



KERNFORSCHUNGSANLAGE JÜLICH GmbH

Zentralabteilung Allgemeine Technologie

**Large Magnetic Coils-
Design Accompanying Calculation and
Optimization
Regarding Orthotropic Interlayers,
Temperature and Elastic Supports-
Derivation of a Special Finite Element**

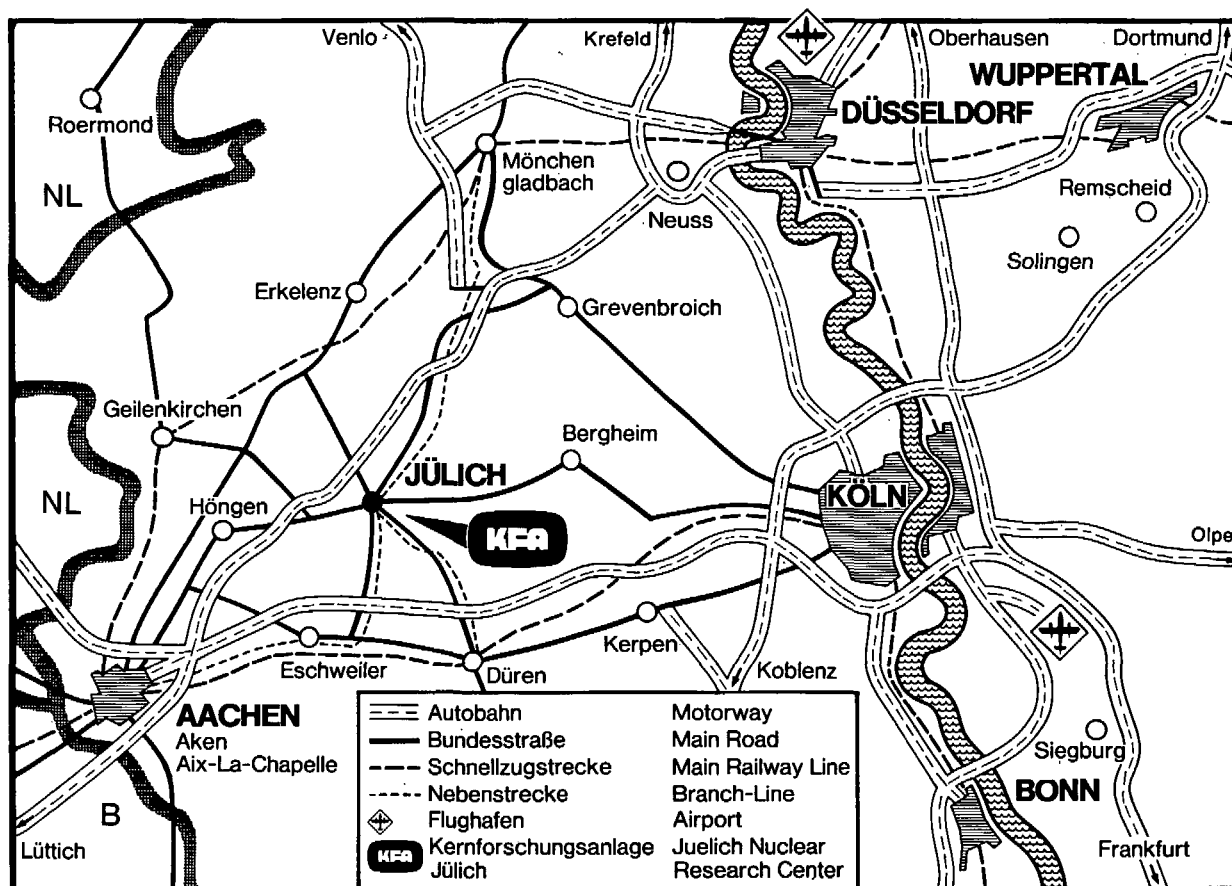
by

J. F. Stelzer, A. Sievers, R. Welzel

The TEXTOR project

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Zentralabteilung Allgemeine Technologie,
coworking with the Institute für Plasmaphysik

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Summary

This paper deals with finite element calculations of large coils as they are used as main coils in tokamaks. They consist of copper layers with glass fibre reinforced resin interlayers inbedded in a strong steel ring. The interlayers behave extremely orthotropic. Therefore cylinder wall elements with radial-tangential orthotropic behavior were developed. Regard is given, too, to the direction dependent thermal expansions because the calculations also include the heated state of the coils.

In a first analysis model the several epoxy layers are condensed to only one the thickness of which is equal to the sum of the single sizes. This fictitious layer is assumed to lie in the middle of the copper and is treated as an orthotropic material. In a following changed model the epoxy layer is situated between the steel ring and the copper. In this location the epoxy was suspected to suffer from the highest shear stresses.

Both models employ springy trusses as supporting features which simulate the real elastic behavior of a sustaining vault.

The results display deformations and stress distributions in all layers. A special attention is given to the shear stresses in the epoxy. Hot and cold states of the coils are considered. For comparing purposes also the behavior of appropriate isotropic coils is examined.

Furthermore, report is given about the forces transferred from the coils to the sustaining vault.

An optimal structure design is carried out concerning the steel ring. An appropriate fully stressed steel ring contour is obtained. Its influence upon the shear stresses in the epoxy layer is considered.

A first appendix deals with the non steady state temperature field during coil operating.

In a second appendix description is given of the development of the special orthotropic finite element.

The third appendix treats the problem of the coherency of the orthotropic properties.

1. Introduction

Former calculations concerning the Textor coils /1/ neglected the orthotropic direction dependent properties of the glass reinforced epoxy insulation imbeddings between the coil windings. Practical experiences /2/, however, showed no satisfying agreement between measured real displacements of loaded coils and these calculations. Therefore the calculations were repeated with a refined finite element model. The several resin layers are now condensed to only one layer the thickness of which is equal to the sum of the single thicknesses. This fictitious layer is assumed to exist in the middle of the whole copper parcel, and it is now possible to treat it as an orthotropic material. For the purposes of the underlying analysis a special finite element of the cylinder wall type with radial-tangential orthotropic material behavior was developed. It also takes regard of the nonisotropic heat expansion properties if exposed to temperaturechangings. A detailed derivation is given in the appendix.

Another disadvantage of the past calculations was the assumption of rigid coil fixing points which caused considerable stress peaks in the attached coil region.

The improved model employs springy trusses as supporting features by which the real elastic behavior of the sustaining vault is simulated.

2. Specifications

The revised model consists of half a coil of 144 elements of the cylinder wall type /3/, see fig.1. Each element extends over an angle of 5 degree. The inner layer represents copper, copying coil layers, the next stratification reproduces the sum of all resin insulation layers. It is situated in the middle of the whole copper, and the following third layer representing copper is of the same thickness as the first one. The outer wall is made of steel.

Subsequent calculations deal with a changed model where the epoxy layer lies between the steel ring and the copper. In this site the epoxy was suspected to suffer the hardest strain.

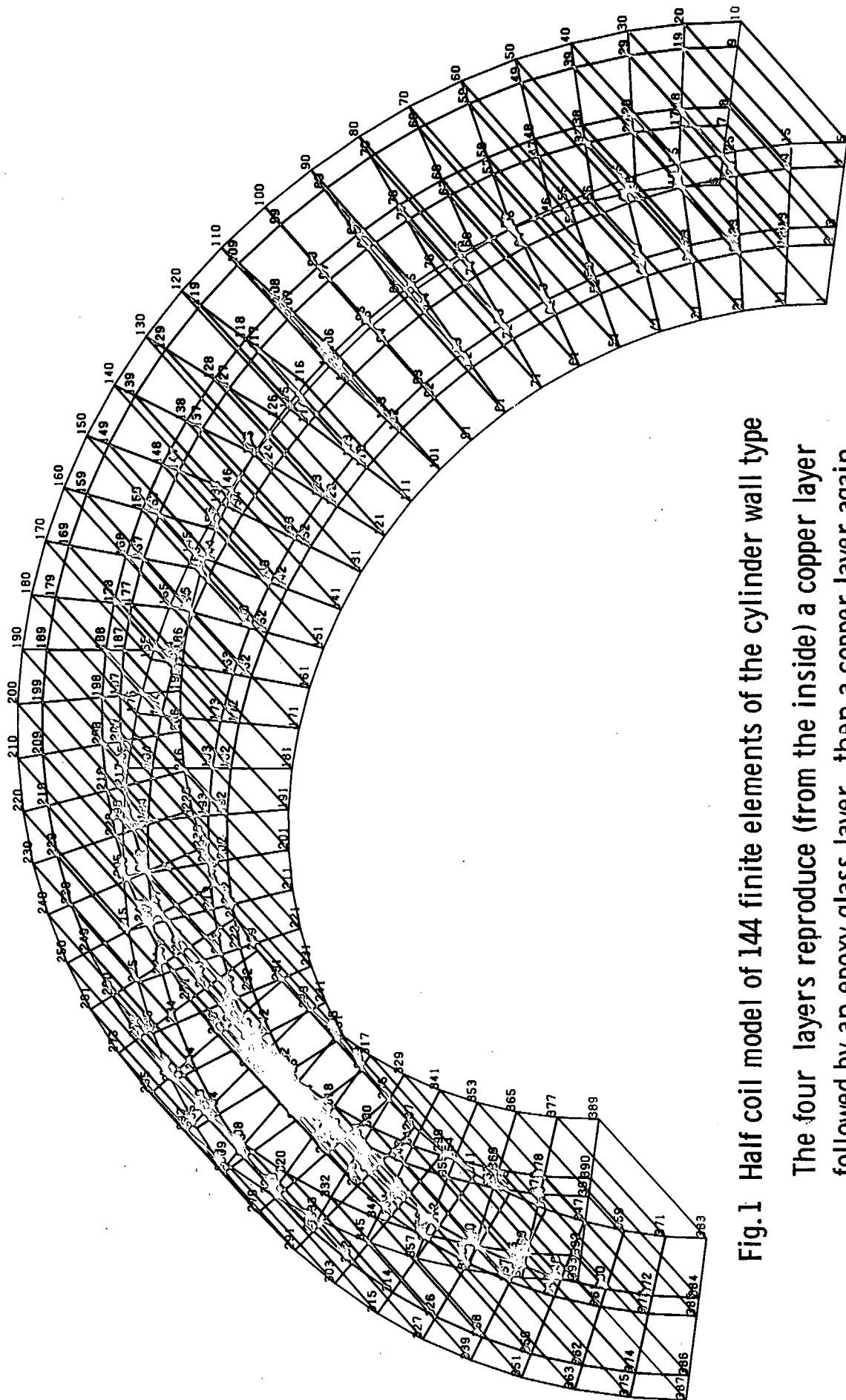


Fig.1 Half coil model of 144 finite elements of the cylinder wall type

The four layers reproduce (from the inside) a copper layer followed by an epoxy glass layer, then a copper layer again and outside a steel ring

Table 1

Underlying dimensions in cm

layer no.	material	inner radius	outer radius	thickness
1	copper	96	108.15	12.15
2	epoxy	108.15	111.85	3.7
3	copper	111.85	124	12.15
4	steel	124	129	5

The coil extension in axial direction is 32.4 cm.

2.1 Magnetic loads

The coils are strained by considerable magnetic loads in the radial direction, see fig. 2, and comparably rather small axial loads, see fig. 3, which occur if the plasma fails. The load distribution calculations were carried out by H.Belitz, KFA-IPP.

An adding of the radial loads results in a vector sequence as shown in fig.4. The resultant forces of the upper and the lower (with symmetric strains) half coil impose an inner tension. Outer forces of the centripetal type remain which press the 16 coils towards the centre of the sustaining vault.

2.2 Simulating of the vault elasticity

According to the design the central vault shows a certain weak elastic behavior near the horizontal center plane. The stiffness increases with the vertical distance. The different stiffnesses were calculated by D.Th.Goering, KFA-IPP, and were introduced into the analysis by means of trusses with analogous stiffness behavior, see fig. 5. There are always two trusses in one plane, one at each axial end.

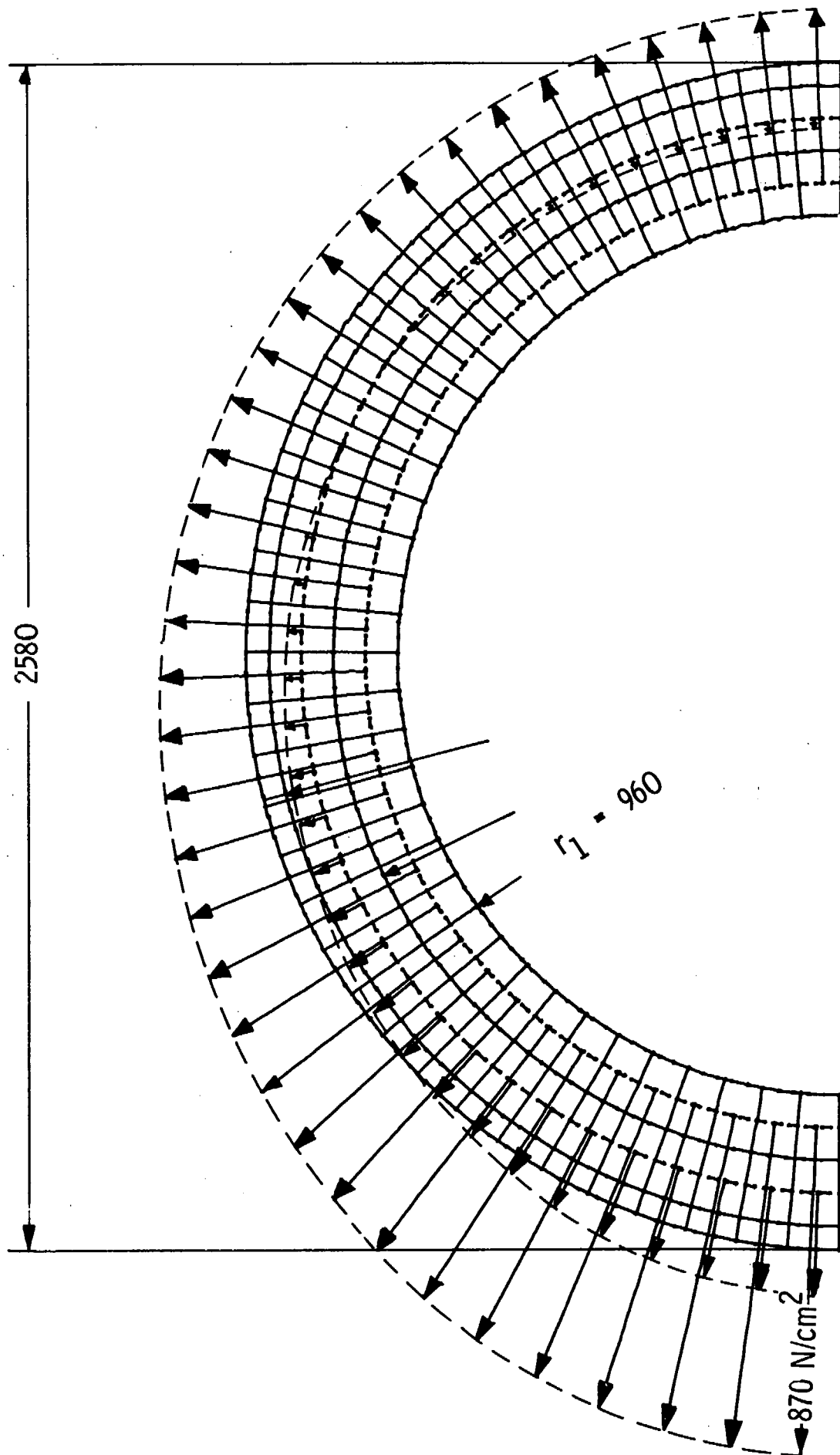


Fig.2 Distribution of the radial magnetic loads (pressure type) acting on two copper layers

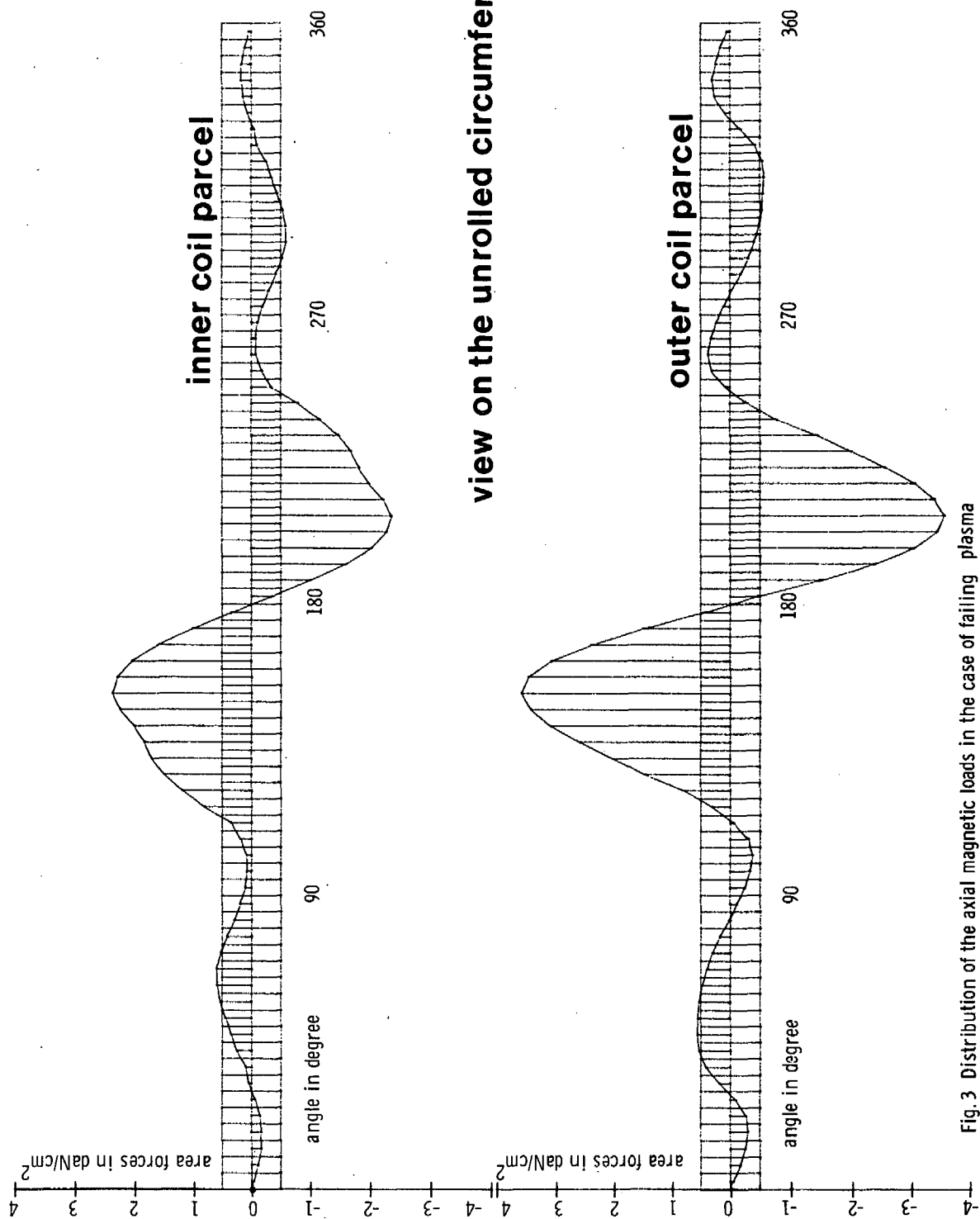
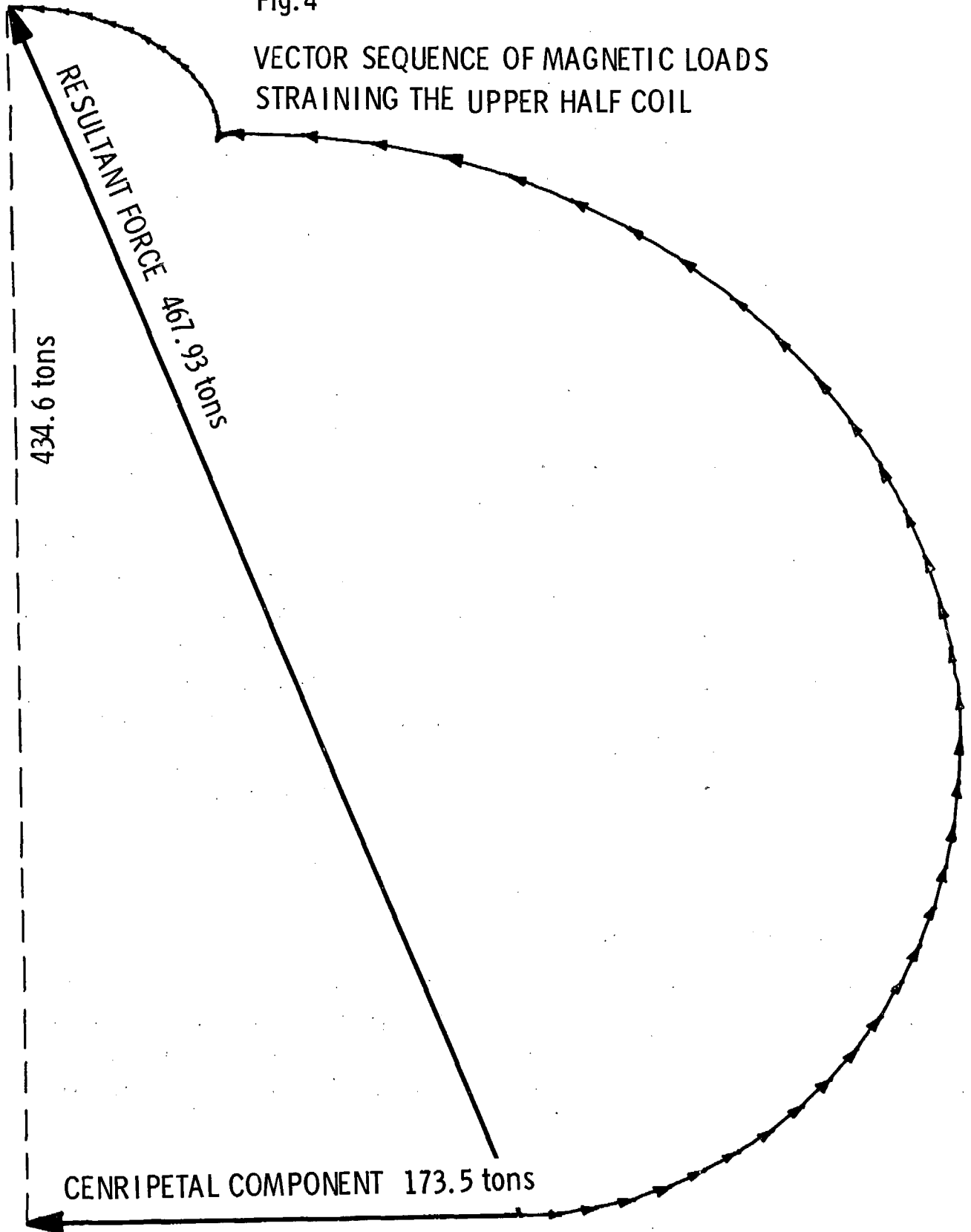


Fig.3 Distribution of the axial magnetic loads in the case of failing plasma

Fig. 4

VECTOR SEQUENCE OF MAGNETIC LOADS
STRAINING THE UPPER HALF COIL



2.3 Properties

In our calculations the copper layers and the outer steel ring get isotropical treatment with the following values.

