13.27
a_{1-}^-	-1.21	-1.46	-1.19 ± 0.08	-1.13
a_{1+}^-	-8.21	-8.14	-8.22 ± 0.07	-8.13

Table 2: Values of the S– and P–wave threshold parameters for the fits 1* and 2* in comparison to the respective data. The results for fits 1 and 2 are similar and thus are not given. Units are appropriate inverse powers of the pion mass times 10^{-2}.

Fig.6 for the KA85 case. While the first order result is only good in the P_{33} partial wave, the second order fits are of comparable quality than the third order ones, although at third order the χ^2/dof is significantly better. Such a behavior does not come completely unexpected since one expects the delta Born graphs to play the most significant role. Such an expectation is build on experience with many models that include the delta or also the explicit calculations of Compton scattering off nucleons in the SSE [18]. This behavior is different from what is found in the chiral expansion, where one still has large corrections when going from second to third order but mostly modest ones from third to fourth order, see [1, 2]. This means that in the channels where the delta plays a significant role, the resummation of higher order terms in the chiral expansion is important and well described in the approach used here. It is also interesting to study the convergence of the S–wave scattering lengths, as it has been done for the chiral expansion in [2]. In the small scale expansion we obtain the results collected in table 3. The convergence for the isovector scattering length is similar to what is obtained in the chiral expansion (as expected from the arguments presented in Ref.[19]). The isoscalar S–wave scattering length receives a large correction when going from second to third order, indicating that certain fourth order pieces related to pion–nucleon physics are still missing (as can also be inferred from the study in Ref.[2]). Some of these results were also found by Ellis and Tang [7], although their approach is based on a relativistic treatment of the fermion fields. Therefore, in their approach all $1/m$ corrections are resummed, but the delta is treated in a less systematic manner. Datta and Pakvasa [20] had already used the one loop representation of Ref.[21] and also added the delta, but not in a systematic fashion as done here. They also found a much improved description of the P_{33} partial wave.

4.3 The sigma term

So far, we have exclusively considered the amplitudes in the physical region. We have noted that there are some strong correlations between some LECs. To further address that problem, we need additional input. This is provided e.g. by the so–called pion–nucleon sigma term, which is the expectation value of the QCD symmetry breaking terms in a proton (or neutron) state. It can be derived from the scalar form factor of the nucleon,

$$\sigma(t) = \langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle , \quad t = (p' - p)^2 , \tag{4.1}$$

		$\mathcal{O}(\varepsilon)$	$\mathcal{O}(\varepsilon^2)$	$\mathcal{O}(\varepsilon^3)$
a_{0+}^+	EM98	0.0	0.24	0.41
	KA85	0.0	0.44	-0.94
a_{0+}^-	EM98	7.90	7.90	7.74
	KA85	7.90	7.90	8.95

Table 3: Convergence of the S–wave scattering lengths. $\mathcal{O}(\varepsilon^n)$ means that all terms up-to-and-including order n are given. Units are $10^{-2}/M_\pi$.

with $|N(p)\rangle$ a nucleon state of four–momentum p and $\hat{m}$ the average light quark mass. The sigma term is nothing but the scalar form factor at $t = 0$, $\sigma \equiv \sigma(t = 0)$. We remark that the sigma term can not be obtained directly from scattering data. One usually considers its value at the Cheng–Dashen point, $t = 2M_\pi^2$, where the chiral corrections are minimized. $\Sigma \equiv \sigma(t = 2M_\pi^2)$ and σ differ by 17 MeV, with 15 MeV stemming from the scalar form factor [22] and (at most) 2 MeV from the so–called remainder [23]. In addition, there exists a whole family of relations between Σ and certain combinations of threshold parameters, as detailed in Ref.[24]. These relations have been worked out to third order in the chiral expansion. We will use here the version given in Ref.[25],

$$\Sigma = \pi F_\pi^2[(4 + 2\mu + \mu^2)a_{0+}^+ - 4M_\pi^2 b_{0+}^+ + 12\mu M_\pi^2 a_{1+}^+] + \Sigma_0 , \qquad (4.2)$$

with $\Sigma_0 = -12.6 \, \text{MeV}$ and $\mu = M_\pi/m \simeq 1/7$. A special variant, which also contains some fourth order pieces, has recently been given by Olsson [26],

$$\Sigma = [F_\pi^2 F(2M_\pi^2)] , \qquad (4.3)$$
$$F(2M_\pi^2) = 14.5 \, a_{0+}^+ - 5.06 \, (a_{0+}^{1/2})^2 - 10.13 \, (a_{0+}^{3/2})^2 - 16.65 \, b_{0+}^+ - 0.06 \, a_{1-}^+ + 5.70 \, a_{1+}^+ - 0.05 ,$$

with the quantities on the right–hand–side being given in units of the pion mass. We will use these sum rules to further constrain our LECs.

To third order in the small scale expansion, the scalar from factor of the nucleon reads (the scalar sector has also been discussed by Kambor [27]):

$$
\begin{aligned}
\sigma(t) &= -4M^2 c_1 - \left(\frac{g_A}{F}\right)^2 \frac{3M^2}{8} \left[-2J_0(0) + (t - 2M^2)K_0(t, 0)\right] \\
&\quad + \left(\frac{g_{\pi N\Delta}}{F}\right)^2 \frac{2M^2}{3} \left[2J_0(-\Delta) + (2M^2 - t - 2\Delta^2)K_0(t, -\Delta) - 2\Delta I_0(t) + \frac{\Delta}{8\pi^2}\right] ,
\end{aligned}
$$
$$(4.4)$$

in terms of the standard loop functions listed in [21]. For $t = 0$, this gives

$$\sigma(t = 0) = -4M^2 c_1 - \left(\frac{g_A}{F}\right)^2 \frac{9M^3}{64\pi} + \left(\frac{g_{\pi N\Delta}}{F}\right)^2 \frac{M^2}{2\pi^2}\sqrt{\Delta^2 - M^2}\ln\left(\frac{\Delta}{M} + \frac{\sqrt{\Delta^2 - M^2}}{M}\right) . \qquad (4.5)$$

Note that the terms $\sim g_{\pi N\Delta}^2$ in Eqs.(4.4,4.5) are of fourth order in the chiral expansion, as already pointed out long time ago [28]. We note that σ only depends on the LEC c_1 (and also the coupling

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$g_{\pi N\Delta}^2$ for the fits where it is left free). If we now use c_1 as determined in fits 1 and 1*, we get an unphysical negative sigma term because c_1 is positive and large. That, however, is an artifact since not all the c_i's can be determined independently. For the fits 2 (2*), we obtain the following values,

$$\sigma = 51.1 \ (47.3) \ \text{MeV} . \tag{4.6}$$

These numbers are slightly larger but consistent within error bars with what has been found before, see Refs.[22, 29]. However, one might question the accuracy of this determination because also in this case one has large correlations between certain LECs. To overcome this, we have performed a different set of fits. As additional input we take the sigma term as determined from the sum rules, Eq.(4.2) and Eq.(4.3) using the threshold parameters from the EM98 and the KA85 analysis as input. More precisely, since the two sum rules differ by some terms of fourth order (and higher) in the chiral expansion, for each set of input threshold parameters we get two numbers for $\sigma = \Sigma - 17\,\text{MeV}$, which we average to obtain the central value and their spread is taken as the theoretical uncertainty. This gives $\sigma = (58.5 \pm 5.4)\,\text{MeV}$ for EM98 and $\sigma = (45.5 \pm 2.7)\,\text{MeV}$ for KA85. The corresponding fits with this additional input are denoted by fit 1* and fit 2*. The resulting LECs are shown in table 4.

LEC	Fit 1*	Fit 2*
c_1	-0.18 ± 0.02	-0.35 ± 0.09
c_2	-5.72 ± 0.03	-1.49 ± 0.66
c_3	6.05 ± 0.03	0.93 ± 0.87
c_4	-8.93 ± 0.04	-3.08 ± 0.81
$(\bar{d}_1 + \bar{d}_2)_\Delta$	5.52 ± 0.07	-0.57 ± 0.64
$(\bar{d}_3)_\Delta$	-4.12 ± 0.07	-0.30 ± 0.64
$(\bar{d}_5)_\Delta$	-0.89 ± 0.04	0.74 ± 0.10
$(\bar{d}_{14} - \bar{d}_{15})_\Delta$	-18.8 ± 0.22	-0.10 ± 1.60
$\bar{d}_{18}$	-0.99 ± 0.20	-1.34 ± 0.24
$g_{\pi N\Delta}$	1.27 ± 0.04	1.00 ± 0.08
$b_3 + b_8$	-7.33 ± 0.12	-0.03 ± 0.89
$f_1 + f_2 - \frac{b_8}{2m}$	-31.3 ± 0.98	-17.9 ± 4.49
$2f_4 - f_5 - \frac{b_8}{4m}$	-68.0 ± 0.77	-23.0 ± 9.78
g_1	-2.05 ± 0.02	-1.05 ± 0.41

Table 4: Values of the LECs in appropriate units of inverse GeV for the fits using as additional input the sigma term.

First, as expected from the numbers given in Eq.(4.6), there are only minor changes in case of fit 2* as compared to fits 2,2*. This is very different for fit 1* compared to fits 1,1*. The value of c_1 is now negative and also, the pion–nucleon dimension two LECs have more natural values. In addition, the correlations between the LECs $c_{1,2,3}$ are somewhat smaller than before. A very important result is the stability of the coupling constant $g_{\pi N\Delta}$, which comes out consistent with what was found in fits 1 and 2, respectively.

Acknowledgments

A   Pion–nucleon–delta vertices

Here, we give the Feynman rules for the relevant $\pi N\Delta$ vertices. Consider first the case on an incoming nucleon (momentum p_1), an outgoing delta (momentum P_μ, isospin i), and an outgoing pion (momentum q, isospin a):

1st order:

$$\frac{g_{\pi N\Delta}}{F}q^\mu\delta^{ia} \ . \tag{A.1}$$

2nd order:

$$-\frac{b_3+b_8}{F}v\cdot qq^\mu\delta^{ia} - \frac{g_{\pi N\Delta}}{mF}P_\mu v\cdot q\delta^{ia} \ . \tag{A.2}$$

3rd order:

$$\delta^{ia}\frac{1}{F}\ \left\{\ -q_\mu\Big((e_1+e_2)(v\cdot q)^2+2M^2(-2e_4+e_5)\Big)\right.$$
$$+\frac{q_\mu}{m}\Big[\frac{b_3}{2}(v\cdot qv\cdot(P+p_1)-q\cdot(P+p_1))-b_8q\cdot p_1-\frac{g_{\pi N\Delta}}{4m}(v\cdot Pv\cdot p_1-P\cdot p_1)\Big]$$
$$+\frac{P_\mu}{m}\Big[\frac{b_3}{2}z((v\cdot q)^2-q^2)+(b_3+b_8)(v\cdot q)^2$$
$$\left.-\frac{g_{\pi N\Delta}}{3m}(-\frac{1}{2}(v\cdot Pv\cdot q-P\cdot q)-(2z_0-1)v\cdot Pv\cdot q+(2z_0-4)\Delta v\cdot q)\Big]\right\} \ . \tag{A.3}$$

Similarly, for the case of an outgoing nucleon (momentum p_2), an incoming delta (momentum P_μ, isospin i), and an outgoing pion (momentum q, isospin a), we have:

1st order:

$$\frac{g_{\pi N\Delta}}{F}q^\mu\delta^{ia} \ . \tag{A.4}$$

2nd order:

$$+\frac{b_3+b_8}{F}v\cdot qq^\mu\delta^{ia} - \frac{g_{\pi N\Delta}}{mF}P_\mu v\cdot q\delta^{ia} \ . \tag{A.5}$$

3rd order:

$$\delta^{ia}\frac{1}{F}\ \left\{\ -q_\mu\Big((e_1+e_2)(v\cdot q)^2+2M^2(-2e_4+e_5)\Big)\right.$$
$$+\frac{q_\mu}{m}\Big[-\frac{b_3}{2}(v\cdot qv\cdot(P+p_2)-q\cdot(P+p_2))+b_8q\cdot p_2-\frac{g_{\pi N\Delta}}{4m}(v\cdot Pv\cdot p_2-P\cdot p_2)\Big]$$
$$+\frac{P_\mu}{m}\Big[-\frac{b_3}{2}z((v\cdot q)^2-q^2)-(b_3+b_8)(v\cdot q)^2$$
$$\left.-\frac{g_{\pi N\Delta}}{3m}(-\frac{1}{2}(v\cdot Pv\cdot q-P\cdot q)-(2z_0-1)v\cdot Pv\cdot q+(2z_0-4)\Delta v\cdot q)\Big]\right\} \ . \tag{A.6}$$

Note that possible $\pi\pi N\Delta$ vertices only start at second order and can therefore not appear in the leading loop graphs of order $\mathcal{O}(\varepsilon^3)$. This explains the vanishing of some diagrams not drawn in Fig.2.

B Expressions for the $1/m$ corrections

In this appendix, we give the lengthy expressions for the various $1/m$ corrections stemming from the Δ–insertions, which are of relevance to our problem.

$1/m$ corrections to $\mathcal{L}_{\pi N}^{(2)}$:

$$\bar{N}\gamma_0\mathcal{B}_{N\Delta}^{\dagger(1)}\gamma_0\mathcal{C}_{\Delta}^{-1(0)}\mathcal{B}_{N\Delta}^{(1)}N = -\frac{g_{\pi N\Delta}^2}{2m}\bar{N}\left\{\frac{4}{3}(1+8z_0+12z_0^2)S\cdot w^i\xi^{ij}S\cdot w^j\right.$$
$$\left.+\ \frac{1}{3}(5-8z_0-4z_0^2)v\cdot w^i\xi^{ij}v\cdot w^j\right\}N\ . \tag{B.1}$$

$1/m$ corrections to $\mathcal{L}_{\pi N}^{(3)}$:

$$\bar{N}\ \left\{\ [\gamma_0(\mathcal{C}_{N\Delta}^{(1)}\mathcal{C}_{\Delta}^{-1(0)}\mathcal{B}_{N\Delta}^{(1)})^{-1}\gamma_0\mathcal{C}_N^{(0)-1}\mathcal{B}_N + \text{h.c.}] + \gamma_0\mathcal{B}_{N\Delta}^{\dagger(1)}\gamma_0\mathcal{C}_{\Delta}^{(1)-1}\mathcal{B}_{N\Delta}^{(1)}\right.$$
$$+\ [\gamma_0\mathcal{B}_{N\Delta}^{\dagger(2)}\gamma_0\mathcal{C}_{\Delta}^{(0)-1}\mathcal{B}_{N\Delta}^{(1)} + \text{h.c.}]\bigg\}N$$
$$= \bar{N}\ \left\{\ \frac{g_{\pi N\Delta}^2}{(2m)^2}\Big[\frac{2}{3}(1+4z_0+12z_0^2)S\cdot w^i\xi^{ij}v\cdot w^j2iS\cdot D + \text{h.c.}\right.$$
$$-\frac{2}{3}(3+4z_0+4z_0^2)v\cdot w^i\xi^{ij}S\cdot w^j2iS\cdot D + \text{h.c.} - \frac{8}{3}(1+4z_0+4z_0^2)S\cdot w^i\xi^{ij}iv\cdot DS\cdot w^j$$
$$+\frac{16}{3}(1+6z_0+8z_0^2)\Delta S\cdot w^i\xi^{ij}S\cdot w^j + \frac{8}{9}(5+8z_0+12z_0^2)S\cdot w^i\xi^{ij}iS\cdot Dv\cdot w^j + \text{h.c.}$$
$$+\frac{2}{3}(-1+4z_0-4z_0^2)v\cdot w^iiv\cdot D\xi^{ij}v\cdot w^j + 4(1-2z_0)\Delta v\cdot w^i\xi^{ij}v\cdot w^j\Big]$$
$$+\ \frac{g_{\pi N\Delta}}{2m}i\Big[-\frac{4}{3}(1+4z+4z_0+12zz_0)(b_3+b_8)S\cdot w^i\xi^{ij}w_{\alpha\beta}^jS_\alpha v_\beta - \text{h.c.}$$
$$+\frac{4}{3}(1+4z+12zz_0)b_3S\cdot w^i\xi^{ij}w_{\alpha\beta}^jv_\alpha S_\beta - \text{h.c.}$$
$$-\frac{2}{3}(6+13z+4zz_0)b_3v\cdot w^i\xi^{ij}w_{\alpha\beta}^jS_\alpha S_\beta - \text{h.c.}$$
$$\left.+\frac{1}{3}(-5+4z+4z_0+4zz_0)(b_3+b_8)v\cdot w^i\xi^{ij}w_{\alpha\beta}^jv_\alpha v_\beta - \text{h.c.}\Big]\right\}N\ . \tag{B.2}$$

