Diffusion and coherence in tilted piecewise linear double-periodic potentials

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In the present paper we study the overdamped motion of Brownian particles in tilted periodic piecewise linear potentials of two maxima per period. This system allows one to observe additional phenomena with respect to the case of single-periodic potentials. We show that for certain types of potentials the effective diffusion coefficient $D(F)$ can be enhanced or suppressed compared to the simple sawtooth case. As the most unexpected result it is found that the curve of diffusion coefficient vs tilt can have two maxima. In the region of Poissonian hopping processes we demonstrate the possibility to have a resonantlike peak in the Péclet number $\mathrm{Pe}(F)$. At the tilting force corresponding to the maximum of $\mathrm{Pe}(F)$, the curve of Péclet number vs temperature possesses two maxima.

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I. INTRODUCTION

Brownian motion in a tilted periodic potential is of relevance in numerous contexts, for it has possible applications in condensed matter physics, nanotechnology, chemical physics, and molecular biology [1–4]. However, the determination of the effective diffusion coefficient of the system for arbitrary temperature, tilting force, and periodic potential remained a challenging task for decades, even at the overdamped limit. For the last case, quite recently, exact analytic treatments were developed [5,6]. Some interesting effects, such as acceleration of free diffusion as a result of tilting [6], and nonmonotonic behavior of diffusion coefficient and coherence level, as a function of temperature [5], were also found.

The approach, developed by Reimann *et al.* [6], has been applied in Ref. [7] to show that a nonhomogeneous dissipation can induce a minimum of the diffusion coefficient vs the applied external force, an enhancement and suppression of the diffusion as a function of temperature, as well as an increase of the coherence level of Brownian motion in a tilted symmetric periodic potential. The approach of Ref. [6] has been applied also in our previous papers [8,9] to the case of a tilted piecewise linear potential. The piecewise linear potential is important for at least two reasons. First, it can be used as a first approximation of the shape of arbitrary potential. Second, it is sufficiently simple to allow an exact algebraic treatment of the relevant quantities. Considering this, we carried out a comprehensive study of the dependence of diffusive and coherent motion of overdamped Brownian particles on temperature and tilting force, for various shapes of tilted sawtooth potential of one maximum per period.

In order to describe various systems in condensed matter physics and biology, more complicated potentials than the simple sawtooth type potential may be required. The role of metastable and bistable potentials in the diffusive motion of particles in periodic structures was pointed out in Refs. [10,11], in the context of superionic conductors. The molecular-dynamics simulation of self-diffusion on metal surfaces [12] and experimental data for superionic conductors [13] provide the evidence that the potential barriers of different heights are important for the understanding of transport processes in corresponding systems. Double-barrier potentials are also of relevance in modeling the kinetics of motor proteins [14] (see also [15]). As emphasized by Asaklil *et al.*, the diffusion problem in symmetric, and especially in asymmetric, double-periodic potentials has not been thoroughly investigated.

The present contribution continues our study [8,9] of Brownian motion in tilted piecewise linear potentials, considering the case when there are two potential barriers per period. The paper is organized as follows. In Sec. II we introduce the model and the quantities of interest, giving in Sec. II B an overview of the choice of the potential. In Sec. III A we study the effective diffusion coefficient. In Sec. III B we analyze how the additional minimum of the potential influences the coherence of the Brownian motion. Finally, our results are summarized in Sec. IV.

II. MODEL AND QUANTITIES OF INTEREST

A. General scheme

We consider the overdamped motion of Brownian particle with the coordinate $x(t)$ in a one-dimensional periodic potential $V_0(x)$, with $0 \leqslant V_0(x) \leqslant A$, and period L, under the influence of a constant external force F, at temperature T. The Langevin equation for such a system reads

$$\eta \frac{dx(t)}{dt} = -\frac{dV_0(x)}{dx} + F + \xi(t), \tag{1}$$

where η is the viscous friction coefficient, and $\xi(t)$ is the zero mean Gaussian white noise with correlation function $\langle \xi(t)\xi(t')\rangle = 2\eta k_B T \delta(t-t')$. The quantity $V(x)=V_0(x)-Fx$ is called the effective potential.