Table 2

Underlying properties of copper and steel

	Young's modulus E daN/mm ²	shear modulus G daN/mm ²	Poisson's ratio ν	heat expansion coefficient α K ⁻¹
copper	12500	4664	0.34	$1,7 \cdot 10^{-5}$
steel	21000	8077	0.3	$1,8 \cdot 10^{-5}$

On account of the stratified glass web imbeddings the epoxy layer has a very orthotropic character. The Young's moduli and the heat expansion coefficients are concerned as fig.6 reveals. Due to a self developed variant of the cylinder wall element it is possible to take into account thus special radial-tangential-axial orthotropic material behavior. The orthotropic properties stem from experimental examinations of BBC Oerlikon. Also the properties which do not depend on the directions, shear modulus and Poisson's ratio, are posted in fig. 6. Unfortunately, there exists a discrepancy between the values of Poisson's ratio contradictory to the theory. The details of this problem are discussed in the appendix 3, p. 74.

2.4 Boundary conditions

Because of the symmetry of loads and size only a half coil is considered. In the end planes, nodal points 1 to 10 and nodal points 383 to 393 respectively, see fig. 1, no angular displacements are permitted. Radial and axial degrees of freedom

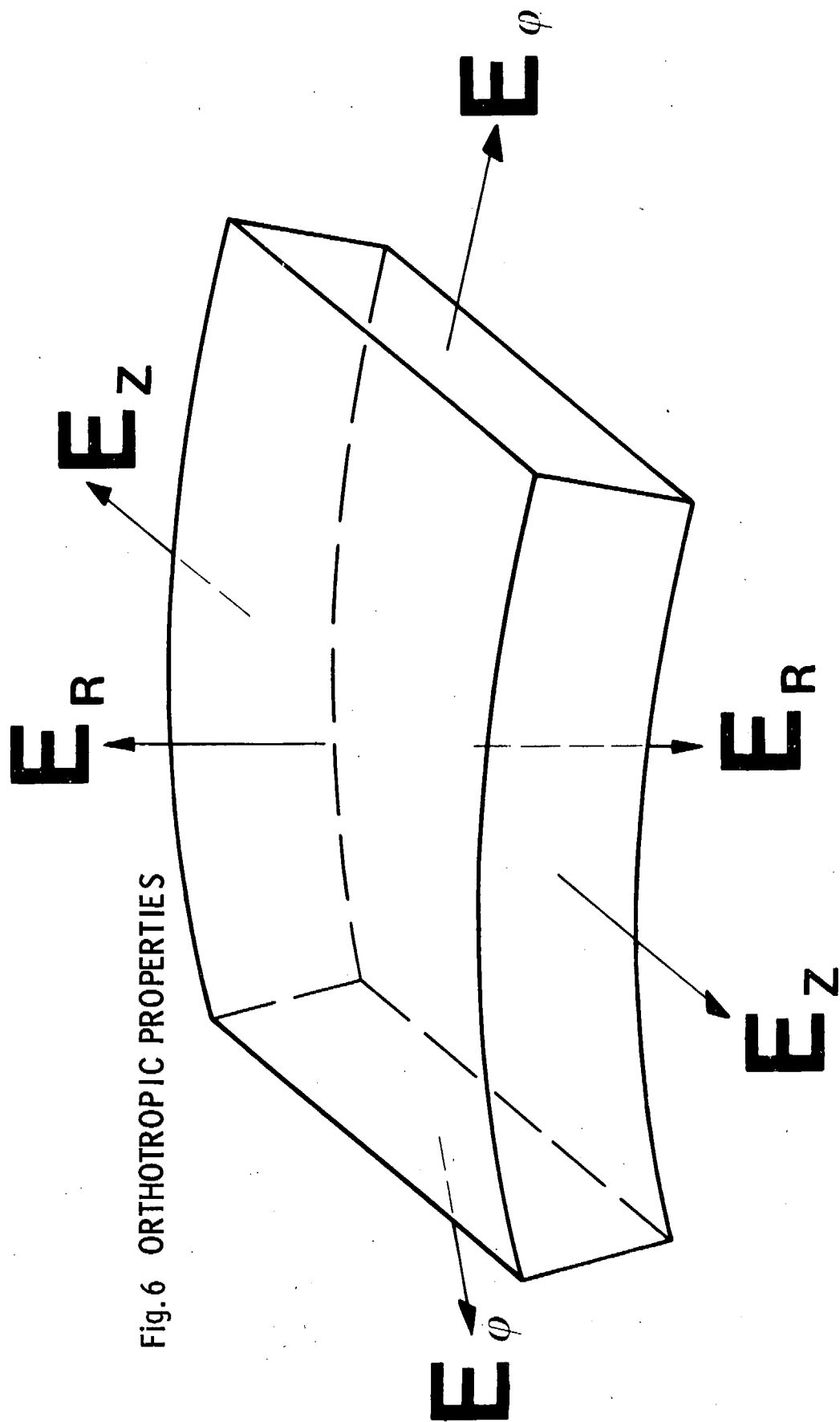


Fig.6 ORTHOTROPIC PROPERTIES

$$\begin{aligned}
 \alpha_\phi = \alpha_Z &= 7 \cdot 10^{-6} \text{K}^{-1} \\
 \alpha_r &= 7 \cdot 10^{-5} \text{K}^{-1} \\
 \nu &= 0.3 \\
 E_\phi = E_Z &= 2500 \text{ daN/mm}^2 \\
 E_r &= 350 \text{ daN/mm}^2 \\
 G_{r\phi} = G_{rZ} = G_{\phi Z} &= 200 \text{ daN/mm}^2
 \end{aligned}$$

are yet unconfined. Other fix points are the outer end points of the trusses, as can be seen in fig.5. Farther restrictions concern the axial displacements of those nodes on the steel ring where the trusses are fixed. This region reaches from 125 to 180 degree. The latter condition is to prevent a lateral yielding of the rather weak trusses as it is prescribed in the reality by the mutual contact of the coils.

3. Examinations

Twelve computer runs were executed. To receive a feeling for the effects of orthotropy to each calculation run of a structure with an orthotropic epoxy layer a parallel run was carried out with an isotropic behaving interlayer. In the isotropic cases we used in deviation to the properties given in chapter 2.3 homogenized properties of copper and resin according to the volume shares and applied them to the two layers of copper and also to the resin layer.

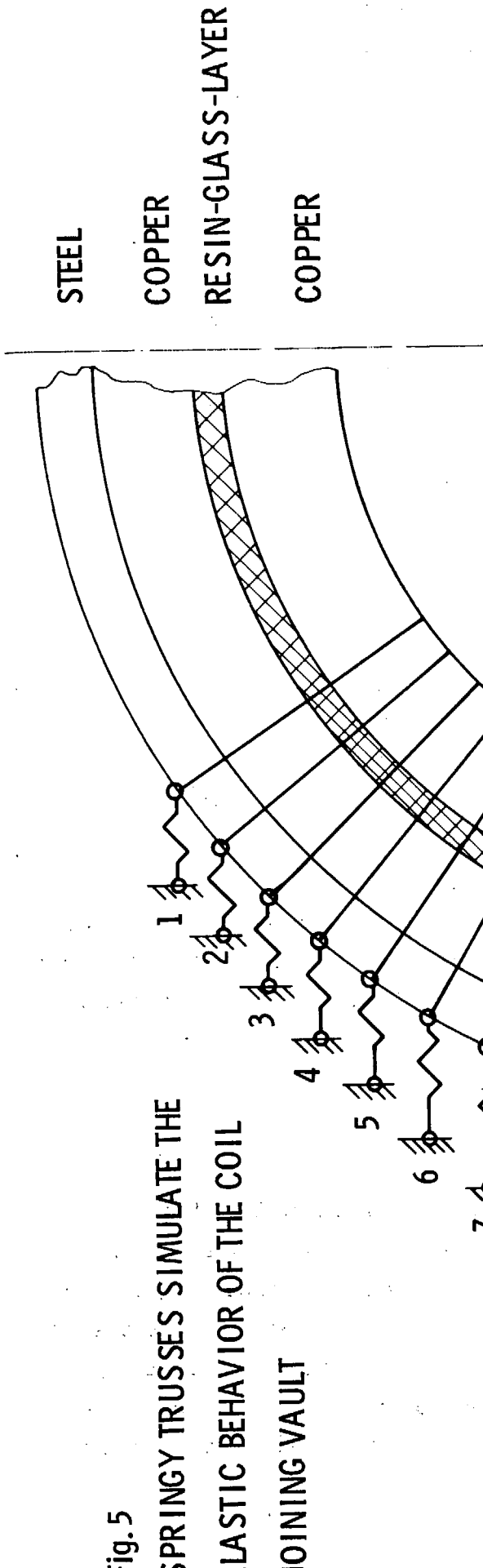
Table 3.

Homogenized properties, isotropic cases

Young's modulus	1100 daN/mm ²
shear modulus	4104 "
Poisson's ratio	0.34
heat expansion	$1.7 \cdot 10^{-5} \text{ K}^{-1}$

Other variations deal with temperature changings. Calculations show, see appendix 1, that the normal heat increase during a shot grows up to 15 K. This will be the overtemperature of the copper over the steel which latter remains cold because of its insulation against the copper. In the case of failing coolant which, strictly speaking, should be excluded all released energy would remain in the coil. This utmost case causes a heat increase of 40 K. A coil surviving this would ever overcome. Leading to the largest tangential positive tensiles in the steel ring and negative tensiles in the copper this worst case was taken into account.

Fig. 5
 SPRINGY TRUSSES SIMULATE THE
 ELASTIC BEHAVIOR OF THE COIL
 JOINING VAULT



Truss no	angle	length in cm	stiffness in daN/cm
1	125°	14.98	1 401 923
2	130	13.95	1 505 769
3	135	14.20	1 478 846
4	140	14.98	1 401 923
5	145	16.37	1 282 692
6	150	18.51	1 134 615
7	155	21.80	963 462
8	160	26.63	788 462
9	165	38.70	623 077
10	170	43.51	482 692
11	175	55.15	380 769
12	180	63.86	328 846

The cross section of all trusses is 10 cm²
 Youngs modulus 21 000 daN/mm²

Additional changings concerned the magnetic loads, i.e. cases with and without plasma, or no magnetic but only thermal load. The examined cases are listed in table 4.

Table 4

Examined cases

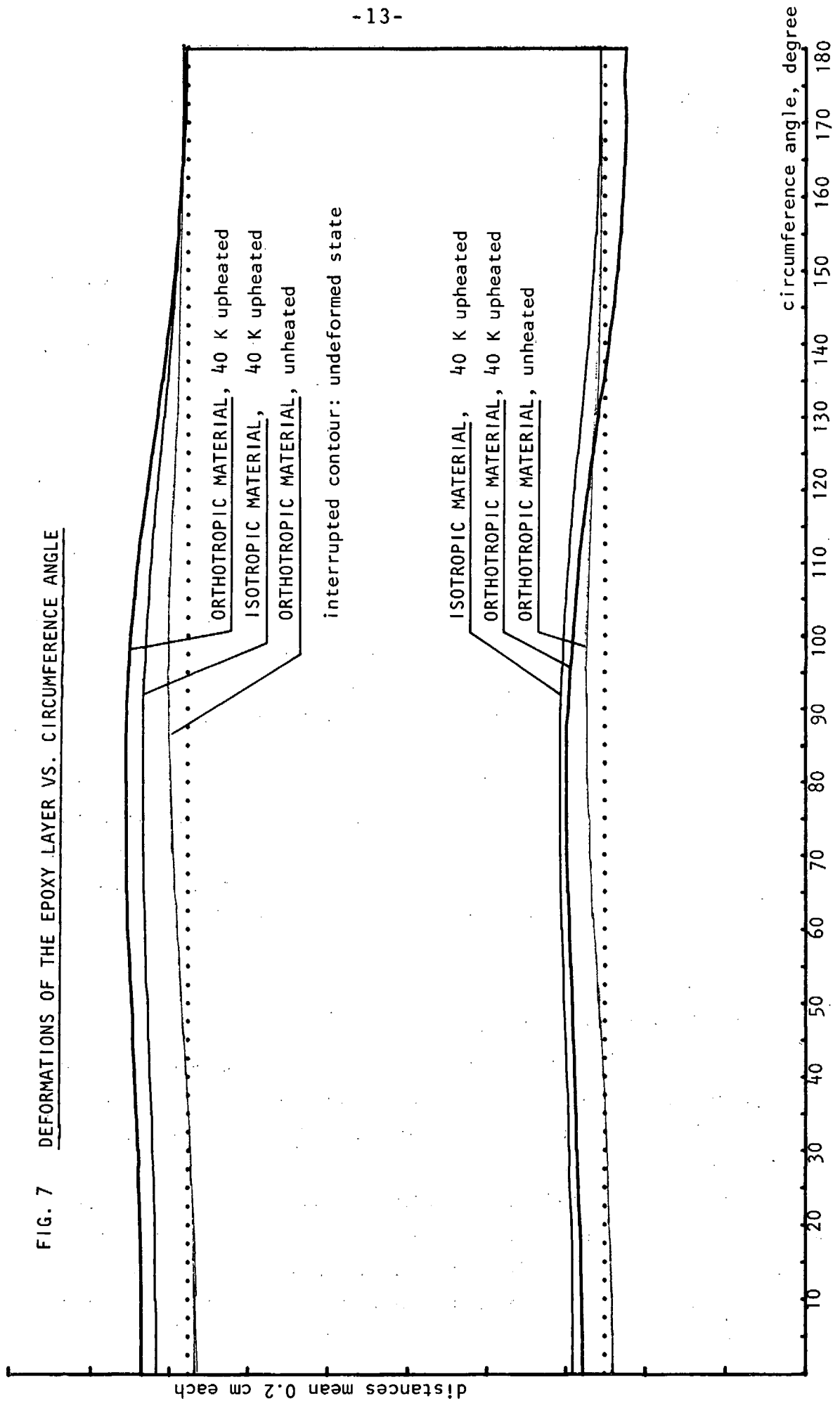
run no.	epoxy behavior	heat increase	plasma current	radial load
1	orthotropic	40 K	0	1
2	"	0	0	1
3	"	40 K	1	1
4	"	0	1	1
5	isotropic	40 K	0	1
6	"	0	0	1
7	"	40 K	1	1
8	"	0	1	1
9	"	40 K	0	1
10	orthotropic	40 K	0	1
11	"	40 K	0	0
12	isotropic	40 K	0	0

The following descriptions yet deal only with the cases 1, 2, 5 and 6, since comparisons of these are of the highest interest. Some look is also given to the samples 11 and 12, where pure thermal loads are assigned.

4. Results

The result lists represent the three displacements of each nodal point, that is in radial, angular and axial direction. In the case of fastened degrees of freedom the charging force in quantity and direction is calculated. As a third information the six basic stresses within each element are printed out, as to say the radial, tangential and axial stress, the three shear stresses and derived from these the reference stress.

FIG. 7 DEFORMATIONS OF THE EPOXY LAYER VS. CIRCUMFERENCE ANGLE



An adapted version of the FEABL /4/ software was applied. First we may turn us towards the deformations.

4.1 Deformations

A first impression of the deformations dependent on isotropy or orthotropy we get from fig.7. Here the radial deformations of the epoxy layer are plotted over the circumference angle. The ordinate is due to the displacements, the layer thickness (3.7 cm) is not in scale. The circumference angle is counted in the mathematical positive sense beginning on the right hand side of fig. 1. Nodal point 1 is at zero degree, point number 383 at 180 degree. The highest deformations appear in the 40 K upheated orthotropic epoxy in the range of low angle coordinates mainly according to the thermal expansion. In the orthotropic case the used radial heat expansion coefficient exceeds the isotropic one by a factor 4.1, compare fig.6 and table 3. The large deflections diminish from about 120° as an influence of the supporting trusses which begin at 125° . According to the magnetic loads the lower contour exhibits even negative radial deformations and goes under the interrupted line representing the undeformed size. Thus the trusses are compressed. The isotropic layer remains thinner, as already pointed out, and the deformations are less. The third contour shows the pattern of the unheated state. The adequate isotropic slope was omitted because it revealed almost no deviation.

4.1.1 Deformations of the steel ring

The steel ring represents the main stiffening member of the coil surpressing too large deformations and transferring the magnetic forces to the vault. Therefore deformations of this ring deserve special attention.

Fig. 8 displays the calculated structural displacements in radial, tangential and axial direction. Each case was examined due to cold and heated (40 K in the copper and the epoxy interlayers) states, and orthotropic respectively isotropic behaving epoxy layers.