$1/m$ corrections to $\mathcal{L}_{\pi N\Delta}^{(2)}$:

$$\bar{T}_\mu^i\gamma_0\mathcal{B}_{\Delta}^{\dagger(1)}\mathcal{C}_{\Delta}^{-1(0)}\mathcal{B}_{N\Delta}^{(1)}N + \text{h.c.} = -\frac{g_{\pi N\Delta}}{2m}2\bar{T}_\mu^iiD_\mu^{ik}\xi^{kl}v\cdot w^lN + \text{h.c.}\ . \tag{B.3}$$

$1/m$ corrections to $\mathcal{L}_{\pi N\Delta}^{(3)}$:

$$\bar{T}_\mu^i \Big\{ \gamma_0 \tilde{\mathcal{D}}_{N\Delta}^{\dagger(2)} \gamma_0 \tilde{\mathcal{C}}_N^{-1(0)} \tilde{\mathcal{B}}_N^{(1)} + \gamma_0 \mathcal{B}_\Delta^{\dagger(1)} \gamma_0 \mathcal{C}_\Delta^{-1(1)} \mathcal{B}_{N\Delta}^{(1)} + \gamma_0 \mathcal{B}_\Delta^{\dagger(1)} \gamma_0 \mathcal{C}_\Delta^{-1(0)} \mathcal{B}_{N\Delta}^{(2)} \Big\} N + \text{h.c.}$$

$$= \bar{T}_\mu^i \Big\{ \frac{-2ib_3}{2m} S_\beta w_{\beta\mu}^i 2iS \cdot D - \frac{4ib_3}{2m} iS \cdot D^{ik} \xi^{kl} w_{\mu\beta}^l S_\beta - \frac{2i(b_3 + b_8)}{2m} iD_\mu^{ik} \xi^{kl} w_{\alpha\beta}^l v_\alpha v_\beta$$

$$- \frac{g_{\pi N\Delta}}{(2m)^2} \Big[2iS \cdot D^{ik} \xi^{kj} w_\mu^j 2iS \cdot D - \frac{8 + 32z_0}{3} iS \cdot D^{ik} \xi^{kl} iD_\mu^{lm} \xi^{mn} S \cdot w^n$$

$$+ \frac{32}{3} z_0 iD_\mu^{ik} \xi^{kl} iS \cdot D^{lm} \xi^{mn} S \cdot w^n - \frac{8z_0 - 4}{3} iD_\mu^{ik} \xi^{kl} iv \cdot D^{lm} \xi^{mn} v \cdot w^n$$

$$+ \frac{8z_0 - 16}{3} \Delta iD_\mu^{ik} \xi^{kl} v \cdot w^l \Big] \Big\} N + \text{h.c.} \ . \tag{B.4}$$

$1/m$ corrections to $\mathcal{L}_{\pi\Delta}^{(2)}$:

$$\bar{T}_\mu^i \gamma_0 \mathcal{B}_\Delta^{\dagger(1)} \gamma_0 \mathcal{C}_\Delta^{-1(0)} \mathcal{B}_\Delta^{(1)} T_\nu^j = \frac{1}{2m} \bar{T}_\mu^i \delta^{ij} 2iS \cdot D g_{\mu\nu} 2iS \cdot T_\nu^j \ . \tag{B.5}$$

$1/m$ corrections to $\mathcal{L}_{\pi\Delta}^{(3)}$:

$$\bar{T}_\mu^i \gamma_0 \mathcal{B}_\Delta^{\dagger(1)} \gamma_0 \mathcal{C}_\Delta^{-1(1)} \mathcal{B}_\Delta^{(1)} T_\nu^j = -\frac{1}{(2m)^2} \bar{T}_\mu^i \delta^{ij} \Big\{ 2iS \cdot D g_{\mu\nu} (iv \cdot D + \Delta) 2iS \cdot D$$
$$+ 4iD_\mu (iv \cdot D - \Delta) iD_\nu \Big\} T_\nu^j \ . \tag{B.6}$$

C   Tree and loop amplitudes

In this appendix, we give the lengthy analytical expressions for the tree and counterterm as well as one loop amplitudes involving intermediate delta states, as depicted in Fig.1 (tree and counterterm graphs) and Fig.2 (loop graphs). We use the loop functions defined in Ref.[21]. All other notation has been defined previously.

Tree diagram contributions:

$$F_\pi^2 g^+(\omega, t) = -g_{\pi N\Delta}^2 \frac{2}{9} (2\omega^2 - 2M_\pi^2 + t) \left(\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta} \right)$$

$$+ g_{\pi N\Delta}^2 \frac{1}{9m} \Big\{ (2\omega^2 - 2M_\pi^2 + t)(\omega^2 - M_\pi^2) \left(\left(\frac{1}{\omega - \Delta} \right)^2 + \left(\frac{1}{\omega + \Delta} \right)^2 \right)$$

$$- (2\omega^2 - 2M_\pi^2 + t)(4\omega^2 - 4M_\pi^2 + t) \left(\frac{1}{\omega + \Delta} \right)^2$$

$$+ 4\omega(4\omega^2 - 4M_\pi^2 + t) \frac{1}{\omega + \Delta}$$

$$- 4\omega^2(3 + 4z_0^2) + (2M_\pi^2 - t)(1 + 8z_0 + 12z_0^2) \Big\}$$

$$- g_{\pi N\Delta}(b_3 + b_8) \frac{4}{9} \omega(2\omega^2 - 2M_\pi^2 + t) \left(\frac{1}{\omega - \Delta} + \frac{1}{\omega + \Delta} \right)$$

$$
\begin{aligned}
&+\quad g_{\pi N\Delta}^2\frac{1}{9m^2}\Big\{-\frac{1}{2}(2\omega^2-2M_\pi^2+t)(\omega^2-M_\pi^2)^2\left(\left(\frac{1}{\omega-\Delta}\right)^3-\left(\frac{1}{\omega+\Delta}\right)^3\right)\\
&\qquad\qquad +\frac{1}{2}(2\omega^2-2M_\pi^2+t)^2(4\omega^2-4M_\pi^2+t)\left(\frac{1}{\omega+\Delta}\right)^3\\
&\qquad\qquad -\omega(4\omega^2-4M_\pi^2+t)(8\omega^2-8M_\pi^2+3t)\left(\frac{1}{\omega+\Delta}\right)^2\\
&\qquad\qquad +\frac{1}{6}(4\omega^2-4M_\pi^2+t)(72\omega^2-24M_\pi^2+7t)\frac{1}{\omega+\Delta}\\
&\qquad\qquad +16\Delta\omega^2(1+2z_0^2)-2\Delta(2M_\pi^2-t)(1+6z_0+8z_0^2)-2\omega(4\omega^2-4M_\pi^2+t)(3+4z_0^2)\Big\}\\
&+\quad g_{\pi N\Delta}(b_3+b_8)\frac{1}{9m}\Big\{2\omega(\omega^2-M_\pi^2)(2\omega^2-2M_\pi^2+t)\left(\left(\frac{1}{\omega-\Delta}\right)^2-\left(\frac{1}{\omega+\Delta}\right)^2\right)\\
&\qquad\qquad +2\omega(2\omega^2-2M_\pi^2+t)(4\omega^2-4M_\pi^2+t)\left(\frac{1}{\omega+\Delta}\right)^2\\
&\qquad\qquad -2[(\omega^2-M_\pi^2)(22\omega^2-6M_\pi^2+5t)+t^2]\frac{1}{\omega+\Delta}\\
&\qquad\qquad -2(\omega^2-M_\pi^2)(2\omega^2-2M_\pi^2+t)\frac{1}{\omega-\Delta}\Big\}\\
&-\quad g_{\pi N\Delta}b_8\frac{2}{9m}(2\omega^2-2M_\pi^2+t)(\omega^2-M_\pi^2)\left(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right)\\
&+\quad [2g_{\pi N\Delta}(e_1+e_2)-(b_3+b_8)^2]\frac{2}{9}\omega^2(2\omega^2-2M_\pi^2+t)\left(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right)\\
&-\quad g_{\pi N\Delta}(2e_4-e_5)\frac{8}{9}M_\pi^2(2\omega^2-2M_\pi^2+t)\left(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right)\,. \qquad\qquad\text{(C.1)}
\end{aligned}
$$

$$
\begin{aligned}
F_\pi^2 h^+(\omega,t)\;=\;&g_{\pi N\Delta}^2\frac{2}{9}\left(\frac{1}{\omega-\Delta}+\frac{1}{\omega+\Delta}\right)\\
-\;&g_{\pi N\Delta}^2\frac{1}{9m}\Big\{(\omega^2-M_\pi^2)\left(\left(\frac{1}{\omega-\Delta}\right)^2-\left(\frac{1}{\omega+\Delta}\right)^2\right)\\
&\qquad +(4\omega^2-4M_\pi^2+t)\left(\frac{1}{\omega+\Delta}\right)^2\\
&\qquad -4\omega\frac{1}{\omega+\Delta}\Big\}\\
+\;&g_{\pi N\Delta}(b_3+b_8)\frac{4}{9}\omega\left(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right)\\
+\;&g_{\pi N\Delta}^2\frac{1}{9m^2}\Big\{\frac{1}{2}(\omega^2-M_\pi^2)^2\left(\left(\frac{1}{\omega-\Delta}\right)^3+\left(\frac{1}{\omega+\Delta}\right)^3\right)\\
&\qquad +\frac{1}{2}(2\omega^2-2M_\pi^2+t)(4\omega^2-4M_\pi^2+t)\left(\frac{1}{\omega+\Delta}\right)^3\\
&\qquad -\omega(10\omega^2-10M_\pi^2+3t)\left(\frac{1}{\omega+\Delta}\right)^2\\
&\qquad +(16\omega^2-12M_\pi^2+3t)\frac{1}{\omega+\Delta}
\end{aligned}
$$

$$-2\omega(1+4z_0^2)\Big\}$$

$$+ \quad g_{\pi N\Delta}(b_3+b_8)\frac{1}{9m}\Big\{-2\omega(\omega^2-M_\pi^2)\Big(\Big(\frac{1}{\omega-\Delta}\Big)^2+\Big(\frac{1}{\omega+\Delta}\Big)^2\Big)$$

$$+2\omega(4\omega^2-4M_\pi^2+t)\Big(\frac{1}{\omega+\Delta}\Big)^2$$

$$-2(7\omega^2-3M_\pi^2+t)\frac{1}{\omega+\Delta}$$

$$+2(\omega^2-M_\pi^2)\frac{1}{\omega-\Delta}\Big\}$$

$$-16\omega z_0$$

$$+ \quad g_{\pi N\Delta}b_8\frac{2}{9m}\Big\{(\omega^2-M_\pi^2)\Big(\frac{1}{\omega-\Delta}+\frac{1}{\omega+\Delta}\Big)-2\omega(1+4z+12zz_0)\Big\}$$

$$- \quad [2g_{\pi N\Delta}(e_1+e_2)-(b_3+b_8)^2]\frac{2}{9}\omega^2\Big(\frac{1}{\omega-\Delta}+\frac{1}{\omega+\Delta}\Big)$$

$$+ \quad g_{\pi N\Delta}(2e_4-e_5)\frac{8}{9}M_\pi^2\Big(\frac{1}{\omega-\Delta}+\frac{1}{\omega+\Delta}\Big)\ . \tag{C.2}$$

$$F_\pi^2 g^-(\omega,t) \quad = \quad g_{\pi N\Delta}^2\frac{1}{9}(2\omega^2-2M_\pi^2+t)\Big(\frac{1}{\omega-\Delta}+\frac{1}{\omega+\Delta}\Big)$$

$$+ \quad g_{\pi N\Delta}^2\frac{1}{9m}\Big\{-\frac{1}{2}(2\omega^2-2M_\pi^2+t)(\omega^2-M_\pi^2)\Big(\Big(\frac{1}{\omega-\Delta}\Big)^2-\Big(\frac{1}{\omega+\Delta}\Big)^2\Big)$$

$$-\frac{1}{2}(2\omega^2-2M_\pi^2+t)(4\omega^2-4M_\pi^2+t)\Big(\frac{1}{\omega+\Delta}\Big)^2$$

$$+2\omega(4\omega^2-4M_\pi^2+t)\frac{1}{\omega+\Delta}\Big\}$$

$$+ \quad g_{\pi N\Delta}(b_3+b_8)\frac{2}{9}w(2\omega^2-2M_\pi^2+t)\Big(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\Big)$$

$$+ \quad g_{\pi N\Delta}^2\frac{1}{9m^2}\Big\{\frac{1}{4}(2\omega^2-2M_\pi^2+t)(\omega^2-M_\pi^2)^2\Big(\Big(\frac{1}{\omega-\Delta}\Big)^3+\Big(\frac{1}{\omega+\Delta}\Big)^3\Big)$$

$$+\frac{1}{4}(2\omega^2-2M_\pi^2+t)^2(4\omega^2-4M_\pi^2+t)\Big(\frac{1}{\omega+\Delta}\Big)^3$$

$$-\frac{1}{2}w(4\omega^2-4M_\pi^2+t)(8\omega^2-8M_\pi^2+3t)\Big(\frac{1}{\omega+\Delta}\Big)^2$$

$$+\frac{1}{12}(4\omega^2-4M_\pi^2+t)(72\omega^2-24M_\pi^2+7t)\frac{1}{\omega+\Delta}$$

$$+w[2(2M_\pi^2-t)(1-2z_0-2z_0^2)-12\omega^2+M_\pi^2(9+4z_0+12z_0^2)]\Big\}$$

$$+ \quad g_{\pi N\Delta}(b_3+b_8)\frac{1}{9m}\Big\{-w(2\omega^2-2M_\pi^2+t)(\omega^2-M_\pi^2)\Big(\Big(\frac{1}{\omega-\Delta}\Big)^2+\Big(\frac{1}{\omega+\Delta}\Big)^2\Big)$$

$$+w(2\omega^2-2M_\pi^2+t)(4\omega^2-4M_\pi^2+t)\Big(\frac{1}{\omega+\Delta}\Big)^2$$

$$+[(\omega^2-M_\pi^2)(-22\omega^2+6M_\pi^2-5t)-t^2]\frac{1}{\omega+\Delta}$$

$$+(2\omega^2 - 2M_\pi^2 + t)(\omega^2 - M_\pi^2)\frac{1}{\omega - \Delta}$$

$$-4\omega(2M_\pi^2 - t)z_0 + \omega^3(16 + 5z - 4zz_0) - \omega M_\pi^2(6 + 13z + 4zz_0)\Big\}$$

$$+ \ g_{\pi N\Delta}b_8\frac{1}{9m}\Big\{(2\omega^2 - 2M_\pi^2 + t)(\omega^2 - M_\pi^2)\left(\frac{1}{\omega - \Delta} + \frac{1}{\omega + \Delta}\right)$$

