Theory of nonoverlapping low frequency modes in axisymmetric toroidal plasmas

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The theory of low frequency instabilities in axisymmetric toroidal plasmas is presented from the point of view of the two-fluids equations, assuming the standard drift wave ordering. Attention is focused on the limit in which neighboring rational surfaces (corresponding to a fixed toroidal mode number) are sufficiently far apart that mode overlapping can be neglected. Owing to field line bending, poloidal side bands $m\pm1$,... coexist with the primary mode m, enhancing noticeably the role of the parallel ion dynamics. The electron and ion branches are investigated systematically under those conditions. It is found that the radial widths of the eigenmodes increase with respect to the slab values; the shear damping rate of the electron branch, respectively, the growth rate of the ion branch increase correspondingly. Other interesting new results are obtained concerning, in particular, the frequency, the growth rate and the poloidal asymmetry of the ion mode fluctuations. It is mentioned that those appear to be directly relevant to various experiments and, in particular, to internal transport barriers. Shortcomings of the standard ballooning formalism are pointed out. [S1070-664X(00)04312-3]

I. INTRODUCTION

We have derived, in the framework of the two-fluids theory, three linear, coupled, two-dimensional (2D) partial differential equations which describe the dynamics of low frequency microinstabilities elongated along the magnetic field lines; this derivation is free of *a priori* assumptions concerning, e.g., the ballooning character of the modes, and of disputable simplifications concerning, e.g., the stress tensor; the role of ion–ion collisions has furthermore been taken systematically into account. The two fluids derivation is attractive because of its immediate physical content, but it fails to include wave-particle resonant interactions and the trapped particles response; those can be taken into account via a kinetic extension of the theory once the target instability parameters have been precisely defined by the two-fluids results.

In view of the curvature of the tokamak axisymmetric confining magnetic field, a crucial parameter in the 2D description of low frequency instabilities is the ratio of the radial width (w) of neighboring eigenmodes (with identical toroidal mode number) to the distance (Δ) between the rational surfaces about which they are respectively localized. Strong overlap occurs if $w/\Delta \gg 1$; under those conditions, new sets of eigenfunctions may be built as sets of linear combinations of the isolated eigenmodes, as first proposed by Taylor.[1] Poloidal coupling, which primarily occurs through the magnetic field inhomogeneity, can then play a decisive role on the linear stability properties: magnetic shear damping of electron drift waves for example is suppressed for a proper phasing of the isolated eigenmodes. The inhomogeneity of the magnetic field plays also an important role in the opposite limit $w/\Delta \ll 1$. Indeed, the sidebands $m\pm1$ of the primary mode m (m is the poloidal mode member) retroact on the dynamics of the latter and modify substantially its radial width and complex frequency. It was shown in Ref. 2 that the width of electron drift modes, respectively, their magnetic shear damping rate, increases by the neoclassical factor $(1+2q^2)^{1/4}$, respectively $(1+2q^2)^{1/2}$, with respect to the slab results;[3] q is the local safety factor.

We concentrate here our efforts on the limiting case where $w/\Delta \ll 1$ (the latter is particularly appropriate to the study of microinstabilities in the vicinity of internal transport barriers associated with small magnetic shear); we note that this inequality entails that the parallel mode number of the side bands is larger than that of the primary mode: the role of the parallel ion dynamics is therefore noticeably enhanced. We shall show how to extract methodically simplified equations for the electron, respectively, the ion drift branch from the above mentioned partial differential equations; for the purpose of the analytical developments, the parameter $|\hat{s}\varepsilon_N/q|$ is assumed to be small here and throughout (ε_N is the ratio of the equilibrium density lenght-scale to the torus major radius and $\hat{s}$ the magnetic shear parameter). The respective eigenfunctions and eigenvalues are thus obtained analytically under conditions for which overlap of neighboring eigenmodes does not occur and, as will be pointed out, the ballooning formalism[4] fails because of inherent restrictive assumptions. Interestingly, the ion branch frequency (measured in the $\mathbf{E}\times\mathbf{B}$ rotating frame) is small—in absolute value—compared to the ion diamagnetic frequency ω_i^* and its sign is opposite to that of ω_i^*. The growth rate is proportional to the absolute value of the magnetic shear parameter, suggesting a staightforward interpretation of the mechanism leading to the formation of internal transport barriers. Finally, the (normalized) density fluctuation in the ion branch is small—in absolute value—compared to the (normalized) ion temperature fluctuation and exhibits an important poloi-

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dal angle variation; this result—which is unrelated to the more usual ballooning topology of the modes obtained in the opposite limit $w/\Delta \gg 1$—is interesting in view of some measured density fluctuations with high spatial resolution. Ion collisions have a negligible effect on the electron branch, but are stabilizing for the ion branch, especially at high mode numbers; collisional damping of ion drift waves can thus play an important role in the transition from the low (L) to the radiative improved (RI) mode triggered in high density discharges by seeding of selected impurities. We note that the ubiquitous mode, which has phase velocities in the ion diamagnetic direction, is not considered here as it requires a finite trapped electron population.

The rest of the paper is organized as follows. The 2D fluid equations for the density, parallel momentum and temperature fluctuations are presented in Sec. II for arbitrary w/Δ (the standard drift wave ordering is assumed throughout). The limiting case $w/\Delta \ll 1$ is considered in Sec. III: particular emphasis is laid on identifying the range of poloidal mode numbers for which the terms introduced by toroidicity compete with the slab terms (maximal complexity ordering). The electron drift branch is identified in Sec. IV and the ion drift branch in Sec. V; in both cases, we consider successively the slab and the actual axisymmetric toroidal geometries. The properties of the electron branch are discussed further in Sec. VI and those of the ion branch in Sec. VII; in particular, the conditions for negligible wave-particle resonant interactions are discussed (the criterion is clearly more stringent for the side bands than for the primary modes). In Sec. VIII, we first provide precise criteria defining the limits of application of the theory and mention some shortcomings of the standard ballooning formalism; we then summarize our most important results and point out salient applications for later detailed consideration.

II. 2D EIGENVALUE EQUATIONS

The conventional drift wave ordering is defined by

$$\omega \sim \omega_j^*, \;\; k_\perp a_i \sim 1, \;\; k_\parallel qR \sim 1 \tag{1a}$$

and

$$\omega_j^*/\Omega_i \sim a_i/L_\perp \sim \mu \ll 1, \tag{1b}$$

where j is the species index, ω_j^* is the diamagnetic frequencies, $a_i = c_i/\Omega_i$, c_i and Ω_i are the ion Larmor radius, thermal velocity and gyrofrequency, respectively. No subscaling (as a fractional power of μ) is formally introduced for the ratio $L_\perp/qR$ of the plasma density/temperature scale length to the connexion length; the equilibrium profiles are then functions of the toroidal flux coordinate only, i.e., $T_i = T_i(\psi) + O(\mu)$, etc. For convenience, we scale, however,

$$c_i/c_e \sim (m_e/m_i)^{1/2} \sim \mu \tag{1c}$$

and consider only one ion species with $Z=1$.

Let ϕ, $n_e = n_i$, t_e, t_i, $\mathbf{u}_e$ and $\mathbf{u}_i$ be the potential, density, temperature, and velocity fluctuations. Within the framework of our ordering the electron energy and parallel momentum equations yield immediately

$$t_e = 0 \tag{2a}$$

and

$$\frac{n_e}{N_e} = \frac{e\phi}{T_e}. \tag{2b}$$

Equations (2a) and (2b) are referred to as the ''adiabatic'' approximation; they are consequential to the large electron mobility along the magnetic field lines. (We note that corresponding results do not obtain for trapped electrons; those are outside the scope of a fluid analysis.) The derivation of the 2D macroscopic equations (and of their kinetic extension) describing the ion dynamics (and that of the trapped electrons) in drift waves under conditions (1a)–(1c) will be presented in a forthcoming paper. Here, we merely state the results after having defined the mode representation appropriate to the 2D toroidal geometry.

We consider fluctuations of the form[5,6]

$$\frac{n(\psi,\chi,\varphi)}{N(\psi)} = \sum_{l,m} \hat{n}_{l,m}(\psi - \psi_{l,m}, \chi; \psi_{l,m}) \times \exp\left\{ il\left[\varphi - \int_0^\chi \nu(\psi_{l,m},\chi')\, d\chi' \right]\right\}, \tag{3}$$

ψ being the coordinate (label) of the toroidal magnetic surfaces, χ a poloidal angle-like coordinate, and φ the toroidal angle;[7] we note that $ds^2 = (h_\psi d\psi)^2 + (h_\chi d\chi)^2 + (h_\varphi d\varphi)^2$ with $h_\varphi = R$ and $h_\psi = (RB_\chi)^{-1}$, so that $\boldsymbol{\nabla}\cdot\mathbf{B} = J^{-1}\partial_\chi(h_\varphi h_\psi B_\chi) \equiv 0$; $h_\chi(\psi,\chi) = JB_\chi$—where J is the Jacobian of the orthogonal transformation $\mathbf{r} \to (\psi,\chi,\varphi)$—is prescribed by the Grad–Shafranov equation and defines the shape of the toroidal cross sections.