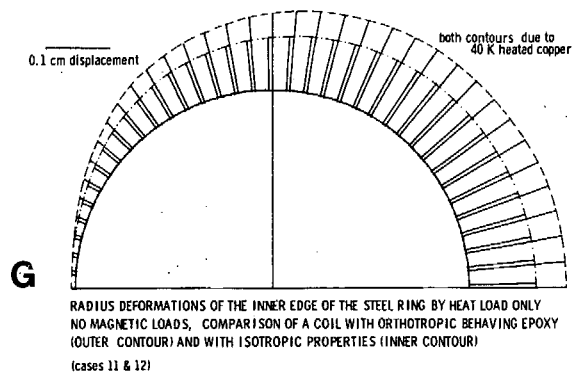
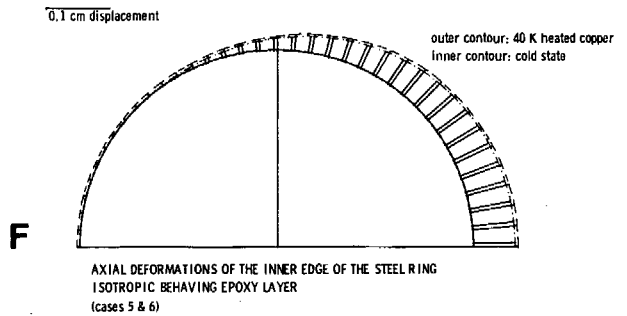
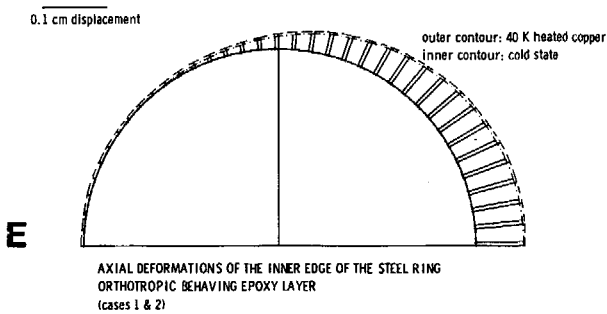
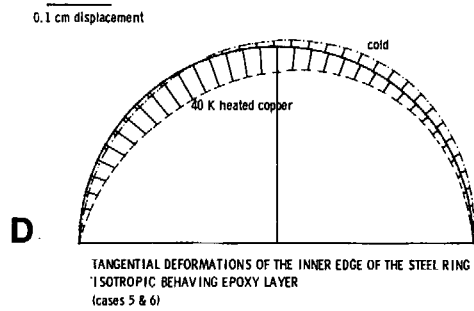
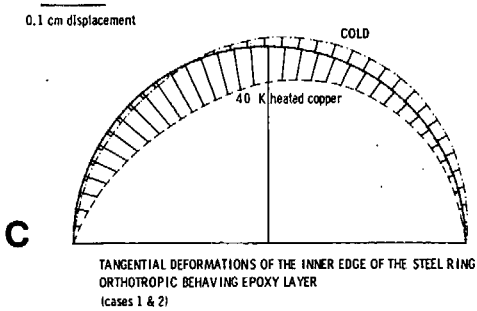
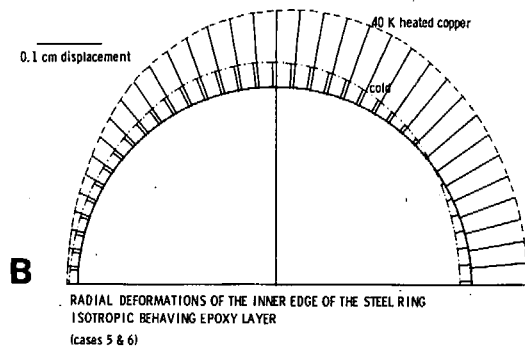
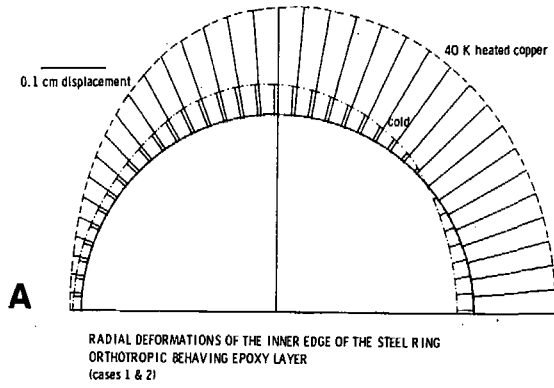


Fig.8
COMPARISON OF DEFORMATION
DISTRIBUTIONS

The largest deflections are to expect in radius direction with orthotropic interlayer and hot state (A). In the following table 5 the locality and the quantity of the maximum deformations are listed. For comparing purposes a figure is added (G) showing radial displacements due to heating only neglecting the magnetic strains. Its pattern shows the uniform expansion of a constantly heated ring fixed with springs in the 180° region.

Table 5

Maximum deformations of the steel ring

direction of deformation	heating state	orthotropic or isotropy	locality in degree	amount in cm
radial	cold	ortho	95	0.0474
radial	cold	iso	95	0.0381
radial	hot	ortho	75	0.1685
radial	hot	iso	70	0.121
tangential	cold	ortho	55	0.0215
tangential	cold	iso	55	0.016
tangential	hot	ortho	120	-0.07
tangential	hot	iso	120	-0.048
axial	cold	ortho	0	0.074
axial	cold	iso	0	0.066
axial	hot	ortho	0	0.075
axial	hot	iso	5	0.074

We recognize that orthotropic material always effects larger deflections.

The maximum radial deflections occur in the cold state at an angle of about 90° displaying the normal behavior of a ring fixed at 180° and mainly pinched from the opposite side. We find appropriate squeezings going forth with negative radial deformations in the angle region from 0° to 55° , see the inner contour of fig. 8A. With isotropic material we observe a similar deformation contour, fig. 8B, with smaller deflections.

The heated case in fig. 8A, outer contour, exhibits nothing but a deflection superposition of the appropriate cold state where the deformations base only on magnetic loads, and the heated state without magnetic forces, fig. 8G, outer contour, as it must be according to the linear theory. This is true in all cases.

The tangential deformations, orthotropic case, cold state, fig. 8C, show in an angle region from 115° to 180° negative values, i.e. the material is drawn along the circumference in the negative angle direction, clockwise sense, according to the large radius deflections in the 90° area. From the other side, from the region of the smaller angles, the material is also drawn towards this radius deflection belly, anticlockwise sense, and therefore shows positive angle deflections. The hot state without loads cannot be seen but can be imagined by a symmetrical half-moon contour because the angle deformations have to vanish at 0° and 180° according to the boundary conditions. The displayed contour of the heated state again grows out of the superposition of the cold state and the sickle contour.

The axial deflections are displayed in fig. 8 E & F. The deflections are the larger the larger the distance is from the fixing points. This is the appropriate deformation of a ring fixed on one side and laterally loaded on the other side. The lateral forces are caused by the absention of the plasma. The small differences between cold and heated state have their cause in the rather small axial extension (32.4 cm) of the ring. It may be of interest to give a list of the ratios of the maximum displacements, orthotropic over isotropic case, see table 6.

Table 6

Ratios of maximum displacements, orthotropic over isotropic case

direction of deformation	heating state	<u>orthotropic deformation</u> <u>isotropic deformation</u>	at angle
radial	cold	1.244	95
radial	hot	1.393	75
tangential	cold	1.343	55
tangential	hot	1.458	120
axial	cold	1.121	0
axial	hot	1.014	0

The highest influence of orthotropy vs. isotropy due to deformations is in the tangential direction. In the axial direction it is rather small.

4.1.2 Cross section deformations

We now may look upon deformations of a cross section and choose a section situated at 90° . The appropriate limiting point numbers can be seen in the figures 9 to 12, the concerned elements carry the numbers (from the inside) 18, 54, 90 and 126. In the captions all deformation quantities are pointwise listed. In the first figure of this serie, no.9, the undeformed cross section is marked by hachures to emphasize the different layers. The deformed cross section, heated case and orthotropic inter-layer, is once drawn in scale scarcely to distinguish from the undeformed state and further in a state where the deformations are 50-fold amplified. We may use the latter for better judging. The structure is shifted to the right in positive z-direction according to the forces resulting in the failing plasma and outwards in positive r-direction. The constricted waist caused by the epoxy is remarkable. Comparing it with the adequate but isotropic case (here now the deformed cross section is characterized by the hachures) we recognize in fig. 10 the smaller deformations especially in r-direction according to the higher stiffness of the iso-

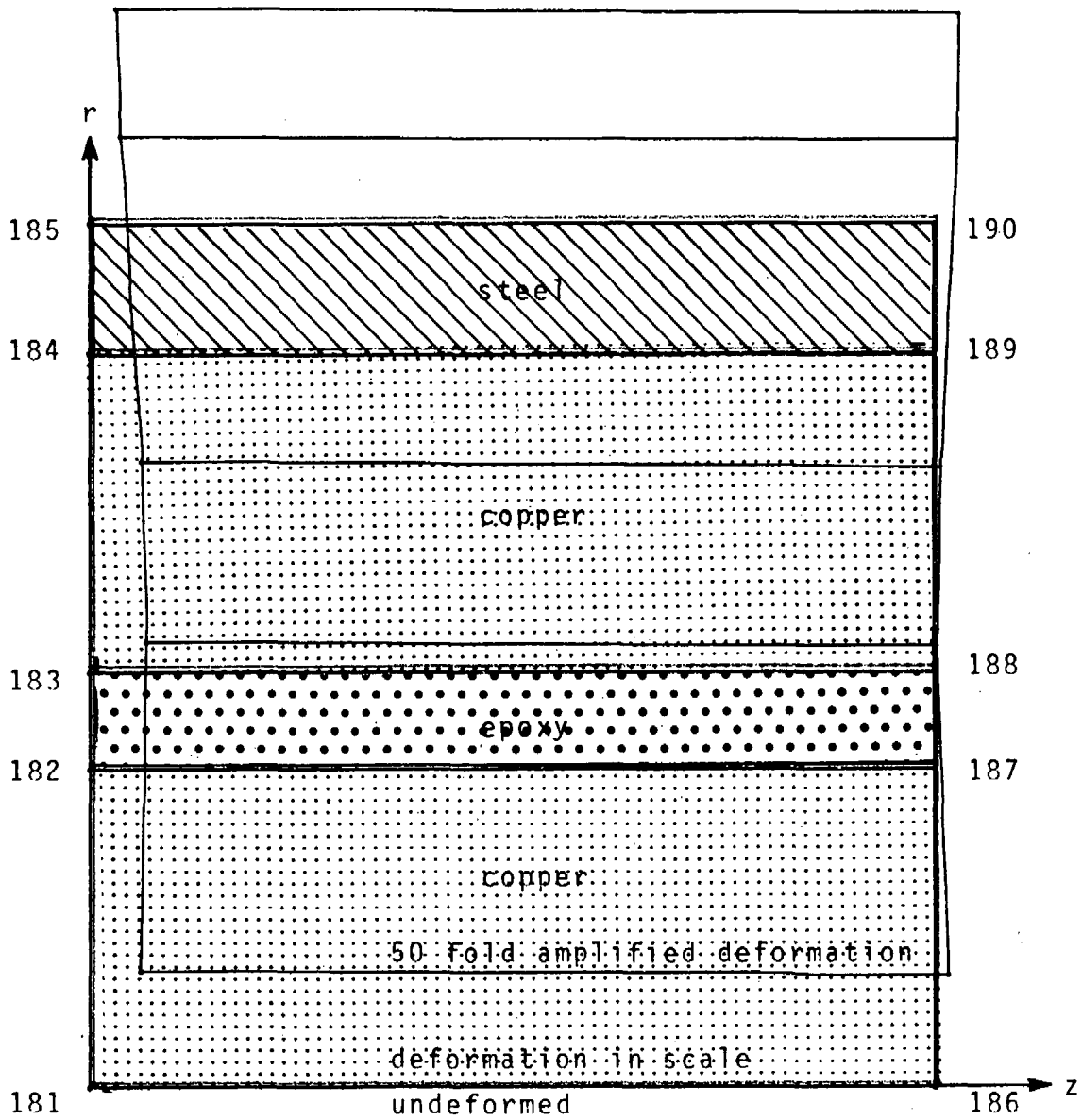


Fig. 9

Deformations of a coil cross section with orthotropic epoxy layer. Heat increase 40 K, no plasma, radial magnetic loads as usual. The section is sited at 90 deg. (Case 1)

Deformations in cm					
Node nr.	r-deform.	z-def.	Node nr.	r-deform.	z-def.
181	0.087	0.037	186	0.083	0.0088
182	0.095	0.042	187	0.091	0.001
183	0.159	0.038	188	0.154	0.0043
184	0.165	0.022	189	0.161	0.017
185	0.163	0.019	190	0.158	0.018

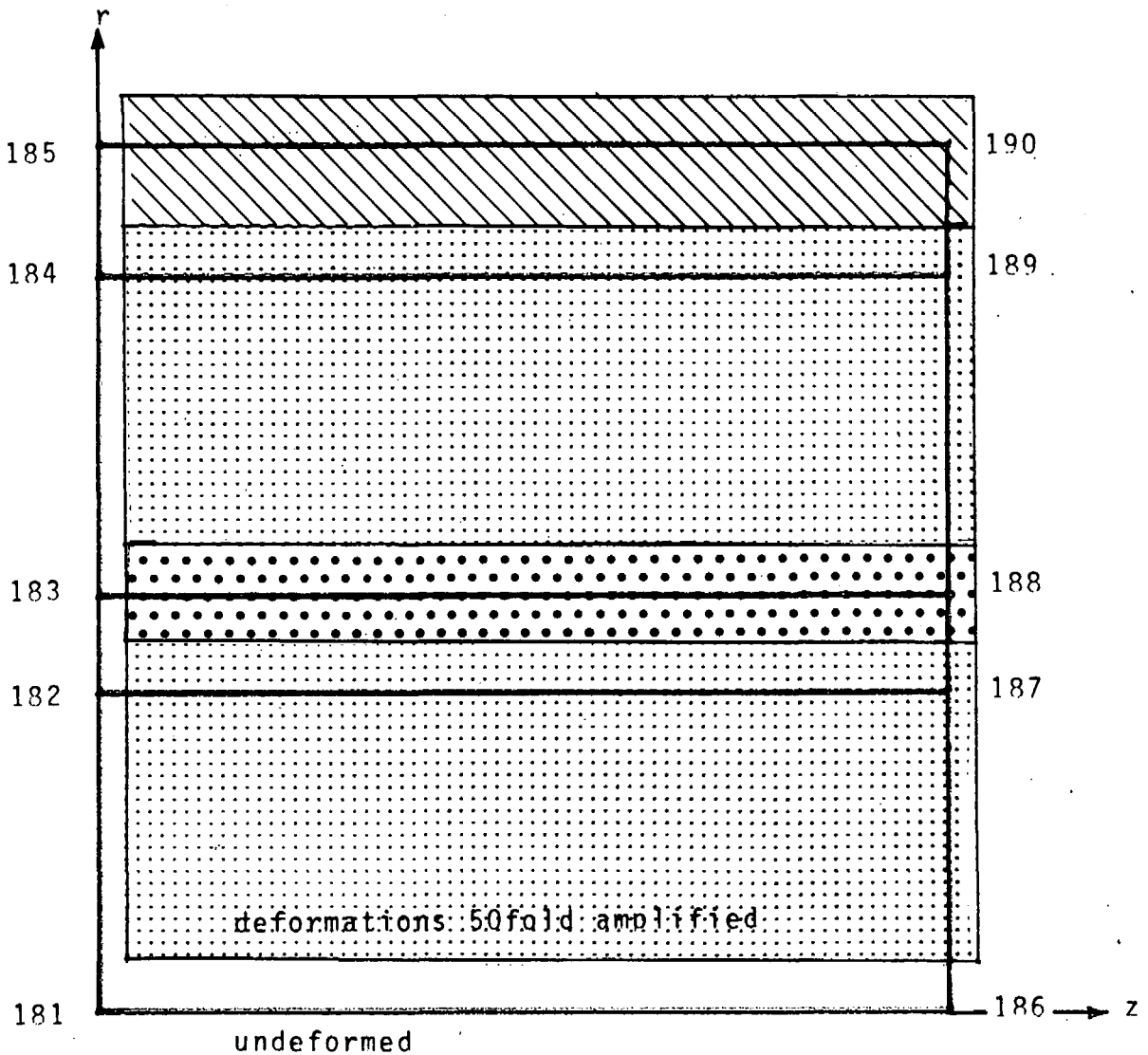


Fig.10

Deformations of a coil cross section with isotropic epoxy layer. Heat increase 40 K, no plasma, usual magnetic loads. Site: 90 deg.

(Case6) Deformations in cm

node nr	r	z	node nr	r	z
181	0.039	0.02	186	0.036	0.021
182	0.039	0.02	187	0.036	0.02
183	0.038	0.019	188	0.036	0.02
184	0.038	0.018	189	0.035	0.019
185	0.037	0.017	190	0.035	0.019

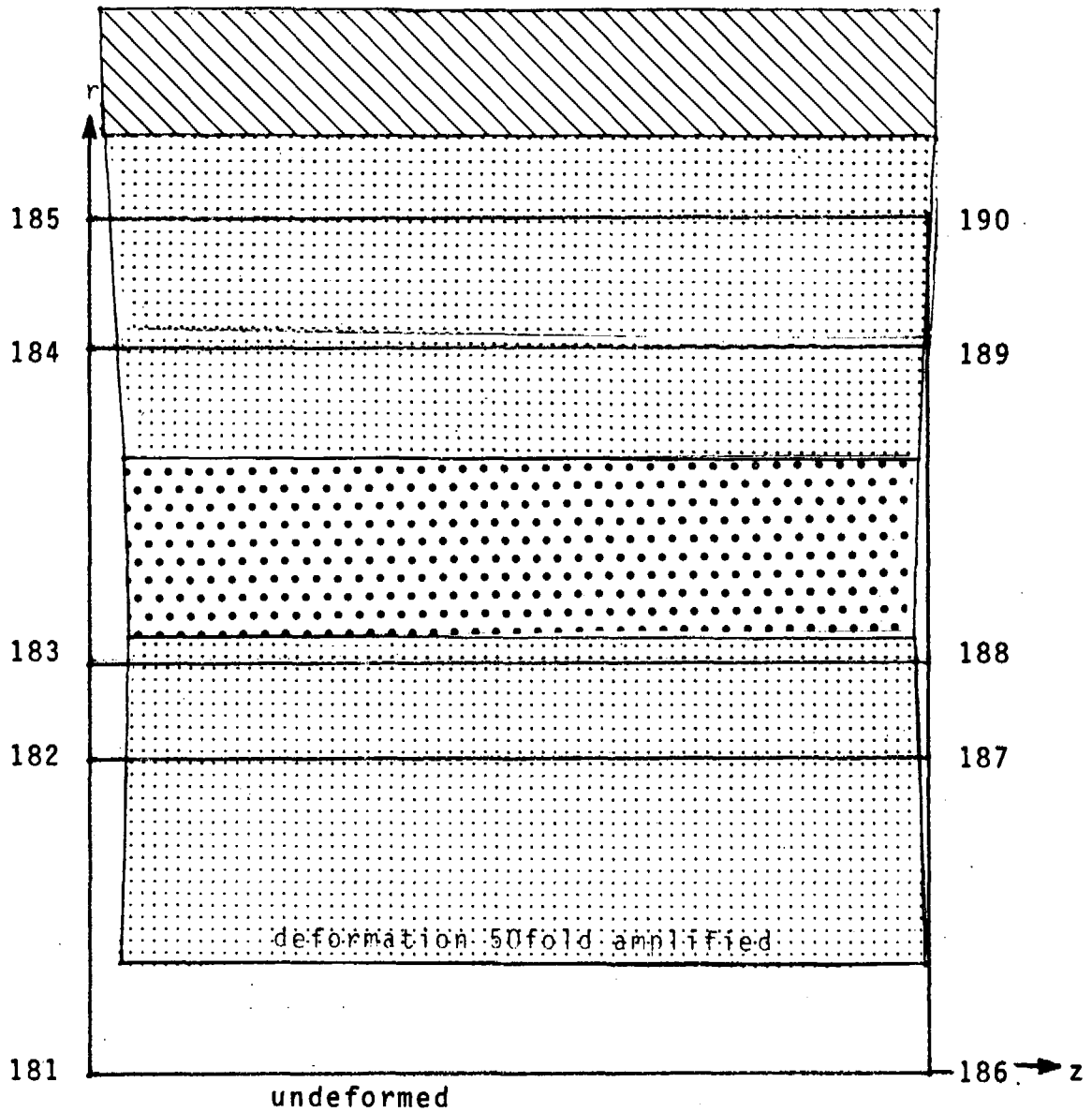


Fig. 11

Deformations of a coil cross section with orthotropic epoxy layer. Heat increase 40 K, operating plasma, usual magnetic loads. Site: 90 deg.