$$-\omega(2M_\pi^2 - t)(1 + 4z + 12zz_0) - \omega^3(4 + 5z - 20zz_0) + \omega M_\pi^2(6 + 13z + 4zz_0)\Big\}$$

$$- \ [2g_{\pi N\Delta}(e_1 + e_2) - (b_3 + b_8)^2]\frac{1}{9}\omega^2(2\omega^2 - 2M_\pi^2 + t)\left(\frac{1}{\omega - \Delta} + \frac{1}{\omega + \Delta}\right)$$

$$+ \ g_{\pi N\Delta}(2e_4 - e_5)\frac{4}{9}M_\pi^2(2\omega^2 - 2M_\pi^2 + t)\left(\frac{1}{\omega - \Delta} + \frac{1}{\omega + \Delta}\right) \ . \tag{C.3}$$

$$F_\pi^2 h^-(\omega, t) \ = \ -g_{\pi N\Delta}^2\frac{1}{9}\left(\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta}\right)$$

$$+ \ g_{\pi N\Delta}^2\frac{1}{9m}\Big\{\frac{1}{2}(\omega^2 - M_\pi^2)\left(\left(\frac{1}{\omega - \Delta}\right)^2 + \left(\frac{1}{\omega + \Delta}\right)^2\right)$$

$$-\frac{1}{2}(4\omega^2 - 4M_\pi^2 + t)\left(\frac{1}{\omega + \Delta}\right)^2$$

$$+2\omega\frac{1}{\omega + \Delta}$$

$$+(1 + 8z_0 + 12z_0^2)\Big\}$$

$$- \ g_{\pi N\Delta}(b_3 + b_8)\frac{2}{9}\omega\left(\frac{1}{\omega - \Delta} + \frac{1}{\omega + \Delta}\right)$$

$$+ \ g_{\pi N\Delta}^2\frac{1}{9m^2}\Big\{-\frac{1}{4}(\omega^2 - M_\pi^2)^2\left(\left(\frac{1}{\omega - \Delta}\right)^3 - \left(\frac{1}{\omega + \Delta}\right)^3\right)$$

$$+\frac{1}{4}(2\omega^2 - 2M_\pi^2 + t)(4\omega^2 - 4M_\pi^2 + t)\left(\frac{1}{\omega + \Delta}\right)^3$$

$$-\frac{1}{2}\omega(10\omega^2 - 10M_\pi^2 + 3t)\left(\frac{1}{\omega + \Delta}\right)^2$$

$$+\frac{1}{2}(16\omega^2 - 12M_\pi^2 + 3t)\frac{1}{\omega + \Delta}$$

$$+\omega(1 + 8z_0 + 12z_0^2) - 2\Delta(1 + 6z_0 + 8z_0^2)\Big\}$$

$$+ \ g_{\pi N\Delta}(b_3 + b_8)\frac{1}{9m}\Big\{\omega(\omega^2 - M_\pi^2)\left(\left(\frac{1}{\omega - \Delta}\right)^2 - \left(\frac{1}{\omega + \Delta}\right)^2\right)$$

$$+\omega(4\omega^2 - 4M_\pi^2 + t)\left(\frac{1}{\omega + \Delta}\right)^2$$

$$-(7\omega^2 - 3M_\pi^2 + t)\frac{1}{\omega + \Delta}$$

$$-(\omega^2 - M_\pi^2)\frac{1}{\omega - \Delta}\Big\}$$

$$- \ g_{\pi N\Delta}b_8\frac{1}{9m}(\omega^2 - M_\pi^2)\left(\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta}\right)$$

$$
+ \quad [2g_{\pi N\Delta}(e_1 + e_2) - (b_3 + b_8)^2]\frac{1}{9}\omega^2\left(\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta}\right)
$$

$$
- \quad g_{\pi N\Delta}(2e_4 - e_5)\frac{4}{9}M_\pi^2\left(\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta}\right) . \tag{C.4}
$$

Loop diagram contributions:

$$
\begin{aligned}
F_\pi^4 g^+(\omega, t) \;=\; & (J_0(\omega) + J_0(\omega))\Big\{ -\frac{8g_{\pi N\Delta}^2 g_A^2}{27(\omega^2 - \Delta^2)}(2M_\pi^4 - 4\omega^2 M_\pi^2 - tM_\pi^2 + 2\omega^4 + \omega^2 t) \\
& +\frac{8g_{\pi N\Delta}^4}{243(\omega^2 - \Delta^2)^2}(8M_\pi^4\omega^2 + 10M_\pi^4\Delta^2 - 16M_\pi^2\omega^4 - 4M_\pi^2\omega^2 t \\
& \quad -20M_\pi^2\omega^2\Delta^2 - 5M_\pi^2 t\Delta^2 + 8\omega^6 + 4\omega^4 t + 10\omega^4\Delta^2 + 5\omega^2 t\Delta^2)\Big\} \\
& + \; J_0(-\Delta)\Big\{\frac{g_{\pi N\Delta}^2}{2187\omega^2(\omega^2 - \Delta^2)^2}(-972M_\pi^2\omega^6 - 972M_\pi^2\omega^2\Delta^4 + 1944M_\pi^2\omega^4\Delta^2 \\
& \quad -3888\omega^4 t\Delta^2 + 1944\omega^2 t\Delta^4 + 1944\omega^6 t) \\
& +\frac{g_{\pi N\Delta}^4}{2187\omega^2(\omega^2 - \Delta^2)^2}(-792M_\pi^4\omega^2\Delta^2 - 504M_\pi^4\omega^4 + 7776M_\pi^2\omega^4\Delta^2 \\
& \quad +252M_\pi^2\omega^4 t + 396M_\pi^2\omega^2 t\Delta^2 - 5688M_\pi^2\omega^2\Delta^4 + 504M_\pi^2\omega^6 \\
& \quad -3492\omega^4 t\Delta^2 + 2844\omega^2 t\Delta^4 + 5688\omega^4\Delta^4 - 6984\omega^6\Delta^2) \\
& +\frac{g_{\pi N\Delta}^2 g_1^2}{2187\omega^2(\omega^2 - \Delta^2)^2}(-50M_\pi^4\omega^4 + 500M_\pi^4\omega^2\Delta^2 - 450M_\pi^4\Delta^4 + 50M^2\omega^6 \\
& \quad -450M_\pi^2\omega^4\Delta^2 + 225M_\pi^2 t\Delta^4 + 25M_\pi^2\omega^4 t - 250M_\pi^2\omega^2 t\Delta^2 - 50M_\pi^2\omega^2\Delta^4 \\
& \quad +450M_\pi^2\Delta^6 + 500\omega^4\Delta^4 - 225t\Delta^6 + 250\omega^2 t\Delta^4 - 50\omega^6\Delta^2 - 450\omega^2\Delta^6 \\
& \quad -25\omega^4 t\Delta^2) \\
& -\frac{50g_{\pi N\Delta}^2 g_A g_1}{243\omega^2}(-2M_\pi^4 + 2\omega^2 M_\pi^2 + M_\pi^2 t + 2M_\pi^2\Delta^2 - 2\omega^2\Delta^2 - t\Delta^2) \\
& +\frac{g_{\pi N\Delta}^2 g_A^2}{27\omega^2(\omega^2 - \Delta^2)}(-2M_\pi^4\omega^2 + 18M_\pi^4\Delta^2 + 2M_\pi^2\omega^4 + M_\pi^2\omega^2 t - 16M_\pi^2\omega^2\Delta^2 \\
& \quad -18M_\pi^2\Delta^4 - 9M_\pi^2 t\Delta^2 - 2\omega^4\Delta^2 - \omega^2 t\Delta^2 + 18\omega^2\Delta^4 + 9t\Delta^4)\Big\} \\
& - \; J_0(\Delta)\frac{4g_{\pi N\Delta}^4}{27(\omega^2 - \Delta^2)^2}(-2M_\pi^4\omega^2 - 2M_\pi^4\Delta^2 + M_\pi^2 t\Delta^2 + 4M_\pi^2\omega^2\Delta^2 + 2M_\pi^2\Delta^4 \\
& \quad +2M_\pi^2\omega^4 + M_\pi^2\omega^2 t - t\Delta^4 - 2\omega^4\Delta^2 - 2\omega^2\Delta^4 - \omega^2 t\Delta^2) \\
& + \; \frac{1}{\omega^2}\left(\frac{J_0(\omega - \Delta)}{(\omega - \Delta)^2} + \frac{J_0(-\omega - \Delta)}{(\omega + \Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{4374}(20M_\pi^4\omega^2 + 18M_\pi^4\Delta^2 - 40M_\pi^2\omega^4 \\
& \quad -10M_\pi^2\omega^2 t - 96M_\pi^2\omega^2\Delta^2 - 9M_\pi^2 t\Delta^2 - 18M_\pi^2\Delta^4 + 20\omega^6 + 78\omega^4\Delta^2 \\
& \quad +10\omega^4 t + 39\omega^2 t\Delta^2 + 18\omega^2\Delta^4 + 9t\Delta^4) \\
& -\frac{25g_{\pi N\Delta}^2 g_A g_1}{243}(4M_\pi^4\omega^2 + 2M_\pi^4\Delta^2 - 8M_\pi^2\omega^4 - 2M_\pi^2\omega^2 t - 16M_\pi^2\omega^2\Delta^2 \\
& \quad -2M_\pi^2\Delta^4 - M_\pi^2 t\Delta^2 + 4\omega^6 + 14\omega^4\Delta^2 + 7\omega^2 t\Delta^2 + 2\omega^2\Delta^4 + t\Delta^4 + 2\omega^4 t)
\end{aligned}
$$

$$
+ \frac{g_{\pi N\Delta}^2 g_A^2}{54} (20 M_\pi^4 \omega^2 + 18 M_\pi^4 \Delta^2 - 40 M_\pi^2 \omega^4 - 10 M_\pi^2 \omega^2 t - 96 M_\pi^2 \omega^2 \Delta^2
$$

$$
- 9 M_\pi^2 t \Delta^2 - 18 M_\pi^2 \Delta^4 + 20 \omega^6 + 78 \omega^4 \Delta^2 + 10 \omega^4 t + 39 \omega^2 t \Delta^2
$$

$$
+ 18 \omega^2 \Delta^4 + 9 t \Delta^4) \Big\}
$$

$$
+ \quad \frac{\Delta}{\omega} \left(\frac{J_0(\omega - \Delta)}{(\omega - \Delta)^2} - \frac{J_0(-\omega - \Delta)}{(\omega + \Delta)^2} \right) \Big\{ \frac{25 g_{\pi N\Delta}^2 g_1^2}{4374} (-20 M_\pi^4 + 80 M_\pi^2 \omega^2
$$

$$
+ 56 M_\pi^2 \Delta^2 + 10 M_\pi^2 t - 60 \omega^4 - 56 \omega^2 \Delta^2 - 30 \omega^2 t - 28 t \Delta^2)
$$

$$
- \frac{25 g_{\pi N\Delta}^2 g_A g_1}{243} (-4 M_\pi^4 + 16 M_\pi^2 \omega^2 + 8 M_\pi^2 \Delta^2 + 2 M_\pi^2 t - 12 \omega^4 - 6 \omega^2 t
$$

$$
- 8 \omega^2 \Delta^2 - 4 t \Delta^2)
$$

$$
+ \frac{g_{\pi N\Delta}^2 g_A^2}{54} (-20 M_\pi^4 + 80 M_\pi^2 \omega^2 + 56 M_\pi^2 \Delta^2 + 10 M_\pi^2 t - 60 \omega^4
$$

$$
- 56 \omega^2 \Delta^2 - 30 \omega^2 t - 28 t \Delta^2) \Big\}
$$

$$
- \quad J_0'(-\Delta) \frac{20 g_{\pi N\Delta}^4}{27(\omega^2 - \Delta^2)} \Delta (-2 M_\pi^4 + 2 M_\pi^2 \omega^2 + M_\pi^2 t + 2 M_\pi^2 \Delta^2 - 2 \omega^2 \Delta^2 - t \Delta^2)
$$

$$
- \quad K_0(t, -\Delta) \frac{2 g_{\pi N\Delta}^2}{9} (2 M_\pi^4 - 2 M_\pi^2 \Delta^2 - 5 M_\pi^2 t + 2 t^2 + 4 t \Delta^2)
$$

$$
+ \quad I_0(t) \frac{4 g_{\pi N\Delta}^2}{9} \Delta (M_\pi^2 - 2t)
$$

$$
+ \quad \frac{g_{\pi N\Delta}^4}{104976 \pi^2 (\omega^2 - \Delta^2)^2} (1296 M_\pi^4 \omega^2 \Delta + 10368 M_\pi^4 \Delta^3 - 14688 M_\pi^2 \Delta^5
$$

$$
- 5184 M_\pi^2 t \Delta^3 - 648 M_\pi^2 \omega^2 t \Delta - 3456 M_\pi^2 \omega^2 \Delta^3 - 1296 M_\pi^2 \omega^4 \Delta + 14688 \omega^2 \Delta^5
$$

$$
- 6912 \omega^4 \Delta^3 - 3456 \omega^2 t \Delta^3 + 7344 t \Delta^5)
$$

$$
+ \frac{g_{\pi N\Delta}^2}{104976 \pi^2 (\omega^2 - \Delta^2)^2} \Delta (1944 M_\pi^4 (\omega^2 - \Delta^2) - 2916 M_\pi^2 \Delta^4 + 7776 M_\pi^2 \omega^2 \Delta^2
$$

$$
+ 972 M_\pi^2 t \Delta^2 - 4860 M_\pi^2 \omega^4 - 972 M_\pi^2 \omega^2 t + 5832 t (\omega^2 - \Delta^2)^2)
$$

$$
+ \frac{g_{\pi N\Delta}^2 g_1^2}{104976 \pi^2 (\omega^2 - \Delta^2)^2} (-2700 \pi M_\pi^5 (\omega^2 + \Delta^2) - 3030 M_\pi^4 \Delta (\omega^2 - \Delta^2)
$$

$$
+ 1350 \pi M_\pi^3 (\omega^2 + \Delta^2)(2\omega^2 + t) + 1515 M_\pi^2 \omega^2 t \Delta - 9840 M_\pi^2 \Delta^5
$$

$$
+ 18410 M_\pi^2 \omega^2 \Delta^3 - 1515 M_\pi^2 t \Delta^3 - 8570 M^2 \omega^4 \Delta - 21440 \omega^4 \Delta^3 + 9840 \omega^2 \Delta^5
$$

$$
+ 4920 t \Delta^5 + 11600 \omega^6 \Delta - 10720 \omega^2 t \Delta^3 + 5800 \omega^4 t \Delta)
$$

$$
- \frac{5 g_{\pi N\Delta}^2 g_A g_1}{1944 \pi^2 (\omega^2 - \Delta^2)^2} (20 \pi M_\pi^5 (\omega^2 + \Delta^2) - 42 M_\pi^4 \Delta (\omega^2 - \Delta^2)
$$

$$
- 10 \pi M_\pi^3 (\omega^2 + \Delta^2)(2\omega^2 + t) - 21 M_\pi^2 t \Delta^3 + 21 M_\pi^2 \omega^2 t \Delta - 112 M_\pi^2 \Delta^5
$$

$$
- 70 M_\pi^2 \omega^4 \Delta + 182 M_\pi^2 \omega^2 \Delta^3 + 112 \omega^6 \Delta + 56 t \Delta^5 + 112 \omega^2 \Delta^5 - 112 \omega^2 t \Delta^3
$$