The basic quantities of interest are the average particle current in the long-time limit

$$\langle \dot{x} \rangle = \lim_{t\to\infty} \frac{\langle x(t)\rangle}{t}, \tag{2}$$

and the effective diffusion coefficient in the same time scale, defined as

$$D = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{2t}. \quad (3)$$

The analytical solution for the particle current (2) goes back to Stratonovich [16] and has subsequently been rederived many times [1,5,6],

$$\langle \dot{x}\rangle = N^{-1}(1 - e^{-LF/k_BT}). \quad (4)$$

The effective diffusion coefficient (3) for the model (1) with $F \geqslant 0$ can be written as [6]

$$D = \frac{D_0}{N^3}\int_{x_0}^{x_0+L} I_+(x)[I_-(x)]^2 \frac{dx}{L}. \quad (5)$$

In the last two equations

$$N = \int_{x_0}^{x_0+L} I_-(x)\frac{dx}{L} \quad (6)$$

and

$$I_\pm(x) = \mp \frac{1}{D_0} e^{\pm V(x)/k_BT}\int_x^{x\mp L} e^{\mp V(y)/k_BT} dy, \quad (7)$$

where x_0 is an arbitrary point and $D_0 = k_BT/\eta$ is the free diffusion coefficient.

The third quantity of interest is the Péclet number

$$\mathrm{Pe} = \frac{L\langle \dot{x}\rangle}{D}, \quad (8)$$

which characterizes the relationship between the directed and diffusive movement of a Brownian particle [5,8,17]. Sometimes it is more convenient to use the randomness parameter [18,19], which is actually measured in experiments, defined as the long time limit of the ratio between the variance of the particle's position and the product of its average position and periodicity,

$$r = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{\langle x(t)\rangle L}. \quad (9)$$

The definitions of the diffusion coefficient and current imply

$$r = \frac{2D}{\langle \dot{x}\rangle L}. \quad (10)$$

Thus $\mathrm{Pe} = 2r^{-1}$, and it is easy to switch between these two quantities, Péclet number and randomness parameter.

B. Choice of potential

The behavior of Brownian particles is determined by the effective potential $V(x)$ and the noise $\xi(t)$. Figure 1 depicts the general shape of the potential used in the present paper (solid line), with no tilting force, and for a finite value of tilting. For comparison, the simple sawtooth potential is given (dashed line), examined by us in Refs. [8,9]. With no loss of generality we have taken the period $L=1$ and the amplitude $A=1$; we assume that $0 < k_1 < k_2 < k < 1$, $0 \leqslant A_1 < A_2 \leqslant 1$, and $\Delta A = A_2 - A_1 < 1$, whereas we are interested in having an additional trap with a smaller potential barrier than the primary barrier. Hence there are two values of the tilt corresponding to the disappearance of the two minima: $F_{ce} = \Delta A/\Delta k$ (where $\Delta k = k_2 - k_1$) for the additional, and $F_c = 1/(1-k)$ for the primary, barrier. We refer to the critical tilt, as the value $f_c = \max(F_{ce}, F_c)$, so that a potential will have no potential barriers.

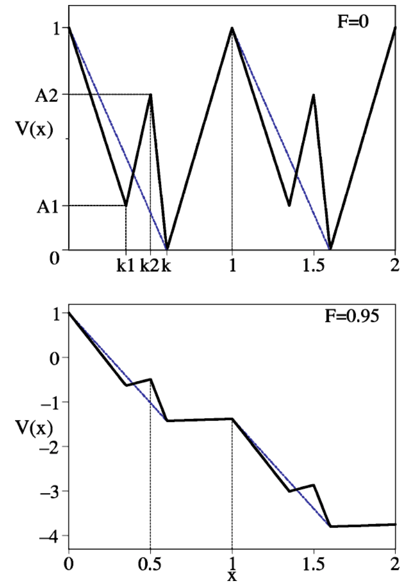

FIG. 1. The general shapes of the effective potentials for tilting forces $F=0$ (above) and $F=0.95$ (below). Solid line: piecewise linear double-periodic potential for $k_1=0.35$, $k_2=0.5$, $k=0.6$, $A_1=0.2$, $A_2=0.7$. Dashed line: simple sawtooth potential, used in Refs. [8,9], for asymmetry parameter $k=0.6$.

All the quantities, we are going to present and plot in this paper, will be in dimensionless units (see Ref. [8]). Furthermore, it is convenient to express the tilt in units of F_c by redefining

$$F_{ce} = \frac{\Delta A(1-k)}{\Delta k}, \quad F_c = 1. \quad (11)$$

The dimensionless double-periodic potential, depicted in Fig. 1, is defined as follows ($n=1,2,\ldots$ is the number of the period):

$$V_{an}(x) = a_{0n} - ax, \quad (n-1) \leqslant x \leqslant k_1 + (n-1),$$

$$V_{bn}(x) = -b_{0n} + bx, \quad k_1 + (n-1) \leqslant x \leqslant k_2 + (n-1),$$

$$V_{cn}(x) = c_{0n} - cx, \quad k_2 + (n-1) \leqslant x \leqslant k + (n-1),$$

$$V_{dn}(x) = -d_{0n} - dx, \quad k + (n-1) \leqslant x \leqslant n, \quad (12)$$

where

$$a_{0n} = A_1 + \frac{1-A_1}{k_1}[k_1 + (n-1)], \quad a = \frac{1-A_1}{k_1} + \frac{F}{1-k},$$

$$b_{0n} = -A_1 + \frac{\Delta A}{\Delta k}[k_1 + (n-1)], \quad b = \frac{\Delta A}{\Delta k} - \frac{F}{1-k},$$