$$\nu(\psi,\chi) = (d\varphi/d\chi)_{\mathbf{B}} = h_\chi B_\varphi / h_\varphi B_\chi \tag{4}$$

is the local pitch angle of the magnetic field lines. l and m are the toroidal and the reference poloidal mode numbers [l,m are of order μ^{-1} according to (1a) and (1b)]. $\psi_{l,m}$ is the coordinate of the rational magnetic surface on which

$$q(\psi_{l,m}) = \oint \nu(\psi_{l,m},\chi)\, d\chi/2\pi = m/l. \tag{5}$$

The functions $\hat{n}_{l,m}(\psi - \psi_{l,m}, \chi; \psi_{l,m})$ are to describe the ''radial'' structure of the modes—those are localized to the neighborhood of the rational surfaces—and, owing to the inhomogeneities of the equilibrium, both a residual poloidal variation which cannot be included in the eikonal and a residual radial variation; the ''residual'' poloidal variation plays hereafter an important role. It is essential that the representation (3) is compatible with both the periodicity requirement—see (5)—and the requirement of long parallel wavelengths[4] (the latter minimizes the stabilizing roles of the parallel ion dynamics and of ion Landau damping). The derivative along the magnetic field lines

$$\mathbf{B}\cdot\boldsymbol{\nabla}(n/N) = (B_\chi/B) h_\chi^{-1} \sum_{l,m} \{ \partial_\chi \hat{n}_{l,m} + il[\nu(\psi,\chi) - \nu(\psi_{l,m},\chi)]\hat{n}_{l,m} \} \times \exp\left\{ il\left[\varphi - \int_0^\chi \nu(\psi_{l,m}\chi')\, d\chi' \right]\right\} \tag{6}$$

is indeed of order $1/qR$ if the radial width of the mode is typically of the order of the inverse of the poloidal mode number.

The equation for the ion density fluctuation is

$$\sum_{l,m} \exp\left\{ il\left[\varphi - \int_0^\chi \nu(\psi_{l,m},\chi')d\chi' \right] - i\omega t \right\} \Big[\{\omega - \omega_{E,l} + \tau_e\omega_{i,l}^* - (1+\tau_e)\omega_{B,i,l} - (1+\tau_e)[\omega - \omega_{E,l} - 1.5\omega_{B,i,l}$$
$$-0.3i\nu_i\Delta_{\perp,l}]\Delta_{\perp,l}\}\hat{n}_{i,l,m} - \{\omega_{B,i,l} + [\omega - \omega_{E,l} - 3\omega_{B,i,l} - 0.775i\nu_i\Delta_{\perp,l}]\Delta_{\perp,l}\}\hat{t}_{i,l,m}$$
$$+ i\omega_{t,i}\{\partial_\chi + il[\nu(\psi,\chi) - \nu(\psi_{l,m},\chi)] - \partial_\chi \ln B\}\hat{u}_{\parallel,i,l,m}\Big] = 0. \tag{7}$$

$\omega_{E,l} = -l d_\psi V$ is the $\mathbf{E}\times\mathbf{B}$ Doppler frequency [$V(\psi)$ is the equilibrium electrostatic potential, the ''radial'' electric field being $E_\psi = -h_\psi^{-1} d_\psi V$], $\tau_e = T_e/T_i$,

$$\omega_{i,l}^* = -l(T_i/e_i)d_\psi \ln N_i \tag{8}$$

is the ion diamagnetic frequency (associated with the diamagnetic drift)

$$\omega_{B,i,l} = -2(T_i/e_i)[l\partial_\psi \ln B - i\nu^{-1}(B_\varphi^2/B^2) \times(\partial_\chi \ln B)\partial_{\psi-\psi_{l,m}}] \tag{9}$$

is the frequency (operator) associated with the magnetic curvature and grad B drifts, $\omega_{t,i} = h_\chi^{-1} c_i B_\chi/B$ is the ion transit frequency,

$$\Delta_{\perp,l} = a_i^2[h_\psi^{-2}\partial^2_{\psi-\psi_{l,m}} - (lBh_\psi)^2] \tag{10}$$

is the perpendicular Laplacian (normalized to a_i^2; we have noted that $\partial_\chi \hat{n}_{l,m} \sim \mu l \hat{n}_{l,m}$) and ν_i is the ion collision frequency (ion collisions are rarely considered in the analysis of low frequency instabilities; they can however play a leading role in some experiments, as shown later).

$\hat{u}_{\parallel,i}$, the fluctuation of parallel ion flow velocity normalized to c_i, is given by the equation

$$\sum_{l,m} \exp\left\{ il\left[\varphi - \int_0^\chi \nu(\psi_{l,m},\chi')d\chi' \right] - i\omega t \right\} \times\Big[(\omega - \omega_{E,l} - 2\omega_{B,i,l} - 1.2i\nu_i\Delta_{\perp,l})\hat{u}_{\parallel,i,l,m} + i\omega_{t,i}\{\partial_\chi + il[\nu(\psi,\chi) - \nu(\psi_{l,m},\chi)]\} \times[(1+\tau_e)\hat{n}_{i,l,m} + \hat{t}_{i,l,m}]\Big] = 0. \tag{11}$$

Finally, the normalized ion temperature fluctuation $\hat{t}_i = t_i/T_i$ is to be obtained from the equation

$$\sum_{l,m} \exp\left\{ il\left[\varphi - \int_0^\chi \nu(\psi_{l,m},\chi')d\chi' \right] - i\omega t \right\} \times\Big[[1.5(\omega - \omega_{E,l}) + (1-1.5\eta_i)\omega_{i,l}^*\Delta_{\perp,l} - 2.5\omega_{B,i,l} - 2i\nu_i\Delta_{\perp,l}]\hat{t}_{i,l,m} - [\omega - \omega_{E,l} + (1-1.5\eta_i)\tau_e\omega_{i,l}^* - 0.5(1+\tau_e)(1-1.5\eta_i)\omega_{i,l}^*\Delta_{\perp,l}]\hat{n}_{i,l,m}\Big] = 0, \tag{12}$$

where $\eta_i = d_\psi \ln T_i / d_\psi \ln N_i$.

III. THE NEGLIGIBLE OVERLAP LIMIT

There are important differences between the equations describing the dynamics of drift waves in a torus and those applicable to the slab and cylindrical models, also in the limit

$$w_{l,m} \ll \Delta_{l,m} = |h_\psi(\psi_{l,m\pm1} - \psi_{l,m})| = |l h_\psi^{-1} d_\psi q|^{-1}, \tag{13}$$

which will be considered here, where the radial width of modes (l, m) and ($l,m\pm1$) is smaller than the distance between the rational surfaces $r_{l,m}$ and $r_{l,m\pm1}$ about which they are localized. Summation over l can be deleted in Eqs. (7), (11), and (12), owing to toroidal axisymmetry of the equilibrium. Summation over m can also be removed if (13) is verified, since the intersection

$$\hat{n}_{i,l,m}(\psi - \psi_{l,m},\chi;\psi_{l,m}) \cap \hat{n}_{i,l,m\pm1}(\psi - \psi_{l,m\pm1},\chi;\psi_{l,m\pm1})$$

is then negligible. The poloidal variation of the magnetic field—which occurs mainly via $\omega_{B,i,l}$—indeed enforces, also in that case, a poloidal dependence of the amplitudes $\hat{n}_{i,l,m}$, $\hat{u}_{\parallel,i,l,m}$, and $\hat{t}_{i,l,m}$!

In the following, we shall consider large aspect ratio tori with circular cross sections; replacing χ by the usual poloidal angle θ, we have $B = B_0(1 - \varepsilon\cos\theta)$, where $\varepsilon = r/R_0$, r is the radius of the toroidal magnetic surface, and $\theta = 0$ corresponds to the position in the outer equatorial plane. We are allowed to replace $\nu(\psi,\chi)$ by $q(r)$ in an ε expansion; hence

$$\omega_{B,i} = 2(T_i/e_iRB_\varphi)[(m/r)\cos\theta + i\sin\theta\partial_{r-r_{l,m}}] + O(\varepsilon) \tag{9'}$$

and

$$\nu(\psi,\chi) - \nu(\psi_{l,m},\chi) = (r - r_{l,m})\partial_r q(r) + O(\varepsilon). \tag{14}$$

The ''residual'' poloidal dependence of $\hat{n}_{i,l,m}$, $\hat{u}_{\parallel,i,l,m}$, and $\hat{t}_{i,l,m}$ will be described by Fourier series, e.g.,

$$\hat{n}_{i,l,m}(r-r_{l,m},\theta)=\sum_{P}\,[\hat{n}_{i,l,m}(r-r_{l,m})]_p e^{ip\theta}. \tag{15}$$

We stress that the above functions are localized to the neighborhood of the sole rational surface $r=r_{l,m}$; the index p corresponds to poloidal sidebands.