$$
- 224 \omega^4 \Delta^3 + 56 \omega^4 t \Delta)
$$

$$
+ \frac{g_{\pi N\Delta}^2 g_A^2}{648 \pi^2 (\omega^2 - \Delta^2)^2} (-\pi M^5 (42 \omega^2 - 150 \Delta^2) - 24 M^4 \Delta (\omega^2 - \Delta^2)
$$

$$
+ \pi M^3 (21 \omega^2 - 75 \Delta^2)(2\omega^2 + t) + 200 M^2 \omega^2 \Delta^3 - 80 M^2 \omega^4 \Delta - 120 M^2 \Delta^5
$$

$$+12M^2\omega^2 t\Delta - 12M^2 t\Delta^3 + 104\omega^6\Delta + 52t\omega^4\Delta - 112t\omega^2\Delta^3 + 60t\Delta^5$$
$$-224\omega^4\Delta^3 + 120\omega^2\Delta^5) \ . \tag{C.5}$$

$$\begin{aligned}
F_\pi^4 h^+(\omega,t) &= (J_0(\omega) + J_0(-\omega))\frac{\omega^2 - M_\pi^2}{\omega^2 - \Delta^2}\Big\{ - \frac{32g_{\pi N\Delta}^4}{81(\omega^2 - \Delta^2)}\omega\Delta + \frac{4g_{\pi N\Delta}^2 g_A^2}{9w}\Delta\Big\} \\
&+ (J_0(\omega) - J_0(-\omega))\frac{\omega^2 - M_\pi^2}{\omega^2 - \Delta^2}\Big\{\frac{8g_{\pi N\Delta}^4}{243(\omega^2 - \Delta^2)}(2w^2 + \Delta^2) - \frac{4g_{\pi N\Delta}^2 g_A^2}{27}\Big\} \\
&+ J_0(\Delta)\frac{8g_{\pi N\Delta}^4}{27(\omega^2 - \Delta^2)^2}\omega\Delta(M_\pi^2 - \Delta^2) \\
&+ J_0(-\Delta)\Big\{\frac{4g_{\pi N\Delta}^4\Delta}{2187\omega(\omega^2 - \Delta^2)^2}(-270M_\pi^2\omega^2 + 918\omega^4 - 648\omega^2\Delta^2) \\
&\qquad + \frac{4g_{\pi N\Delta}^2 g_1^2\Delta}{2187\omega(\omega^2 - \Delta^2)^2}(50M_\pi^2(\omega^2 - \Delta^2) + 125\Delta^4 - 200\omega^2\Delta^2 + 75\omega^4) \\
&\qquad - \frac{200g_{\pi N\Delta}^2 g_A g_1}{243\omega}\Delta + \frac{4g_{\pi N\Delta}^2 g_A^2}{27\omega\Delta(\omega^2 - \Delta^2)}(8M_\pi^2\omega^2 - 4M_\pi^2\Delta^2 + w^2\Delta^2 - 5\Delta^4)\Big\} \\
&+ \frac{1}{\omega}\left(\frac{J_0(\omega - \Delta)}{(\omega - \Delta)^2} + \frac{J_0(-\omega - \Delta)}{(\omega + \Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{4374}(20M_\pi^2\Delta - 30\omega^2\Delta - 32\Delta^3) \\
&\qquad - \frac{25g_{\pi N\Delta}^2 g_A g_1}{243}(4M_\pi^2\Delta - 6\omega^2\Delta - 8\Delta^3) \\
&\qquad + \frac{g_{\pi N\Delta}^2 g_A^2}{27}(10M_\pi^2\Delta - 15\omega^2\Delta - 28\Delta^3)\Big\} \\
&+ \frac{1}{\omega^2}\left(\frac{J_0(\omega - \Delta)}{(\omega - \Delta)^2} - \frac{J_0(-\omega - \Delta)}{(\omega + \Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{4374}(-5M_\pi^2\omega^2 - 6M_\pi^2\Delta^2 + 5\omega^4 \\
&\qquad + 51\omega^2\Delta^2 + 6\Delta^4) \\
&\qquad - \frac{25g_{\pi N\Delta}^2 g_A g_1}{243}(-M_\pi^2\omega^2 - 2M_\pi^2\Delta^2 + \omega^4 + 11\omega^2\Delta^2 + 2\Delta^4) \\
&\qquad + \frac{g_{\pi N\Delta}^2 g_A^2}{54}(-5M_\pi^2\omega^2 - 18M_\pi^2\Delta^2 + 5\omega^4 + 63\omega^2\Delta^2 + 18\Delta^4)\Big\} \\
&+ J_0'(-\Delta)(M_\pi^2 - \Delta^2)\Big\{\frac{50g_{\pi N\Delta}^2 g_1^2}{729\omega} + \frac{68\omega g_{\pi N\Delta}^4}{81(\omega^2 - \Delta^2)} - \frac{100g_{\pi N\Delta}^2 g_A g_1}{243\omega} \\
&\qquad + \frac{2g_{\pi N\Delta}^2 g_A^2}{3\omega}\Big\} \\
&- \frac{g_{\pi N\Delta}^4\omega}{26244\pi^2(\omega^2 - \Delta^2)^2}\Delta^2(-2916M_\pi^2 + 1944\Delta^4) \\
&- \frac{g_{\pi N\Delta}^2 g_1^2\omega}{52488\pi^2(\omega^2 - \Delta^2)^2}(1350\pi M_\pi^3\Delta + 645M_\pi^2(w^2 - \Delta^2) + 1700\omega^4 + 1860\Delta^4 \\
&\qquad - 3560\omega^2\Delta^2) \\
&- \frac{g_{\pi N\Delta}^2 g_A g_1\omega}{54\pi^2(\omega^2 - \Delta^2)}M_\pi^2 \\
&+ \frac{5g_{\pi N\Delta}^2 g_A g_1\omega}{972\pi^2(\omega^2 - \Delta^2)^2}(-10\pi M_\pi^3\Delta + 3M_\pi^2(\omega^2 - \Delta^2) + 12(\omega^2 - \Delta^2)^2)
\end{aligned}$$

$$+\frac{g_{\pi N\Delta}^2 g_A^2}{324\pi^2\omega\Delta(\omega^2-\Delta^2)^2}(\pi M_\pi^3(48\omega^4-93\omega^2\Delta^2+36\Delta^4)-42M_\pi^2\omega^2\Delta(\omega^2-\Delta^2)$$

$$+4\omega^4\Delta^3-10\omega^2\Delta^5+6\Delta\omega^6)\ . \tag{C.6}$$

$$
\begin{aligned}
F_\pi^4 g^-(\omega,t) &= (J_0(\omega)+J_0(-\omega))\frac{1}{\omega^2-\Delta^2}\Big\{\frac{16g_{\pi N\Delta}^4}{81(\omega^2-\Delta^2)}\omega\Delta(-2M_\pi^4+4M_\pi^2\omega^2+M_\pi^2 t-\omega^2 t-2\omega^4)\\
&\quad -\frac{2g_{\pi N\Delta}^2 g_A^2}{27}(2M_\pi^4-4M_\pi^2\omega^2-M_\pi^2 t+2\omega^4+\omega^2 t)\Big\}\\
&+ (J_0(\omega)-J_0(-\omega))\frac{1}{\omega^2-\Delta^2}\Big\{\frac{4g_{\pi N\Delta}^4}{243(\omega^2-\Delta^2)}(2M_\pi^4\Delta^2+4M_\pi^4\omega^2-8M_\pi^2\omega^4\\
&\quad -4M_\pi^2\omega^2\Delta^2-2M_\pi^2\omega^2 t-M_\pi^2 t\Delta^2+2\omega^4\Delta^2+4\omega^6+2\omega^4 t+\omega^2 t\Delta^2)\\
&\quad -\frac{2g_{\pi N\Delta}^2 g_A^2}{9\omega}\Delta(-2M_\pi^4+4M_\pi^2\omega^2+M_\pi^2 t-2\omega^4-\omega^2 t)\Big\}\\
&+ J_0(\Delta)\frac{4g_{\pi N\Delta}^4}{27(\omega^2-\Delta^2)^2}\omega\Delta(-2M_\pi^4+2M_\pi^2\omega^2+M_\pi^2 t+2M_\pi^2\Delta^2-2\omega^2\Delta^2-t\Delta^2)\\
&+ J_0(-\Delta)\Big\{\frac{2g_{\pi N\Delta}^2 g_1^2}{2187\omega(\omega^2-\Delta^2)^2}\Delta(-100M_\pi^4(\omega^2-\Delta^2)+50M_\pi^2\omega^2 t-250M_\pi^2\Delta^4\\
&\quad -50M_\pi^2\omega^4-50M_\pi^2 t\Delta^2+300M_\pi^2\omega^2\Delta^2+125t\Delta^4-400\omega^4\Delta^2\\
&\quad +75\omega^4 t+250\omega^2\Delta^4-200\omega^2 t\Delta^2+150\omega^6)\\
&\quad +\frac{2g_{\pi N\Delta}^4}{2187\omega(\omega^2-\Delta^2)^2}\Delta(540M_\pi^4\omega^2+1296M_\pi^2\omega^2\Delta^2-270M_\pi^2\omega^2 t\\
&\quad -2376M_\pi^2\omega^4+918\omega^4 t-648\omega^2 t\Delta^2-1296\omega^4\Delta^2+1836\omega^6)\\
&\quad -\frac{100g_{\pi N\Delta}^2 g_A g_1}{243w}\Delta(-2M_\pi^2+2\omega^2+t)\\
&\quad +\frac{2g_{\pi N\Delta}^2 g_A^2}{27\omega\Delta(\omega^2-\Delta^2)}(-16M_\pi^4\omega^2+8M_\pi^4\Delta^2+16M_\pi^2\omega^4+8M_\pi^2\omega^2 t+10M_\pi^2\Delta^4\\
&\quad -10M_\pi^2\omega^2\Delta^2-4M_\pi^2 t\Delta^2+2\omega^4\Delta^2+\omega^2 t\Delta^2-10\omega^2\Delta^4-5t\Delta^4)\Big\}\\
&+ \frac{\Delta}{\omega}\left(\frac{J_0(\omega-\Delta)}{(\omega-\Delta)^2}+\frac{J_0(-\omega-\Delta)}{(\omega+\Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{8748}(-40M_\pi^4+100M_\pi^2\omega^2+64M_\pi^2\Delta^2\\
&\quad +20M_\pi^2 t-60\omega^4-30\omega^2 t-64\omega^2\Delta^2-32t\Delta^2)\\
&\quad -\frac{25g_{\pi N\Delta}^2 g_A g_1}{486}(-8M_\pi^4+20M_\pi^2\omega^2+4M_\pi^2 t+16M_\pi^2\Delta^2-12\omega^4-16\omega^2\Delta^2\\
&\quad -6\omega^2 t-8t\Delta^2)\\
&\quad +\frac{g_{\pi N\Delta}^2 g_A^2}{108}(-40M_\pi^4+100M_\pi^2\omega^2+112M_\pi^2\Delta^2+20M_\pi^2 t-60\omega^4-112\omega^2\Delta^2\\
&\quad -30\omega^2 t-56t\Delta^2)\Big\}\\
&+ \frac{1}{\omega^2}\left(\frac{J_0(\omega-\Delta)}{(\omega-\Delta)^2}-\frac{J_0(-\omega-\Delta)}{(\omega+\Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{8748}(10M_\pi^4\omega^2+12M_\pi^4\Delta^2-20M_\pi^2\omega^4\\
&\quad -114M_\pi^2\omega^2\Delta^2-5M_\pi^2\omega^2 t-6M_\pi^2 t\Delta^2-12M_\pi^2\Delta^4+10\omega^6+5\omega^4 t\\
&\quad +102\omega^4\Delta^2+51\omega^2 t\Delta^2+12\omega^2\Delta^4+6t\Delta^4)
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{25g_{\pi N\Delta}^2 g_A g_1}{486}(2M_\pi^4\omega^2 + 4M_\pi^4\Delta^2 - 4M_\pi^2\omega^4 - 26M_\pi^2\omega^2\Delta^2 - M_\pi^2 w^2 t\\
&\qquad -2M_\pi^2 t\Delta^2 - 4M_\pi^2\Delta^4 + 2\omega^6 + \omega^4 t + 22\omega^4\Delta^2 + 4\omega^2\Delta^4 + 11\omega^2 t\Delta^2\\
&\qquad +2t\Delta^4)\\
&+\frac{g_{\pi N\Delta}^2 g_A^2}{108}(10M_\pi^4\omega^2 + 36M^4\Delta^2 - 20M_\pi^2\omega^4 - 5M_\pi^2\omega^2 t - 162M_\pi^2\omega^2\Delta^2\\
&\qquad -18M_\pi^2 t\Delta^2 - 36M_\pi^2\Delta^4 + 10\omega^6 + 126\omega^4\Delta^2 + 5\omega^4 t + 63\omega^2 t\Delta^2\\
&\qquad +36\omega^2\Delta^4 + 18t\Delta^4)\Big\}\\
+\ & J_0'(-\Delta)\Big\{\left(\frac{25g_{\pi N\Delta}^2 g_1^2}{729\omega} + \frac{34g_{\pi N\Delta}^4}{81(\omega^2-\Delta^2)}\omega\right)(\Delta^2 - M^2)(2M^2 - 2\omega^2 - t)\\
&+\frac{4g_{\pi N\Delta}^2}{9}\omega(-M_\pi^2 + \Delta^2)\\
&-\frac{50g_{\pi N\Delta}^2 g_A g_1}{243\omega}(-2M_\pi^4 + 2M_\pi^2\omega^2 + M_\pi^2 t + 2M_\pi^2\Delta^2 - 2\omega^2\Delta^2 - t\Delta^2)\\
&+\frac{g_{\pi N\Delta}^2 g_A^2}{3\omega}(-2M_\pi^4 + 2\omega^2 M_\pi^2 + M_\pi^2 t + 2M_\pi^2\Delta^2 - 2\omega^2\Delta^2 - t\Delta^2)\Big\}\\
-\ & K_0(t,-\Delta)\frac{4g_{\pi N\Delta}^2}{9}\Delta\omega(-2M_\pi^2 + t + 2\Delta^2)\\
-\ & I_0(t)\frac{2g_{\pi N\Delta}^2}{27}\omega(-8M_\pi^2 + 12\Delta^2 + 5t)\\
-\ & \frac{g_{\pi N\Delta}^2}{209952\pi^2(\omega^2-\Delta^2)^2}\omega(1944M_\pi^4(\omega^2-\Delta^2) - 5832M_\pi^2\omega^2\Delta^2 - 972M_\pi^2\omega^2 t\\
&\qquad +1944M_\pi^2\omega^4 + 972M_\pi^2 t\Delta^2 + 3888M_\pi^2\Delta^4 - 2268t(\omega^2-\Delta^2)^2)\\
-\ & \frac{g_{\pi N\Delta}^2 g_1^2}{209952\pi^2(\omega^2-\Delta^2)^2}\omega(-5400\pi M_\pi^5\Delta - 3030M_\pi^4(\omega^2-\Delta^2)\\
&\qquad +2700\pi M_\pi^3\Delta(t+2\omega^2) - 3790M_\pi^2\omega^2\Delta^2 + 1260M_\pi^2\Delta^4 + 2530M_\pi^2\omega^4\\
&\qquad -1515M_\pi^2 t\Delta^2 + 1515M_\pi^2\omega^2 t + 500\omega^6 + 760\omega^4\Delta^2 - 1260\omega^2\Delta^4\\
&\qquad -630t\Delta^4 + 380\omega^2 t\Delta^2 + 250\omega^4 t)\\
-\ & \frac{g_{\pi N\Delta}^4}{209952\pi^2(\omega^2-\Delta^2)^2}\omega(16200M_\pi^4\Delta^2 - 4536M_\pi^4\omega^2 + 2268M_\pi^2\omega^2 t\\
&\qquad -5400M_\pi^2\omega^2\Delta^2 - 18576M_\pi^2\Delta^4 - 8100M_\pi^2 t\Delta^2 + 4536M_\pi^2\omega^4\\
&\qquad -5400\omega^2 t\Delta^2 + 9288t\Delta^4 - 10800\omega^4\Delta^2 + 18576\omega^2\Delta^4)\\
+\ & \frac{5g_{\pi N\Delta}^2 g_A g_1}{3888\pi^2(\omega^2-\Delta^2)^2}\omega(40\pi M_\pi^5\Delta - 42M_\pi^4(\omega^2-\Delta^2) - 20\pi M_\pi^3\Delta(t+2\omega^2)\\
&\qquad +M_\pi^2(\omega^2-\Delta^2)(21t + 14\omega^2 - 14\Delta^2) + 14(\omega^2-\Delta^2)^2(2\omega^2 + t))\\
-\ & \frac{g_{\pi N\Delta}^2 g_A^2}{1296\pi^2\omega\Delta(\omega^2-\Delta^2)^2}(\pi M^5(144\Delta^4 + 192w^4 - 372\omega^2\Delta^2) - 96M_\pi^4\omega^2\Delta(\omega^2-\Delta^2)\\
&\qquad +\pi M_\pi^3(t+2\omega^2)(186\omega^2\Delta^2 - 96\omega^4 - 72\Delta^4) + 48M_\pi^2\omega^4 t\Delta - 48M_\pi^2\omega^2 t\Delta^3\\
&\qquad -106M_\pi^2\omega^2\Delta^5 + 100M_\pi^2\omega^4\Delta^3 + 6M_\pi^2\omega^6\Delta - 98\omega^4 t\Delta^3 + 53\omega^2 t\Delta^5\\
&\qquad +106\omega^4\Delta^5 + 90\omega^8\Delta - 196\omega^6\Delta^3 + 45\omega^6\Delta t)\ . \qquad\qquad\text{(C.7)}
\end{aligned}
$$