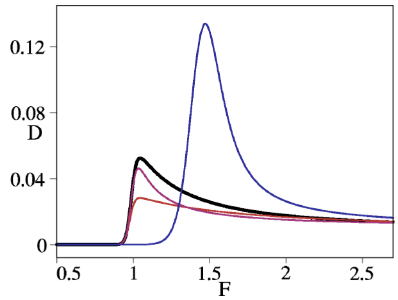

FIG. 2. The enhancement and suppression of the diffusion coefficient, compared to the simple sawtooth potential, due to the additional trap. $T=0.01$, $k=0.6$, $\Delta A=0.45$. In decreasing order of the maximal values of $D(F)$: (1), $F_{ce}=1.5$; (2), simple sawtooth potential; (3), $F_{ce}=0.5$; (4), $F_{ce}=F_c=1$.

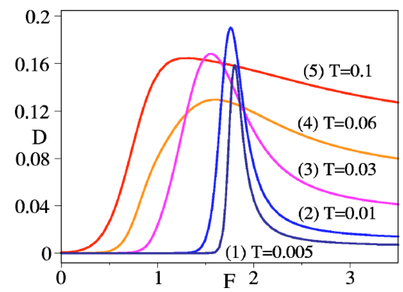

FIG. 3. The plot of the diffusion coefficient $D(F)$ for the different values of temperature. Potential parameters: $k_1=0.4$, $k_2=0.5$, $k=0.6$, $A_1=0.55$, $A_2=1$, $F_{ce}=1.8$.

$$c_{0n}=\frac{A_2}{k-k_2}[k+(n-1)], \quad c=\frac{A_2}{k-k_2}+\frac{F}{1-k},$$

$$d_{0n}=\frac{1}{1-k}[k+(n-1)], \quad d=\frac{1-F}{1-k}. \tag{13}$$

We emphasize that the general character of the transport process is determined by the value of F_{ce}, and thus for a fixed k by the differences ΔA and Δk. The values of single parameters k_1, k_2, A_1, A_2 are of no importance, while the differences ΔA and Δk play a role, even if the values of F_{ce}, determined by them, are the same.

All the dependencies we are going to plot in this paper are calculated on the basis of the explicit algebraic expressions for the diffusion coefficient, current, and Péclet number. The calculations are done analogously to the ones carried out in Ref. [8] for the simple sawtooth potential and are revealed in the Appendix.

III. INFLUENCE OF AN EXTRA TRAP

A. Diffusion

At low temperature a double-periodic potential gives a possibility to favor or suppress the maximal value of the diffusion coefficient $D(F)$, compared to the case of the simple sawtooth potential. The situation is illustrated in Fig. 2. In the case $F_{ce}<F_c$ the maximal value of $D(F)$ is decreased due to the additional potential minima, which can be easily understood if we think of climbing. The decrease is, at fixed k and ΔA, largest if $F_{ce}=F_c$. For $F_{ce}>F_c$ diffusion starts to increase. If F_{ce} is larger than the value of the tilting force, which corresponds to the maximum of $D(F)$ in the case of the simple sawtooth potential, then the maximal value of diffusion increases due to the extra trap. The two cases $F_{ce}<F_c$ and $F_{ce}>F_c$ are in a sense equal, whereas the tilt corresponding to the maximum of $D(F)$ is close to the value of f_c and thus one of the minima is stretched out. If $F_{ce}=F_c$ then the effective potential contains at $F_{\max}\approx f_c$ more segments where the value of deterministic force is approximately zero and thus the spreading is suppressed compared to the one in single-barrier potential as well as in a double-periodic potential with $F_{ce}\neq F_c$. At higher temperature the existence of an additional potential trap diminishes the diffusion.

Henceforth our main interest will be focused on the potentials with $F_{ce}>F_c$, which provide additional phenomena with respect to the case of simple periodic potentials. The case $F_{ce}<F_c$ does not differ, if $F\to F_c$, much from the case of the simple sawtooth potential. However, we remark that in biological systems potentials for which $F_{ce}<F_c$, often play a role [14,15].