The operator $\partial_\chi+il[\nu(\psi,\chi)-\nu(\psi_{l,m},\chi)]$ leads to the algebraic expression $i(p+\hat{s}k_\theta x)$, where $\hat{s}=rd_r\ln q$ is the local magnetic shear parameter, $k_\theta=m/r$ the poloidal wave number, and $x=r-r_{l,m}$. The characteristic value of x is the radial width $w_{l,m}$ of the eigenmode. It results from the working hypothesis (13) that

$$|\hat{s}k_\theta x|\ll 1. \tag{13$'$}$$

The differential operator ∂_x is also attributed a characteristic value $\partial_x\sim w_{l,m}^{-1}$; hence

$$|k_\theta\ll|\hat{s}^{-1}\partial_x|. \tag{13$''$}$$

These inequalities will be used later on to simplify the analysis. For the time being, we rewrite the expressions under the sums in Eqs. (7), (11), and (12) in the strictly equivalent forms

$$\begin{aligned}&[\omega'+\tau_e\omega_i^*-(1+\tau_e)(\omega'-0.3i\nu_i\Delta_\perp)\Delta_\perp](\hat{n}_i)_p-(\omega'-0.775i\nu_i\Delta_\perp)\Delta_\perp(\hat{t}_i)_p-\omega_{t,i}(p+\hat{s}k_\theta x)(\hat{u}_{\parallel,i})_p\\&\quad=k_\theta\overline{u}_{B,i}\{(1+\tau_e)(1-1.5\Delta_\perp)[(\hat{n}_i)_{p-1}+(\hat{n}_i)_{p+1}]+(1-3\Delta_\perp)[(\hat{t}_i)_{p-1}+(\hat{t}_i)_{p+1}]\}\\&\qquad+\overline{u}_{B,i}\partial_x\{(1+\tau_e)(1-1.5\Delta_\perp)[(\hat{n}_i)_{p-1}-(\hat{n}_i)_{p+1}]+(1-3\Delta_\perp)[(\hat{t}_i)_{p-1}-(\hat{t}_i)_{p+1}]\},\end{aligned} \tag{16}$$

$$\begin{aligned}&(\omega'-1.2i\nu_i\Delta_\perp)(\hat{u}_{\parallel,i})_p-\omega_{t,i}(p+\hat{s}k_\theta x)[(1+\tau_e)(\hat{n}_i)_p+(\hat{t}_i)_p]\\&\quad=2k_\theta\overline{u}_{B,i}[(u_{\parallel,i})_{p-1}+(u_{\parallel,i})_{p+1}]\quad+2\overline{u}_{B,i}\partial_x[(u_{\parallel,i})_{p-1}-(u_{\parallel,i})_{p+1}]\end{aligned} \tag{17}$$

and

$$\begin{aligned}&[1.5\omega'+(1-1.5\eta_i)\omega_i^*\Delta_\perp-2i\nu_i\Delta_\perp](\hat{t}_i)_p-[\omega'+(1-1.5\,\eta_i)\tau_e\omega_i^*-0.5(1+\tau_e)(1-1.5\eta_i)\omega_i^*\Delta_\perp](\hat{n}_i)_p\\&\quad=2.5k_\theta\overline{u}_{B,i}[(\hat{t}_i)_{p-1}+(\hat{t}_i)_{p+1}]+2.5\overline{u}_{B,i}\partial_x[(\hat{t}_i)_{p-1}-(\hat{t}_i)_{p+1}].\end{aligned} \tag{18}$$

We have defined $\omega'=\omega-\omega_E$, $\overline{u}_{B,i}=T_i/e_iRB_\varphi$ and dropped superfluous indices. Consistently with the ε expansion, we must neglect the term $\hat{u}_{\parallel,i,l,m}\partial_\chi\ln\mathrm{B}$ by comparison to $\partial_\chi\hat{u}_{\parallel,i,l,m}$ on the right-hand side of Eq. (7).

We shall solve Eqs. (16)–(18) in the next sections, assuming

$$\varepsilon_N\hat{s}/q\ll 1, \tag{19a}$$

where

$$\varepsilon_N\equiv L_N/R=(Rd\ln N/dr)^{-1}; \tag{19b}$$

this condition is particularly well verified at the plasma edge and near transport barriers.

Prior going into the details of the solutions, we identify at this point the range of poloidal mode numbers for which the terms introduced by toroidicity compete with the slab terms. To this end, we assume (and shall verify later) that the angular frequency ω' is smaller than or of order of ω^*; we assume furthermore (and shall also verify) that $(\hat{n}_i)_{\pm1}\sim(\hat{t}_i)_{\pm1}$. The dissimilarity between the equations for $(\hat{n}_i)_{\pm1}$, respectively $(\hat{u}_{\parallel,i})_{\pm1}$, and those for $(\hat{n}_i)_0$, respectively $(\hat{u}_{\parallel,i})_0$, arises from the factor $p=\pm1$ replacing $\hat{s}k_\theta x\ll1$. We therefore argue that the toroidal drive (on the right-hand sides) must match the slab drives (on the left-hand sides) rather for $p=0$ than for $p=\pm1$. Equations (16) and (17) then yield the order of magnitude relations

$$\omega_{t,i}\hat{s}k_\theta x(\hat{u}_{\parallel,i})_0\sim\overline{u}_{B,i}\partial_x(\hat{n}_i)_1 \tag{20}$$

and

$$\omega_{t,i}\hat{s}k_\theta x(\hat{t}_i)_0\sim\overline{u}_{B,i}\partial_x(\hat{u}_{\parallel,i})_1. \tag{21}$$

The magnitude of $(\hat{n}_i)_1$, is obtained from Eq. (18),

$$\omega^*(\hat{n}_i)_1\sim\overline{u}_{B,i}\partial_x(\hat{t}_i)_0. \tag{22}$$

It follows from (20) and (22) that

$$(\hat{s}k_\theta x)^2\sim q^2(a_i/x)(\varepsilon_N\hat{s}/q)[(\hat{t}_i)_0/(\hat{u}_{\parallel,i})_0]; \tag{23}$$

the ratios (x/a_i) and $(\hat{t}_i)_0/(\hat{u}_{\parallel,i})_0$ will be obtained in relation to $(\varepsilon_N\hat{s}/q)$ by considering the slab model solutions. Further, the ordering relations (20) and (21) lead to

$$(\hat{t}_i)_1\sim(\hat{n}_i)_1\sim q(\hat{s}k_\theta x)^{-1}(\varepsilon_N\hat{s}/q)(\hat{t}_i)_0 \tag{24}$$

and

$$(\hat{u}_{\parallel,i})_1\sim q^{-1}(\hat{s}k_\theta x)(x/a_i)(\hat{t}_i)_0. \tag{25}$$

We have selected only a few terms of Eqs. (16)–(18) for the purpose of the above discussion. We have verified that the terms not considered are not dominant indeed. It will appear from (23)–(25) and the solution of the slab equations that a single expansion parameter arises, namely $\varepsilon_N\hat{s}/q$, if we consider the safety factor q as of order unity. We have however assumed $k_\theta\ll\partial_x$ here above [so that $k_\theta\overline{u}_{B,i}\ll\overline{u}_{B,i}\partial_x$—compare, e.g., (20) and (16)—and $\Delta_\perp\sim a_i^2\partial_x^2$]. Since the shear parameter appears separately in the scaling of $k_\theta a_i$, cf. (23), we shall assume

$$\hat{s}^2 \in [q^2(a_i/x)(\varepsilon_N \hat{s}/q)(\hat{t}_i)_0/(\hat{u}_{\parallel,i})_0,1], \tag{26a}$$

hereafter; k_θ^2/∂_x^2 will then be in the range

$$k_\theta^2/\partial_x^2 \in [1,q^2(a_i/x)(\varepsilon_N \hat{s}/q)(\hat{t}_i)_0/(\hat{u}_{\parallel,i})_0]. \tag{26b}$$

The scaling relations (23)–(26a) define the maximum complexity ordering, i.e., they allow to obtain the most general solutions of Eqs. (16)–(18) by expansion if $\varepsilon_N \hat{s}/q \ll 1$.

IV. THE ELECTRON DRIFT BRANCH FOR $\hat{s}\varepsilon_N/q \ll 1$ AND $w \ll \Delta$

A. The slab model

We set $\overline{u}_{B,i}=0$ in Eqs. (16)–(18). The electron drift branch is characterized by $\omega' \sim \omega^*$ and $\Delta_\perp \ll 1$. Neglecting ion collisions, Eq. (18) then yields

$$1.5\omega'^{(0)} t^{(0)} = [\omega'^{(0)} - (1-1.5\eta_i)\omega_e^*] n_i^{(0)} \tag{27}$$

in leading order. Introducing this result in Eq. (17) shows that $\hat{u}_{\parallel,i}^{(0)} \propto \hat{s} k_\theta x \hat{n}_i^{(0)}$. [Note that we may always define $p + \hat{s}k_\theta x = \hat{s}k_\theta x'$, $x' = x - p\Delta_{l,m}$.] Accordingly, Eq. (16) with $\Delta_\perp \ll 1$ cannot be satisfied in leading order, unless $\omega_{t,i}\hat{s}k_\theta x \hat{u}_{\parallel,i}^{(0)} \ll \omega' \hat{n}_i^{(0)}$ also. It follows that

$$\omega'^{(0)} = \omega_e^*, \tag{28}$$

$$(\hat{t}_i)_0 = \eta_i (\hat{n}_i)_0, \tag{27$'$}$$

and

$$\hat{u}_{\parallel,i}^{(0)} = (\omega_{t,i}/\omega_e^*)\hat{s}k_\theta x(1+\tau_e+\eta_i)\hat{n}_i^{(0)}. \tag{29}$$

Introducing these results into (16) yields the slab eigenvalue equation,

$$[\tau_e \Delta_\perp - \tau_e(1+\tau_e+\eta_i)^{-1}(\omega'^{(1)}/\omega_e^*) + (\varepsilon_N \hat{s} x/q a_s)^2]\hat{n}_i^{(0)} = 0, \tag{30}$$

where $a_s^2 = \tau_e a_i^2$ and $\omega'^{(1)}$ is the first order correction to the eigenvalue $\omega' = \omega'^{(0)} + \omega'^{(1)}$.