$$
\begin{aligned}
F_\pi^4 h^-(\omega,t) &= (J_0(\omega)+J_0(-\omega))\frac{\omega^2-M_\pi^2}{\omega^2-\Delta^2}\Big\{\frac{4g_{\pi N\Delta}^4}{243(\omega^2-\Delta^2)}(9\Delta^2+8\omega^2)-\frac{8g_{\pi N\Delta}^2 g_A^2}{27}\Big\} \\
&+ (J_0(\omega)-J_0(-\omega))\frac{\omega^2-M_\pi^2}{\omega^2-\Delta^2}\Big\{-\frac{32g_{\pi N\Delta}^4}{243(\omega^2-\Delta^2)}\,w\Delta+\frac{4g_{\pi N\Delta}^2 g_A^2}{27\omega}\Delta\Big\} \\
&- J_0(\Delta)\frac{2g_{\pi N\Delta}^4}{27(\omega^2-\Delta^2)^2}(M_\pi^2-\Delta^2)(\omega^2+\Delta^2) \\
&+ J_0(-\Delta)\Big\{\frac{g_{\pi N\Delta}^4}{2187(\omega^2-\Delta^2)^2}(414M_\pi^2\omega^2+486M_\pi^2\Delta^2+1710\Delta^4-2610\omega^2\Delta^2) \\
&\qquad +\frac{g_{\pi N\Delta}^2}{9}-\frac{50g_{\pi N\Delta}^2 g_1^2}{2187(\omega^2-\Delta^2)\omega^2}(\omega^2+\Delta^2)(M_\pi^2-\Delta^2) \\
&\qquad -\frac{100g_{\pi N\Delta}^2 g_A g_1}{243\omega^2}(M_\pi^2-\Delta^2)+\frac{2g_{\pi N\Delta}^2 g_A^2}{9\omega^2(\omega^2-\Delta^2)}(M_\pi^2-\Delta^2)(w^2-3\Delta^2)\Big\} \\
&+ \frac{1}{\omega^2}\left(\frac{J_0(\omega-\Delta)}{(\omega-\Delta)^2}+\frac{J_0(-\omega-\Delta)}{(\omega+\Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{8748}(-5M_\pi^2\omega^2-4M_\pi^2\Delta^2+5\omega^4 \\
&\qquad +9\omega^2\Delta^2+4\Delta^4) \\
&\qquad -\frac{25g_{\pi N\Delta}^2 g_A g_1}{486}(-M_\pi^2\omega^2-4M_\pi^2\Delta^2+\omega^4+13\omega^2\Delta^2+4\Delta^4) \\
&\qquad +\frac{g_{\pi N\Delta}^2 g_A^2}{108}(-5M_\pi^2\omega^2-36M_\pi^2\Delta^2+5\omega^4+121\omega^2\Delta^2+36\Delta^4)\Big\} \\
&+ \frac{\Delta}{\omega}\left(\frac{J_0(\omega-\Delta)}{(\omega-\Delta)^2}-\frac{J_0(-\omega-\Delta)}{(\omega+\Delta)^2}\right)\Big\{\frac{25g_{\pi N\Delta}^2 g_1^2}{4374}(-5\omega^2-4\Delta^2) \\
&\qquad -\frac{25g_{\pi N\Delta}^2 g_A g_1}{243}(2M_\pi^2-3\omega^2-6\Delta^2)+\frac{g_{\pi N\Delta}^2 g_A^2}{54}(20M_\pi^2-25\omega^2-56\Delta^2)\Big\} \\
&- J_0'(-\Delta)\frac{122g_{\pi N\Delta}^4}{243(\omega^2-\Delta^2)}\Delta(M_\pi^2-\Delta^2) \\
&- K_0(t,-\Delta)\frac{g_{\pi N\Delta}^2}{18}(-4M_\pi^2+4\Delta^2+t) \\
&- I_0(t)\frac{2g_{\pi N\Delta}^2}{9}\Delta \\
&- \frac{g_{\pi N\Delta}^4}{52488\pi^2(\omega^2-\Delta^2)^2}\Delta(1458M_\pi^2(\omega^2+\Delta^2)-108\Delta^4-2700\Delta^2\omega^2+864\omega^4) \\
&\qquad +\frac{g_{\pi N\Delta}^2 g_1^2}{104976\pi^2(\omega^2-\Delta^2)^2}(675\pi M_\pi^3(\omega^2+\Delta^2)+645M_\pi^2\Delta(\omega^2-\Delta^2)+4850\Delta\omega^4 \\
&\qquad +5010\Delta^5-9860\omega^2\Delta^3) \\
&\qquad +\frac{g_{\pi N\Delta}^2}{108\pi^2(\omega^2-\Delta^2)}\Delta(M_\pi^2-5\omega^2+5\Delta^2) \\
&\qquad -\frac{5g_{\pi N\Delta}^2 g_A g_1}{1944\pi^2(\omega^2-\Delta^2)^2}(\pi M_\pi^3(15\omega^2-25\Delta^2)+3M_\pi^2\Delta(\omega^2-\Delta^2)+70\Delta(\omega^2-\Delta^2)^2) \\
&\qquad +\frac{g_{\pi N\Delta}^2 g_A^2}{1296\pi^2(\omega^2-\Delta^2)^2}(\pi M_\pi^3(117\omega^2-123\Delta^2)+72M_\pi^2\Delta(\omega^2-\Delta^2)+210\Delta^5
\end{aligned}
$$

$$-388\omega^2\Delta^3 + 178\omega^4\Delta) \ . \tag{C.8}$$

D   Threshold parameters

Here, we collect the analytical expressions of the Δ–contribution for the S– and P–wave scattering lengths and effective ranges. The corresponding pure nucleon terms are given in Ref.[2]. As before, we explicitly display the off–shell parameters but we note that these are fixed in the actual calculation. We have:

$$a_{0+}^- = \frac{g_{\pi N\Delta}^2 M_\pi^3}{36\pi(m+M_\pi)mF^2}(1-4z_0+4z_0^2) - \frac{M_\pi^3(b_3+b_8)}{18\pi(m+M_\pi)F^2}(-5+4z+4z_0+4zz_0) \ , \tag{D.1}$$

$$
\begin{aligned}
b_{0+}^- =& \ \frac{g_{\pi N\Delta}^2 M_\pi}{144\pi(m+M_\pi)m^3(M_\pi+\Delta)F^2}\Big\{16m^3+m^2[M_\pi^2(2-20z_0-24z_0^2)-\Delta(30+40z_0+24z_0^2)]\\
&-M_\pi(M_\pi+\Delta)(2m-M_\pi)(1-4z_0+4z_0^2)\Big\}\\
&-\frac{g_{\pi N\Delta}M_\pi(b_3+b_8)}{72\pi(m+M_\pi)m^2(M_\pi+\Delta)F^2}\Big\{m^2[M_\pi(-26-2z+24z_0+16zz_0)\\
&+\Delta(-42-2z+24z_0+16zz_0)]+M_\pi(M_\pi+\Delta)(2m-M_\pi)(5-4z-4z_0-4zz_0)\Big\}\\
&-\frac{g_{\pi N\Delta}M_\pi b_8}{36\pi(m+M_\pi)F^2}(6+13z+14zz_0)\\
&-\frac{g_{\pi N\Delta}^2 mM_\pi}{648\pi^3(m+M_\pi)\sqrt{\Delta^2-M_\pi^2}F^4}\Big\{15\Delta\ln\frac{\Delta+\sqrt{\Delta^2-M_\pi^2}}{M_\pi}+13\sqrt{\Delta^2-M_\pi^2}\Big\}
\end{aligned} \tag{D.2}
$$

$$
\begin{aligned}
a_{0+}^+ =& \ \frac{g_{\pi N\Delta}^2 M_\pi^2}{18\pi(m+M_\pi)mF^2}\Big\{m(-5+8z_0+4z_0^2)+6\Delta(1-2z_0)\Big\}\\
&-\frac{g_{\pi N\Delta}^2 mM_\pi^2\sqrt{\Delta^2-M_\pi^2}}{24\pi^3(m+M_\pi)F^4}\ln\frac{\Delta+\sqrt{\Delta^2-M_\pi^2}}{M_\pi} \ ,
\end{aligned} \tag{D.3}
$$

$$
\begin{aligned}
b_{0+}^+ =& \ \frac{g_{\pi N\Delta}^2}{72\pi(m+M_\pi)m^2(M_\pi+\Delta)F^2}\Big\{4m^3[M_\pi(-1+8z_0+4z_0^2)+\Delta(-5+8z_0+4z_0^2)]\\
&+2m^2[M_\pi^2(9-8z_0-20z_0^2)+M_\pi\Delta(5-32z_0-20z_0^2)+12\Delta^2(1-2z_0)]\\
&+m[M_\pi^3(-5+8z_0+4z_0^2)+M_\pi^2\Delta(-17+32z_0+4z_0^2)+12M_\pi\Delta^2(-1+2z_0)]\\
&+6M_\pi^2(M_\pi+\Delta)\Delta(1-2z_0)\Big\}\\
&-\frac{4g_{\pi N\Delta}(b_3+b_8)M_\pi^2}{9\pi(m+M_\pi)(M_\pi+\Delta)F^2}\\
&-\frac{g_{\pi N\Delta}^2}{864\pi^3(m+M_\pi)m\sqrt{\Delta^2-M_\pi^2}F^4}\ln\frac{\Delta+\sqrt{\Delta^2-M_\pi^2}}{M_\pi}\Big\{m^2(-154M_\pi^2+144\Delta^2)\\
&+9M_\pi(2m-M_\pi)(M_\pi^2-\Delta^2)\Big\} \ ,
\end{aligned} \tag{D.4}
$$

$$a_{1-}^- = \frac{g_{\pi N\Delta}^2}{54\pi(m+M_\pi)m(M_\pi+\Delta)F^2}\Big\{2m^2+m[M_\pi(5+8z_0+12z_0^2)+\Delta(1+8z_0+12z_0^2)]$$

$$+M_\pi^2(5 + 12z_0 + 16z_0^2) - 3M_\pi\Delta - 2\Delta^2(1 + 6z_0 + 8z_0^2)\}$$

$$+\frac{2g_{\pi N\Delta}M_\pi(b_3 + b_8)}{27\pi(m + M_\pi)(M_\pi + \Delta)F^2}\Big\{-m + M_\pi(-2 + z_0) + \Delta z_0\Big\}$$

$$+\frac{g_{\pi N\Delta}M_\pi b_8}{54\pi(m + M_\pi)F^2}(1 + 4z + 12zz_0)$$

$$-\frac{m[2g_{\pi N\Delta}(e_1 + e_2) - 4g_{\pi N\Delta}(2e_4 - e_5) - (b_3 + b_8)^3]}{27\pi(m + M_\pi)(M_\pi + \Delta)F^2}\Big\}$$

$$+\quad\frac{g_{\pi N\Delta}^2 m}{3888\pi^3(m + M_\pi)(\Delta^2 - M_\pi^2)^2\sqrt{\Delta^2 - M_\pi^2}M_\pi^2 F^4}$$

$$\Big\{\sqrt{\Delta^2 - M_\pi^2}\sqrt{\Delta(2M_\pi - \Delta)}\left(\arcsin\frac{M_\pi - \Delta}{M_\pi} + \frac{\pi}{2}\right)$$

$$\Big[\frac{125g_1^2}{27}(\Delta - M_\pi)\Delta(-2M_\pi^4 - M_\pi^3\Delta + 3M_\pi^2\Delta^2 + M_\pi\Delta^3 - \Delta^4)$$

$$-\frac{50g_A g_1}{3}(\Delta - M_\pi)\Delta(-2M_\pi^4 + 3M_\pi^3\Delta + 9M_\pi^2\Delta^2 + M_\pi\Delta^3 - 3\Delta^4)$$

$$-\frac{g_A^2}{2}\Delta(-60M_\pi^5 + 270M_\pi^4\Delta + 216M_\pi^3\Delta^2 - 456M_\pi^2\Delta^3 - 180M_\pi\Delta^4 + 162\Delta^5)\Big]$$

$$+\sqrt{\Delta^2 - M_\pi^2}\Big[-(\Delta^2 - M_\pi^2)M_\pi^2(29M_\pi^3 - 24M_\pi^2\Delta - 26M_\pi\Delta^2 + 30\Delta^3)$$

$$+\frac{5g_1^2}{324}(\Delta - M_\pi)M_\pi[(533 - 270\pi)M_\pi^5 + (-1485 + 270\pi)M_\pi^4\Delta - 2702M_\pi^3\Delta^2$$

$$+1410M_\pi^2\Delta^3 + 2454M_\pi\Delta^4 + 360\Delta^5] + \frac{g_{\pi N\Delta}^2}{3}M_\pi^2(-123M_\pi^5 - 28M_\pi^4\Delta + 221M_\pi^3\Delta^2$$

$$+38M_\pi^2\Delta^3 - 80M_\pi\Delta^4 - 46\Delta^5) - \frac{5g_A g_1}{6}(\Delta - M_\pi)M_\pi[(55 - 30\pi)M_\pi^5$$

$$-(31 + 50\pi)M_\pi^4\Delta - 170M_\pi^3\Delta^2 + 6M_\pi^2\Delta^3 + 130M_\pi\Delta^4 + 40\Delta^5]$$

$$-\frac{g_A^2}{2\Delta}M_\pi[-96\pi M_\pi^7 + (243 - 117\pi)M_\pi^6\Delta - (184 - 186\pi)M_\pi^5\Delta^2$$

$$-(545 - 123\pi)M_\pi^4\Delta^3 + (331 - 72\pi)M_\pi^3\Delta^4 + 410M_\pi^2\Delta^5 - 147M_\pi\Delta^6 - 108\Delta^7]\Big]$$

$$+\sqrt{\Delta^2 - M_\pi^2}\sqrt{\Delta(\Delta + 2M_\pi)}\ln\frac{\Delta + M_\pi + \sqrt{\Delta(\Delta + 2M_\pi)}}{M_\pi}$$