In Fig. 3 we have depicted the diffusion coefficient vs tilting force for the different values of temperature, for a potential for which $F_{ce}>F_c$ [20]. This figure highlights a counterintuitive phenomenon: at lower noise intensities, the maximal value of the diffusion coefficient $D(F)$ can be bigger than at higher noise intensities (compare curves 2, 3, and 4 with each other, or 1 with 4, or 2 with 5). At low and high values of temperature the situation is back to usual (compare curves 1 with 2, and 4 with 5). In Ref. [9] we have observed a similar situation for the simple sawtooth potential in the case $k>k_E\approx 0.8285$ (see Fig. 2 in Ref. [9]). Such a behavior is explained by the ratio between escape and relaxation times at different noise intensities and the relaxation time dependence on the temperature (see Ref. [5] for more details). At lower and intermediate temperatures the maximum of diffusion coefficient $D(F)$ occurs around the tilt f_c, thus the relaxation time is larger for the potentials with $F_{ce}>F_c$ [21] and one can obtain the effect also for the potentials with smaller asymmetry parameter than k_E. If $k>k_E$ the effect is just more remarkable if there is also an additional trap with $F_{ce}>F_c$. For the two cases, the single-periodic and double-periodic potentials just discussed, the general behavior of the diffusion coefficient $D(T,F)$ is the same.

Figure 4 represents the dependence $D(F)$ in the case of the same potential as used in Fig. 3, but in a logarithmic scale. In this plot one can distinguish two acceleration rates for the diffusion. The two rates are the more different, the lower the noise intensity, and associate with two different Poissonian processes (the latter fact will be discussed in more details in Sec. III B). Thereby the Poissonian process in the first region coincides with the one which takes place in the corresponding simple sawtooth potential. The picture for the current is similar.

The presence of two potential barriers leads one to think that there can be two maxima of the diffusion coefficient vs

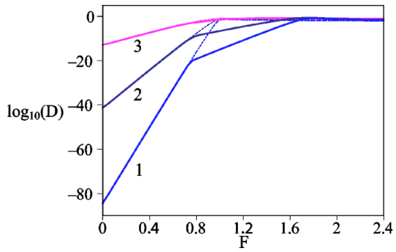

FIG. 4. The plot of the diffusion coefficient for different noise intensities. For solid lines the potential parameters are the same as in Fig. 3: (1), $T=0.005$; (2), $T=0.01$; (3), $T=0.03$. Dashed lines: diffusion coefficients for the sawtooth potential with asymmetry parameter $k=0.6$ at the same temperatures.

tilting force, however, in practice such a situation is not trivial. Nevertheless, for a certain type of potential shape it is possible to obtain a situation, for which the diffusion coefficient $D(F)$ possesses two maxima and passes a considerable minimum under the critical tilt (see Fig. 5). The minimum of $D(F)$ is the deepest in the temperature region where the amplification of diffusion is minimal and the maxima of $D(F)$ are equal. At higher and lower temperatures one of the maxima starts to dominate and the other one to decrease. To obtain two maxima in the dependence of $D(F)$ a small but sharp additional potential barrier is needed, which is followed by a steep fall. However, the situation is extremely sensitive to the potential parameters and to the noise intensity and needs to be studied further.

B. Coherence of motion

The transport of Brownian particles is characterized by the average motion in the direction of the bias and the spreading due to the noise. By coherent motion one means large particle current with minimal diffusion; hence the greater the Péclet number, the greater the coherence of Brownian transport. When one is speaking of the coherence of motion, one mostly speaks in the context of noise intensity. However, the Péclet number as a function of bias offers also interesting properties.

In Refs. [8,9] it was shown in the case of a simple sawtooth potential that at low temperatures and for subcritical tilt the coherence level stabilizes at the value of Péclet number $\mathrm{Pe}(F)=2$. In this region the acceleration of diffusion is most

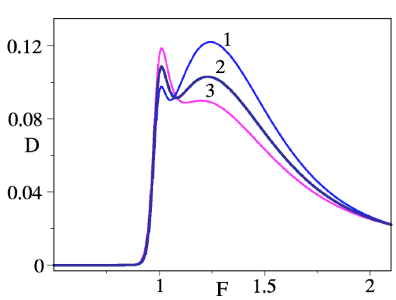

FIG. 5. The existence of two maxima for diffusion coefficient *vs* tilting force: (1), $T=0.0095$; (2), $T=0.01$; (3), $T=0.0105$. The potential parameters are $k_1=0.79$, $k_2=0.8$, $k=0.81$, $A_1=0.888$, $A_2=1$.