The solutions of Eq. (30) are

$$\hat{n}_i^{(0)} \propto H_n(K_s^{1/2} x/a_s)\exp(-K_s x^2/2a_s^2), \tag{31}$$

where

$$K_s^2 = -(\varepsilon_N \hat{s}/q)^2, \tag{32}$$

$$\omega'^{(1)}/\omega_e^* = -[k_\theta^2 a_s^2 + (2n+1)K_s](1+\tau_e+\eta_i)\tau_e^{-1} \tag{33}$$

and the H_n's ($n=0,1,2,...$) are Hermite polynomials,

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0. \tag{34}$$

The two values of K_s obtained from Eq. (31) lead to oscillating solutions as $x \to \pm\infty$. Had we not approximated ω' by ω_e^* in writing the last term of Eq. (30) would however have lead to $K_s^2 = -\tau_e(k_\theta a_s \hat{s} \omega_{t,i}/\omega')^2$ instead of Eq. (32). Requiring unstable solutions ($\mathrm{Im}\,\omega' > 0$) to decay spatially (causal relation) shows that we must select the root

$$K_s = i|\varepsilon_N \hat{s}/q|\,\mathrm{sign}\,\omega_e^* \tag{32$'$}$$

of Eq. (32); here, sign $\omega_e^* = \omega_e^*/|\omega_e^*|$. It follows that

$$\mathrm{Im}\,\omega'^{(1)} = -(2n+1)|\varepsilon_N \hat{s}/q|(1+\tau_e+\eta_i)\tau_e^{-1}|\omega_e^*| \tag{33$'$}$$

corresponding to damped solutions. Damping of electron drift waves by transiting ions (we note that $\tau_e^{-1}\varepsilon_N \omega_e^*/q = k_\theta a_i c_i/qR$) in a sheared magnetic field is a consequence of the wave energy being radiated away from the rational surface, as shown by Pearlstein and Berk.[3]

B. The toroidal eigenvalue equation

The slab results suggest that we search for modes with frequencies

$$\omega' \sim \omega^* \tag{35}$$

and characteristic radial scales

$$x/a_s \sim \Lambda^{-1} \equiv (\hat{s}\varepsilon_N/q)^{-1/2} \tag{36}$$

(thus $\Lambda \ll 1$). We assume furthermore,

$$(\hat{n}_i)_0 \sim (\hat{t}_i)_0 \tag{37}$$

and

$$(\hat{u}_{\parallel,i})_0 \sim \Lambda(\hat{t}_i)_0 \tag{38}$$

in compliance with (27′) and (29). The maximum ordering relations (23), (24), and (25) then imply that

$$\hat{s}k_\theta x \sim \Lambda, \tag{23$'$}$$

$$(\hat{t}_i)_1 \sim (\hat{n}_i)_1 \sim \Lambda(\hat{t}_i)_0, \tag{24$'$}$$

$$(\hat{u}_{\parallel,i})_1 \sim (\hat{t}_i)_0 \sim \Lambda^{-1}(\hat{u}_{\parallel,i})_0 \tag{25$'$}$$

for the toroidal electron drift branch. Relations (26a) and (26b) finally show that $k_\theta \partial_x \ll 1$ provided that the shear parameter be in the range

$$\hat{s} \in [\Lambda,1]. \tag{26$'$}$$

We scale the collision frequency as

$$\nu_i \sim \omega^*, \tag{39}$$

in order that the collisional diffusion rate $\nu_i \Delta_\perp$ be comparable to magnetic shear damping. The system of equations to consider in the two leading orders is then

$$\omega'^{(0)} = \omega_e^*, \tag{28$'$}$$

$$\omega'^{(1)}(\hat{n}_i)_0 - \omega_e^* \Delta_\perp[(1+\tau_e)(\hat{n}_i)_0 + (\hat{t}_i)_0] - \omega_{t,i}\hat{s}k_\theta x(\hat{u}_{\parallel,i})_0 = \overline{u}_{B,i}\sum_{\sigma=\pm 1}(k_\theta - \sigma\partial_x)[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma], \tag{40}$$

$$(\hat{u}_{\parallel,i})_\sigma^{(0)} = 0 \tag{41}$$

[continuity equation for $p=0$ and $p=\sigma=\pm 1$, respectively; we note that Eq. (41) implies that $(\hat{u}_{\parallel,i})_1 = O(\Lambda)(\hat{t}_i)_0$; this result is *not* inconsistent with (25′)],

$$\omega_e^*(\hat{u}_{\parallel,i})_0 - \omega_{t,i}\hat{s}k_\theta x[(1+\tau_e)(\hat{n}_i)_0 + (\hat{t}_i)_0] = 0, \tag{42}$$

$$\omega_e^*(\hat{u}_{\parallel,i})_\sigma^{(1)} - \sigma\omega_{t,i}[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma] = 0 \tag{43}$$

[parallel momentum equations; here above, $(\hat{u}_{\parallel,i})_\sigma^{(1)} \sim \Lambda(\hat{t}_i)_0$; the right-hand side of Eq. (42) vanishes in view of (41)] and

$$(\hat{t}_i)_0 - \eta_i(\hat{n}_i)_0 = 0, \tag{44}$$

$$1.5\omega_e^*[(\hat{t}_i)_\sigma - (\hat{n}_i)_\sigma] = 2.5\overline{u}_{B,i}(k_\theta + \sigma\partial_x)(\hat{t}_i)_0 \tag{45}$$

(energy equations).

Equations (44) and (42) relating $(\hat{t}_i)_0$ and $(\hat{u}_{\parallel,i})_0$ to $(\hat{n}_i)_0$ are identical to the slab equations (27′) and (29). In order to express the right-hand side of Eq. (40) as a function of, e.g., $(\hat{n}_i)_0$, we require a third relation for the $\sigma = \pm 1$ variables; this is provided by the continuity equation in next order,

$$-\sigma\omega_{t,i}(\hat{u}_{\parallel,i})_\sigma^{(1)} = \overline{u}_{B,i}(k_\theta + \sigma\partial_x)[(1+\tau_e)(\hat{n}_i)_0 + (\hat{t}_i)_0]. \tag{46}$$

Combining Eqs. (46) and (43) yields

$$[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma] = -(\omega_e^*/\omega_{t,i})qa_i(k_\theta + \sigma\partial_x)[(1+\tau_e)(\hat{n}_i)_0 + (\hat{t}_i)_0], \tag{43′}$$

which inserted in (40) yields the toroidal eigenvalue equation

$$[\tau_e(1+2q^2)\Delta_\perp - \tau_e(1+\tau_e+\eta_i)^{-1}(\omega'^{(1)}/\omega_e^*) + (\varepsilon_N\hat{s}x/qa_s)^2](\hat{n}_i)_0 = 0. \tag{47}$$

Equation (47) has been derived earlier from a kinetic approach.[2] It is identical to Eq. (30) except for the neoclassical factor $1+2q^2$. The solutions are

$$(\hat{n}_i)_0 \propto H_n(K_t^{1/2}x/a_s)\exp(-K_t x^2/2a_s^2), \tag{48}$$

where

$$K_t = i(1+2q^2)^{-1/2}|\varepsilon_N\hat{s}/q|\,\mathrm{sign}\,\omega_e^* \tag{49}$$

and

$$\omega'^{(1)} = -[(1+2q^2)k_\theta^2 a_s^2\,\mathrm{sign}\,\omega_e^* + i(1+2q^2)^{1/2} \times(2n+1)|\varepsilon_N\hat{s}/q|](1+\tau_e+\eta_i)\tau_e^{-1}|\omega_e^*|. \tag{50}$$

Comparing Eqs. (48) and (49) to (32′) and (33′) shows that the toroidal eigenmodes are broader and more stable than the slab modes.

In view of (36) and (23′), the ordering (39) of the collision frequency corresponds to

$$\hat{\nu}_i \equiv qR\nu_i/c_i \sim 1, \tag{39′}$$

i.e., to a plasma at the border between the high and the intermediate collisionality regime. Collisions have no effect on the stability of the electron drift branch at these relatively high collisionalities.

The toroidal eigenmodes $\hat{t}_{i,l,m}(r-r_{l,m},\theta), \hat{n}_{i,l,m}(r-r_{l,m},\theta), \hat{u}_{\parallel,i,l,m}(r-r_{l,m},\theta)$ are given by series similar to Eq. (15), where the $p=0$ terms are defined by Eqs. (48), (44), and (42), the $p=\pm 1$ terms by Eqs. (46), (43′), and (45), and higher order sidebands are obtained by iteration.