$$\Big[\frac{100g_1^2}{108}(\Delta - M_\pi)\Delta^2(20M_\pi^3 - 8M_\pi^2\Delta - 11M_\pi\Delta^2 - \Delta^3)$$

$$-\frac{50g_A g_1}{3}(\Delta - M_\pi)\Delta^3(-2M_\pi^2 + M_\pi\Delta + \Delta^2)$$

$$-3g_A^2\Delta^2(-20M_\pi^4 + 12M_\pi^3\Delta + 27M_\pi^2\Delta^2 - 10M_\pi\Delta^3 - 9\Delta^4)\Big]$$

$$+\ln\frac{\Delta + \sqrt{\Delta^2 - M_\pi^2}}{M_\pi}\Big[-6(\Delta^2 - M_\pi^2)M_\pi^2(-6M_\pi^4 + 5M_\pi^3\Delta + 12M_\pi^2\Delta^2 - 5M_\pi\Delta^3 - 6\Delta^4)$$

$$+\frac{100g_1^2}{108}(\Delta - M_\pi)(-4M_\pi^7 + 22M_\pi^6\Delta + 30M_\pi^5\Delta^2 - 48M_\pi^4\Delta^3$$

$$-48M_\pi^3\Delta^4 + 30M_\pi^2\Delta^5 + 22M_\pi\Delta^6 - 4\Delta^7) + \frac{2g_{\pi N\Delta}^2}{3}M_\pi^2(-64M_\pi^6 - 210M_\pi^5\Delta$$

$$+422M_\pi^4\Delta^2 + 420M_\pi^3\Delta^3 - 652M_\pi^2\Delta^4 - 210M_\pi\Delta^5 + 294\Delta^6)$$

$$-\frac{100g_Ag_1}{3}(\Delta - M_\pi)(2M_\pi^7 + 5M_\pi^6\Delta - 3M_\pi^5\Delta^2 - 12M_\pi^4\Delta^3 + 9M_\pi^2\Delta^5 + M_\pi\Delta^6 - 2\Delta^7)$$

$$-\frac{6g_A^2}{\Delta}(16M_\pi^9 + 6M_\pi^8\Delta - 29M_\pi^7\Delta^2 - 36M_\pi^6\Delta^3 - 9M_\pi^5\Delta^4 + 72M_\pi^4\Delta^5 + 41M_\pi^3\Delta^6$$

$$\left.\left.-60M_\pi^2\Delta^7 - 19M_\pi\Delta^8 + 18\Delta^9)\right]\right\}, \tag{D.5}$$

$$a_{1-}^+ = -\frac{g_{\pi N\Delta}^2}{54\pi(m + M_\pi)m(M_\pi + \Delta)F^2}\Big\{-4m^2 + m[M_\pi(-7 + 8z_0 + 12z_0^2) + \Delta(1 + 8z_0 + 12z_0^2)]$$

$$+4M_\pi^2(-1 + 4z_0^2) + 6M_\pi\Delta(1 - 2z_0) - 2\Delta^2(1 + 6z_0 + 8z_0^2)\Big\}$$

$$-\frac{4g_{\pi N\Delta}M_\pi(b_3 + b_8)}{27\pi(m + M_\pi)(M_\pi + \Delta)F^2}(m + 2M_\pi(1 + z_0) + 2\Delta z_0)$$

$$-\frac{2g_{\pi N\Delta}M_\pi b_8}{27\pi(m + M_\pi)F^2}(1 + 4z + 12zz_0)$$

$$-\frac{2M_\pi^2 m[2g_{\pi N\Delta}(e_1 + e_2) - 4g_{\pi N\Delta}(2e_4 - e_5) - (b_3 + b_8)^2]}{27\pi(m + M_\pi)(M_\pi + \Delta)F^2}$$

$$+\frac{g_{\pi N\Delta}^2 m}{324\pi^3(m + M_\pi)(\Delta^2 - M_\pi^2)^2\sqrt{\Delta^2 - M_\pi^2}M_\pi^2 F^4}$$

$$\Big\{\ln\frac{\Delta + \sqrt{\Delta^2 - M_\pi^2}}{M_\pi}\Big[\frac{1}{4}M_\pi^2(\Delta^2 - M_\pi^2)(77M_\pi^4 - 149M_\pi^2\Delta^2 + 72\Delta^4)$$

$$-\frac{25g_1^2}{162}(\Delta - M_\pi)(-M_\pi^7 - 27M_\pi^6\Delta - 15M_\pi^5\Delta^2 + 63M_\pi^4\Delta^3 + 33M_\pi^3\Delta^4 - 45M_\pi^2\Delta^5$$

$$-17M_\pi\Delta^6 + 9\Delta^7) - \frac{2g_{\pi N\Delta}^2}{9}M_\pi^2(16M_\pi^6 + 105M_\pi^5\Delta - 147M_\pi^4\Delta^2 - 210M_\pi^3\Delta^3$$

$$+246M_\pi^2\Delta^4 + 105M_\pi\Delta^5 - 115\Delta^6) + \frac{25g_Ag_1}{9}(\Delta - M_\pi)(-M_\pi^7 - 7M_\pi^6\Delta - 3M_\pi^5\Delta^2$$

$$+15M_\pi^4\Delta^3 + 9M_\pi^3\Delta^4 - 9M_\pi^2\Delta^5 - 5M_\pi\Delta^6 + \Delta^7) + \frac{g_A^2}{2\Delta}(-32M_\pi^9 - M_\pi^8\Delta + 58M_\pi^7\Delta^2$$

$$+12M_\pi^6\Delta^3 + 18M_\pi^5\Delta^4 - 30M_\pi^4\Delta^5 - 82M_\pi^3\Delta^6 + 28M_\pi^2\Delta^7 + 38M_\pi\Delta^8 - 9\Delta^9)\Big]$$

$$+\sqrt{\Delta^2 - M_\pi^2}\Big[\frac{1}{2}M_\pi^4(\Delta^2 - M_\pi^2)(2M_\pi + \Delta) - \frac{5g_1^2}{1944}(\Delta - M_\pi)M_\pi[(-1118 + 270\pi)M_\pi^5$$

$$-(285 + 270\pi)M_\pi^4\Delta + 2387M_\pi^3\Delta^2 + 930M_\pi^2\Delta^3 - 984M_\pi\Delta^4 - 360\Delta^5]$$

$$-\frac{g_{\pi N\Delta}^2}{9}M_\pi^2(51M_\pi^5 - 26M_\pi^4\Delta - 102M_\pi^3\Delta^2 + 44M_\pi^2\Delta^3 + 33M_\pi\Delta^4 - 9\Delta^5)$$

$$+\frac{5g_Ag_1}{36}(\Delta - M_\pi)M_\pi[-(50 + 10\pi)M_\pi^5 - (23 - 10\pi)M_\pi^4\Delta + 121M_\pi^3\Delta^2 + 78M_\pi^2\Delta^3$$

$$-56M_\pi\Delta^4 - 40\Delta^5] + \frac{g_A^2}{12\Delta}M_\pi[96\pi M_\pi^7 - (222 - 21\pi)M_\pi^6\Delta + (1 - 186\pi)M_\pi^5\Delta^2$$

$$+(491 - 75\pi)M_\pi^4\Delta^3 - (25 - 77\pi)M_\pi^3\Delta^4 - 377M_\pi^2\Delta^5 + 24M_\pi\Delta^6 + 108\Delta^7\Big]$$

$$+\sqrt{\Delta^2 - M_\pi^2}\sqrt{\Delta(2M_\pi - \Delta)}\left(\arcsin\frac{M_\pi - \Delta}{M_\pi} + \frac{\pi}{2}\right)$$

$$\Big[-\frac{125g_1^2}{108}(\Delta - M_\pi)\Delta(2M_\pi^4 + M_\pi^3\Delta - 3M_\pi^2\Delta^2 - M_\pi\Delta^3 + \Delta^4)$$

$$+\frac{25g_Ag_1}{6}(\Delta - M_\pi)\Delta(2M_\pi^4 + M_\pi^3\Delta - 3M_\pi^2\Delta^2 - M_\pi\Delta^3 + \Delta^4)$$

$$+\frac{3g_A^2}{8}\Delta(20M_\pi^5 - 10M_\pi^4\Delta - 24M_\pi^3\Delta^2 + 44M_\pi^2\Delta^3 + 20M_\pi\Delta^4 - 18\Delta^5)\Big]$$

$$+\sqrt{\Delta^2 - M_\pi^2}\sqrt{\Delta(\Delta + 2M_\pi)}\ln\frac{\Delta + M_\pi + \sqrt{\Delta(\Delta + 2M_\pi)}}{M_\pi}$$

$$\Big[-\frac{25g_1^2}{324}(\Delta - M_\pi)\Delta(10M_\pi^4 - 25M_\pi^3\Delta + 11M_\pi^2\Delta^2 + 7M_\pi\Delta^3 - 3\Delta^4)$$

$$+\frac{25g_Ag_1}{18}(\Delta - M_\pi)\Delta(2M_\pi^4 - 5M_\pi^3\Delta - M_\pi^2\Delta^2 + 3M_\pi\Delta^3 + \Delta^4)$$

$$+\frac{g_A^2}{4}\Delta(10M_\pi^5 - 35M_\pi^4\Delta + 12M_\pi^3\Delta^2 + 32M_\pi^2\Delta^3 - 10M_\pi\Delta^4 - 9\Delta^5)\Big]\Big\}\,, \tag{D.6}$$

$$a_{1+}^- = -\frac{g_{\pi N\Delta}^2}{108\pi(m + M_\pi)m(\Delta^2 - M_\pi^2)F^2}\Big\{m^2(-4M_\pi - 2\Delta) + m[M_\pi^2(-1 + 8z_0 + 12z_0^2)$$

$$+2M_\pi\Delta - \Delta^2(1 + 8z_0 + 12z_0^2)] + M_\pi^3(-1 + 4z_0^2) + 4M_\pi^2\Delta(1 - 3z_0 - 4z_0^2)$$

$$-M_\pi\Delta^2(5 + 4z_0^2) + 2\Delta^3(1 + 6z_0 + 8z_0^2)\Big\}$$

$$+\frac{g_{\pi N\Delta}M_\pi(b_3 + b_8)}{27\pi(m + M_\pi)(\Delta^2 - M_\pi^2)F^2}\Big\{m(M_\pi + 2\Delta) + M_\pi^2(2z_0 - 1) + M_\pi\Delta - 2\Delta^2z_0\Big\}$$

$$+\frac{g_{\pi N\Delta}M_\pi b_8}{54\pi(m + M_\pi)F^2}(1 + 4z + 12zz_0)$$

$$+\frac{M_\pi^2m[2g_{\pi N\Delta}(e_1 + e_2) - 4g_{\pi N\Delta}(2e_4 - e_5) - (b_3 + b_8)^2]}{54\pi(m + M_\pi)(\Delta^2 - M_\pi^2)F^2}(2M_\pi + \Delta)$$

$$+\ \frac{g_{\pi N\Delta}^2m}{3888\pi^3(m + M_\pi)(\Delta^2 - M_\pi^2)^2\sqrt{\Delta^2 - M_\pi^2}M_\pi^2F^4}$$

$$\Big\{\sqrt{\Delta^2 - M_\pi^2}\Big[M_\pi^2(\Delta^2 - M_\pi^2)(\Delta - M_\pi)(29M_\pi^2 + 41M_\pi\Delta + 15\Delta^2)$$

$$-\frac{5g_1^2}{324}M_\pi[(533 + 136\pi)M_\pi^6 + (1009 + 540\pi)M_\pi^5\Delta - (1217 - 135\pi)M_\pi^4\Delta^2$$

$$-2056M_\pi^3\Delta^3 + 1044M_\pi^2\Delta^4 + 1047M_\pi\Delta^5 - 360\Delta^6]$$

$$+\frac{g_{\pi N\Delta}^2}{3}(M_\pi + \Delta)M_\pi^2(-123M_\pi^4 + 137M_\pi^3\Delta + 84M_\pi^2\Delta^2 - 103M_\pi\Delta^3 + 23\Delta^4)$$

$$-\frac{5g_Ag_1}{6}M_\pi(-(55 + 15\pi)M_\pi^6 - (43 - 20\pi)M_\pi^5\Delta + (139 + 25\pi)M_\pi^4\Delta^2 + 88M_\pi^3\Delta^3$$

$$-124M_\pi^2\Delta^4 - 45M_\pi\Delta^5 + 40\Delta^6) - \frac{g_A^2}{4\Delta}M_\pi(-192\pi M_\pi^7 + (486 + 117\pi)M_\pi^6\Delta$$

$$+(184+372\pi)M_\pi^5\Delta^2 - (1090+123\pi)M_\pi^4\Delta^3 - (331+144\pi)M_\pi^3\Delta^4 + 820M_\pi^2\Delta^5$$
$$+147M_\pi\Delta^6 - 216\Delta^7\Big)\Big]$$

$$+\ln\frac{\Delta+\sqrt{\Delta^2-M_\pi^2}}{M_\pi}\Big[6M_\pi^2(\Delta^2-M_\pi^2)(\Delta-M_\pi)(3M_\pi^3+8M_\pi^2\Delta+2M_\pi\Delta^2-3\Delta^3)$$

$$-\frac{50g_1^2}{27}(M_\pi^8+13M_\pi^7\Delta-2M_\pi^6\Delta^2-39M_\pi^5\Delta^3+39M_\pi^3\Delta^5+2M_\pi^2\Delta^6-13M_\pi\Delta^7-\Delta^8)$$

$$+\frac{2g_{\pi N\Delta}^2}{3}(M_\pi+\Delta)M_\pi^2(32M_\pi^5-242M_\pi^4\Delta+31M_\pi^3\Delta^2+389M_\pi^2\Delta^3-63M_\pi\Delta^4-147\Delta^5)$$

$$-\frac{100g_Ag_1}{3}(M_\pi^8-3M_\pi^7\Delta-4M_\pi^6\Delta^2+9M_\pi^5\Delta^3+6M_\pi^4\Delta^4-9M_\pi^3\Delta^5-4M_\pi^2\Delta^6$$

$$+3M_\pi\Delta^7+\Delta^8)-\frac{6g_A^2}{\Delta}(16M_\pi^9-3M_\pi^8\Delta-29M_\pi^7\Delta^2+18M_\pi^6\Delta^3-9M_\pi^5\Delta^4$$

$$-36M_\pi^4\Delta^5+41M_\pi^3\Delta^6+30M_\pi^2\Delta^7-19M_\pi\Delta^8-9\Delta^9)\Big]$$

$$+\sqrt{\Delta^2-M_\pi^2}\sqrt{\Delta(2M_\pi-\Delta)}\left(\arcsin\frac{M_\pi-\Delta}{M_\pi}+\frac{\pi}{2}\right)$$

$$\Big[-\frac{25g_1^2}{108}\Delta(-10M_\pi^5+65M_\pi^4\Delta+104M_\pi^3\Delta^2-19M_\pi^2\Delta^3-40M_\pi\Delta^4+8\Delta^5)$$

$$-\frac{25g_Ag_1}{6}M_\pi\Delta(2M_\pi^4-5M_\pi^3\Delta-12M_\pi^2\Delta^2-M_\pi\Delta^3+4\Delta^4)$$

$$-\frac{15g_A^2}{4}M_\pi^2\Delta(-2M_\pi^3-3M_\pi^2\Delta+\Delta^3)\Big]$$

$$+\sqrt{\Delta^2-M_\pi^2}\sqrt{\Delta(\Delta+2M_\pi)}\ln\frac{\Delta+M_\pi+\sqrt{\Delta(\Delta+2M_\pi)}}{M_\pi}$$

$$\Big[-\frac{25g_1^2}{108}\Delta(30M_\pi^5+35M_\pi^4\Delta-88M_\pi^3\Delta^2-33M_\pi^2\Delta^3+40M_\pi\Delta^4+16\Delta^5)$$

$$-\frac{25g_Ag_1}{6}\Delta(-6M_\pi^5-15M_\pi^4\Delta+20M_\pi^3\Delta^2+21M_\pi^2\Delta^3-12M_\pi\Delta^4-8\Delta^5)$$

$$-\frac{3g_A^2}{4}\Delta(30M_\pi^5+115M_\pi^4\Delta-96M_\pi^3\Delta^2-201M_\pi^2\Delta^3+80M_\pi\Delta^4+72\Delta^5)\Big]\Big\}\,,\qquad\text{(D.7)}$$

$$a_{1+}^+ = \frac{g_{\pi N\Delta}^2}{54\pi(m+M_\pi)m(\Delta^2-M_\pi^2)F^2}\Big\{2m^2(M_\pi+2\Delta)+m[M_\pi^2(-1+8z_0+12z_0^2)$$

$$+2M_\pi\Delta-\Delta^2(1+8z_0+12z_0^2)]+M_\pi^3(-1+4z_0^2)+4M_\pi^2\Delta(1-3z_0-4z_0^2)$$

$$-M_\pi\Delta^2(5+4z_0^2)+2\Delta^3(1+6z_0+8z_0^2)\Big\}$$

$$+\frac{2g_{\pi N\Delta}M_\pi(b_3+b_8)}{27\pi(m+M_\pi)(\Delta^2-M_\pi^2)F^2}\Big\{m(2M_\pi+\Delta)+M_\pi^2(1-2z_0)-M_\pi\Delta+2\Delta^2z_0\Big\}$$

$$+\frac{g_{\pi N\Delta}M_\pi b_8}{27\pi(m+M_\pi)F^2}(1+4z+12zz_0)$$

$$-\frac{M_\pi^2m(M_\pi+2\Delta)[2g_{\pi N\Delta}(e_1+e_2)-2g_{\pi N\Delta}(2e_4-e_5)-(b_3-b_8)^2]}{27\pi(m+M_\pi)(\Delta^2-M_\pi^2)F^2}$$