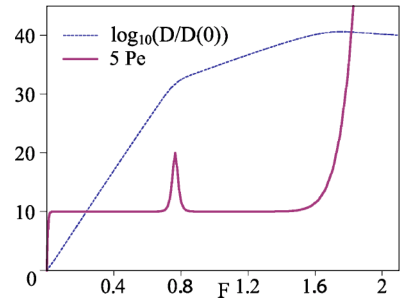

FIG. 6. The comparison of the dependencies of the diffusion coefficient and Péclet number on the tilting force. Dashed line, $\log_{10}[D(F)/D(0)]$; solid line, $5\times\mathrm{Pe}$. Temperature $T=0.01$.

essential. The situation corresponds to the case when particles are mainly localized around the potential minima and transport can be described with great accuracy by the Poissonian hopping process.

Considering the double-periodic potentials, the average distribution of Brownian particles can change at low temperatures drastically for different values of the subcritical tilting force. For the potentials with $F_{ce}>1$ there exists a threshold value of the tilting force

$$F_0=\frac{(1-\Delta A)(1-k)}{1-k-\Delta k}, \qquad (14)$$

at which the main potential barrier becomes smaller than the additional barrier. If $F<F_0$, particles are mainly localized near the primary traps, whereas if $F>F_0$, near the extra traps. As a result the acceleration of diffusion *vs* tilting force is realized through two different Poissonian processes: The first one takes place if $F<F_0$, while the second one if $F>F_0$. As seen in Fig. 6, the two regions of the acceleration of diffusion in Fig. 4 correspond to these different Poisson processes.

In the region of crossover between the two regimes of the enhancement of diffusion, the Péclet number passes through a sharp maximum (a minimum in randomness parameter) with the characteristic value $\mathrm{Pe}=4(r=1/2)$. The observed enhancement of coherence—decrease of randomness—appears in the region where the acceleration regime of diffusion and current changes, whereas the increase of diffusion slows down compared to the increase of current (see Fig. 7). In this case the average populations of the primary traps and the extra traps are close to each other and the possibility of the localization of Brownian particles near the minima of both types is considerable, leading to the relative suppression of diffusion. The suppression is the largest if both of the potential traps are switched on with equal weights. Such a doubling of the effective number of the localization centers in the region of Poissonian process gives a qualitative explanation for the universal value of Péclet number $\mathrm{Pe}=4$. It agrees also with the result of Ref. [18] obtained for two-step hopping processes applicable for the approximate description of transport in present situation. In Fig. 7 one can see also that the tilting force F_0 lies approximately in the beginning of the domain of crossover ($F_0=0.733$).

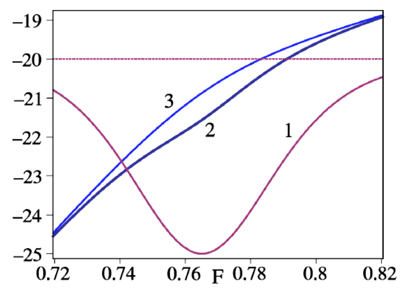

FIG. 7. Appearance of the minimum in the randomness parameter $r(F)$ in the region of the crossover. The plot of the randomness parameter, current, and diffusion coefficient vs tilting force at $T=0.01$. (1), $10\times(r-3)$; (2), $\ln(2D)$; (3), $\ln(\langle\dot{x}\rangle)$; dashed line corresponds to the coherence level with $r=1$.

For the existence of the extremum in the coherence of Brownian motion vs tilt, the condition $F_{ce}>1$ must be satisfied. The tilting force F_0 has a physical meaning only if the latter inequality is satisfied, having the value in the range $0<F_0<1$. This circumstance follows [22] from Eq. (14) together with Eq. (11) for F_{ce} which leads to the relation

$$(1-F_0)=\frac{\Delta k}{1-k-\Delta k}(F_{ce}-1). \tag{15}$$

If $F_{ce}<1$, the Péclet number $\mathrm{Pe}(F)$ does not have a maximum [23]. On the other hand, if $F_0<1$ is sufficiently close to unity, the peak of coherence merges into the region where the motion cannot be described as the Poissonian process anymore and $\mathrm{Pe}(F)$ increases monotonically. In particular, this case is actual for the potentials for which the diffusion coefficient $D(F)$ possesses two maxima.

With the rise of temperature the peak of the coherence disappears. We have illustrated the situation in Fig. 8 for the randomness parameter. At higher temperature the minimum of the randomness parameter broadens, whereas the region of tilting force where the two potential minima have a comparable weight enlarges, and the posterior part of the plateau diminishes (the transport between extra traps is not describable with a Poissonian hopping process anymore) and finally randomness decreases and coherence increases monotonically.