V. THE ION DRIFT BRANCH FOR $\hat{s}\varepsilon_N/q \ll 1$ AND $w \ll \Delta$

A. The slab model

We set $\overline{u}_{B,i}=0$ in Eqs. (16)–(18). The ion drift branch is characterized by $\omega' \ll \omega^*$ and $\Delta_\perp \ll 1$. Equation (18) then yields

$$\hat{n}_i = \{\omega'[(2/3)-\eta_i]^{-1}(\tau_e\omega_i^*)^{-1} + \tau_e^{-1}\Delta_\perp\}\hat{t}_i \tag{51}$$

in leading order. Introducing this result and [see Eq. (17)]

$$\hat{u}_{\parallel,i} = (\omega_{t,i}/\omega')\hat{s}k_\theta x\hat{t}_i \tag{52}$$

in the continuity equation (16) leads to the slab eigenvalue equation

$$\{\Delta_\perp + [(2/3)-\eta_i]^{-1}(\omega'/\omega_i^*) - (\omega_i^*/\omega')(\varepsilon_N\hat{s}x/qa_i)^2\}\hat{t}_i = 0 \tag{53}$$

(where $\varepsilon_N/qa_i = -\omega_{t,i}k_\theta/\omega_i^*$), whose solutions are

$$\hat{t}_1 \propto H_n(K_s^{1/2}x/a_i)\exp(-K_s x^2/2a_i^2). \tag{54}$$

K_s is any solution of

$$K_s^2[k_\theta^2 a_i^2 + (1+2n)K_s] = [(2/3)-\eta_i]^{-1}(\varepsilon_N\hat{s}/q)^2 \tag{55}$$

subject to the constraint $\mathrm{Re}\,K_s > 0$ and ω' is given by

$$\omega'/\omega_i^* = [(2/3)-\eta_i][k_\theta^2 a_i^2 + (1+2n)K_s]. \tag{56}$$

We note that the eigenvalue equation (53) is different from that obtained by Coppi *et al.*[8] who failed to take finite Larmor radius corrections to the perpendicular components of the fluctuation velocity field systematically into account. The latter play an essential role in the ion energy equation (18), where they provide the term $(1-1.5\eta_i)\omega_i^*\Delta_\perp\hat{t}_i$. As a consequence, the slab eigenmode is here broader and the growth rate larger.

B. The toroidal eigenvalue equation

The slab results suggest that we search for modes with frequencies

$$\omega' \sim (\varepsilon_N\hat{s}/q)^{2/3}\omega^* \equiv \lambda^2\omega^* \tag{57}$$

and characteristic radial scales

$$x/a_i \sim (\varepsilon_N\hat{s}/q)^{-1/3} \equiv \lambda^{-1}, \tag{58}$$

if $k_\theta^2 \leqslant \partial_x^2$ (the above relations define λ). We shall assume furthermore

$$(\hat{n}_i)_0 \sim \lambda^2(\hat{t}_i)_0 \tag{59}$$

and

$$(\hat{u}_{\parallel,i})_0 \sim (\hat{t}_i)_0 \tag{60}$$

following Eqs. (51) and (52). The maximum ordering relations (23), (24), and (25) then imply that

$$\hat{s}k_\theta x \sim \lambda^2, \tag{23″}$$

$$(\hat{t}_i)_1 \sim (\hat{n}_i)_1 \sim \lambda(\hat{t}_i)_0 \sim \lambda^{-1}(\hat{n}_i)_0, \tag{24″}$$

$$(\hat{u}_{\parallel,i})_1 \sim \lambda(\hat{t}_i)_0 \sim \lambda(\hat{u}_{\parallel,i})_0, \tag{25″}$$

for the toroidal ion drift branch. Relations (26a) and (26b) finally show that $k_\theta \partial_x \leqslant 1$ provided that the shear parameter be in the range

$$s \in [\lambda^2, 1]. \tag{26''}$$

We again scale the ion collision frequency as

$$\nu_i \sim \omega^* \tag{39'}$$

and note that $\nu_i \Delta_\perp$ is then of order ω'. The system of equations to consider in the two leading orders is then

$$\tau_e \omega_i^* (\hat{n}_i)_0 - \omega_{t,i} \hat{s} k_\theta x (\hat{u}_{\parallel,i})_0 = \overline{u}_{B,i} \sum_{\sigma=\pm 1} (k_\theta - \sigma \partial_x)[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma], \tag{61}$$

$$\tau_e \omega_i^* (\hat{n}_i)_\sigma - \sigma \omega_{t,i} (\hat{u}_{\parallel,i})_\sigma = \overline{u}_{B,i} (k_\theta + \sigma \partial_x)(\hat{t}_i)_0 \tag{62}$$

(continuity equations for $p=0$ and $p=\sigma=\pm 1$, respectively),

$$(\omega' - 1.2 i \nu_i \Delta_\perp)(u_{\parallel,i})_0 - \omega_{t,i} \hat{s} k_\theta x (\hat{t}_i)_0 = 2\overline{u}_{B,i} \sum_\sigma (k_\theta - \sigma \partial_x)(\hat{u}_{\parallel,i})_\sigma, \tag{63}$$

$$-\sigma \omega_{t,i}[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma] = 2\overline{u}_{B,i}(k_\theta + \sigma \partial_x)(\hat{u}_{\parallel,i})_0 \tag{64}$$

(parallel momentum equations), and

$$[1.5\omega' + (1 - 1.5\eta_i)\omega_i^* \Delta_\perp - 2 i \nu_i \Delta_\perp](\hat{t}_i)_0 - (1 - 1.5\eta_i)\tau_e \omega_i^* (\hat{n}_i)_0 = 2.5\overline{u}_{B,i} \sum_\sigma (k_\theta - \sigma \partial_x)(\hat{t}_i)_\sigma, \tag{65}$$

$$-(1 - 1.5\eta_i)\tau_e \omega_i^* (\hat{n}_i)_\sigma = 2.5\overline{u}_{B,i}(k_\theta + \sigma \partial_x)(\hat{t}_i)_0 \tag{66}$$

(energy equations). We note that we have made use of $(\hat{t}_i)_2 \leqslant \lambda (\hat{t}_i)_0, (\hat{u}_{\parallel,i})_2 \leqslant \lambda (\hat{u}_{\parallel,i})_0$ and also $(\hat{n}_i)_2 \leqslant \lambda (\hat{t}_i)_0$ in Eqs. (66), (64), and (62), respectively.

Equation (66) provides $(\hat{n}_i)_0$ in terms of $(\hat{t}_i)_0$. Equation (62) then shows that $(\hat{u}_{\parallel,i})_0 \propto \sigma (k_\theta + \sigma \partial_x)(\hat{t}_i)_0$ so that the right-hand side of Eq. (63) vanishes. Equation (64) shows that $[(1+\tau_e)(\hat{n}_i)_\sigma + (\hat{t}_i)_\sigma] \propto \sigma (k_\theta + \sigma \partial_x)(\hat{u}_{\parallel,i})_0$; accordingly, the right-hand side of Eq. (61) also vanishes. Finally, Eqs. (64) and (66) yield $(\hat{t}_i)_\sigma$ which inserted in Eq. (65) provides a third relation between the $p=0$ variables.

Since

$$(\hat{u}_{\parallel,i})_0 = (\omega' - 1.2 i \nu_i \Delta_\perp)^{-1} \omega_{t,i} \hat{s} k_\theta x (\hat{t}_i)_0, \tag{63'}$$

one has

$$(\hat{n}_i)_0 = (\tau_e \omega_i^*)^{-1} \omega_{t,i}^2 \hat{s} k_\theta x (\omega' - 1.2 i \nu_i \Delta_\perp)^{-1} \hat{s} k_\theta x (\hat{t}_i)_0. \tag{61'}$$

Furthermore,

$$(\hat{t}_i)_\sigma = (1 - 1.5\eta_i)^{-1} (\tau_e \omega_i^*)^{-1} (1+\tau_e) 2.5\overline{u}_{B,i}(k_\theta + \sigma \partial_x) \times (\hat{t}_i)_0 - \sigma \omega_{t,i}^{-1} 2\overline{u}_{B,i}(k_\theta + \sigma \partial_x)(\hat{u}_{\parallel,i})_0. \tag{67}$$

The toroidal eigenvalue equation for the ion drift branch is accordingly [see Eq. (65)]

$$\left[D\Delta_\perp + \frac{1}{(2/3) - \eta_i} \frac{\omega'}{\omega_i^*} - \left(\frac{\varepsilon_N \hat{s}}{q} \right)^2 (x/a_i) \times \frac{\omega_i^*}{\omega' - 1.2 i \nu_i \Delta_\perp} (x/a_i) \right] (\hat{t}_i)_0 = 0, \tag{68a}$$

where the coefficient in front of the Laplacian is

$$D = 1 + 12.5(1 + \tau_e^{-1})[\varepsilon_N/(1 - 1.5\eta_i) k_\theta a_i]^2 - 2 i \nu_i/(1 - 1.5\eta_i)\omega_i^*. \tag{68b}$$

Equation (68a) is of fourth order, owing to collisional diffusion; that effect will later on be considered via an expansion procedure only. The leading equation is then of second order. Its solutions are

$$(\hat{t}_i)_0 \propto H_n(K_t^{1/2} x/a_i) \exp(-K_t x^2/2a_i^2), \tag{69}$$

where K_t is any root of

$$D_0^2 K_t^2 [k_\theta^2 a_i^2 + (1+2n)K_t] = [(2/3) - \eta_i]^{-1} (\varepsilon_N \hat{s}/q)^2, \tag{70}$$

subject to the constraint $\mathrm{Re}\, K_t > 0$; the eigenvalues ω' are given by

$$\omega'/\omega_i^* = [(2/3) - \eta_i] D_0 [k_\theta^2 a_i^2 + (1+2n)K_t]. \tag{71}$$

D_0 obtains by neglecting the last term in Eq. (68b).