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$$+ \frac{g_{\pi N\Delta}^2 m}{648\pi^3(m+M_\pi)(\Delta^2-M_\pi^2)^2\sqrt{\Delta^2-M_\pi^2}M_\pi^2 F^4}$$

$$\left\{\sqrt{\Delta^2-M_\pi^2}\Big[M_\pi^4(\Delta-M_\pi)(\Delta^2-M_\pi^2)+\frac{5g_1^2}{972}M_\pi[(559+270\pi)M_\pi^6+(833+270\pi)M_\pi^5\Delta\right.$$

$$-(1336-270\pi)M_\pi^4\Delta^2-1457M_\pi^3\Delta^3+957M_\pi^2\Delta^4+624M_\pi\Delta^5-180\Delta^6]$$

$$+\frac{g_{\pi N\Delta}^2}{9}(\Delta+M_\pi)M_\pi^2(51M_\pi^4+M_\pi^3\Delta-103M_\pi^2\Delta^2+15M_\pi\Delta^3+18\Delta^4]$$

$$-\frac{5g_Ag_1}{18}M_\pi[(25-10\pi)M_\pi^6+(27-10\pi)M_\pi^5\Delta-(77+10\pi)M_\pi^4\Delta^2-43M_\pi^3\Delta^3$$

$$+67M_\pi^2\Delta^4+16M_\pi\Delta^5-20\Delta^6]+\frac{g_A^2}{12\Delta}M_\pi[-96\pi M_\pi^7+(222+42\pi)M_\pi^6\Delta$$

$$+(2+186\pi)M_\pi^5\Delta^2-(491+150\pi)M_\pi^4\Delta^3-(50+72\pi)M_\pi^3\Delta^4+377M_\pi^2\Delta^5+48M_\pi\Delta^6$$

$$-108\Delta^7]\Big]$$

$$+\ln\frac{\Delta+\sqrt{\Delta^2-M_\pi^2}}{M_\pi}\Big[-\frac{1}{2}(\Delta^2-M_\pi^2)(\Delta-M_\pi)\Delta M_\pi^2(77M_\pi^2-72\Delta^2)+\frac{25g_1^2}{81}(-M_\pi^8+13M_\pi^7\Delta$$

$$+12M_\pi^6\Delta^2-39M_\pi^5\Delta^3-30M_\pi^4\Delta^4+39M_\pi^3\Delta^5+28M_\pi^2\Delta^6-13M_\pi\Delta^7-9\Delta^8)$$

$$+\frac{2g_{\pi N\Delta}^2}{9}(\Delta+M_\pi)M_\pi^2(-32M_\pi^5+137M_\pi^4\Delta+157M_\pi^3\Delta^2-367M_\pi^2\Delta^3-125M_\pi\Delta^4$$

$$+230\Delta^5)-\frac{50g_Ag_1}{9}(-M_\pi^8+3M_\pi^7\Delta+4M_\pi^6\Delta^2-9M_\pi^5\Delta^3-6M_\pi^4\Delta^4+9M_\pi^3\Delta^5$$

$$+4M_\pi^2\Delta^6-3M_\pi\Delta^7-\Delta^8)+\frac{g_A^2}{\Delta}(16M_\pi^9-M_\pi^8\Delta-29M_\pi^7\Delta^2+12M_\pi^6\Delta^3-9M_\pi^5\Delta^4$$

$$-30M_\pi^4\Delta^5+41M_\pi^3\Delta^6+28M_\pi^2\Delta^7-19M_\pi\Delta^8-9\Delta^9)\Big]$$

$$+\sqrt{\Delta^2-M_\pi^2}\sqrt{\Delta(2M_\pi-\Delta)}\left(\arcsin\frac{M_\pi-\Delta}{M_\pi}+\frac{\pi}{2}\right)$$

$$\Big[\frac{25g_1^2}{324}\Delta(30M_\pi^5+45M_\pi^4\Delta+24M_\pi^3\Delta^2+21M_\pi^2\Delta^3-12\Delta^5)$$

$$-\frac{25g_Ag_1}{6}M_\pi^2\Delta(2M_\pi^3+3M_\pi^2\Delta-\Delta^3)+\frac{15g_A^2}{4}M_\pi^2\Delta(2M_\pi^3+3M_\pi^2\Delta-\Delta^3)\Big]$$

$$+\sqrt{\Delta^2-M_\pi^2}\sqrt{\Delta(\Delta+2M_\pi)}\ln\frac{\Delta+M_\pi+\sqrt{\Delta(\Delta+2M_\pi)}}{M_\pi}$$

$$\Big[\frac{25g_1^2}{324}\Delta(50M_\pi^5+5M_\pi^4\Delta-72M_\pi^3\Delta^2-47M_\pi^2\Delta^3+40M_\pi\Delta^4+24\Delta^5)$$

$$-\frac{25g_Ag_1}{18}\Delta(10M_\pi^5+M_\pi^4\Delta-16M_\pi^3\Delta^2-7M_\pi^2\Delta^3+8M_\pi\Delta^4+4\Delta^5)$$

$$\left.+\frac{g_A^2}{4}\Delta(50M_\pi^5+5M_\pi^4\Delta-48M_\pi^3\Delta^2-83M_\pi^2\Delta^3+40M_\pi\Delta^4+36\Delta^5)\Big]\right\}. \tag{D.8}$$

We note that some of these terms are clearly of fourth order in the chiral expansion, because in that case one counts Δ as order $\mathcal{O}(p^0)$.

References

[1] N. Fettes, Ulf-G. Meißner and S. Steininger, Nucl. Phys. A640 (1998) 199.

[2] N. Fettes and Ulf-G. Meißner, `hep-ph/0002162`, Nucl. Phys. A (2000), in print.

[3] E. Jenkins and A.V. Manohar, Phys. Lett. B259 (1991) 353.

[4] T.R. Hemmert, B.R. Holstein and J. Kambor, J. Phys. G: Nucl. Part. Phys. 24 (1998) 1831.

[5] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. A615 (1997) 483.

[6] T. Becher and H. Leutwyler, Eur. Phys. J.C9 (1999) 643.

[7] P.J. Ellis and H.-B. Tang, Phys. Rev. C57 (1998) 3356.

[8] J. Gasser and A. Zepeda, Nucl. Phys. B174 (1980) 445.

[9] E. Jenkins and A.V. Manohar, Phys. Lett. B255 (1991) 558.

[10] V. Bernard, N. Kaiser, J. Kambor and Ulf-G. Meißner, Nucl. Phys. B388 (1992) 315.

[11] V. Bernard, N. Kaiser and Ulf-G. Meißner, Int. J. Mod. Phys. E4 (1995) 193.

[12] V. Bernard, H.W. Fearing, T.R. Hemmert and Ulf-G. Meißner, Nucl. Phys. A635 (1998) 121; (E) Nucl. Phys. A642 (1998) 563.

[13] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. B364 (1991) 283.

[14] R. Koch, Nucl. Phys. A448 (1986) 707.

[15] E. Matsinos, Phys. Rev. C56 (1997) 3014; E. Matsinos, private communication.

[16] SAID on-line program, R.A. Arndt et al., see website http://said.phys.vt.edu/.

[17] V. Bernard, T.R. Hemmert and Ulf-G. Meißner, *forthcoming*.

[18] T.R. Hemmert, B.R. Holstein, J. Kambor and G. Knöchlein, Phys. Rev. D57 (1998) 5746.

[19] V. Bernard, N. Kaiser and Ulf-G. Meißner, Phys. Rev. C52 (1995) 2185.

[20] A. Datta and S. Pakvasa, Phys. Rev. D56 (1997) 4322.

[21] M. Mojžiš, Eur. Phys. J. C2 (1998) 181.

[22] J. Gasser, H. Leutwyler and M.E. Sainio, Phys. Lett. B253 (1990) 252.

[23] V. Bernard, N. Kaiser and Ulf-G. Meißner, Phys. Lett. B389 (1996) 144.

[24] J. Gasser, Nucl. Phys. B279 (1987) 65; J. Gasser, Proc. 2nd Intern. Workshop on πN physics (Los Alamos 1987), eds. W.R. Gibbs and B.M.K. Nefkens, Los Alamos report LA-11184-C (1987), p. 266.

[25] J. Gasser, H. Leutwyler, M. Locher and M.E. Sainio, Phys. Lett. B213 (1988) 85.

[26] M. Olsson, `hep-ph/0001203`, Phys. Lett. B (2000), in print.

[27] J. Kambor, in "Chiral Dynamics: Theory and Experiment", A.M. Bernstein, D. Drechsel and Th. Walcher (eds.) (Springer, Berlin, 1998).

[28] V. Bernard, N. Kaiser and Ulf-G. Meißner, Z. Phys. C60 (1993) 111.

[29] P. Büttiker and Ulf-G. Meißner, Nucl. Phys. A668 (2000) 97.
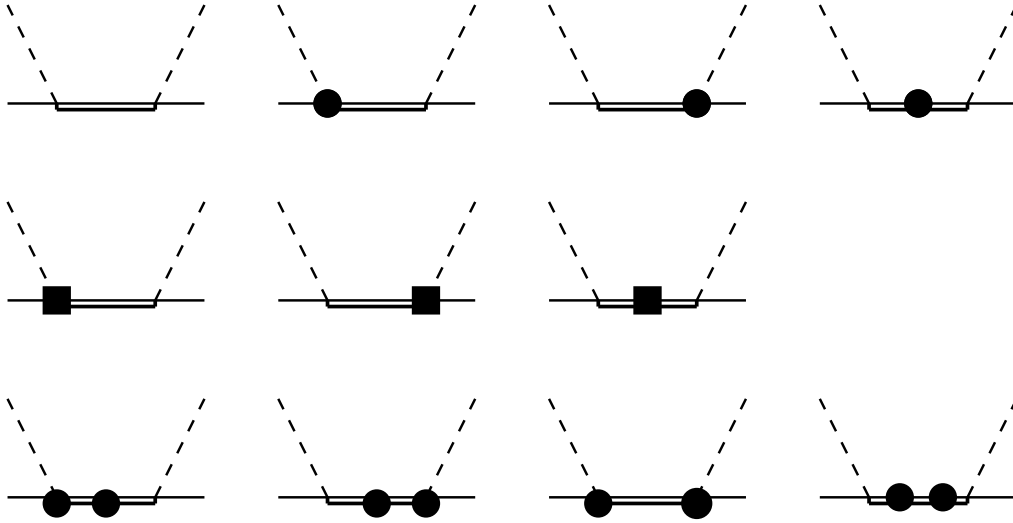
Figures