(Case 3) Deformations in cm

node nr	r	z	node nr	r	z
181	0.087	0.025	186	0.084	-0.0027
182	0.094	0.031	187	0.092	-0.01
183	0.158	0.027	188	0.156	-0.0067
184	0.165	0.012	189	0.162	0.0064
185	0.162	0.009	190	0.159	0.0083

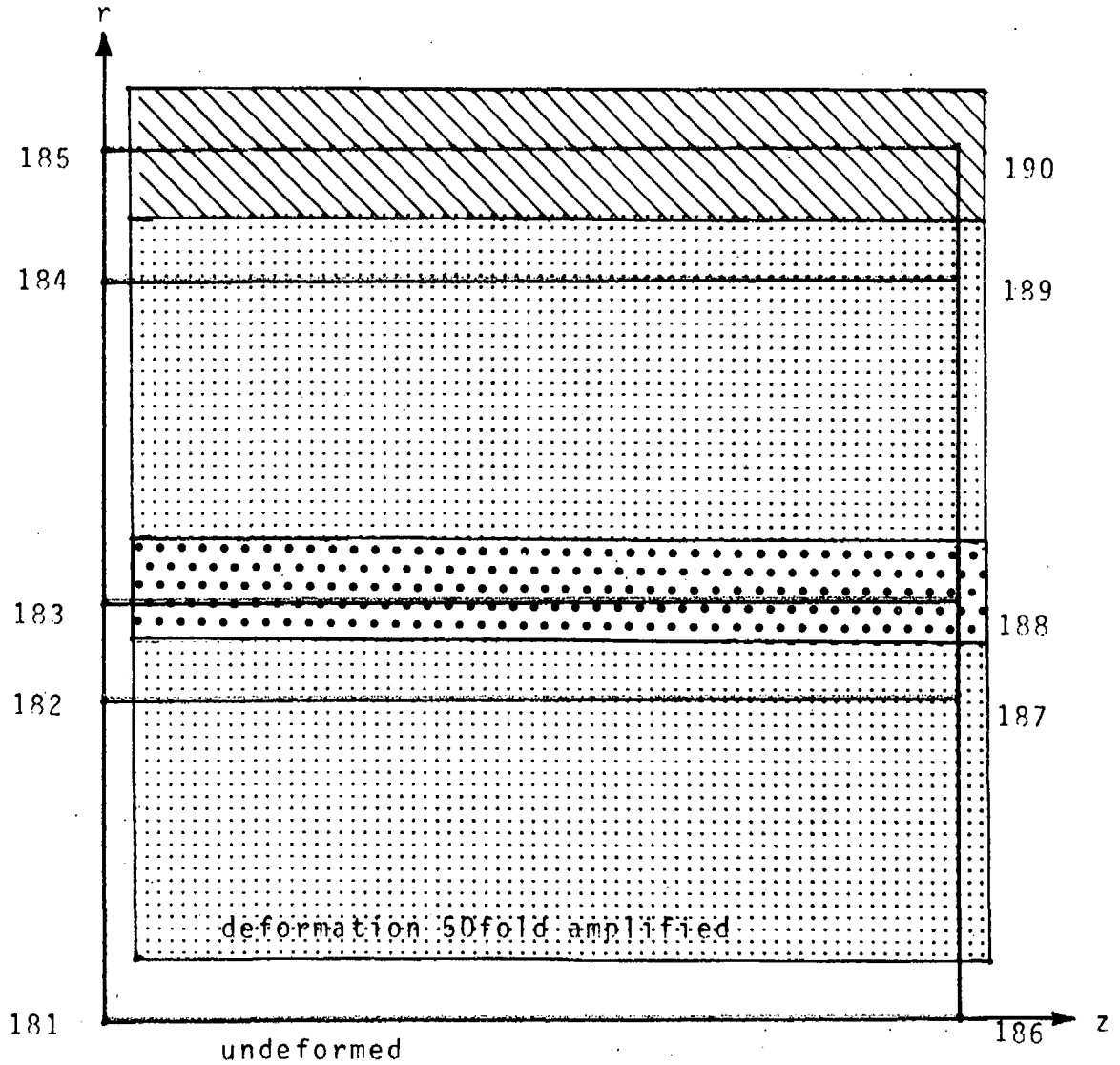


Fig.12 Deformations of a coil cross section with orthotropic epoxy layer. No heat increase, no plasma, usual radial magnetic loads. Site: 90 deg

(Case 2) Deformations in cm

Node nr.	r-deform.	z-def.	node nr.	r-deform.	z-def.
181	0.046	0.023	186	0.042	0.023
182	0.045	0.021	187	0.041	0.022
183	0.047	0.021	188	0.043	0.021
184	0.047	0.019	189	0.042	0.020
185	0.046	0.018	190	0.042	0.020

tropic material. Also the waist constriction has vanished as was to expect.

Fig. 11 treats the case of small axial forces due to normally operating plasma. The orthotropic and heated coil moves radially in the same way as in fig.9 is shown but the axial, the z-deformation, is appropriate less.

The waist constriction only happens if the coil is in the hot state. That is revealed by fig. 12 where the cross section, orthotropic interlayer, cold state, exhibits quite straight sidewalls.

4.2 Stresses

In all cases according to table 4 seven stresses as usually were calculated. However, the results displayed that of the three shear stresses only the one in the radius angle ($r\phi$) direction was of a considerable quantity. Therefore only this one is taken into account in the following. Special regard to this shear stress is inevitable for there exists a critical value of 0.5 daN/mm^2 maximum shear stress within the epoxy layer which should not be exceeded unless the coil lifetime would be restricted.

In the result displays the reference stresses are omitted, too, because they depend in an overwhelming manner on the tangential stresses.

4.2.1 Stress distributions of the cold state

Let us turn towards the comparisons of the radial, tangential, axial and r-phi stress all in the cold state between the cases of orthotropic and isotropic interlayers. The results are manifested in fig. 13.

No differences to speak about appear in the radial stress distributions, cases A respectively B. The influence of the fixing area can be seen in the angle region from 125° to 180° but it does not concern the inner copper layer which is protected by the rather weak epoxy layer.

The figures C and D deal with the tangential stresses. The differences between the orthotropic (C) and the isotropic case

(d) are not so large as the first impression effects because of different scales in the ordinates as to see in the quantities:

steel ring, element no.126
 angle: between 90 and 95°
 tangential stress, orthotropic case 3.68 daN/mm²
 " " , isotropic " 3.49 " .

The oscillating pattern of the stress in the epoxy layer is a consequence of the noncoherency of the Poisson's ratios. A detailed discussion of this problem is done in the appendix 3, p. 75.

The curious slope of the tangential stress in the steel needs some remarks. According to the stress tensor the tangential stress is built together out of shares coming from the three displacements in radial, tangential and axial direction, u , v and w

$$\sigma_{\phi} = G \left\{ A \frac{\partial u}{\partial r} + B \left(\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \phi} \right) + A \frac{\partial w}{\partial z} \right\} ,$$

whereby

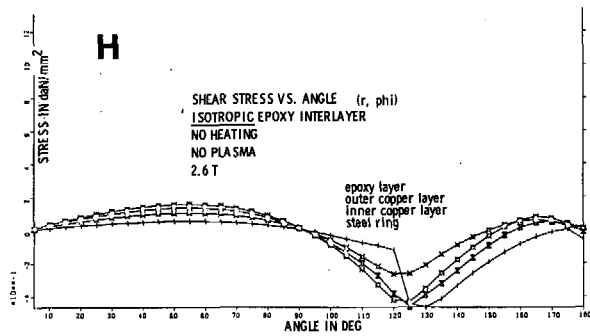
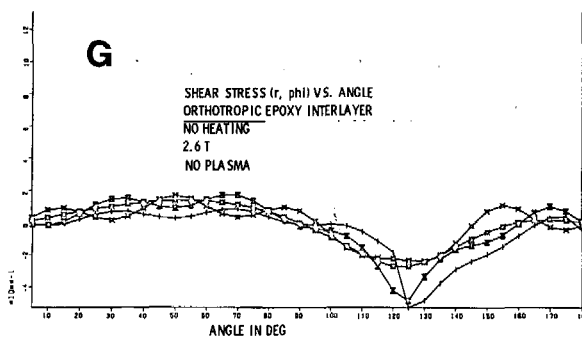
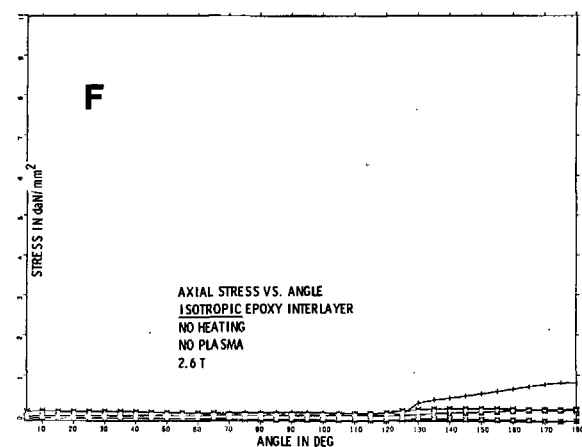
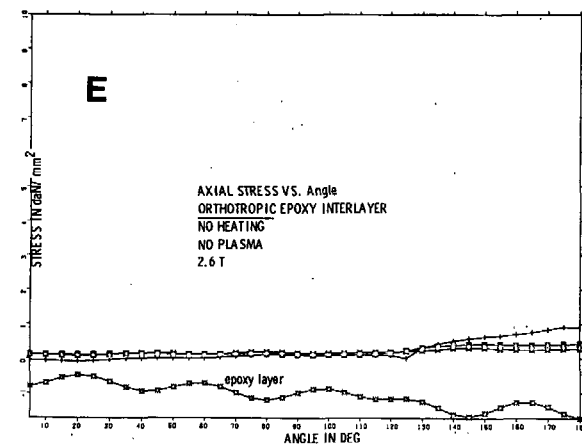
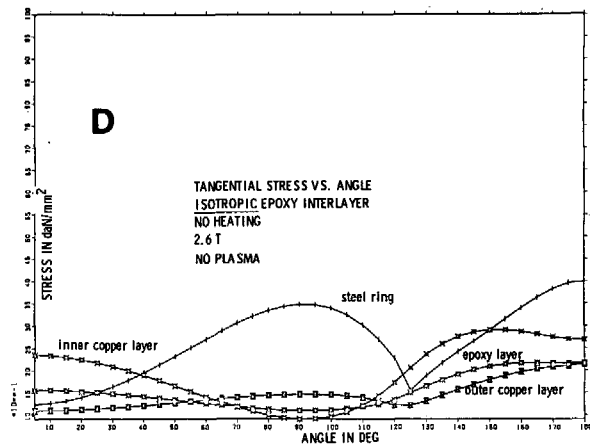
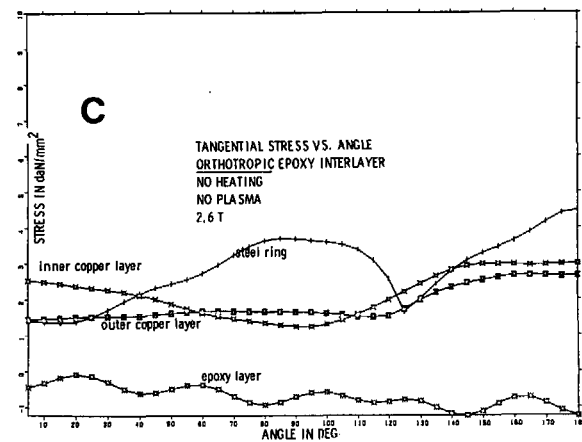
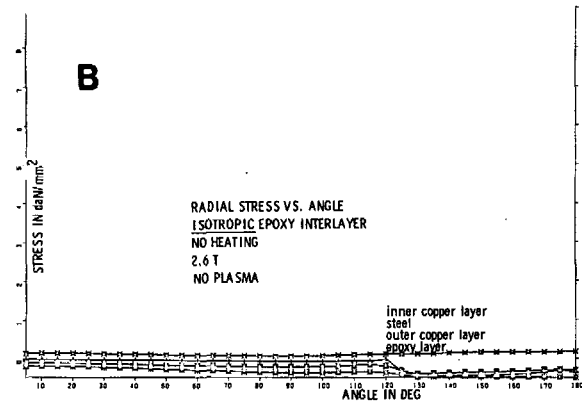
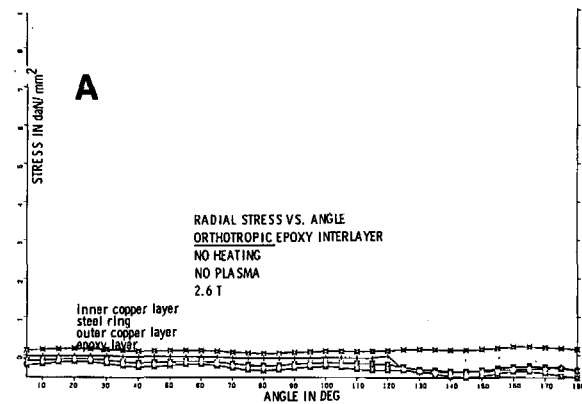
$$A = \frac{2\nu}{1-2\nu} ; \quad B = \frac{2(1-\nu)}{1-2\nu} \quad (\text{isotropy})$$

$$G = \frac{E}{2(1+\nu)} .$$

The derivations of the displacements can be realized from fig.8 where it can be seen whether a derivation of a displacement is zero, that is at the localities of maxima and minima, compare table 5, or steep, or flat. This somewhat complicated correlation darkens the situation and does not permit a quick judgement starting from the displacement slopes only.

The tangential stress maximum in the area of about 90° is mainly a consequence of the local large radial deformation u/r , whereas the main weight in the stress maximum at 180° has its origin in the slope of the derivation $\partial u/\partial r$.

The minimum at 125° has obviously its reason in the influence of the vault which begins here with a rather hard stiffness, see fig.5.



ORTHOTROPIC **ISOTROPIC EPOXY LAYER**
COMPARISON OF STRESS DISTRIBUTIONS, COLD STATE

FIG 13

In the case of the axial stresses there occur different behaviors only in the epoxy layer, E & F, which show negative values in the orthotropic case. The steel ring exhibits increasing stresses in the fixing area whereas the copper layers show no influence.

The shear stresses, here only the $r-\phi$ stresses, depend on tangential, v , and radial, u , deformations

$$\sigma_{r\phi} = G \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \phi} \right).$$

The maximum values which are expected in the epoxy layer are the following. We also include the heated state for better comparing reasons.

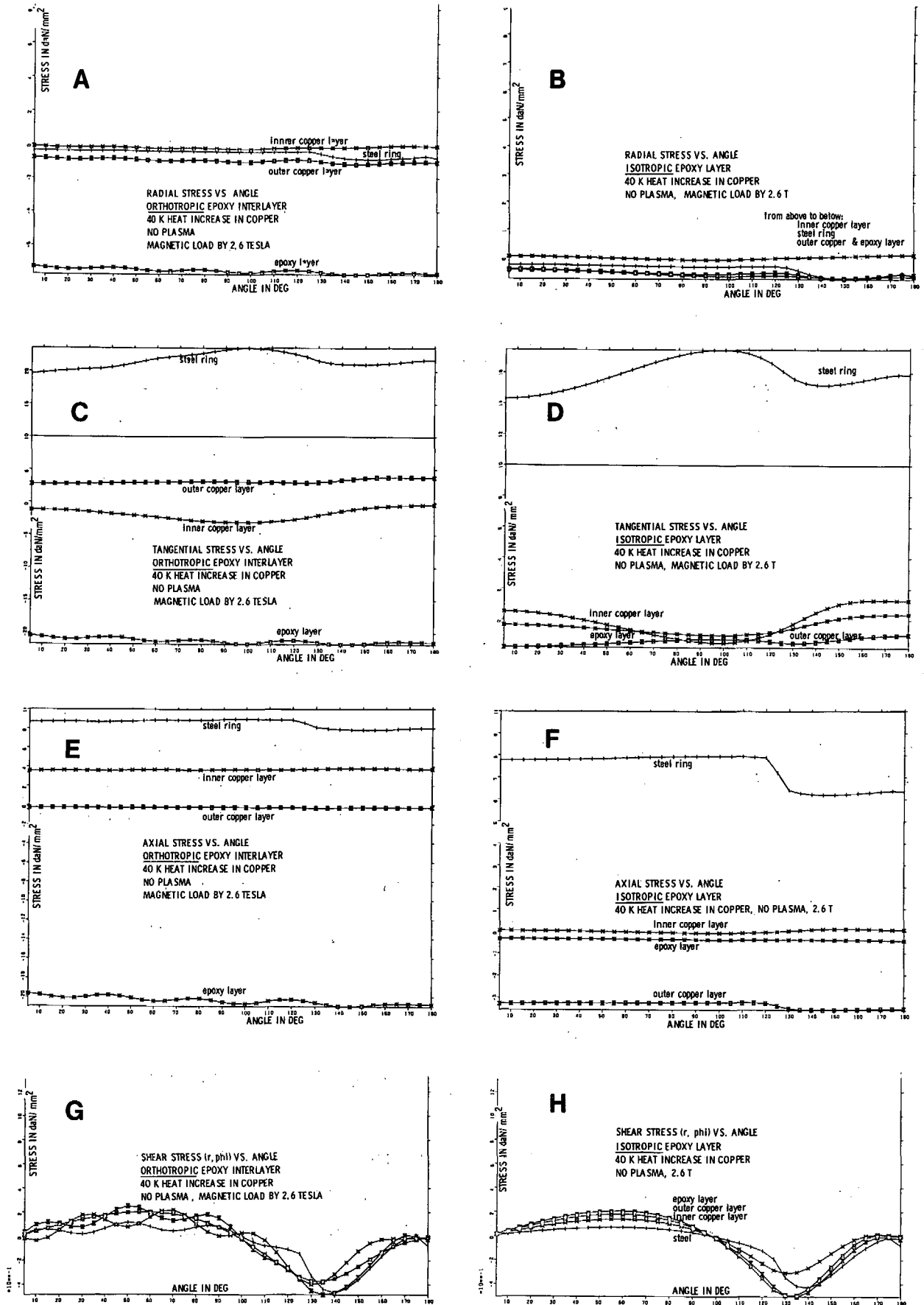
Table 7

Maximum shear stresses in the epoxy layer, daN/mm²

case	material	temperature	maximum	locality	minimum	locality
2	orthotropic	cold	0.1396	55°	-0.27	125°
6	isotropic	cold	0.1582	55°	-0.4307	125°
1	orthotropic	hot	0.2005	60°	-0.3755	135°
5	isotropic	hot	0.2084	60°	-0.4912	130°

We learn that the minima have higher absolute quantities, and that the orthotropy softens the relationships. The influence of the heat is not very severe and not even in the worst case, no.5, the critical limit of 0.5 daN/mm^2 is exceeded. The normal operating state will be case 2 and we enjoy a safety factor of nearly 2.

In the fig. 13 G & H also as in all the other figures the shear stress pattern in the epoxy layer is represented by the starry contour (*).



ORTHOTROPIC **ISOTROPIC EPOXY LAYER**
FIG 14 **COMPARISON OF STRESS DISTRIBUTIONS, UPHEATED STATE**

4.2.2 Stress distributions of the hot state

Looking at the radial stresses, see fig. 14 A & B, we observe a considerable difference due to the epoxy layer between orthotropic (A) and isotropic (B) material. In the orthotropic case the layer is compressed to a negativestress of about -8 daN/mm^2 , whereas the isotropic layer shows almost no difference to the appropriate two cold states, fig. 13 A & B. The metal layers are scarcely afflicted by this deviation.

The hardest influence of the heating is exercised on the tangential stresses, C & D. In the case of the orthotropic epoxy layer the stress in the steel ring jumps up to a maximum of 20 daN/mm^2 because it is strained by the expanding hot copper. This stress value has been the reason for dimensioning the steel ring to 5 cm thickness. In the orthotropic case the stresses are even higher than in the case of isotropic material. The distinct extrema from the cold state are here softened. On the contrary to the steel the epoxy layer suffers from large negative stresses which of course do not occur in the isotropic case.

The axial stresses, E & F, exhibit some similarity to the foregoing ones. The height of the compression stress in the epoxy layer in the orthotropic case of even more than -20 daN/mm^2 is remarkable.

The results concerning the shear stresses are yet listed in table 7. The interpretation is already carried out in the foregoing chapter.

4.2.3 Summarized knowledges

The perceptions we gain from the stress calculations may now be summarized.

1. The shear stresses which had to be cautiously observed remain in all cases low enough with a sufficient distance to the critical value. The orthotropic case is even less severe.
2. The hot state manifests rather high tangential and axial stresses within the epoxy layer.

4.2.4 Comparisons with former results

It may be of interest to compare the just introduced results with former ones. Fig. 15 shows the tangential stresses in a coil model consisting of three layers, one of steel and two of copper-resin mixture with isotropical properties. This coil was considered to be fixed by rigid points in the hachured area. This led to the displayed stress peaks in that region. These peaks do no longer appear in such a coined matter in the new stress patterns, fig. 14 C & D, where the elasticity of the supporting vault is involved especially in the case of warm copper layers, fig. 13 C & D.

4.3 Force transfer to the central vault

The centripetal forces impress the central vault, see fig.4. They operate only in purely horizontal direction. The vertical component of the resultant effects merely the coil itself in the horizontal cross sections.

The horizontal centripetal forces are transferred in our model by trusses of an elasticity analogous to the real amount of the supporting components. Though we know the whole quantity of the load which is to conduct by all trusses, 173.5 tons, we furthermore ought to know the spatial force distribution over all trusses. This is of importance for the design.

The simplest load case is the one of a cold coil with isotropic interlayer which is only strained by radial forces, see fig.16. Because of the only radial load the backwards arranged 12 trusses bear forces equal to the 12 trusses supporting the 12 nodal points of the front edge. The sum of all the single forces, of course, should be of the amount equal to the applied magnetic forces. For reasons of this test this sum is printed out in the figures, too. The small deviations are according to a truncation error.

We perceive that the larger forces are transferred in the upper region, the region of the stiffer trusses.