In view of (58) and (23″), the ordering (39′) of the collision frequency again corresponds to

$$\hat{\nu}_i \equiv qR\nu_i/c_i \sim 1. \tag{39''}$$

The effect of collisions on the stability, as obtained by a perturbative approach of Eq. (68a), is given by the equation

$$(\Delta\omega')_n \int_{-\infty}^{\infty} (\hat{t}_i)_0 \left[\frac{1}{(2/3) - \eta_i} + \left(\frac{\varepsilon_N \hat{s}}{q} \right)^2 \frac{x^2}{a_i^2} \left(\frac{\omega_i^*}{\omega'} \right)^2 \right] (\hat{t}_i)_0 dx = 2 i \nu_i \int_{-\infty}^{\infty} (\hat{t}_i)_0 \left\{ \left[\frac{1}{1 - 1.5\eta_i} + 0.6 \left(\frac{\varepsilon_N \hat{s}}{q} \right)^2 \frac{x^2}{a_i^2} \left(\frac{\omega_i^*}{\omega'} \right)^2 \right] \Delta_\perp + 1.2 \left(\frac{\varepsilon_N \hat{s}}{q} \right)^2 \left(\frac{\omega_i^*}{\omega'} \right)^2 x \frac{\partial}{\partial x} \right\} (\hat{t}_i)_0, \tag{72}$$

where $(\hat{t}_i)_0$ and ω' are given by Eqs. (69)–(71). Note that

$$\int_{-\infty}^{\infty} (\hat{t}_i)_0 x \partial_x (\hat{t}_i)_0 dx = -(1/2) \int_{-\infty}^{\infty} (\hat{t}_i)_0^2 dx.$$

The toroidal eigenmodes $\hat{t}_{i,l,m}(r - r_{l,m}, \theta)$, $\hat{n}_{i,l,m}(r - r_{l,m}, \theta)$, $\hat{u}_{\parallel,i,l,m}(r - r_{l,m}, \theta)$ are given by series similar to Eq. (15), where the $p=0$ terms are defined by Eqs. (69), (61′), and (63′), the $p=\pm 1$ terms by Eqs. (67), (64), and (62), and the higher order side band amplitudes can be obtained recursively. Of particular interest, in view of the ordering relation $(\hat{n}_i)_{\pm 1} \sim \lambda^{-1} (\hat{n}_i)_0$—*the latter is characteristic of the ion branch* [see (24″)]—is the expression of the density fluctuation. Equations (61′) and (66) lead to

$$\hat{n}_i(x,\theta)=(\tau_e\omega_i^*)^{-1}\{\omega_{t,i}^2\hat{s}k_\theta x(\omega'-1.2i\nu_i\Delta_\perp)^{-1}\hat{s}k_\theta x + (5/3)[\eta_i-(2/3)]^{-1}\omega_{B,i}\}(\hat{t}_i)_0, \tag{73}$$

where the $\cos\theta$ and $\sin\theta$ dependent frequency operator $\omega_{B,i}$ is defined in (9′) and $(\hat{t}_i)_0$ is given in (69). Writing $\hat{n}_i(x,\theta)$ in the more compact form

$$\hat{n}_i(x,\theta)=(\hat{n}_i)_0+(\hat{n}_i)_c\cos\theta+(\hat{n}_i)_s\sin\theta, \tag{73′}$$

we find the ratios

$$\frac{(\hat{n}_i)_c}{(\hat{n}_i)_0}=\frac{10}{3}D_0(k_\theta^2a_i^2+K_t)\frac{q^2}{\hat{s}^2\varepsilon_N}\frac{a_i^2}{x^2} \tag{74a}$$

and

$$\frac{(\hat{n}_i)_s}{(\hat{n}_i)_0}=-\frac{10}{3}iD_0(k_\theta^2a_i^2+K_t)\frac{q^2}{\hat{s}^2\varepsilon_N}\frac{K_t}{k_\theta x} \tag{74b}$$

for $H_n\equiv1$ and $\nu_i\Delta_\perp\ll\omega^t$; we have made use of Eq. (71). Those results will be discussed in Sec. VIII.

VI. PROPERTIES OF THE ELECTRON BRANCH

Equation (50) shows that electron drift waves are damped in the absence of kinetic and trapped particle effects. The shear damping rate of nonoverlapping toroidal eigenmodes is

$$\gamma_n=-(2n+1)(1+2q^2)^{1/2}\tau_e^{-1}(1+\tau_e+\eta_i)|k_\theta a_s||\hat{s}|c_s/qR, \tag{75}$$

where n is the order of the Hermite polynomial in Eq. (48). This value is larger than the corresponding rate obtained in the plasma slab model[3] by the factor $(1+2q^2)^{1/2}$; it is independent of ε_N but is proportional to the modulus of the magnetic shear parameter; it is further proportional to $|k_\theta a_i|$.

The characteristic radial scale length of those modes is, according to Eq. (49),

$$w=a_s|K_t|^{-1/2}=a_s(1+2q^2)^{1/4}|q/\varepsilon_N\hat{s}|^{1/2}. \tag{76}$$

This is again larger than the corresponding slab model result by the factor $(1+2q^2)^{1/4}$. The radial width is also proportional to qa_s if $2q^2>1$ and inversely proportional to both $|\varepsilon_N|^{1/2}$ and $|\hat{s}|^{1/2}$. It can therefore become comparable to the equilibrium density length scale in regions of weak shear and/or small ε_N.

The premise of negligible mode overlapping $w/\Delta\ll1$ is satisfied if

$$k_\theta^2a_s^2\ll(1+2q^2)^{-1/2}q^{-1}|\varepsilon_N/\hat{s}|; \tag{77}$$

the range of poloidal wave numbers allowed by the present theory thus decreases if ε_N decreases and increases if either q or $\hat{s}$ decreases.

The ratio of the parallel phase velocity to the ion thermal velocity is

$$|\omega'/k_\parallel c_i|_{p=0}=\tau_e^{1/2}q^{1/2}(1+2q^2)^{-1/4}|\varepsilon_N\hat{s}|^{-1/2}, \tag{78a}$$

respectively,

$$|\omega'/k_\parallel c_i|_{p=\pm1}=\tau_e^{1/2}|k_\theta a_s|q|\varepsilon_N|^{-1}, \tag{78b}$$

if we consider the main mode, respectively the sidebands. Ion Landau damping on the former is generally negligible but thermal ions can be in resonance with the sidebands (i.e., $|\omega'/k_\parallel c_i|_{p=\pm1}\leq1$) if

$$|k_\theta a_s|\leq\tau_e^{-1/2}q^{-1}|\varepsilon_N|. \tag{79}$$

A kinetic treatment is imperatively required for mode numbers comparable to or smaller than (79).

It follows from conditions (77) and (79) that *the range of poloidal mode numbers over which both conditions of negligible overlap* $(w/\Delta<1)$ *and negligible wave-particle resonant interaction are simultaneously fulfilled* is given by

$$\tau_e^{-1}q^{-2}\varepsilon_N^2<k_\theta^2a_s^2<(1+2q^2)^{-1/2}q^{-1}|\varepsilon_N/\hat{s}|; \tag{80a}$$

compatibility requires that

$$\tau_e^{-1}(1+2q^2)^{1/2}|\varepsilon_N\hat{s}/q|\ll1. \tag{80b}$$

Finally, ion collisions do not affect the stability of the electron branch if $\nu_i\Delta_\perp\ll\omega^*$, i.e., replacing $\Delta_\perp$ by K_t^{-1}, if

$$\hat{\nu}_i\equiv qR\nu_i/c_i\ll|k_\theta a_s|q^2(1+2q^2)^{1/2}|\varepsilon_N^2\hat{s}|^{-1}, \tag{81}$$

which condition is experimentally always satisfied under the condition for which ion Landau damping on the side bands is not operative.

VII. PROPERTIES OF THE ION BRANCH

It is shown in the Appendix that there is a spatially bounded marginally stable eigenmode if $\eta_i<2/3$ and both a spatially bounded growing mode and a spatially bounded damped mode if $\eta_i>2/3$. Only the latter will be considered below, successively for $\hat{s}\sim1$ and in the limit $\hat{s}\ll1$. The role of ion collisions will be discussed separately.

A. The case with finite magnetic shear

If $\hat{s}\sim1$, condition (13) of negligible overlap requires that $k_\theta^2a_i^2\ll|K_t|$. It follows that

$$K_t=0.5(1\pm i\sqrt{3})[(1+2n)(\eta_i-2/3)]^{-1/3}|\varepsilon_N\hat{s}/qD_0|^{2/3} \tag{82}$$

for the spatially bounded complex conjugate (one unstable, the other stable) solutions of Eq. (70) corresponding to $\eta_i>2/3$. Condition (13) sets the following upper bound requirement on $k_\theta^2a_i^2$,

$$k_\theta^2a_i^2\left[1+12.5(1+\tau_e^{-1})\frac{\varepsilon_N^2}{(1.5\eta_i-1)^2k_\theta^2a_i^2}\right]^{2/3} \ll[(1+2n)(\eta_i-2/3)]^{-1/3}\left|\frac{\varepsilon_N}{q\hat{s}^2}\right|^{2/3} \tag{83}$$

(we note that the left-hand side of this inequality is a monotonically increasing function of $k_\theta^2a_i^2$).

The growth and the damping rates of the two modes under consideration are given by

$$\gamma_n=\frac{\sqrt{3}}{2}\left[\left(\eta_i-\frac{2}{3}\right)(1+2n)\right]^{2/3}D_0^{1/3}\left|\frac{\varepsilon_N\hat{s}}{q}\right|^{2/3}(\mp\omega_i^*), \tag{84}$$

whereas their (real) angular frequency is

$$\mathrm{Re}\,\omega' = -\frac{1}{2}\left[\left(\eta_i - \frac{2}{3}\right)(1+2n)\right]^{2/3} D_0^{1/3}\left|\frac{\varepsilon_N \hat{s}}{q}\right|^{2/3}\omega_i^* . \tag{85}$$

$\mathrm{Re}\,\omega'$ and ω_i^* have therefore opposite signs: with respect to the $\mathbf{E}\times\mathbf{B}$ rotating framework, the modes of the ion drift branch propagate poloidally in the electron drift direction, as those of the electron branch. The frequencies of the former are however smaller than those of the latter by, typically, the (expansion) factor $|\varepsilon_N \hat{s}/q|^{2/3}$.