Figure 1: Tree and counterterm graphs involving the delta. Dashed, solid and double lines refer to pions, nucleons and deltas, in order. Crossed graphs and diagrams that vanish are not shown. The vertex dot and vertex square refer to an insertion from the dimension two, respectively three effective $\pi\Delta$ or $\pi N\Delta$ Lagrangian.
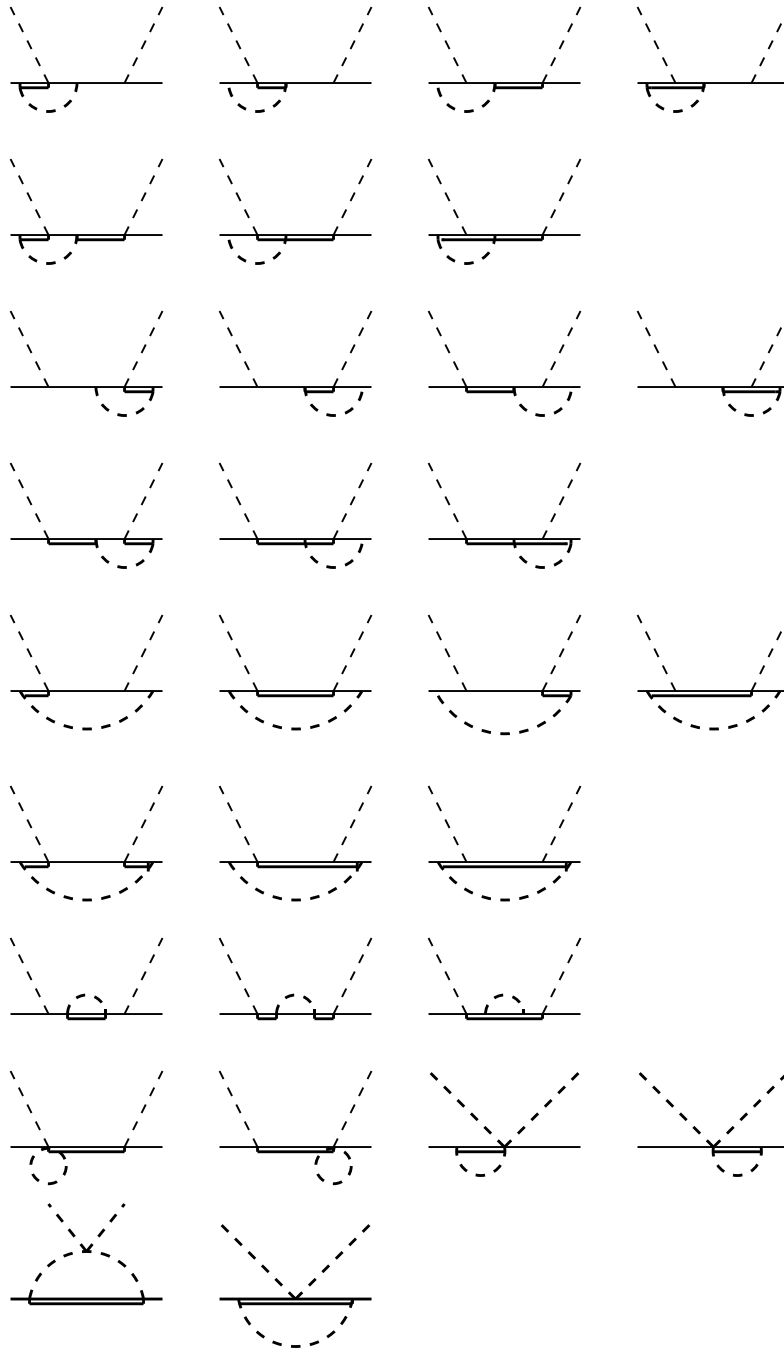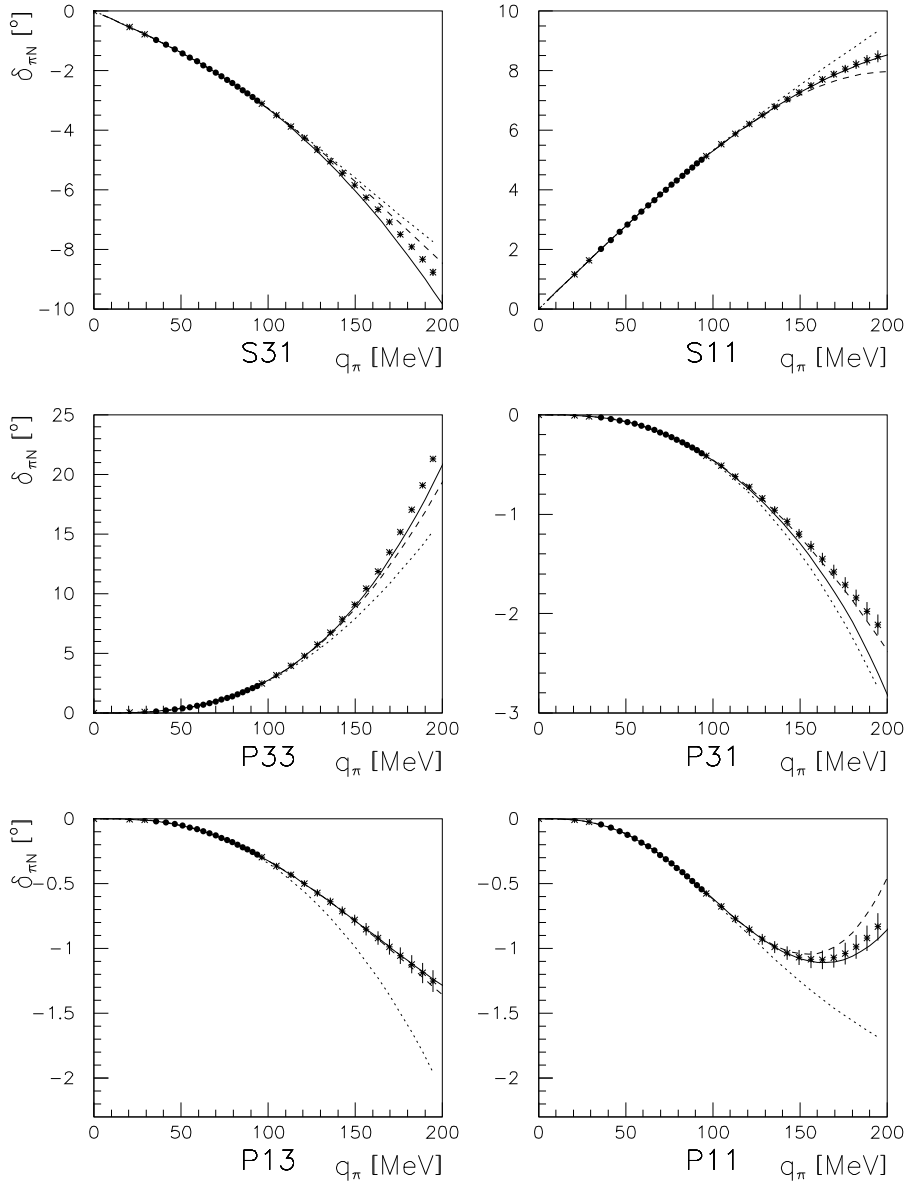
Figure 2: Leading one loop graphs involving the delta. Dashed, solid and double lines refer to pions, nucleons and deltas, in order. Crossed graphs and diagrams that vanish are not shown.

Figure 3: Fits and predictions for the EM98 phase shifts as a function of the pion laboratory momentum q_π. Fitted in each partial wave are the data between 41 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. Dotted and dashed lines: Third and fourth order calculation based on the chiral expansion [1, 2].
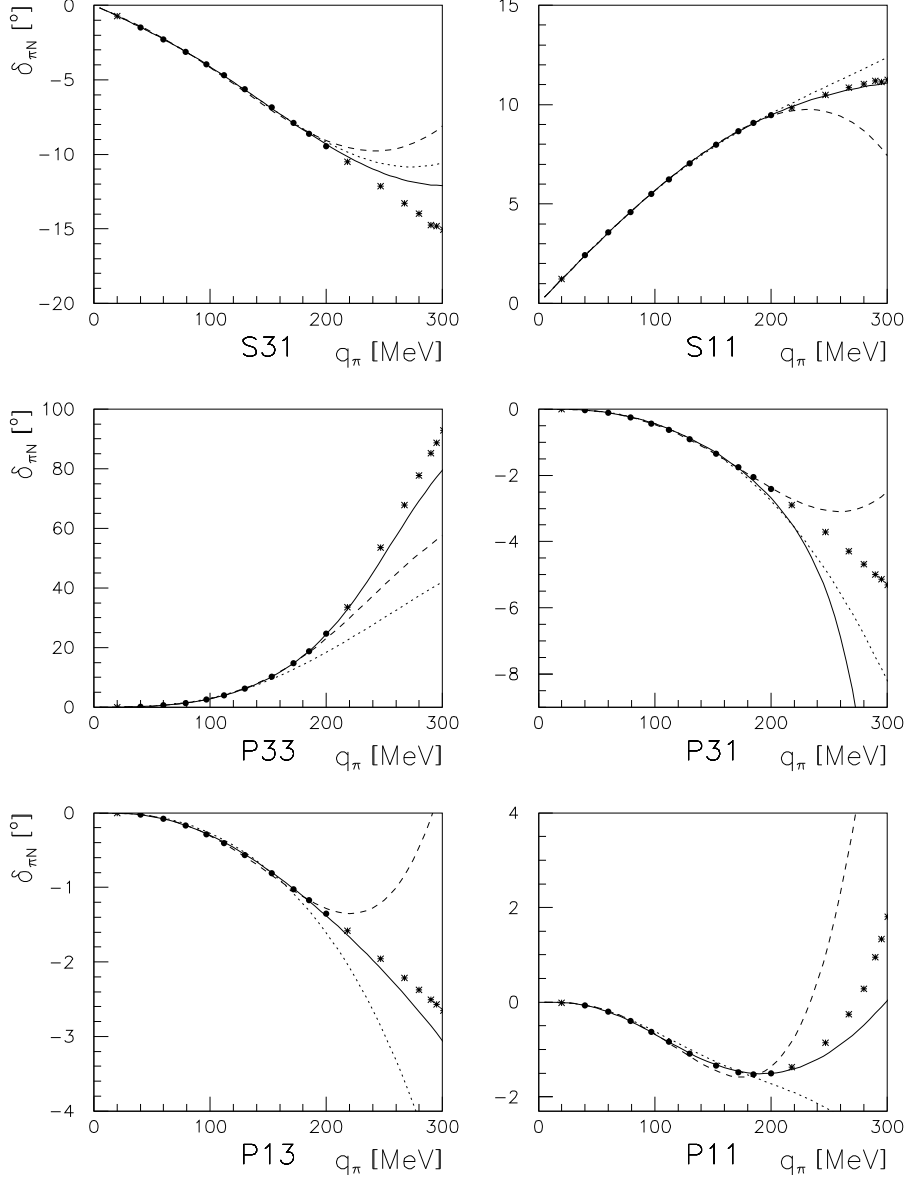
Figure 4: Fits and predictions for the KA85 phase shifts as a function of the pion laboratory momentum q_π. Fitted in each partial wave are the data between 40 and 200 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. Dotted and dashed lines: Third and fourth order calculation based on the chiral expansion [1, 2].
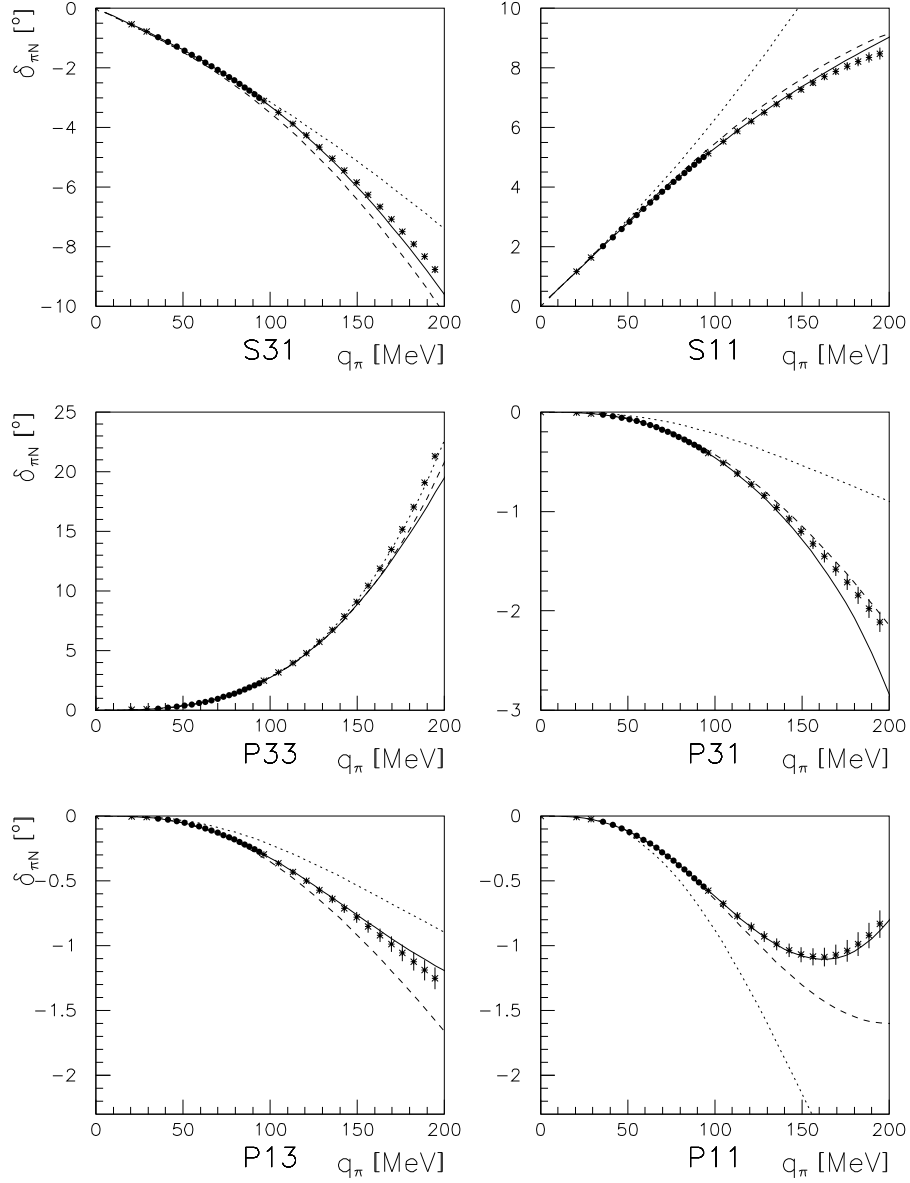
Figure 5: Fits and predictions for the EM98 phase shifts as a function of the pion laboratory momentum q_π to first (dotted lines), second (dashed lines) and third (solid lines) order in the small scale expansion. Fitted in each partial wave are the data between 41 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted.
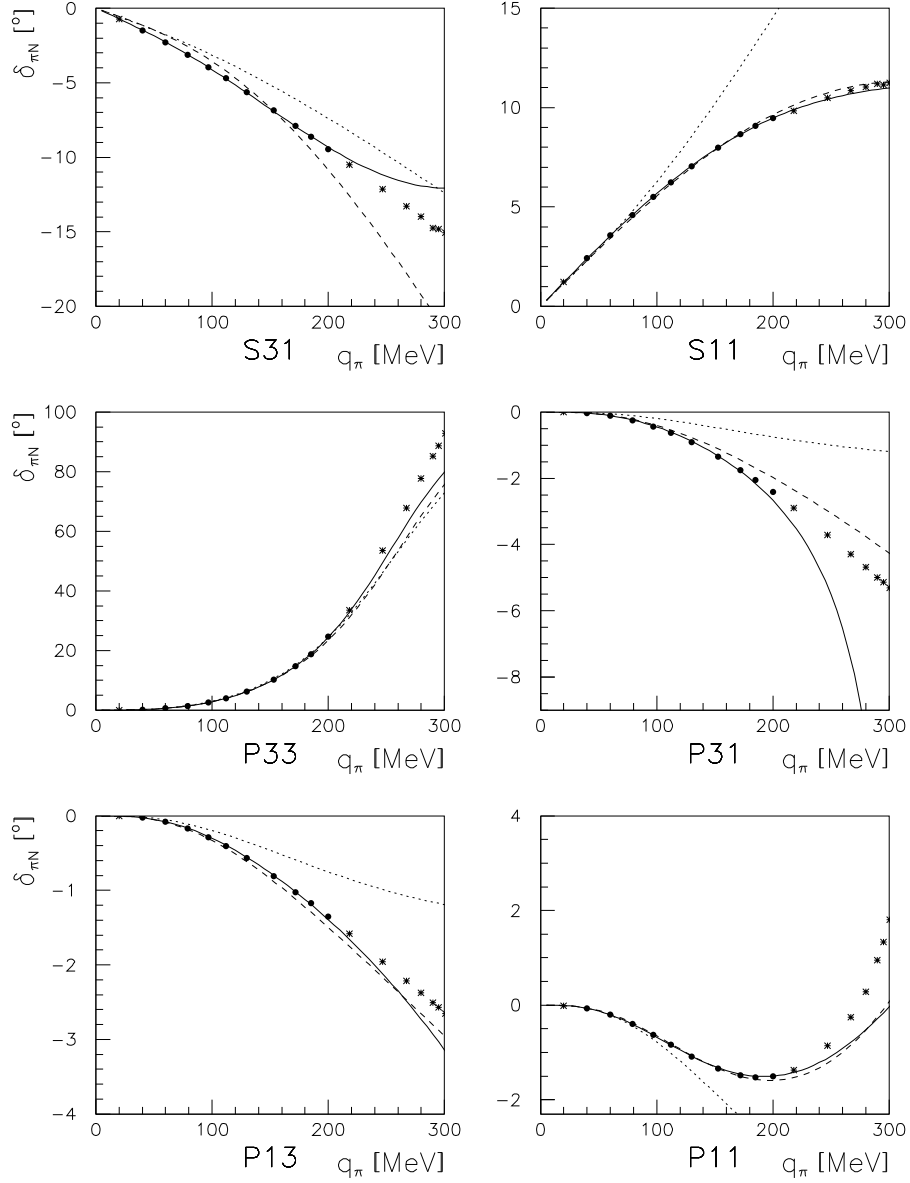
Figure 6: Fits and predictions for the KA85 phase shifts as a function of the pion laboratory momentum q_π to first (dotted lines), second (dashed lines) and third (solid lines) order in the small scale expansion. Fitted in each partial wave are the data between 40 and 200 MeV (filled circles). For higher and lower energies, the phases are predicted.