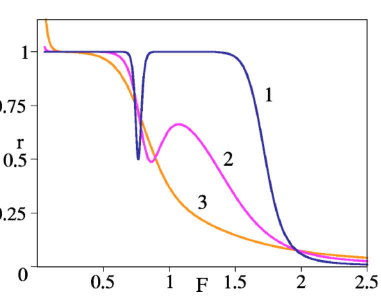

FIG. 8. The plot of the randomness parameter r vs tilt F for different noise intensities: (1), $T=0.01$; (2), $T=0.03$; (3), $T=0.06$.

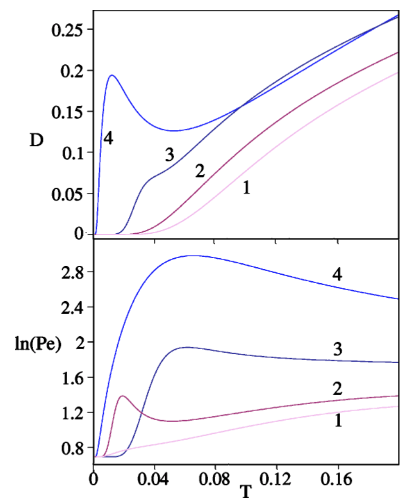

FIG. 9. Diffusion coefficient and Péclet number vs temperature for different tilts: (1), $F=0.7$; (2), $F=0.8$; (3), $F=1.1$; (4), $F=1.75$.

In our previous papers [8,9] on diffusion and current in tilted piecewise linear potentials, we have demonstrated the possibility to obtain the existence of a maximum in the Péclet number vs temperature for a simple sawtooth potential (cf. [5]), in connection with the minimum in the diffusion coefficient, for increasing noise intensity. In Refs. [5,7] it is pointed out that for a homogeneous system the Péclet number can show a maximum, although neither the diffusion coefficient nor the average current density shows an extremum. The present model allows us to observe for different tilts both the phenomena as one can see in Fig. 9. The situation is actually valid also for the simple sawtooth potential; however, the latter effect is small. Thus we can say that the characteristic features of the transport of Brownian particles in tilted inhomogeneous systems can be reproduced also in the framework of the minimal scheme (cf. Refs. [5,7,24]). Furthermore, as one can see in Fig. 10, in the region of static external force, where $\mathrm{Pe}(F)$ exhibits a maximum (minimum in randomness factor), the Péclet number vs temperature has two maxima, and is extremely sensitive to noise intensity. The observed behavior of the Péclet number reflects first of all the complicated properties of diffusion coefficient as a function of the tilting force and temperature.

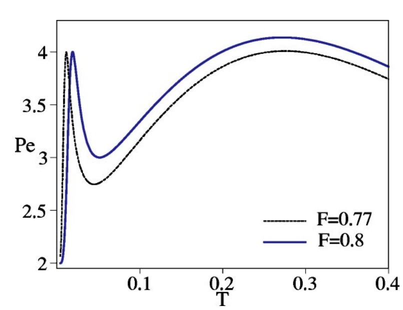

FIG. 10. The existence of two maxima in Pe vs T.

IV. CONCLUSIONS

In the present paper we have studied the overdamped Brownian motion in tilted double-periodic piecewise linear potentials, being obviously more realistic and flexible for possible applications in condensed matter physics and biology, in the presence of white thermal noise. It proves that due to an additional potential barrier, for certain parameter values, many new effects occur in the transport processes of Brownian particles, in particular if $F_{ce} > F_c$.

The general dependence of the diffusion coefficient vs tilting force obeys as a rule the typical behavior found earlier. However, in the present case, the acceleration of diffusion is characterized by two regions, related to the two potential barriers and different Poissonian processes. In the region of the crossover we have demonstrated the possibility to have a resonantlike peak in the Péclet number $\mathrm{Pe}(F)$. For the values of tilting force characteristic of the enhancement of the coherence, the Péclet number vs noise intensity possesses two maxima. Furthermore, for a certain type of potential, the effective diffusion coefficient $D(F)$ can have two maxima.

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APPENDIX: ANALYTICAL RESULTS FOR DOUBLE-PERIODIC POTENTIAL

In dimensionless units the expressions for the diffusion coefficient, current, and Péclet factor have the following form [8]:

$$D = TYZ^{-3}, \tag{A1}$$

$$\langle \dot{x} \rangle = \varphi_0 Z^{-1}, \tag{A2}$$

$$\mathrm{Pe} = \varphi_0 Z^2 (TY)^{-1}, \tag{A3}$$

where

$$\varphi_0 = 1 - \exp\left(-\frac{F}{T(1-k)}\right), \tag{A4}$$

and in the case of the potential determined by Eqs. (12) and (13),

$$Z = \int_0^{k_1} dx\, H_{-a}(x) + \int_{k_1}^{k_2} dx\, H_{-b}(x) + \int_{k_2}^{k} dx\, H_{-c}(x) + \int_k^1 dx\, H_{-d}(x), \tag{A5}$$

$$Y = \int_0^{k_1} dx\, H_{+a}(x) H_{-a}^2(x) + \int_{k_1}^{k_2} dx\, H_{+b}(x) H_{-b}^2(x) + \int_{k_2}^{k} dx\, H_{+c}(x) H_{-c}^2(x) + \int_k^1 dx\, H_{+d}(x) H_{-d}^2(x). \tag{A6}$$