The radial width of the nonoverlapping toroidal eigenmodes,

$$w = a_i|\mathrm{Re}\,K_t|^{-1/2} = a_i\sqrt{2}[(1+2n)(\eta_i - 2/3)]^{1/6}|qD_0/\varepsilon_N\hat{s}|^{1/3}, \tag{86}$$

their growth (damping) rates and their real frequencies are larger than the corresponding slab values by the factor $D_0^{1/3}$ [cf. Eq. (68b)]. This effect is particularly significant for small values of $k_\theta^2 a_i^2$. As a result of the equilibrium poloidal asymmetry, w, γ_n/ω_i^* and $\mathrm{Re}\,\omega'/\omega_i^*$ actually behave as $|k_\theta a_i|^{-2/3}$ for $k_\theta a_i \to 0$; in the same asymptotic limit, D_0 is furthermore proportional to $\varepsilon_N^2/(\eta_i - 2/3)^2$ so that w is also proportional to $|\varepsilon_N|^{1/3}(\eta_i - 2/3)^{-1/2}$ and both γ_n and $\mathrm{Re}\,\omega'$ are proportional to $|\varepsilon_N|^{4/3}(\eta_i - 2/3)^0\omega_i^* \propto |L_N|^{1/3}$. Finally, the width is inversely proportional to $|\hat{s}|^{1/3}$, whereas both γ_n and $\mathrm{Re}\,\omega'$ are proportional to $|\hat{s}|^{2/3}$ for any $k_\theta a_i$; the growth rate thus decreases if the absolute value of the magnetic shear decreases.

The ratio of the parallel phase velocity to the ion thermal velocity is

$$|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0} = 2^{-3/2}[(1+2n)(\eta_i - 2/3)]^{1/2}, \tag{87a}$$

respectively,

$$\begin{aligned}|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm 1} &= |\mathrm{Re}\,\omega'|qR/c_i \\ &= 0.5[(1+2n)(\eta_i - 2/3)]^{2/3}|qD_0\hat{s}^2/\varepsilon_N|^{1/3}|k_\theta a_i|\end{aligned} \tag{87b}$$

if we consider the main mode, respectively, the sidebands. The strength of wave particle resonant interaction is characterized by the exponentials $\exp[-0.5(\mathrm{Re}\,\omega'/k_\parallel c_j)^2]$, $j \equiv e,i$. In view of (87a), ion resonance with the main mode will therefore lead to a significant modification of the instability η_i threshold value. Resonant interaction with the sidebands will also affect significantly the instability criterion since $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm1} = |w/\Delta||\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0}$, where $|w/\Delta| \ll 1$ by assumption, cf. Eq. (13). Under the premise $k_\theta^2 a_i^2 \ll |K_t|$, a kinetic treatment is thus necessary in complement to the more intuitive two-fluids approach.

B. The case with weak magnetic shear

If $\hat{s} \ll 1$, condition (13) of negligible overlap can also be satisfied when $k_\theta^2 a_i^2 > K_t$. Only this case is considered below since the opposite limit has been analyzed in Sec. VII A for arbitrary $\hat{s}$ values. Under those conditions, Eq. (70) yields $K_t = K_t^{(0)} + K_t^{(1)}$, where

$$K_t^{(0)} = \pm i(\eta_i - 2/3)^{-1/2}|\varepsilon_N\hat{s}/qk_\theta a_i D_0|, \tag{88a}$$

$$K_t^{(1)} = -(1+2n)(K_t^{(0)})^2/2k_\theta^2 a_i^2. \tag{88b}$$

$\mathrm{Re}\,K_t$ is therefore positive, corresponding to bounded eigenmodes, in agreement with the results of the Appendix for $\eta_i > 2/3$. Condition (13) sets the following upper bound requirement on $|k_\theta a_i|$,

$$D_0|k_\theta a_i|^3 \ll (\eta_i - 2/3)^{-1/2}|\varepsilon_N/\hat{s}q|, \tag{89}$$

whereas the premise $k_\theta^2 a_i^2 > (1+2n)|K_t|$ leading to (88a) and (88b) requires that

$$D_0|k_\theta a_i|^3 \gg (1+2n)(\eta_i - 2/3)^{-1/2}|\varepsilon_N\hat{s}/q|. \tag{90}$$

These criteria can be further simplified either if $D_0 - 1 \ll 1$ or if $D_0 - 1 \propto (\varepsilon_N/k_\theta a_i)^2 \gg 1$; substituting D_0 in the latter case leads to replacing $|k_\theta a_i|$ by $|k_\theta a_i|^3$ on the left-hand sides and ε_N, respectively $(\eta_i - 2/3)^{-1/2}$, by ε_N^{-1}, respectively, $(\eta_i - 2/3)^{3/2}$ on the right-hand sides.

The growth and the damping rates of the two modes under consideration are given by

$$\gamma_n = (\mp\,\mathrm{sign}\,\omega_i^*)(1+2n)(\eta_i - 2/3)^{1/2}|\hat{s}|c_i/qR \tag{91}$$

(where $\mathrm{sign}\,\omega_i^* = \omega_i^*/|\omega_i^*|$) whereas their (real) angular frequency is

$$\mathrm{Re}\,\omega' = -(\eta_i - 2/3)D_0 k_\theta^2 a_i^2 \omega_i^* . \tag{92}$$

$\mathrm{Re}\,\omega'$ and ω_i^* have again opposite signs.

The characteristic radial scale of nonoverlapping toroidal eigenmodes,

$$w = a_i|K_t|^{-1/2} = a_i(\eta_i - 2/3)^{1/4}|qk_\theta a_i D_0/\varepsilon_N\hat{s}|^{1/2}, \tag{93}$$

is here larger than that of the corresponding slab model by the factor $D_0^{1/2}$; the real part of the frequency is multiplied by D_0 and the growth (the damping) rate is unchanged. Finally, the radial scale is proportional to $|\hat{s}|^{-1/2}$, the growth rate to $|\hat{s}|$ and $\mathrm{Re}\,\omega'$ is independent of $\hat{s}$.

The ratio of the parallel phase velocity to the ion thermal velocity is

$$|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0} = (\eta_i - 2/3)^{3/4} D_0^{1/2}|k_\theta a_i|^{3/2}|q/\varepsilon_N\hat{s}|^{1/2}, \tag{94a}$$

respectively,

$$|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm1} = (\eta_i - 2/3)D_0|k_\theta a_i|^3 q|\varepsilon_N|^{-1} \tag{94b}$$

if we consider the main mode, respectively, the sidebands. It follows from conditions (89) and (94b) that *the range of poloidal mode numbers over which both conditions of negligible overlap* $(w/\Delta < 1)$ *and negligible wave particle resonant interaction are simultaneously fulfilled* is given by

$$\begin{aligned}[\eta_i - (2/3)]^{-1}q^{-1}|\varepsilon_N| &< D_0|k_\theta a_i|^3 \\ &< [\eta_i - (2/3)]^{-1/2}|\epsilon_N/\hat{s}q|;\end{aligned} \tag{95a}$$

compatibility requires that

$$|\hat{s}|[\eta_i - (2/3)]^{-1/2} < 1. \tag{95b}$$

[We note that it follows from (95a) and (95b) that (90) is automatically verified unless the profiles are very far above instability threshold.]

C. The role of ion collisions

The effect of ion collision on the stability of the ion branch is described perturbatively by Eq. (72). We shall here consider only the limit where $k_\theta^2 a_i^2 > (1+2n)|K_t|$ since wave-particle resonant interactions play otherwise a leading role. It is easily shown, making use of (88a), (92), and the inequality (90), that Eq. (72) then simplifies into

$$-i\Delta\omega' = \Delta\gamma = -(4/3)k_\theta^2 a_i^2 \nu_i, \tag{96}$$

which result can also be obtained nonperturbatively. Equation (96) shows that ion collisions play a stabilizing role proportional to $k_\theta^2 a_i^2$.

VIII. SUMMARY AND DISCUSSION

We have presented a systematic derivation of the equations appropriate to the description of low frequency oscillations in axisymmetric toroidal plasmas. In the following, we discuss successively the domain of validity of the theory and summarize our most important results.

A. Domain of validity

Our theoretical approach is justified if the following conditions are fulfilled:

(1) The eigenmode radial widths are small compared to the distance between the successive rational surfaces $\cdots r_{l,m-1}, r_{l,m}, r_{l,m+1}, \ldots$;
(2) Resonant wave particle interactions are negligible;
(3) The populations of trapped particles play a negligible role;
(4) The parameter $|\varepsilon_n \hat{s}/q|$ is small compared to unity.