In fig. 17 the force distributions are displayed with additional regard to orthotropic interlayers, heat adding and axial forces. In all considered cases the trusses of the frontal edge are

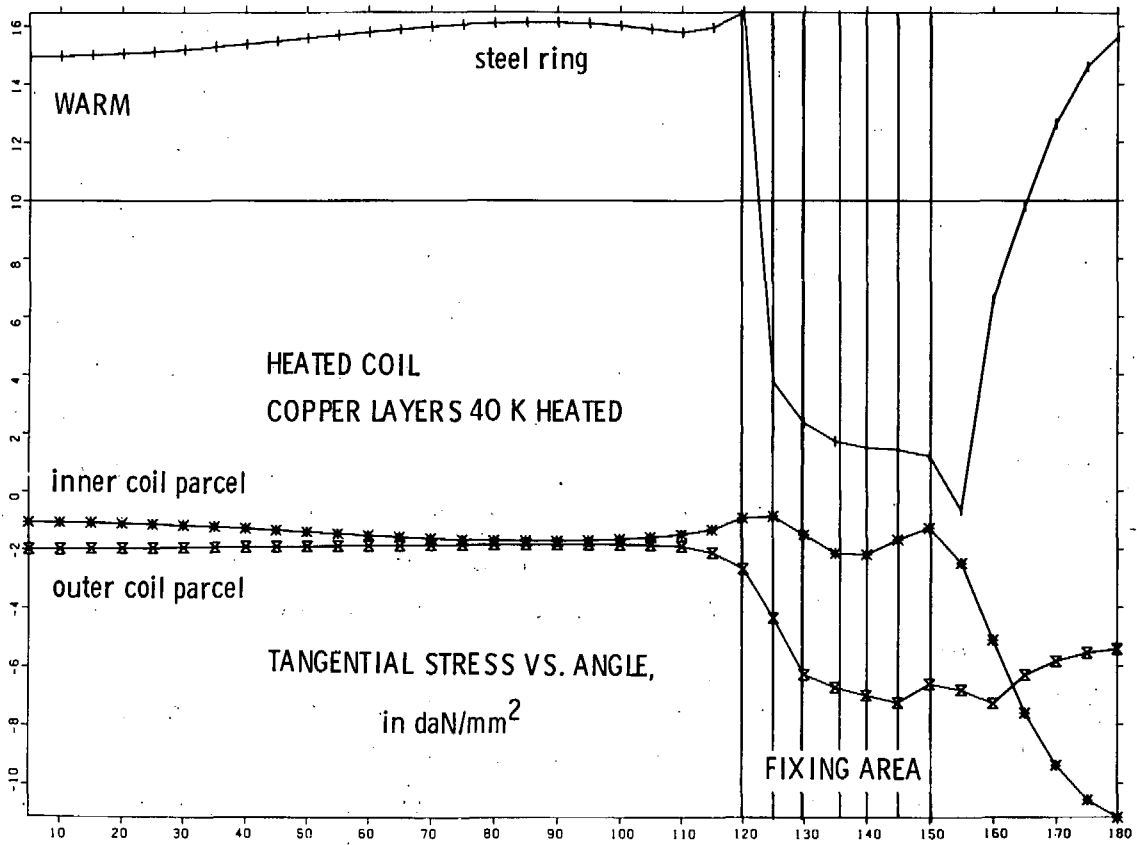
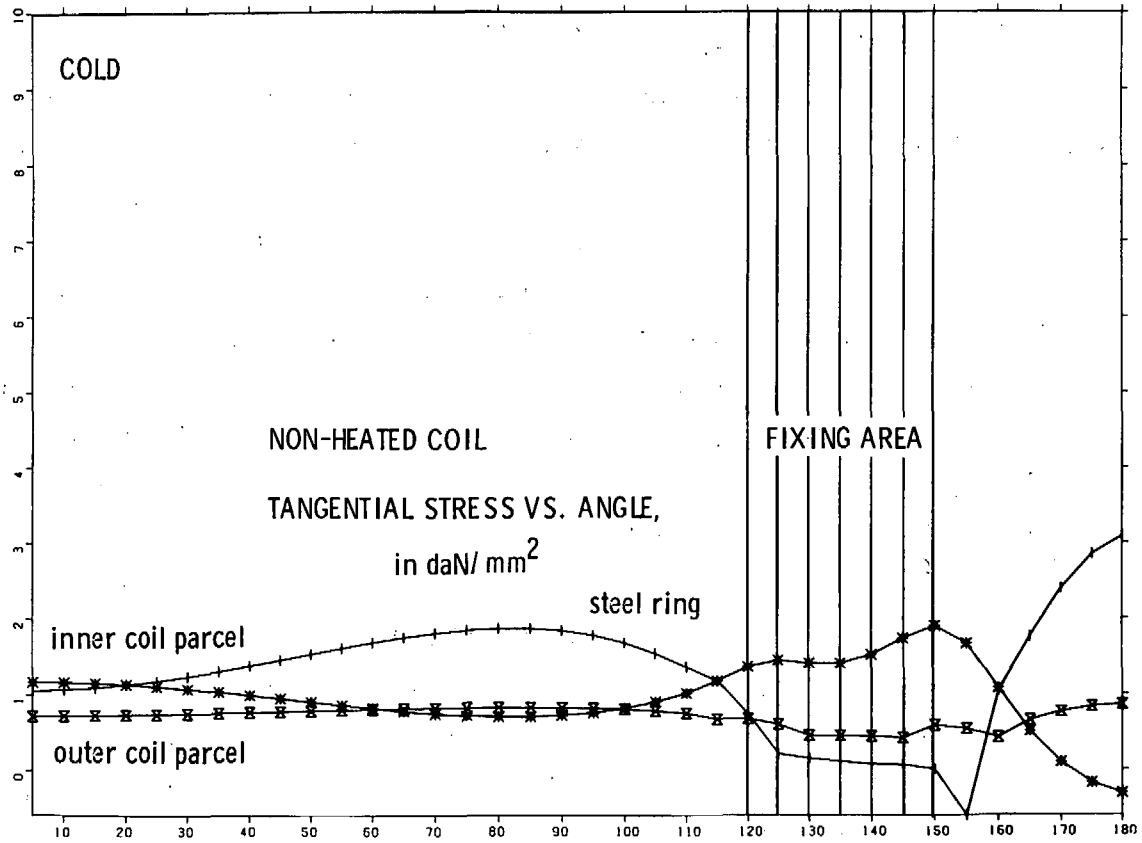


FIG. 15 TANGENTIAL STRESS DISTRIBUTIONS FROM A 3-LAYER ISOTROPIC MODEL WITH RIGID SUPPORTING POINTS

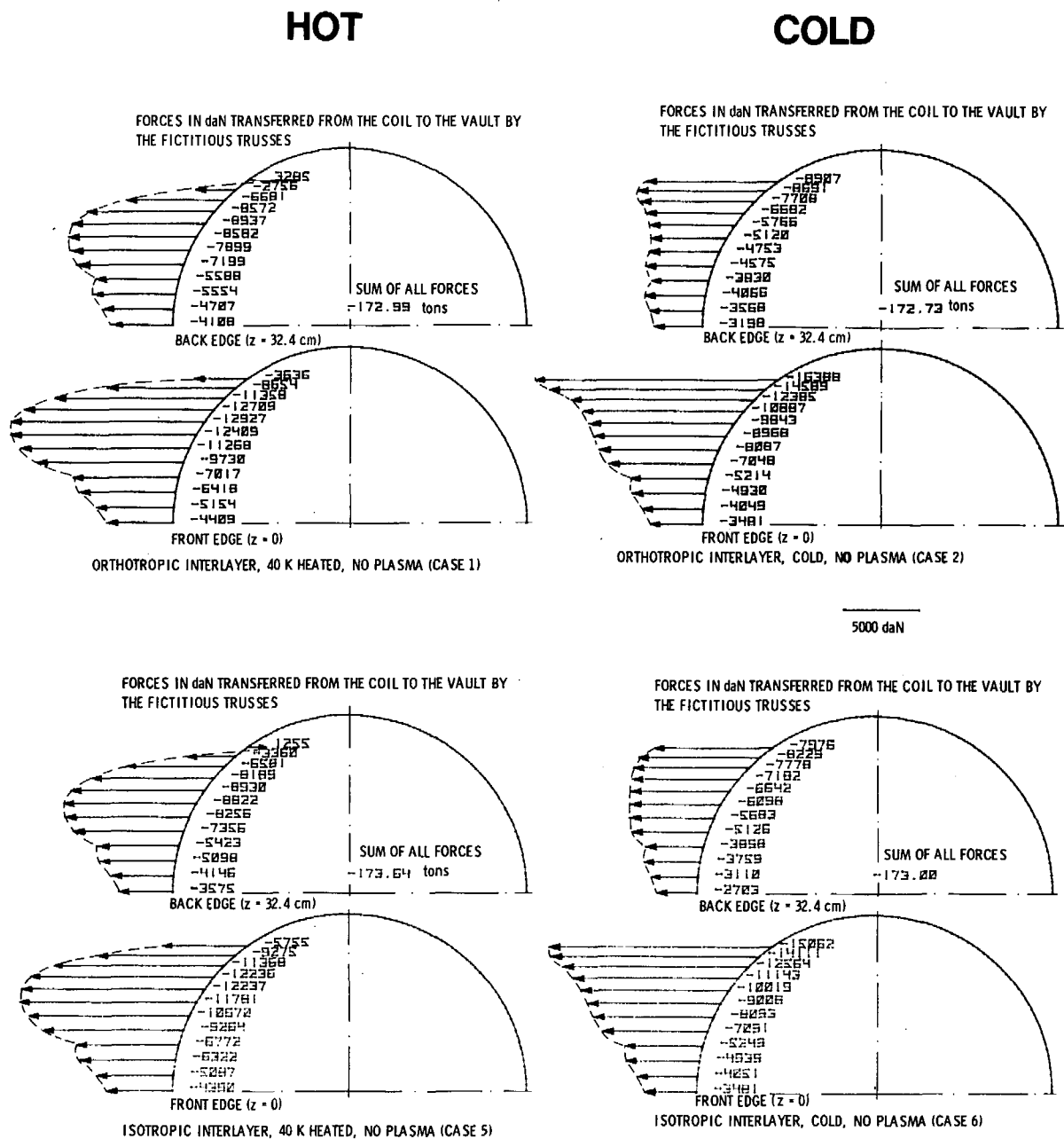
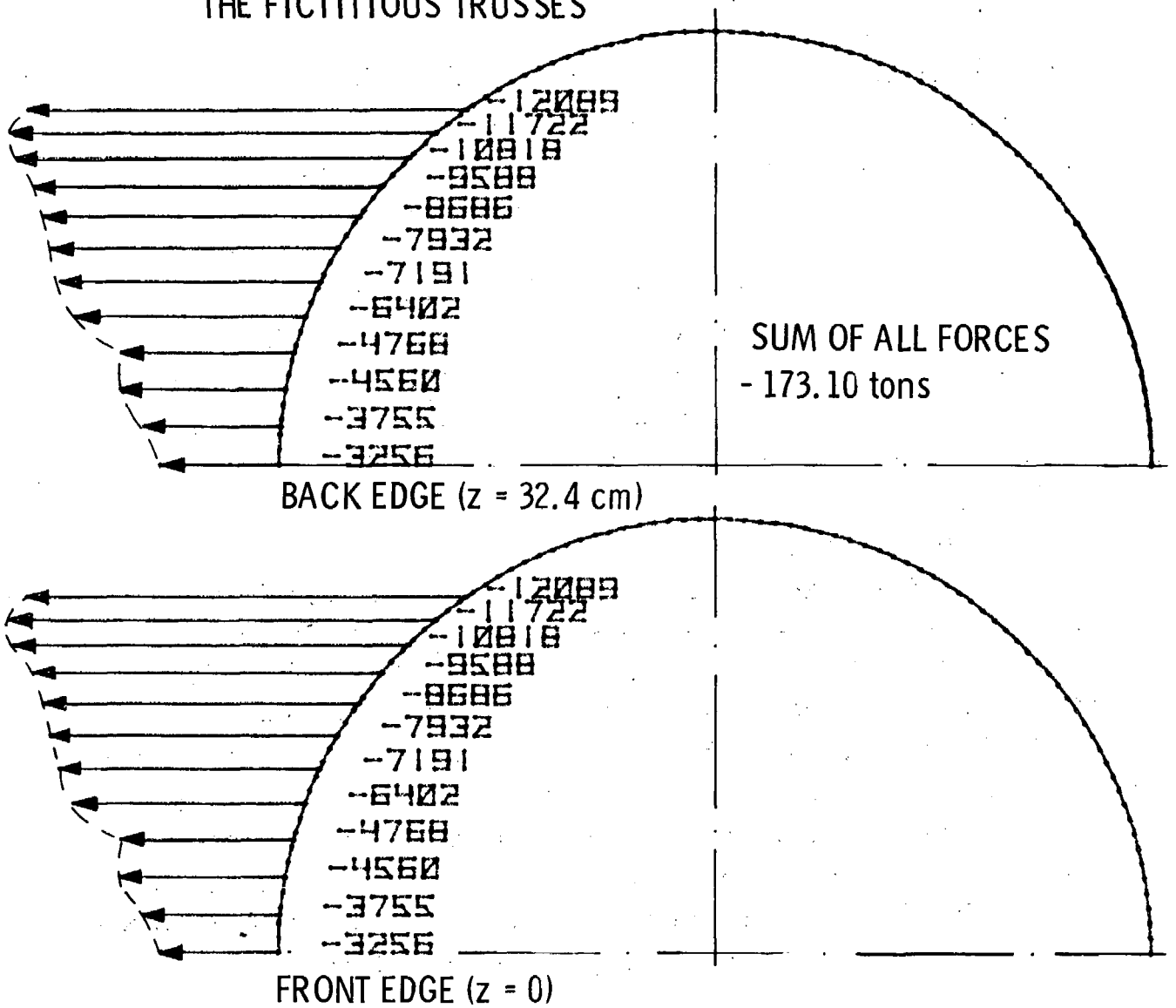


FIG.17 FORCE TRANSFER FROM THE COIL TO THE CENTRAL VAULT

FIG. 16 FORCES IN daN TRANSFERRED FROM THE COIL TO THE VAULT BY THE FICTITIOUS TRUSSES



ISOTROPIC INTERLAYER, COLD, NO AXIAL FORCES

higher strained than those of the back edge. While the force distribution in the cold state shows fundamentally a similar behavior as in the case of fig.16 in the hot state the force maximum is shifted down to the upper third part. The differences between an orthotropic and isotropic behaving coil are not so very considerable.

4.4 Changing the locality of the epoxy layer

Our considerations until now concerned the model where the epoxy was assumed to exist amidst of the copper. In reality there are several epoxy layers between the copper and also between the copper and the steel. The latter layer was suspected to be higher strained than the others. It is impossible to judge this on account of our described model. Therefore the model was changed in such a way that the epoxy layer is now between the steel ring and the copper. Its thickness is assumed with 1 cm according to a medium value of the reality.

4.4.1 Stress distributions

First the shear stresses within the epoxy layer may be considered, see fig. 18 B, B & F. The influence of the shifted locality is not very severe as we learn by comparing it with fig. 13 and 14. The shear stresses are in all cases not too different. Fig. 18 D reveals a hard oscillating in the steel ring presumably to explain with the noncoherency of the orthotropic properties, since no oscillations happen in the isotropic case, F. The cold state, D, is even harder than the hot one, B, where frequent deviations are rather smoothed. In table 8 the maximum shear stresses of the epoxy layer are listed.

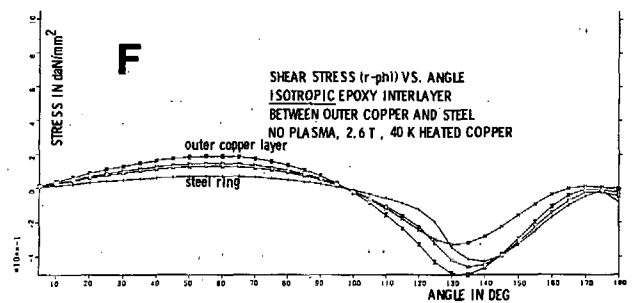
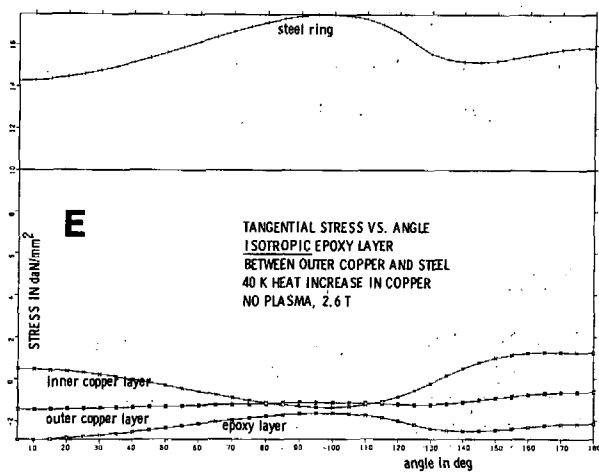
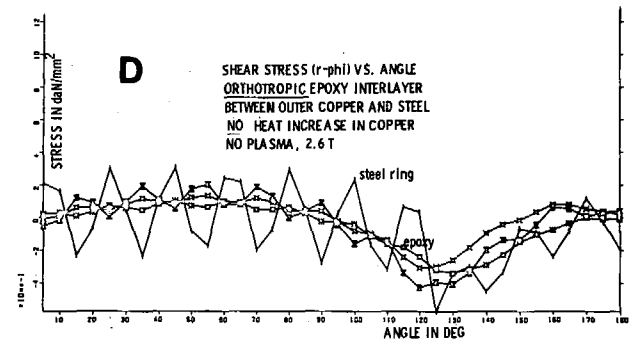
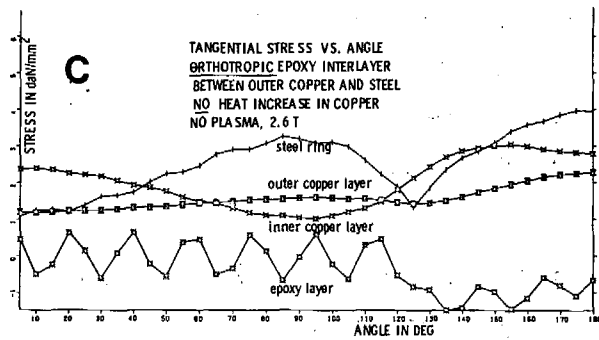
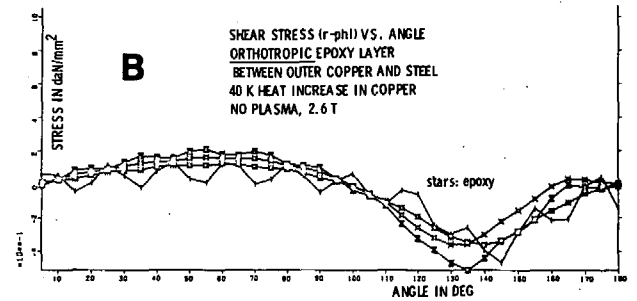
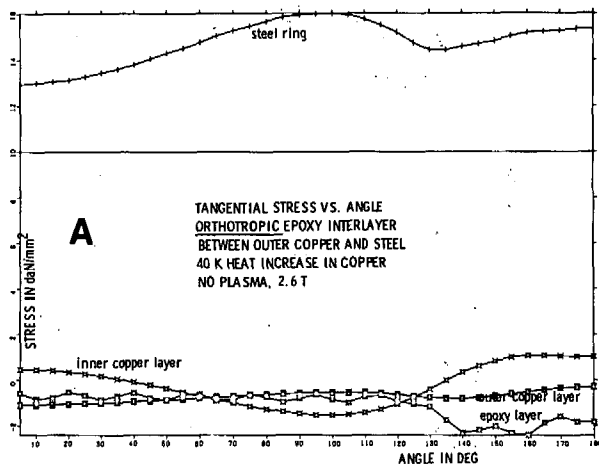


Fig.18 STRESSES WITH INSULATION LAYER BETWEEN STEEL AND COPPER

Table 8

Maximum shear stresses in the epoxy layer, daN/mm^2 , epoxy layer of 1 cm thickness between steel ring and copper

material	temperature	maximum	locality	minimum	locality
orthotropic	cold	0.103	35°	-0.336	130°
isotropic	cold	0.01	55°	-0.44	125°
orthotropic	hot (40 K)	0.123	60°	-0.36	140°
isotropic	hot (40 K)	0.13	60°	-0.46	135°

In the orthotropic cold case the absolute maximum shear stress is now 0.336 versus 0.27 daN/mm^2 in the case of the epoxy layer in the middle plane between the copper, table 7. This means an increase of 24 %. In the hot case the changed model shows a shear stress decrease.

The tangential stresses within the steel ring are considerably influenced by the location of the epoxy layer which exercises a dampening effect in this regard if arranged between copper and steel. Table 9 gives a confrontation of the maximum tangential stresses in the steel in the case of direct contact of copper and steel and epoxy in some distance between the copper, and in the case where epoxy exists between copper and steel.

Table 9

Maximum tangential stresses in the steel ring in daN/mm^2

no	material	temperature	epoxy layer between steel and copper		epoxy layer amidst of the copper	
			maximum	locality	maximum	locality
1	orthotropic	cold	3.26	85°	3.69	85°
2	isotropic	cold	3.50	90°	3.50	90°
3	orthotropic	hot (40K)	16.04	100°	23.59	100°
4	isotropic	hot (40K)	17.50	105°	17.50	100°

In the orthotropic hot case, no.3, the epoxy layer between steel and copper reduces the stress to 68%. Also in the cold case a stress decrease is to observe if epoxy is between the different adjacencies. In the isotropic cases 2 and 4, however, the situation of the epoxy is of no importance because of the homogenisation.

4.4.2 Deformations

Comparisons between the first model and the changed one concerning the absolute amounts of the deformations show lessened quantities in the latter case. The reason is the improved stiffness since the thickness of the epoxy layer is now only 1 cm. The former model had a 3.7 cm thick epoxy layer.