Using the notation $v(x) = V(x)/T$, we have

$$H_{\pm a}(x) = \frac{e^{-F(1\pm1)/2T(1-k)}}{D_0} e^{\pm v_{a1}(x)} \left(\int_x^{k_1} dy\, e^{\mp v_{a1}(y)} + \int_{k_1}^{k_2} dy\, e^{\mp v_{b1}(y)} + \int_{k_2}^{k} dy\, e^{\mp v_{c1}(y)} + \int_k^1 dy\, e^{\mp v_{d1}(y)} + \int_1^{x+1} dy\, e^{\mp v_{a2}(y)} \right),$$

$$H_{\pm b}(x) = \frac{e^{-F(1\pm1)/2T(1-k)}}{D_0} e^{\pm v_{b1}(x)} \left(\int_x^{k_2} dy\, e^{\mp v_{b1}(y)} + \int_{k_2}^{k} dy\, e^{\mp v_{c1}(y)} + \int_k^1 dy\, e^{\mp v_{d1}(y)} + \int_1^{k_1+1} dy\, e^{\mp v_{a2}(y)} + \int_{k_1+1}^{x+1} dy\, e^{\mp v_{b2}(y)} \right),$$

$$H_{\pm c}(x) = \frac{e^{-F(1\pm1)/2T(1-k)}}{D_0} e^{\pm v_{c1}(x)} \left(\int_x^{k} dy\, e^{\mp v_{c1}(y)} + \int_k^1 dy\, e^{\mp v_{d1}(y)} + \int_1^{k_1+1} dy\, e^{\mp v_{a2}(y)} + \int_{k_1+1}^{k_2+1} dy\, e^{\mp v_{b2}(y)} + \int_{k_2+1}^{x+1} dy\, e^{\mp v_{c2}(y)} \right),$$

$$H_{\pm d}(x) = \frac{e^{-F(1\pm1)/2T(1-k)}}{D_0} e^{\pm v_{d1}(x)} \left(\int_x^{1} dy\, e^{\mp v_{d1}(y)} + \int_1^{k_1+1} dy\, e^{\mp v_{a2}(y)} + \int_{k_1+1}^{k_2+1} dy\, e^{\mp v_{b2}(y)} + \int_{k_2+1}^{k+1} dy\, e^{\mp v_{c2}(y)} + \int_{k+1}^{x+1} dy\, e^{\mp v_{d2}(y)} \right). \tag{A7}$$

Performing the integrations in Eqs. (A7) one obtains after some cumbersome calculations the algebraic expressions for the quantities Z and Y, given by Eqs. (A5) and (A6):

$$Z = \varphi_0 \left(k_1 g_{ab} - k_2 g_{bc} + k\, g_{cd} - \frac{1}{d} \right) + \frac{T}{a} S_1 (1 - \lambda_1) + \frac{T}{b} S_2 (1 - \lambda_2) - \frac{T}{c} S_3 (1 - \lambda_3) + \frac{T}{d} S_4 (1 - \lambda_4), \tag{A8}$$

$$Y = \varphi_0^3\left[k_1\left(\frac{1}{a^3}+\frac{1}{b^3}\right) - k_2\left(\frac{1}{b^3}+\frac{1}{c^3}\right) + k\left(\frac{1}{c^3}+\frac{1}{d^3}\right) - \frac{1}{d^3}\right] + T\varphi_0^2\left(\frac{(1-\lambda_1)}{a^3}(2S_1+S_1') + \frac{(1-\lambda_2)}{b^3}(2S_2+\lambda_2^{-1}S_2') - \frac{(1-\lambda_3)}{c^3}(2S_3+S_3') + \frac{(1-\lambda_4)}{d^3}(2S_4+\lambda_4^{-1}S_4')\right) + T\left(\frac{(1-\lambda_1)}{a}S_1^2\left[\lambda_1 S_1' + \frac{\varphi_0}{2a}(1+\lambda_1)\right] + \frac{1-\lambda_2}{b}S_2^2\left[S_2' - \frac{\varphi_0}{2b}(1+\lambda_2)\right] - \frac{(1-\lambda_3)}{c}S_3^2\left[\lambda_3 S_3' + \frac{\varphi_0}{2c}(1+\lambda_3)\right] + \frac{(1-\lambda_4)}{d}S_4^2\left[S_4' - \frac{\varphi_0}{2d}(1+\lambda_4)\right]\right) + 2\varphi_0\left(\frac{k_1}{a}\lambda_1 S_1 S_1' - \frac{\Delta k}{b}S_2 S_2' + \frac{k-k_1}{c}\lambda_3 S_3 S_3' - \frac{1-k}{d}S_4 S_4'\right). \tag{A9}$$