The limit opposite to (1) is most often considered—implicitly or explicitly—in the literature.[9–11] Inequalities (77), respectively, (83) and (89), indicate the parameter ranges for which condition (1) is satisfied for the electron, respectively the ion branch; (83) refers to the case $k_\theta^2 a_i^2 < K_t$ and (89) to $k_\theta^2 a_i^2 > K_t$, K_t being a measure of $a_i^2 \partial_x^2$. 2D eigenvalue equations for the functions $\hat{n}_i(x,\theta)$, $\hat{t}_i(x,\theta)$, $\hat{u}_{\parallel,i}(x,\theta)$ are obtained in the limit here under consideration; equivalently a system of nine ordinary differential equations for $[\hat{n}_i(x)]_p$, $[\hat{t}_i(x)]_p$, $[\hat{u}_{\parallel,i}(x)]_p$, $p=0,\pm 1$ can be written after Fourier decomposition. [Higher order sidebands play a negligible role if $|\varepsilon_N \hat{s}/q| \ll 1$; the primary oscillation ($p=0$) and the sidebands ($p=\pm 1$) are of course centered on the same rational surface!] The slab eigenvalue equation is recovered only for the range of poloidal mode numbers defined by

$$k_\theta^2 a_i^2 \geqslant 5.56(1+\tau_e^{-1})\varepsilon_N^2[\eta_i-(2/3)]^{-2} \tag{97}$$

in the case of the ion mode [see (68b)] and in no circumstances in the case of the electron mode.

It is important to note that the ballooning mode formalism[4] cannot grasp the toroidal effects obtained here, insofar as it assumes $\phi(x,\theta)$ to be of the form

$$\phi(x,\theta) = \sum_p \phi_p(x)\exp\{i[-(m_0+p)\theta + l\varphi - \omega t]\}$$

with[1,12]

$$\phi_p(x) = A(x)\phi_0(x-p/lq'),$$

$A(x)$ being a slowly varying envelope. Our analysis actually demonstrates that the above form is too restrictive; the assumptions inherent to the ballooning mode representation have been clarified in a paper by Kim and Wakatani.[13]

The strength of wave-particle resonant interaction is proportional to $\exp(-\omega'^2/2k_\parallel^2 c_j^2)$. The fluid description of ions is thus adequate if $\omega'/k_\parallel c_i$ is larger than unity. The characteristic parallel wave number however takes on different values according to whether one considers the primary oscillation or the sidebands. It is thus important to ascertain first the relative contribution of both. We note to this end that the coefficient $(1+2q^2)$, respectively D_0, multiplying the differential operator $\Delta_\perp$ in Eq. (47), respectively, (68a), is the sum of the contributions from finite Larmor radius (FLR) and curvature induced effects (1 vs $2q^2$, respectively, 1 vs D_0-1). The contributions of the sidebands and of the primary oscillation are thus in the ratio $2q^2/1$ for the electron branch and $(D_0-1)/1$ for the ion branch. [That the former is independent of k_θ sheds concern on the validity of the assumption $\phi_p(x) = A(x)\phi_0(x-p/lq')$ introduced together with the ballooning formalism.]

Resonant interaction of the primary electron drift oscillation with ions is weak as $|q/\varepsilon_N \hat{s}|^{1/2}$ is usually—and has been assumed in our derivation—large compared to unity [cf. Eq. (78a); the neoclassical factor $(1+2q^2)^{-1/4}$, however reduces $|\omega'/k_\parallel c_i|_{p=0}$ somewhat]; resonant interaction with the sidebands plays an important role at small $|k_\theta a_s|$ satisfying the inequality (79); it has been shown in Ref. 2 that ion transit resonance damping then exceeds shear damping. In conclusion, conditions (1) and (2) above are fulfilled simultaneously for the electron branch only if $(k_\theta a_s)^2$ is in the range given by Eq. (80a).

Considering now the ion branch, the ratio $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0}$ is always of order unity when $k_\theta^2 a_i^2 \ll K_t$, cf. Eq. (86a). Since $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm 1} < |\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0}$, the fluid description is, as a consequence, never appropriate in this case. The ratios $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=0}$ and $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm 1}$ are given by Eqs. (94a) and (94b) when $k_\theta^2 a_i^2 \geqslant K_t$. Inequalities (95a) indicate the range of poloidal mode numbers over which both conditions (1) and (2) are fulfilled simultaneously; Inequality (95b) shows that there is room if $|\hat{s}| \ll 1$. It is interesting to note that the right-hand side of Eq. (94b) has a minimum with respect to $|\varepsilon_N|/[\eta_i-(2/3)]$ for

$$\{|\varepsilon_N/[\eta_i-(2/3)]\}_{\min} = 1.5|k_\theta a_i|/[12.5(1+\tau_e^{-1})]^{1/2}; \tag{98}$$

thus

$$|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm 1} \geqslant 4.71(1+\tau_e^{-1})^{1/2} q k_\theta^2 a_i^2. \tag{94b$'$}$$

At a given poloidal mode number, the resonant cut-off condition $|\mathrm{Re}\,\omega'/k_\parallel c_i|_{p=\pm 1} = 1$ is thus reached simultaneously for two different values of $|\varepsilon_N|/[\eta_i-(2/3)]$, those being respectively, larger and smaller than $\{|\varepsilon_N/[\eta_i-(2/3)]\}_{\min}$.

Since the population of trapped electrons has been neglected in this two-fluids approach, the ubiquitous mode[14] could not be recovered; contrary to the modes discussed here, the latter has phase velocities in the ion diamagnetic direction.

B. Important results and forthcoming applications

(1) The angular frequency of the ion drift mode is given by Eq. (92) if $k_\theta^2 a_i^2 \gg K_t$; its sign is opposite to that of ω_i^*. The growth rate, given by Eq. (91), is proportional to the absolute value of the magnetic shear parameter. *This result should bring a simple explanation to the formation of internal transport barriers with minimum q profiles*[15,16] (we note that the assumption of negligible overlapping is particularly appropriate here; *a contrario*, the strong overlapping limit implicitly considered in most other theoretical works does not apply). The growth rate is furthermore independent of $|k_\theta a_i|$. As a result, Landau damping may suppress the instability at long wavelengths, see Eq. (94b) and the above discussion. At the other end of the spectrum, i.e., for finite $|k_\theta a_i| \leqslant 1$ values, ion collisions may stabilize the system if ν_i is large enough, as Eq. (96) shows (we recall that the description of FLR by the gyrostress tensor is appropriate only if the preceeding inequality is verified). *Shrinking of the instability range owing to both Landau and collisional damping appears to adequately explain the reduction of conductive/convective anomalous transport at the edge of (the high density) radiative improved confinement mode discharges*[17] (the analysis of Sec. VIII B requiring $\hat{s} \ll 1$ is however not directly applicable here). The experimental applications outlined in this paragraph will be investigated more thoroughly in other papers.

(2) The amplitude of the density fluctuations in the ion drift mode is characterized by important poloidal asymmetries with two maxima and two minima [see Eqs. (73′), (74a), and (74b)]. Under the conditions for which the results of Sec. VIII B are valid, i.e., $K_t < k_\theta^2 a_i^2$ *and low shear*, the maxima are located near the equatorial plane, both on the low and the high field sides; indeed, $\tan\theta_{\max} \approx [K_t/(k_\theta a_i)^2]$, hence $\theta_{\max} \approx 0$ or π. The minima are close to $\theta = \pm\pi/2$. Those results may be relevant to observations made in the core of TEXT-U (Texas Experimental Tokamak-Upgrade).[18] The ion temperature fluctuations are larger than the density fluctuations and only weakly θ dependent.

(3) By comparison to the slab results, toroidal curvature of the magnetic field lines leads to a broadening of both the electron and the ion drift eigenmodes. Concomitantly, the shear damping rate of the electron branch and the growth rate of the ion branch increase. Those increments are characterized by fractional powers of the factors $1+2q^2$ and D [the latter is defined in (68b)], respectively. Finite Larmor radius corrections to ω_e^* are proportional to $(1+2q^2)$ in the electron branch [Eq. (50)].

APPENDIX: PROPERTIES OF THE ROOTS OF THE ION BRANCH DISPERSION RELATION

The ion branch dispersion relation (70) can be cast into the form

$$z^3 + z^2 = \alpha, \tag{A1}$$

where

$$z = (1+2n) K_t / k_\theta^2 a_i^2 \tag{A2}$$

and

$$\alpha = (1+2n)^2 (\varepsilon_N \hat{s}/q)^2 / [(2/3) - \eta_i] D_0^2 k_\theta^6 a_i^6. \tag{A3}$$

Let $z = u + iv$. The real numbers u and v are solutions of either the equations

$$v^2 = 3u^2 + 2u, \tag{A4}$$

$$u(2u+1)^2 = -\alpha/2, \tag{A5}$$

or of

$$v = 0, \tag{A6}$$

$$u^3 + u^2 = \alpha. \tag{A7}$$

If α is positive, i.e., if $\eta_i < 2/3$, then the real solution (A6) and (A7) is characterized by $u > -1$, i.e., $\mathrm{Re}\, K_t > 0$. The mode is spatially bounded but the growth rate vanishes [Eq. (71) with $\mathrm{Im}\, K_t = 0$]. The complex roots (A4), (A5) are characterized by $u < 0$, i.e., $\mathrm{Re}\, K_t < 0$. These modes are spatially unbounded, whether $v > 0$ or $v < 0$ (i.e., whether $\mathrm{Im}\, \omega' < 0$ or $\mathrm{Im}\, \omega' > 0$), and therefore of no interest.

If α is negative, i.e., if $\eta_i > 2/3$, then the real solution (A6) and (A7) is characterized by $u < -1$, i.e., $\mathrm{Re}\, K_t < 0$. This marginally stable mode ($\mathrm{Im}\, \omega' = 0$) is spatially unbounded and therefore unphysical. The complex roots (A4) and (A5) are characterized by $u > 0$, i.e., $\mathrm{Re}\, K_t > 0$. The two corresponding modes are spatially bounded, one growing and the other decaying in time.

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