Fig. 19 shows a cross section deformation in the cold state. The hot case can be seen in fig.20. The epoxy layer reveals a trapezoidal shape yielding the transition from the copper to the steel.

5. Optimum structural design of the steel ring

A certain kind of structural design optimization deals with the changing of the shape of a structure pursuing the aim of an unique stress in all cross sections. This method is known as fully stressed design, FSD, reference /5/. It is an iterative procedure coupled with a finite element analysis.

Proposed is an ideal stress, σ_{id} , which finally shall occur all over the whole structure. In an iteration step $k+1$ the cross section of the i -th element is changed from the foregoing, the k -th step, by the procedure

$$A_i^{k+1} = \frac{\sigma_i^K}{\sigma_{id}} A_i^K .$$

Weakly stressed regions get thus reduced cross sections and vice versa. The procedure generally converges quickly.

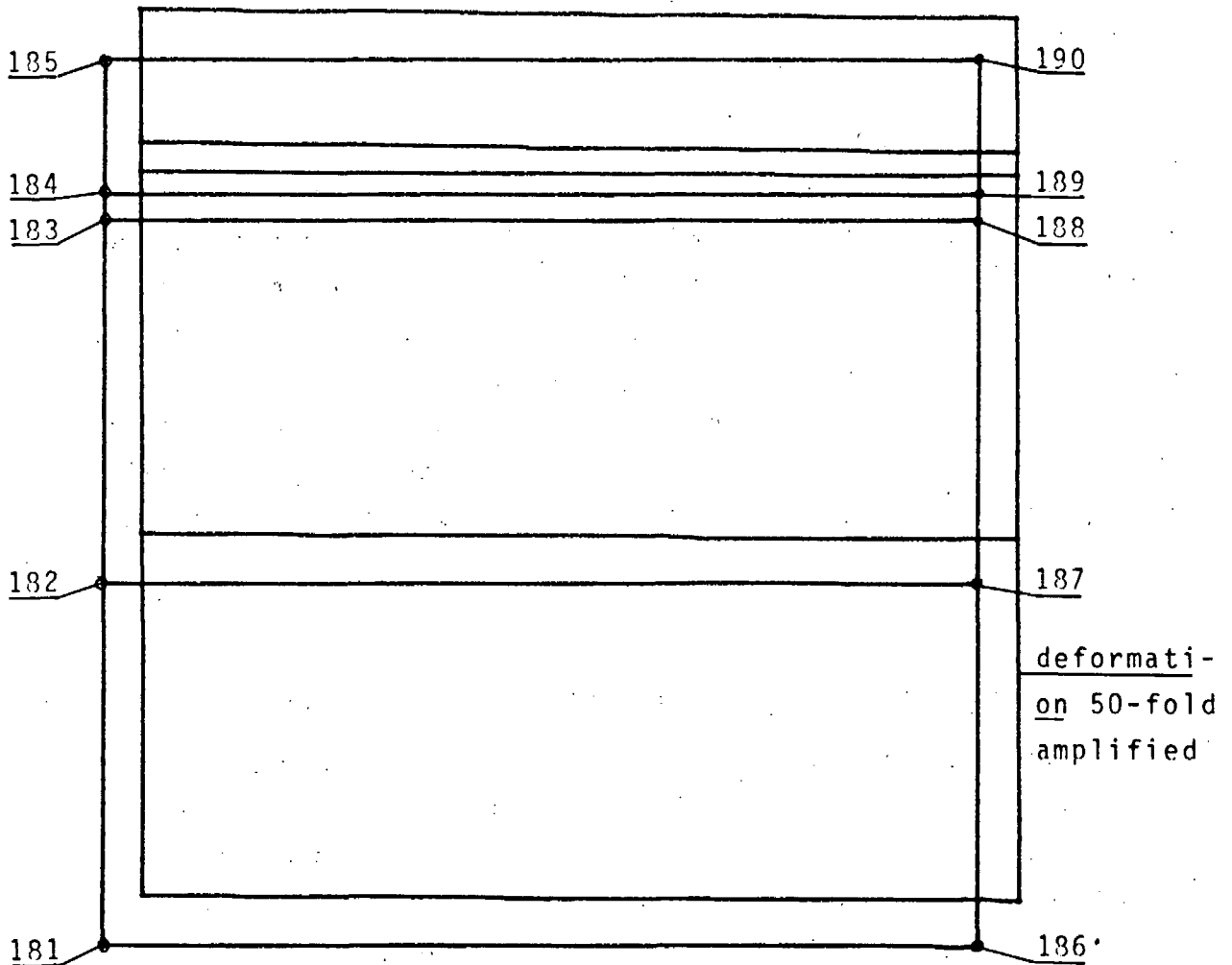


Fig.19

Deformations of a cross section at 90 deg. The orthotropic epoxy layer is between the steel ring and the copper (nodal points 183-184-189-188). Cold state, no plasma, 2.6 T.

Deformations in cm:

Node nr.	r	z	Node nr.	r	z
181	0.038	0.0305	186	0.0344	0.0313
182	0.0378	0.0288	187	0.034	0.0299
183	0.0368	0.0271	188	0.0335	0.0285
184	0.0383	0.027	189	0.031	0.0284
185	0.038	0.0259	190	0.0308	0.0273

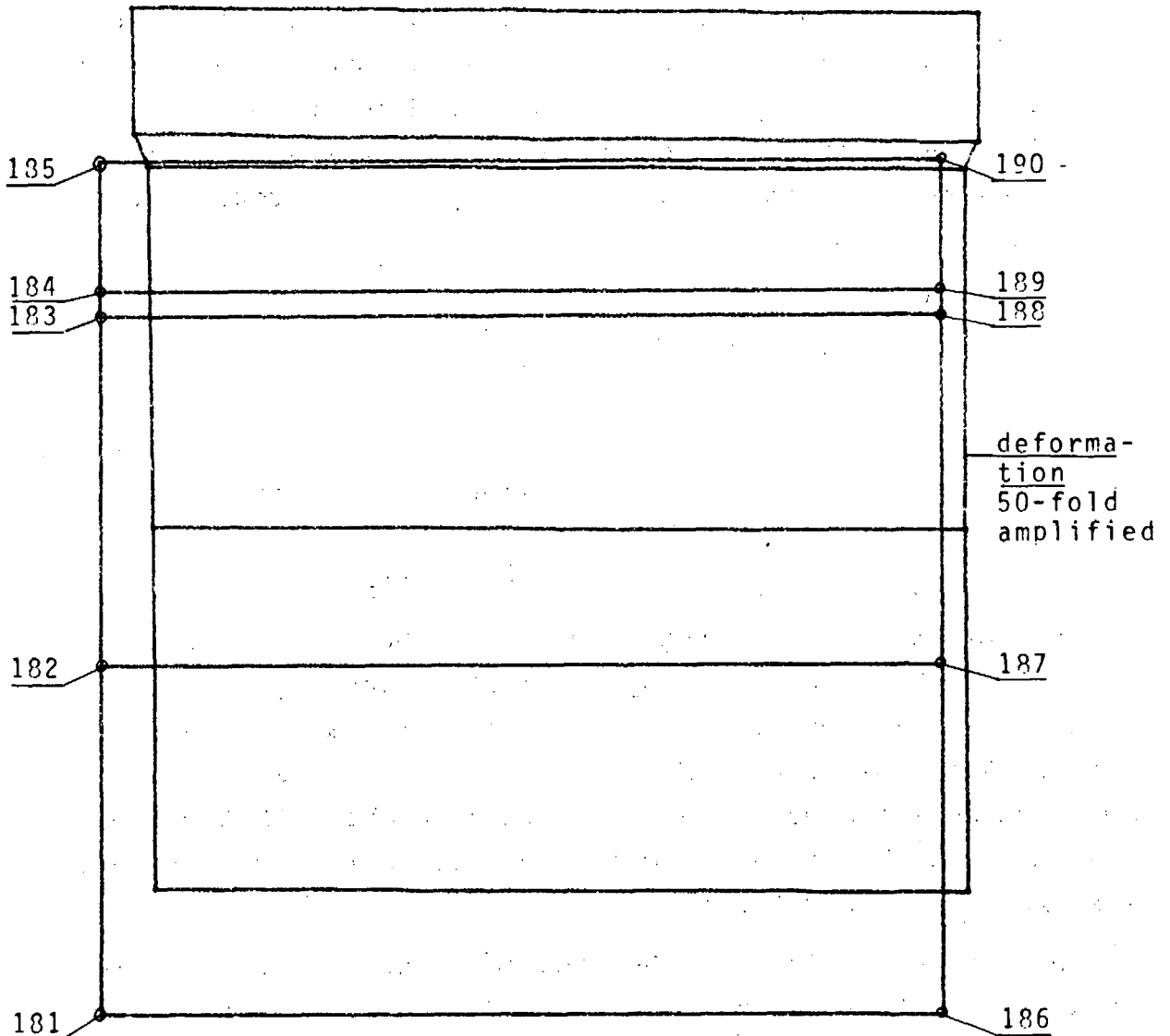


Fig.20

Deformations of a cross section at 90 deg. The epoxy layer of orthotropic character lies between the steel ring and the copper (limiting nodal points 183-184-189-188). 40 K upheated state, no plasma, 2.6 T.

Deformations in cm:

Node nr.	r	z	Node nr.	r	z
181	0.098	0.0428	186	0.094	0.0189
182	0.108	0.0408	187	0.104	0.0178
183	0.117	0.0372	188	0.113	0.01835
184	0.122	0.0263	189	0.114	0.029
185	0.12	0.0246	190	0.113	0.0287

As shown above the steel ring reveals in its cold state rather unique radial and axial stresses but very changing tangential stresses. Therefore it was suggested that the FSD should be applied to the tangential stress changing the ring thickness until the tangential stress is all over unique. Then the tangential deformations of the steel ring would be unique. It was supposed that this would reduce the shear stresses in the epoxy layers since the shear stress depends vigorously on the tangential deformations v

$$\sigma_{r\phi} = G \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \phi} \right).$$

The reduction of the shear stress in the epoxy is of importance because these layers are the weak link of the design.

5.1 Results concerning the stress unification

The FSD procedure was first executed on the coil model with a thick orthotropic epoxy layer amidst of the copper, cold state. Fig. 21 shows the results. The start pattern is rather rugged. With the number of iteration steps the pattern is smoothed more and more, and is about straight with the step eleven. Here the procedure was interrupted. The ideal stress was proposed with 200 daN/cm^2 .

Now we may have a look on the created contour. Fig. 22 displays it after the first iteration step. After the eleventh step, however, the FSD effects a contour which was unexpected because of its very unequal shape, see fig. 23.

The question is to put, of course, whether the executed procedure is useful and effective. Additional considerations are necessary. Optimization procedures of structures are just at the beginning. Their application on tokamak coils is a challenging field, for these structures belong to the very high strained mechanical devices.

The asymptotic character of the FSD is to recognize when regarding the thickness variation of the steel ring at a single location, see fig. 24. It is to see that only small changings are to observe after the 6th or 7th iteration step.

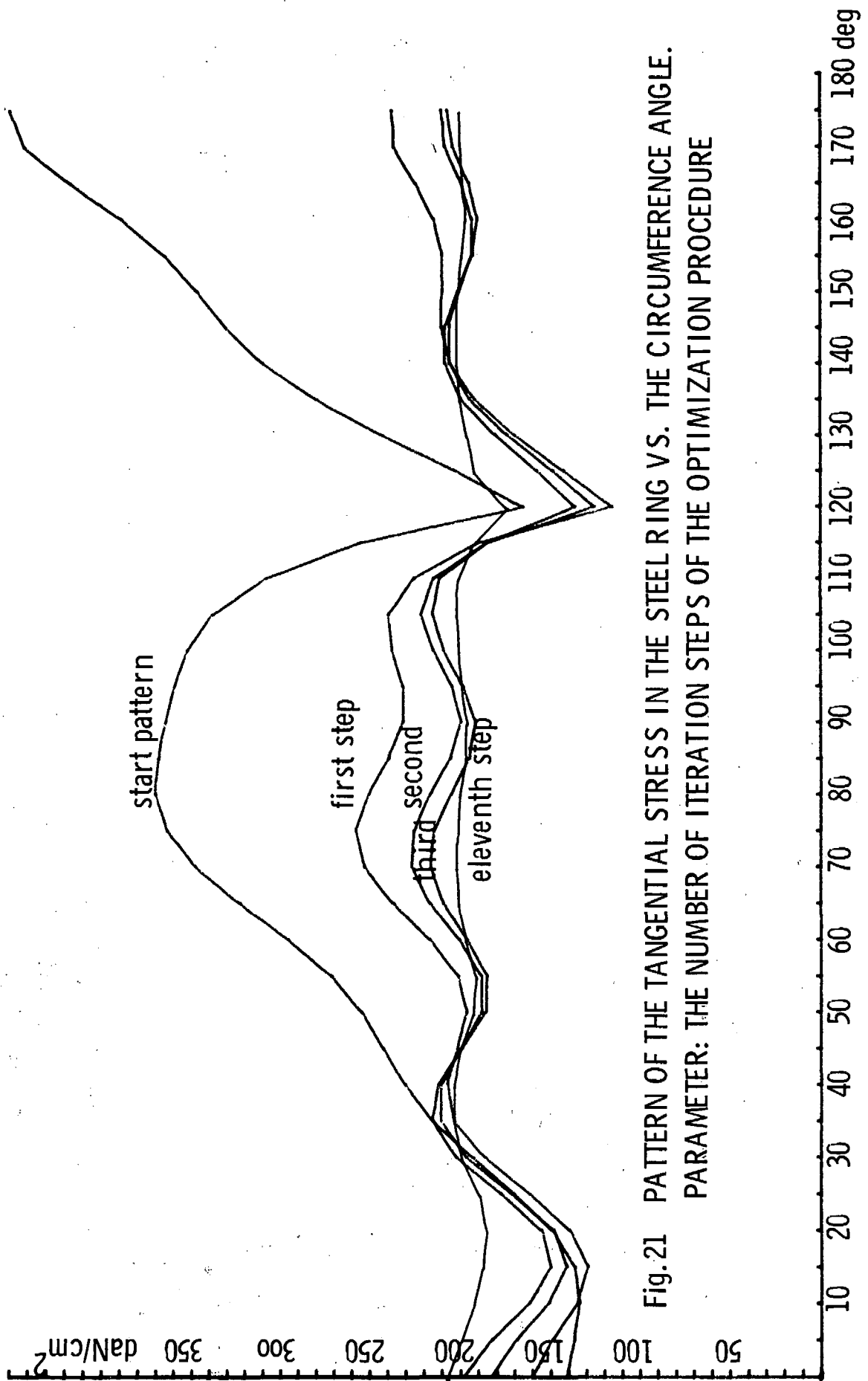


Fig.21 PATTERN OF THE TANGENTIAL STRESS IN THE STEEL RING VS. THE CIRCUMFERENCE ANGLE.
PARAMETER: THE NUMBER OF ITERATION STEPS OF THE OPTIMIZATION PROCEDURE

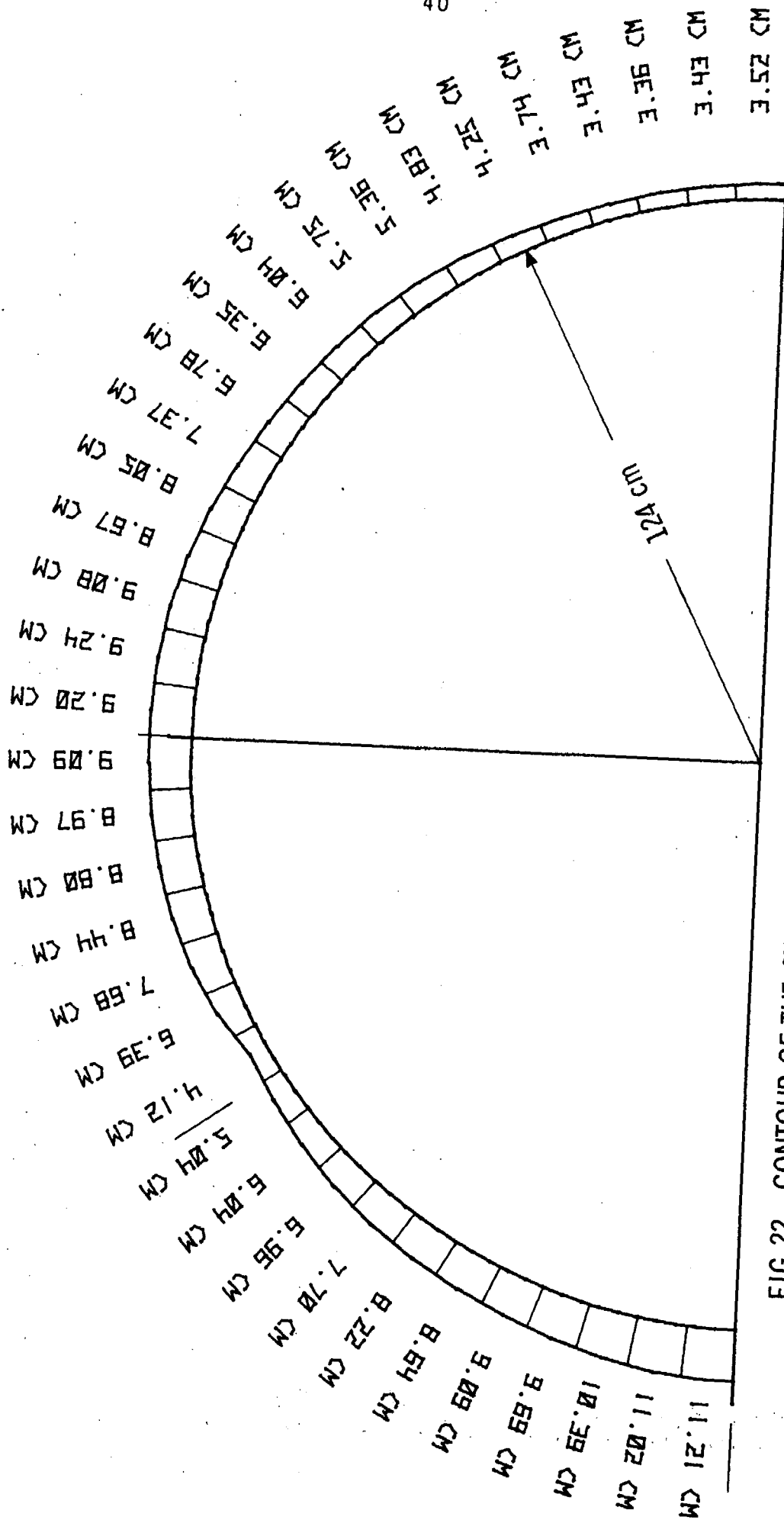


FIG.22 CONTOUR OF THE STEEL RING ACCORDING TO A FULLY STRESSED DESIGN OPTIMIZATION PROCEDURE. ATTACHED ARE THE LOCAL THICKNESSES STATE AFTER THE FIRST ITERATION STEP