Here

$$g_{ab} = \frac{1}{a}+\frac{1}{b},\quad g_{bc} = \frac{1}{b}+\frac{1}{c},$$

$$g_{cd} = \frac{1}{c}+\frac{1}{d},\quad g_{ad} = \frac{1}{a}+\frac{1}{d}; \tag{A10}$$

$$\lambda_1 = \exp\left(-\frac{Fk_1}{T(1-k)} - \frac{1-A_1}{T}\right),$$

$$\lambda_2 = \exp\left(\frac{F\Delta k}{T(1-k)} - \frac{\Delta A}{T}\right),$$

$$\lambda_3 = \exp\left(\frac{F(k-k_2)}{T(1-k)} + \frac{A_2}{T}\right),$$

$$\lambda_4 = \exp\left(-\frac{1-F}{T}\right),$$

$$\lambda_5 = \exp\left(\frac{F(k-k_1)}{T(1-k)} + \frac{A_1}{T}\right),$$

$$\lambda_6 = \exp\left(\frac{F(1-k_1)}{T(1-k)} - \frac{1-A_1}{T}\right),$$

$$\lambda_7 = \exp\left(\frac{F(1-k_2)}{T(1-k)} - \frac{1-A_2}{T}\right); \tag{A11}$$

$$S_1 = -g_{ab} + \frac{g_{bc}}{\lambda_2} - \frac{g_{cd}}{\lambda_5} + \frac{g_{ad}}{\lambda_6},$$

$$S_1' = \frac{g_{ab}}{\lambda_6} - \frac{g_{bc}}{\lambda_7} + \frac{g_{cd}}{\lambda_4} - g_{ad},$$

$$S_2 = -g_{ab}(1-\varphi_0) + \frac{g_{bc}}{\lambda_2} - \frac{g_{cd}}{\lambda_5} + \frac{g_{ad}}{\lambda_6},$$

$$S_2' = g_{ab} - g_{bc}\lambda_2(1-\varphi_0) + g_{cd}\lambda_5(1-\varphi_0) - g_{ad}\lambda_6(1-\varphi_0),$$

$$S_3 = -g_{ab}\lambda_2(1-\varphi_0) + g_{bc}(1-\varphi_0) - \frac{g_{cd}}{\lambda_3} + \frac{g_{ad}}{\lambda_7},$$

$$S_3' = \frac{g_{ab}}{\lambda_5} - \frac{g_{bc}}{\lambda_3} + g_{cd}(1-\varphi_0) - g_{ad}\lambda_4(1-\varphi_0),$$

$$S_4 = -g_{ab}\lambda_5(1-\varphi_0) + g_{bc}\lambda_3(1-\varphi_0) - g_{cd}(1-\varphi_0) + \frac{g_{ad}}{\lambda_4},$$

$$S_4' = \frac{g_{ab}}{\lambda_5} - \frac{g_{bc}}{\lambda_3} + g_{cd} - g_{ad}\lambda_4(1-\varphi_0). \tag{A12}$$

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[20] If not marked otherwise in the figure caption, we henceforth calculate all the graphics for the same values of potential parameters: $k_1=0.4$, $k_2=0.5$, $k=0.6$, $A_1=0.55$, $A_2=1$; $F_{ce}=1.8$.
[21] We also note that for a double-barrier potential with $F_{ce}>F_c$ in the region of tilting force $F_c<F<F_{ce}$ the length of the path of relaxation from a potential maximum into the next minimum is determined as $s_{rel}=1-\Delta k$ being always larger than the length $s_{rel}=k$ in a corresponding single-barrier potential. At the same time, if $F_{ce}<F_c$, the length of the path of relaxation in a double-barrier potential at $F_{ce}<F<F_c$ coincides with the one in a single-barrier potential.
[22] Note that for $F_{ce}>1$ the condition $\Delta k<1-k$ must be always satisfied as one can see from Eq. (11).
[23] Inequality $F_{ce}<1$ is valid always if $\Delta k>1-k$ [see Eq. (11)] and then one can see from Eq. (14) that $F_0<0$. However, condition $F_{ce}<1$ can be satisfied also for $\Delta k<1-k$ which yield on the basis of Eq. (15) $F_0>1$.
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