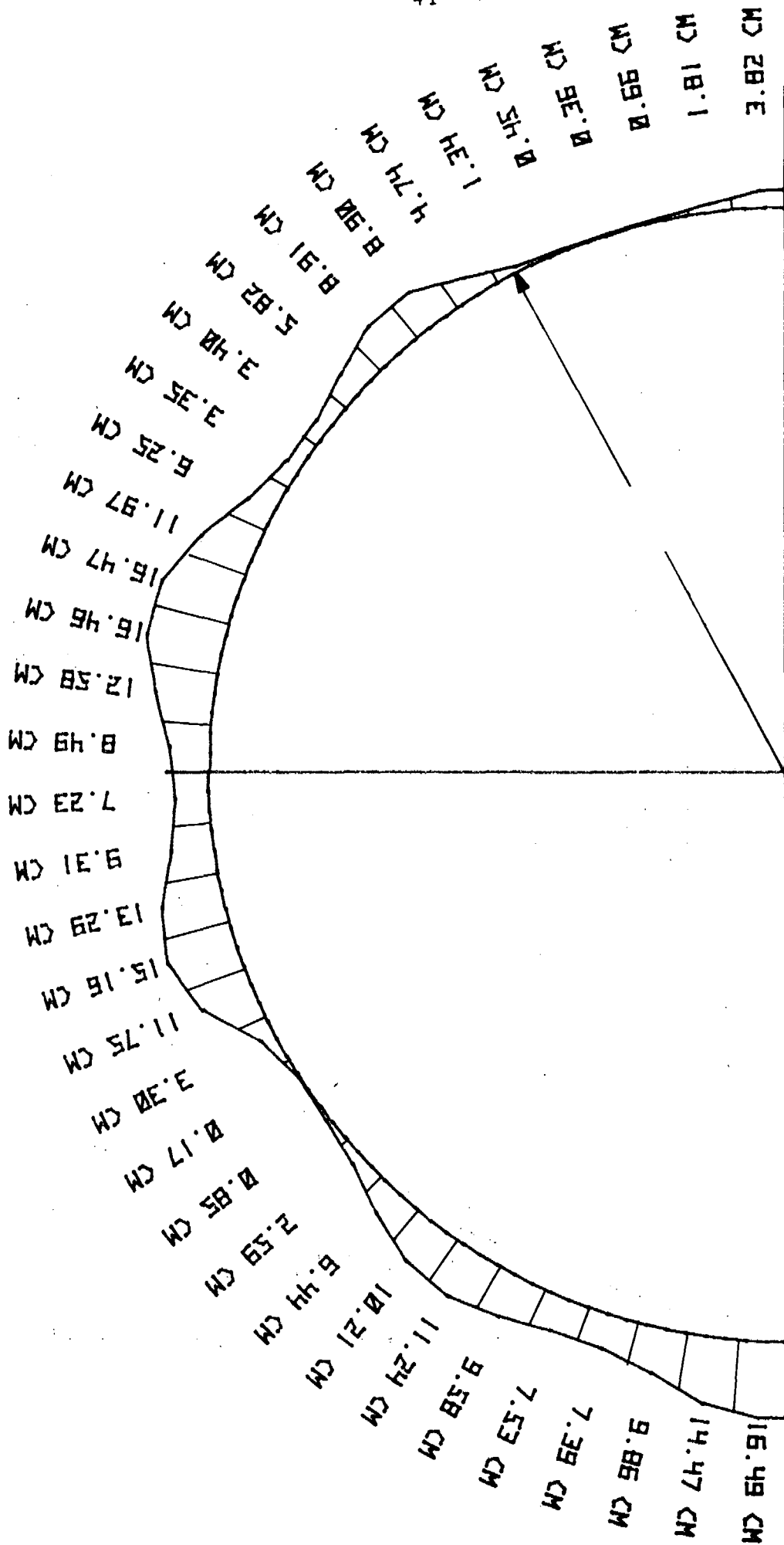
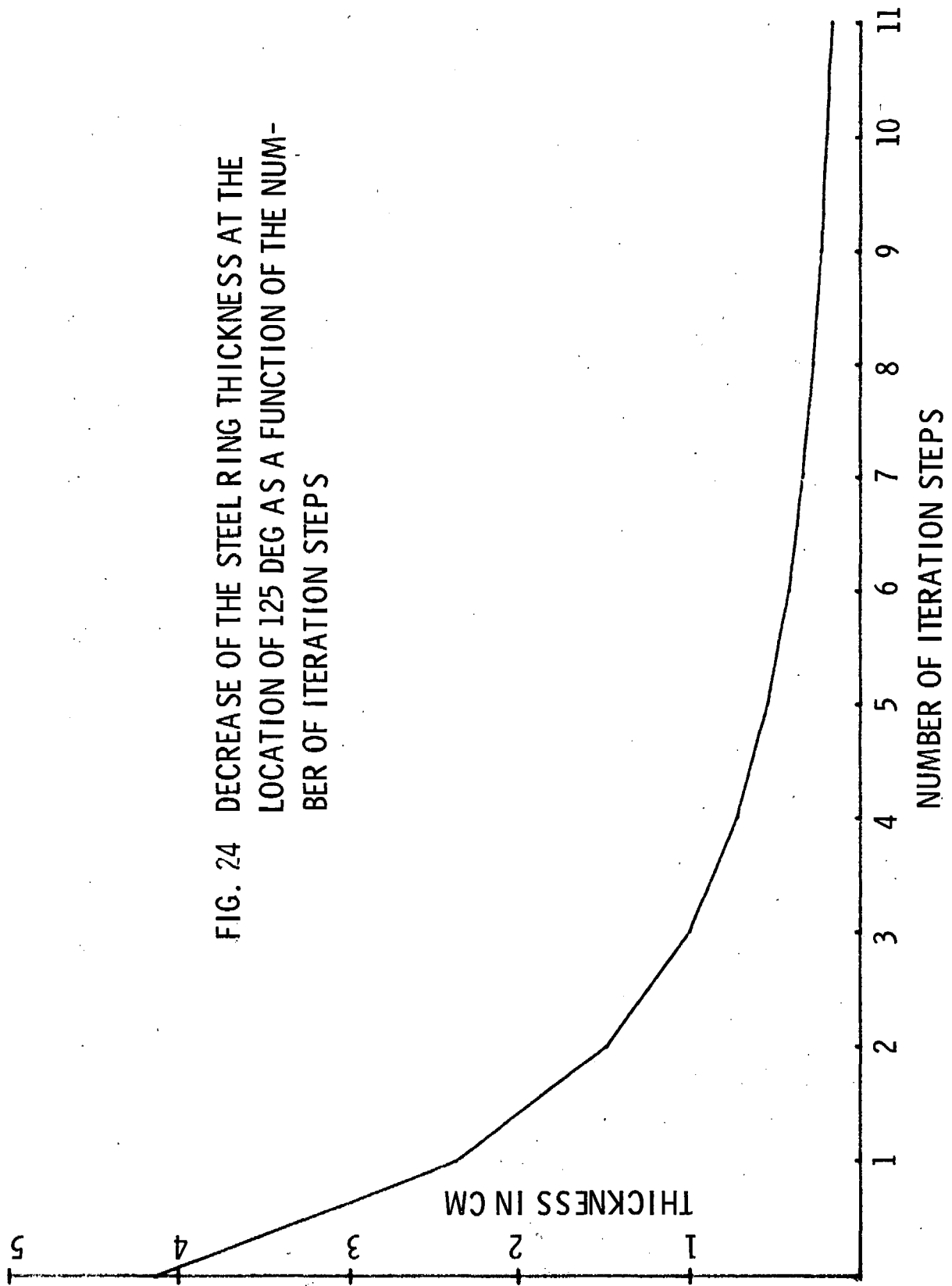


FIG. 23 CONTOUR OF THE STEEL RING AFTER 11 STEPS OF FULLY STRESSED DESIGN

FIG. 24 DECREASE OF THE STEEL RING THICKNESS AT THE
LOCATION OF 125 DEG AS A FUNCTION OF THE NUM-
BER OF ITERATION STEPS



5.2 Influence on the shear stress in the epoxy

Unfortunately, the desired decrease of this stress did not take place, as fig. 25 displays. There is almost no influence to observe.

It was suspected that a higher influence would be seen if we use the changed model. In this case an epoxy layer is situated between the steel ring and the copper.

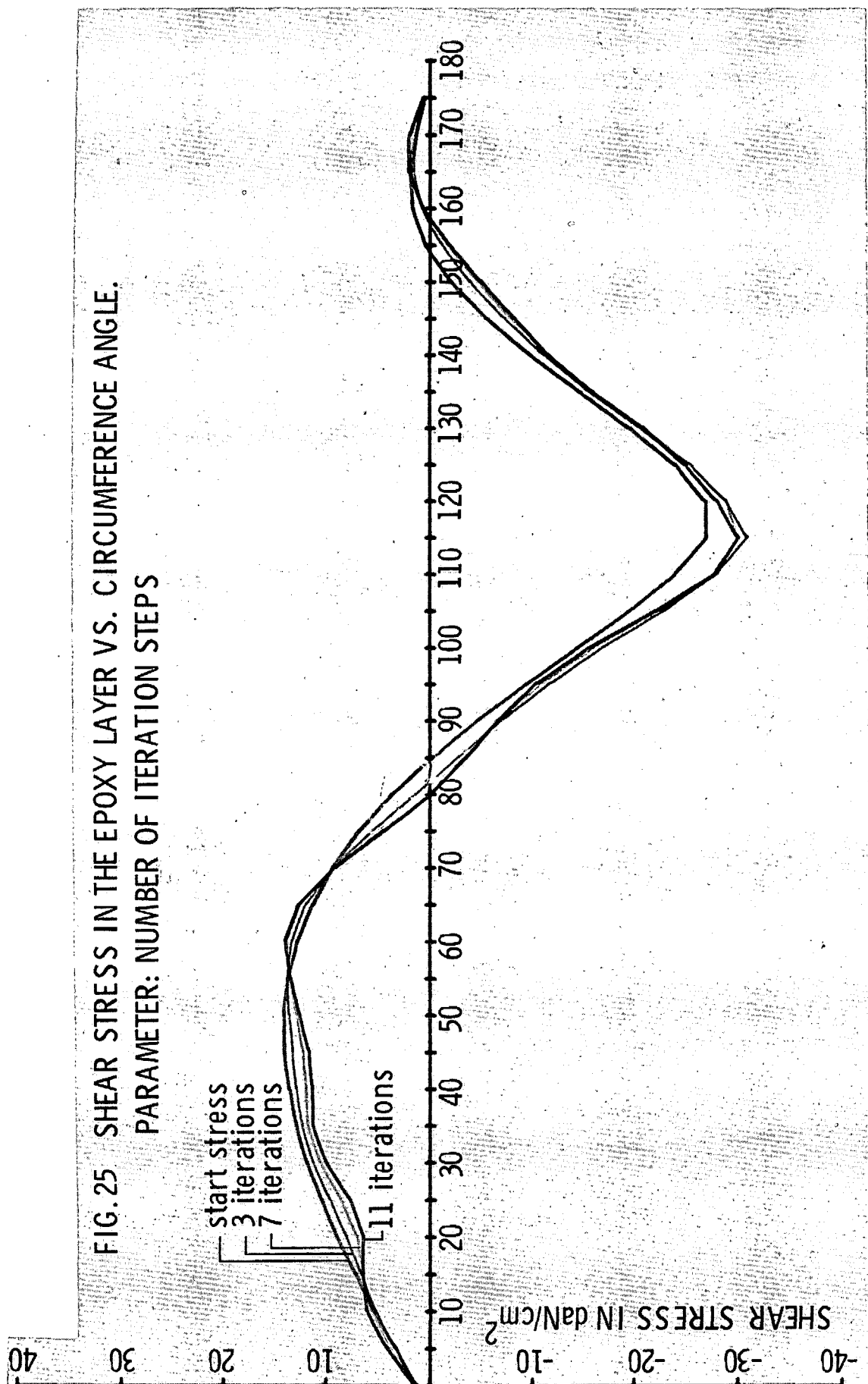
5.3 Fully stressed design on the changed model

The calculation model with an epoxy interlayer of 1 cm thickness between steel ring and copper was subjected the FSD procedure. In fig. 26 the effect can be seen concerning the tangential stress in the steel ring. The smoothing effect runs well again with the exception of the 120 deg region. Improvements would occur with additional iterations.

The resulting steel ring contour is displayed in fig. 27. The thickness variations are now less wavy compared to fig. 23 and therefore they seem more reasonable.

There is again only a poor influence on the shear stress in the epoxy, as fig. 28 shows. The desired goal of a smoothing or removing of the $r-\phi$ -shear stress is not achieved.

FIG. 25 SHEAR STRESS IN THE EPOXY LAYER VS. CIRCUMFERENCE ANGLE.
PARAMETER: NUMBER OF ITERATION STEPS



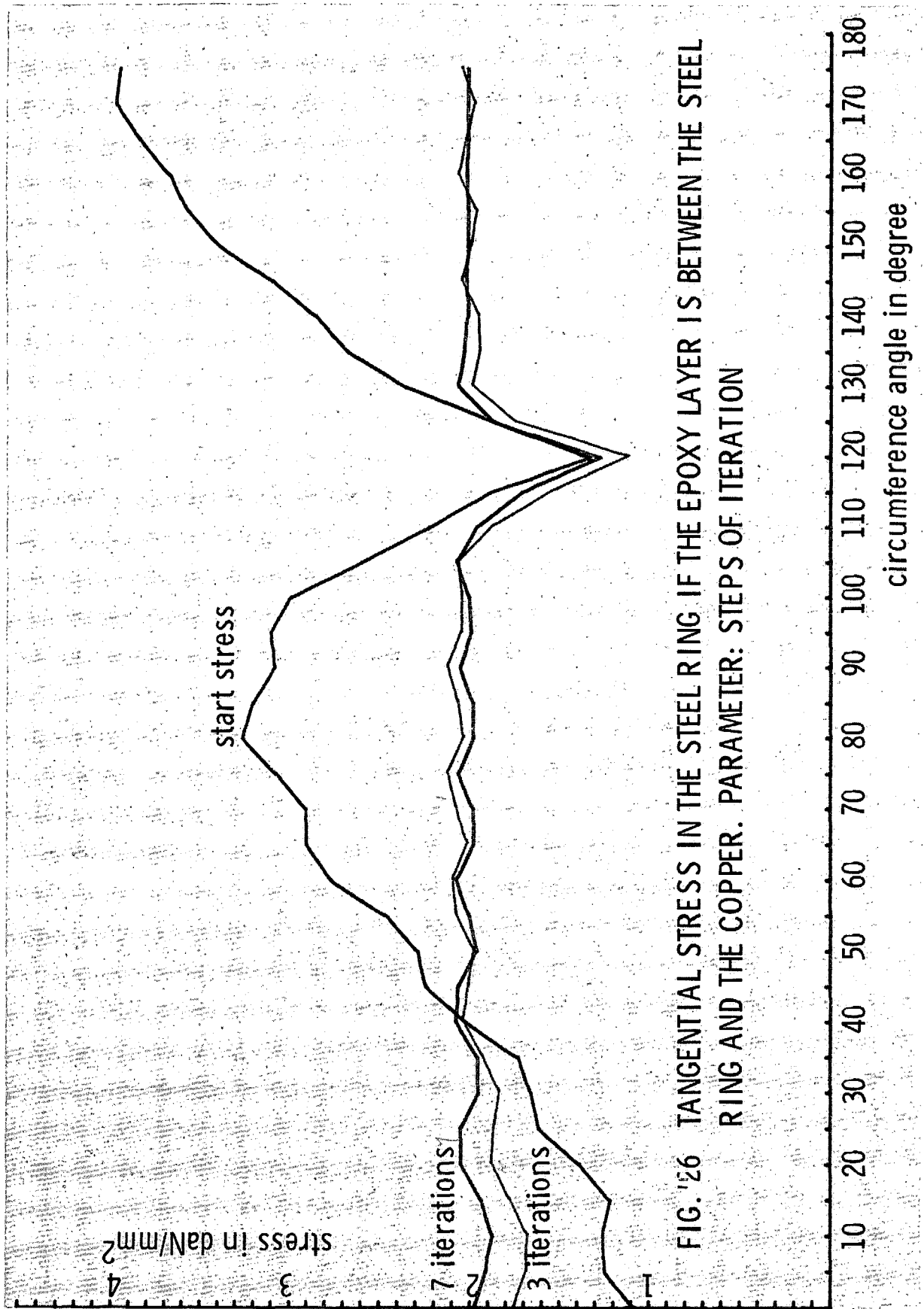


FIG. '26 TANGENTIAL STRESS IN THE STEEL RING IF THE EPOXY LAYER IS BETWEEN THE STEEL RING AND THE COPPER. PARAMETER: STEPS OF ITERATION

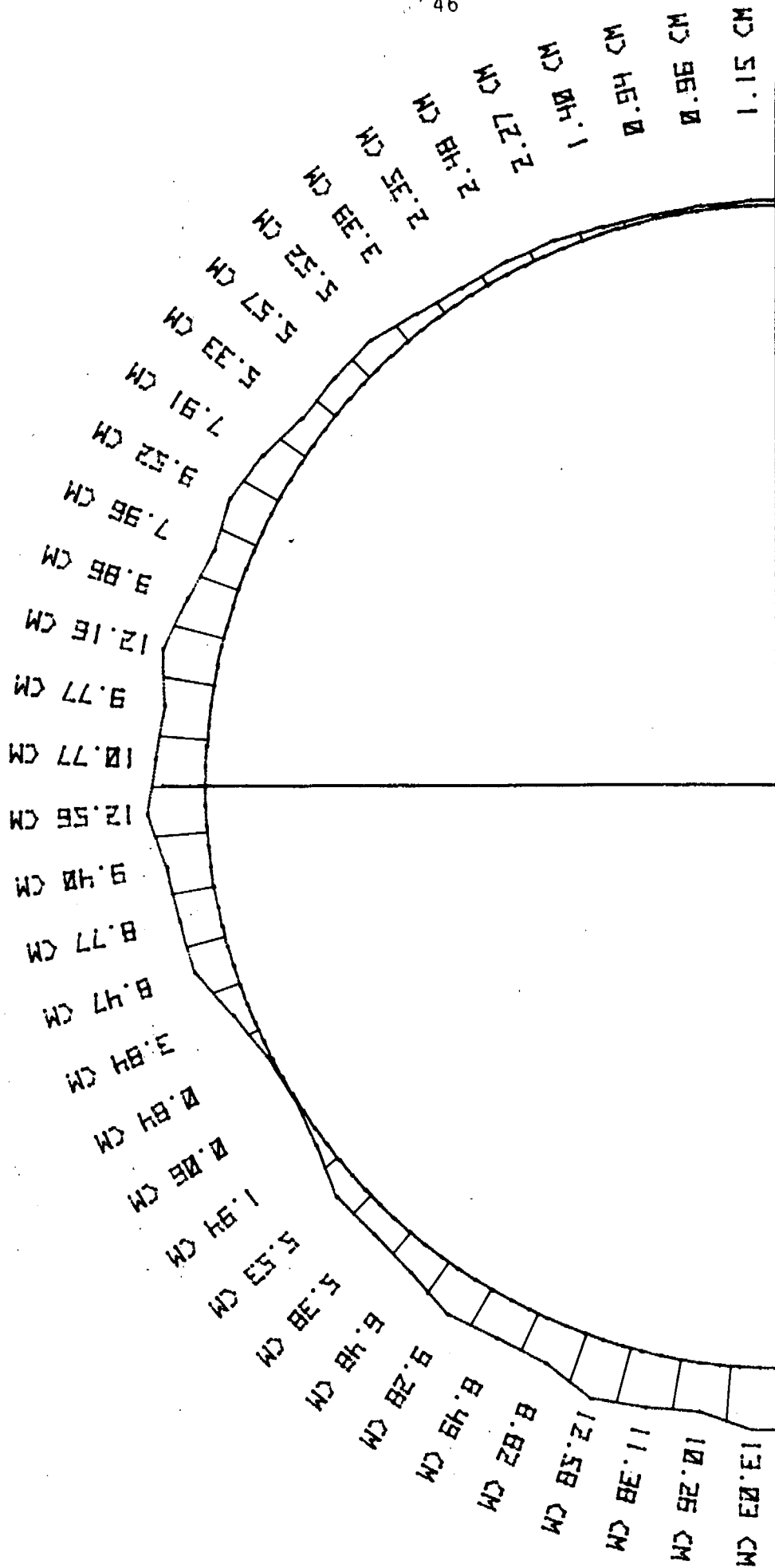
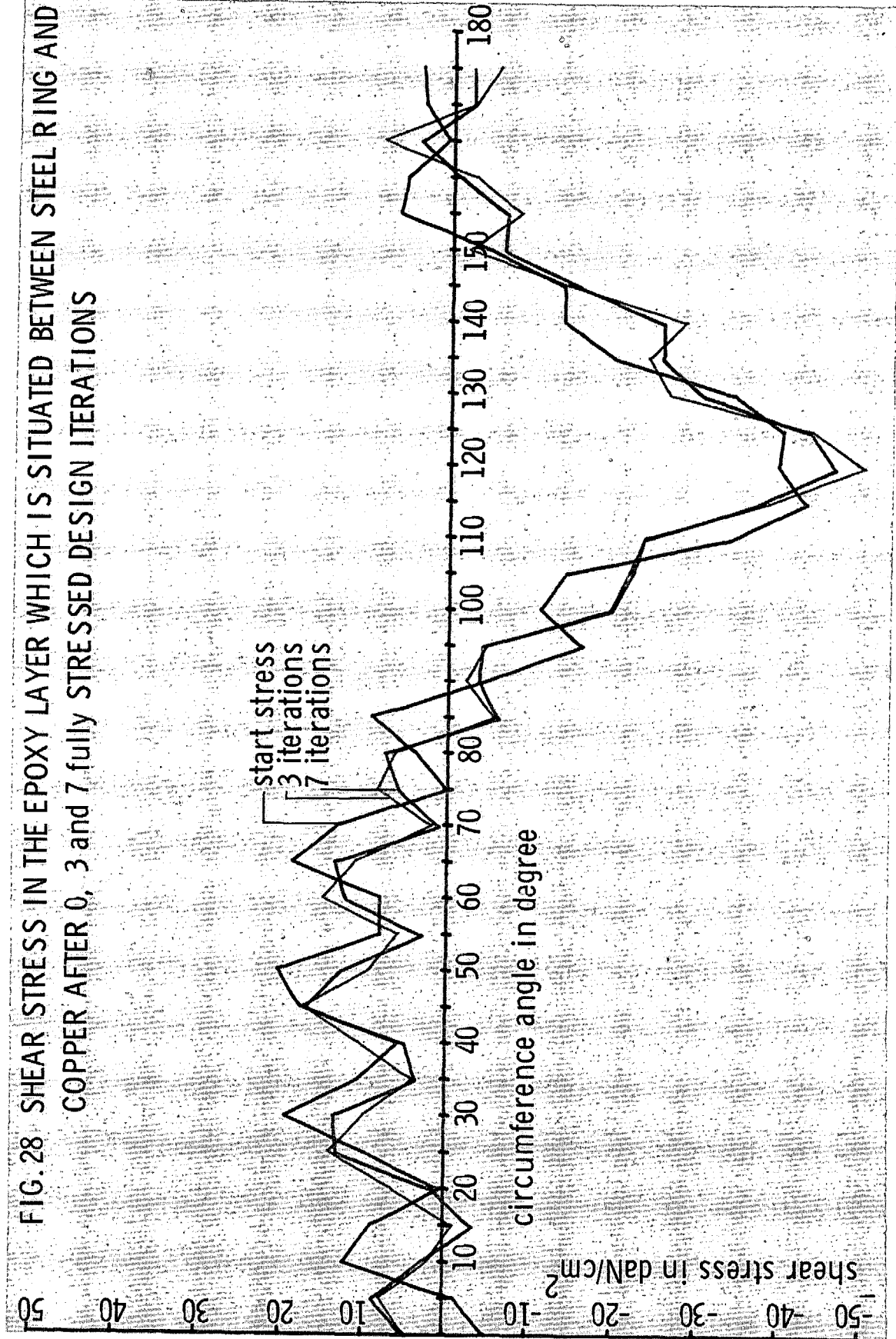


FIG. 27 EPOXY BETWEEN STEEL AND COPPER: CONTOUR OF THE STEEL RING AFTER 7 STEPS OF FULLY STRESSED DESIGN

FIG. 28 SHEAR STRESS IN THE EPOXY LAYER WHICH IS SITUATED BETWEEN STEEL RING AND COPPER AFTER 0, 3 and 7 fully STRESSED DESIGN ITERATIONS



Appendix 1

Time dependent temperature fields in the coil

1. Heat release

Dealing with the temperatures in the coil copper and the flow of cooling water we have to regard to the time dependent power release, see fig.29, basing on calculations of B.Giesen, KFA-IPP. The maximum of power release is reached 14 seconds after the beginning. It is finished after 25 s. We consider only the harder case of 2.6 T.

2. Model and boundary conditions

The coil is interpreted as an developed copper bar of 70 m length which is subdivided in calculation slices with lengths depending on the cooling flow velocity and an arbitrary time step. The temperatures of the copper and the cooling water are introduced as functions of locality and time. The boundary conditions are as following.

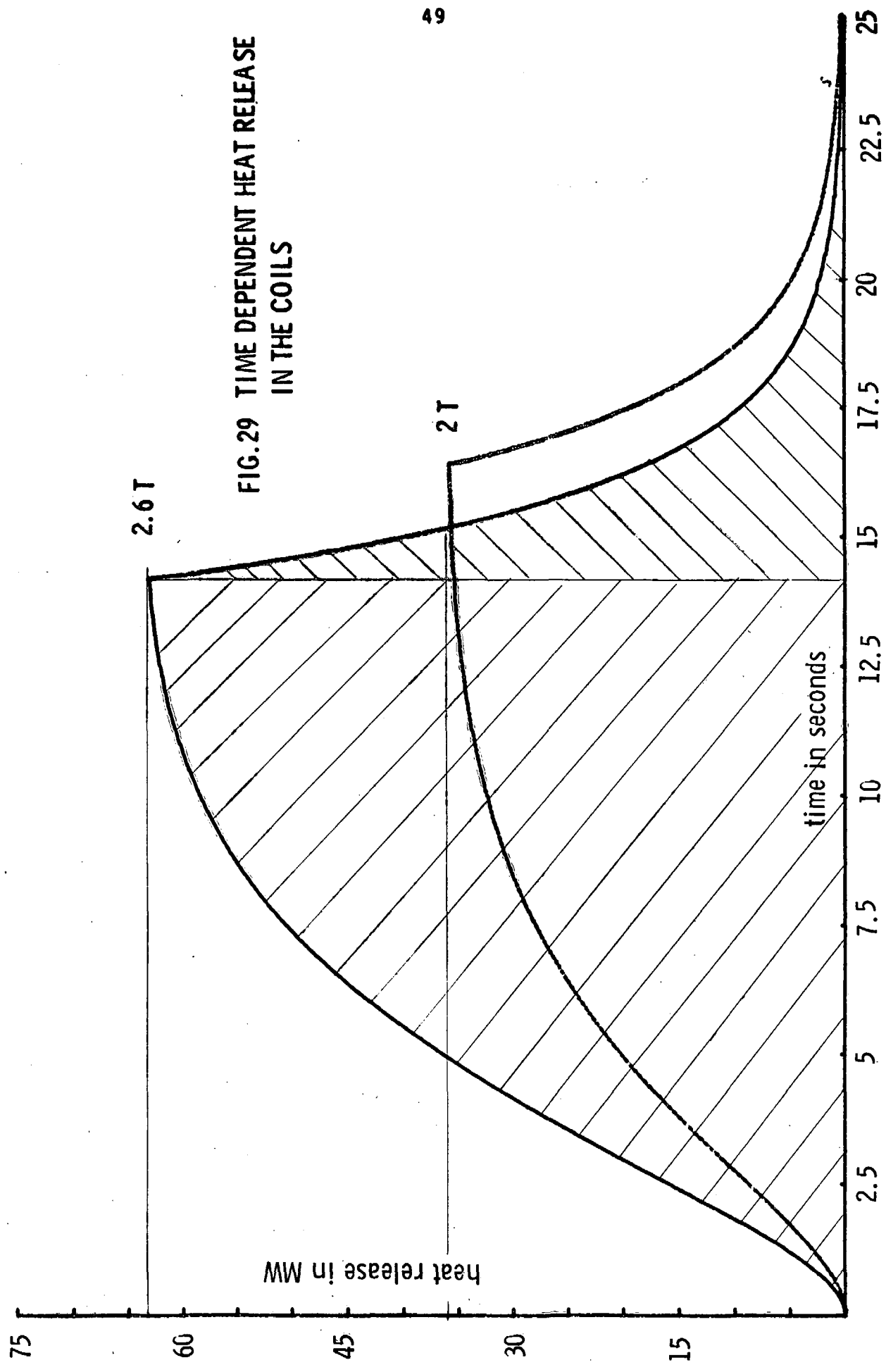
At the beginning copper and water are throughout on the starting temperature, and the water always enters the first element with a constant temperature.

3. Calculating modalities

Two program loops consider the relationships. An inner one runs over a single element stepwise enlarging the time. After each time step the local temperature is raised to a fictitious amount according to the released time dependent electrical power. This fictitious value is now reduced by that temperature difference which matches the heat amount transferred to the cooling water. The appropriate heat increase in this water is stored and serves in a later step as the entrance temperature to the adjacent element. First the inner loop works off all time dependent temperatures of the entry element. Then an outer loop shifts over to the next adjacent element which is subdued to the inner loop again, etc.

The following properties are involved:

FIG.29 TIME DEPENDENT HEAT RELEASE
IN THE COILS



specific heat capacity of water	4715 J/(kgK)
" " " " copper	400 "
heat transfer coefficient	13000 W/(m ² K)
water flow per hole	0.654 kg/s

4. Results

The results are exhibited in fig. 30 & 31. We may first consider the copper temperatures, fig.30. In the period from 0 to 5 s the temperature increase is approximately unique. During the space of time from 7 to 14 s the cooling influence of the water becomes effectively, as to see from the entrance temperatures which remain low. According to the trespassing water coolant heat accepting and appropriately temperature increasing, also the copper temperatures are raised.

From the 14th second the power release of the coil decreases very rapidly. Despite, the copper reaches the maximum temperature not before 15 s, since the water already preheated has to pass through. In the neighbourhood of the water entry, however, the power decrease is to discover already by lowering temperatures.

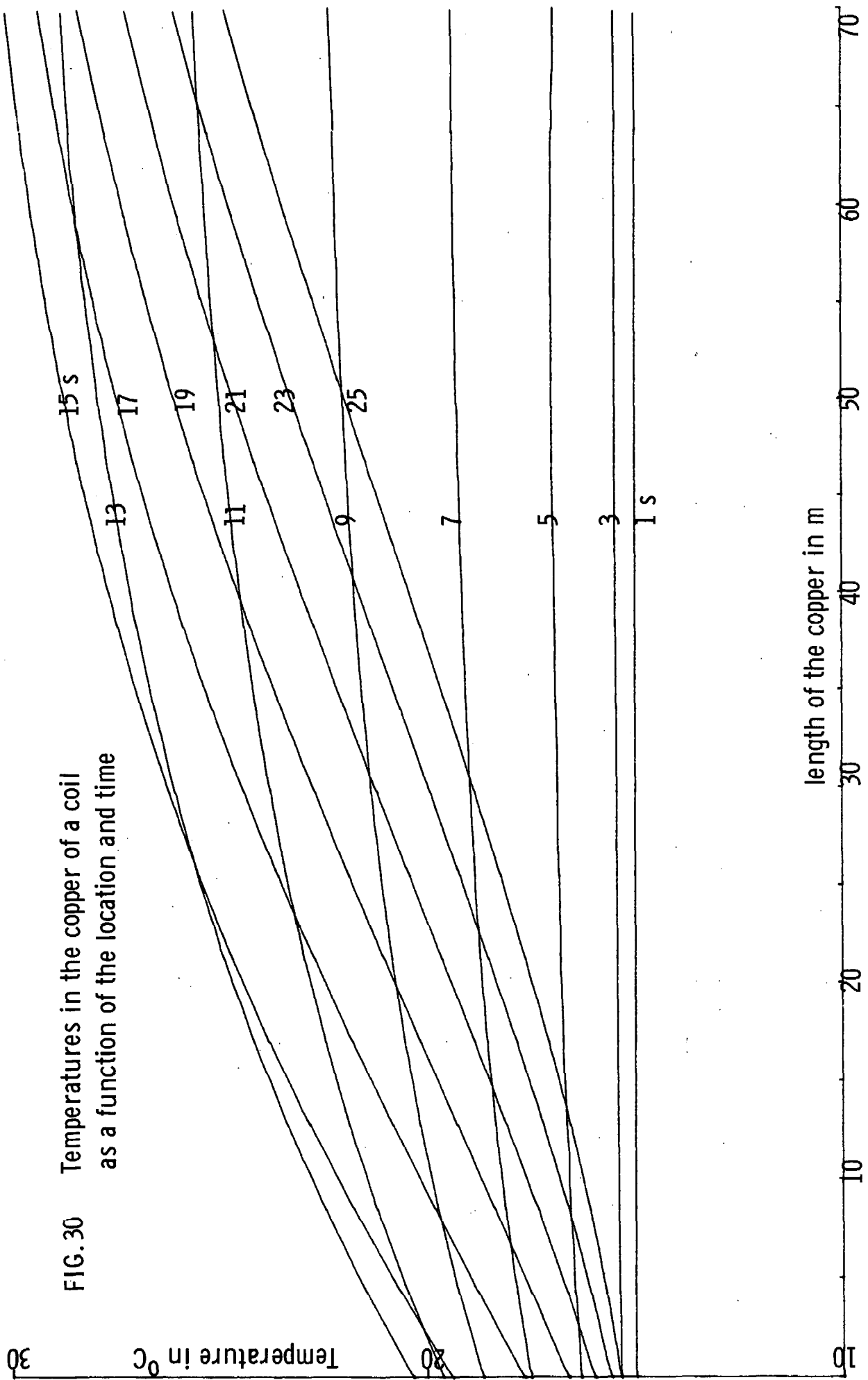
In the sequence we recognize during a proceeding fall of temperatures an S-shaped temperature pattern.

Though the power release has finished after 25 s the cooling down procedure still continues.

For the calculation of heat stresses the temperature difference between two adjacent copper layers is of importance. Every coil winding is about 7 m long and therefore we have to compare two local temperatures in this distance at the same moment.

The largest temperature difference is to recognize on the 15-s isochronic contour between the localities 7 and 14 m. representing the region of the steepest ascent of the contours in that diagram. The amount of this difference is 1.8 K.

The highest copper temperature, 30°C, is to expect at the outlet 15 s after the start. Regarding the assumed start temperature of 15°C that means an maximum overtemperature of 15 K.



It is to recommend to arrange the coil in a way that the coolant entrance is situated adjacent to the steel ring. The outlet is then on the inner side of the coil. Consequently the temperature difference between the steel and the neighboured copper remains small and the pressure due to the expanding warm copper is partly taken over by the colder copper. The cooling water temperatures, fig.31, don't display comparably very different contours because of the good heat transfer from the copper to the water. A dissimilarity is to find at the entry for the water gets in with a constant temperature.

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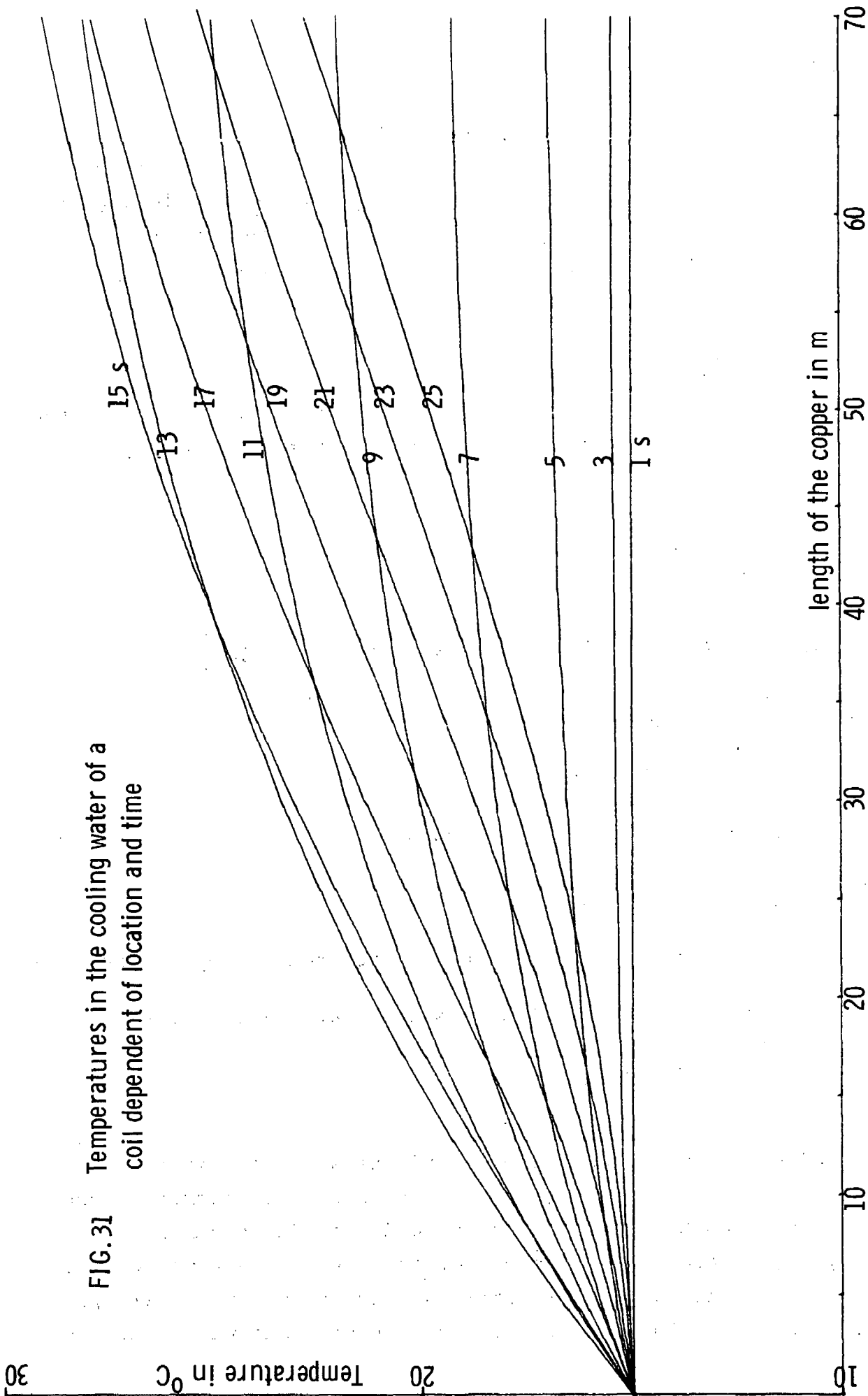
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Appendix 2

Derivation of a special finite element

This appendix gives the derivation procedure of a modified finite element of the cylinder wall type with radial-tangential orthotropic behavior. We begin with a short survey of the theory of finite elements as far as it is thought to be useful in this context.

1. Principles

The principle of minimum potential energy may be stated as the vanishing of the first variation of the potential energy functional π_p (ref. /1/, p.202):

$$\delta \pi_p = \sum_{n=1}^N \delta \pi_p^n = 0 \quad (1)$$

$$= \sum_{n=1}^N \left[\iiint_{V_n} (\sigma^{ij} \delta \epsilon_{ij} - \bar{F}^i \delta u_i - \Delta \theta \lambda^{ij} \delta \epsilon_{ij}) dV - \iint_{S_{\sigma_n}} (\bar{T}^i \delta u_i) dS \right]$$

σ^{ij} stress tensor component

ϵ_{ij} strain tensor component

$\bar{F}^i$ prescribed body force component

u_i displacement component

$\Delta \theta$ $\theta - \theta_r$ temperature in K above a reference temperature θ_r

λ^{ij} tensor component of stress-temperature-coefficients

$\bar{T}^i$ prescribed surface tractions

A solid body with the volume V and the surface S is divided into a finite number (N) of elements with the volumes V_n and the surfaces S_n . σ^{ij} , ϵ_{ij} are regarded as functions of u_i .

u_1, u_2, u_3 are the components of the displacement vector $\vec{u}$.

$S_{\sigma n}$ are those parts of S_n where surface tractions $\bar{T}^i$ are prescribed (including zero-tractions). On the remaining parts of

S_n : S_{un} the u_i have to meet the Dirichlet boundary conditions

$$u_i = \bar{u}_i \quad \text{on } S_{un}. \quad (2)$$

The statement of equ. (1): $\sum_{n=1}^N \delta \Pi_p^n = \delta \Pi_p$ imposes appropriate continuity conditions on u_i (ref. /2/, p.109, 114 ff, 135). Before further development we wish to give some remarks to the tensors ϵ_{ij} , σ_{ij} in the special case of orthotropy.

1.1 Relations between deformations, stress and strain

In the theory of small displacements (infinitesimal theory) (ref. /3/, p.148; ref. /4/, p.9; ref. /5/, p.132, 137, 138) the strain displacement relations are:

$$\epsilon_{ij} = \frac{1}{2} (u_{i/j} + u_{j/i}). \quad (3)$$

The stress-strain relation is as following (/6/, P.180):

$$\epsilon_{ij} = a_{ijkl} \sigma^{kl} + \alpha_{ij} \Delta \theta, \quad (4)$$

a_{ijkl} is the compliance tensor $\alpha_{ij} = \alpha_{ji}$ the coefficients of linear thermal expansion.

We will treat a special case of homogeneous (a_{ijkl} are constants) anisotropic body (ref. /7/, p.63) a body with cylindrical orthotropy. We leave the tensor-notation and instead of equ.(3) we receive (ref. /8/, p.5; ref. /5/, p. 141):

$$\begin{aligned} \epsilon_1 = \epsilon_r &= \frac{\partial u}{\partial r} \\ \epsilon_2 = \epsilon_\varphi &= \frac{1}{r} \frac{\partial r}{\partial \varphi} + \frac{u}{r} \\ \epsilon_3 = \epsilon_z &= \frac{\partial w}{\partial z} \\ \epsilon_4 = \gamma_{r\varphi} &= \frac{1}{r} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{r} \\ \epsilon_5 = \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \epsilon_6 = \gamma_{\varphi z} &= \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \varphi} \end{aligned} \quad (5)$$

u, v, w are the 'physical' components (ref. /5/, p. 34,45) of the displacement vector $\vec{v}$, which are functions of r, ϕ, z . It can be shown because of symmetry relations that in the case of orthotropy there are only 9 independent non-zero components of the tensor a_{ijkl} . The tensor of thermal expansion α_{ij} has only three components (ref. /8/, p.66; ref. /6/, p.184). Equ. (4) can therefore be written (ref. /8/, p.9; ref. /9/, p.251):

$$\begin{aligned}
 \epsilon_1 = \epsilon_r &= \frac{1}{E_r} \sigma_r - \frac{V_{\phi r}}{E_\phi} \sigma_\phi - \frac{V_{zr}}{E_z} \sigma_z + \alpha_r \Delta\theta \\
 \epsilon_2 = \epsilon_\phi &= -\frac{V_{r\phi}}{E_r} \sigma_r + \frac{1}{E_\phi} \sigma_\phi - \frac{V_{z\phi}}{E_z} \sigma_z + \alpha_\phi \Delta\theta \\
 \epsilon_3 = \epsilon_z &= -\frac{V_{rz}}{E_r} \sigma_r - \frac{V_{\phi z}}{E_\phi} \sigma_\phi + \frac{1}{E_z} \sigma_z + \alpha_z \Delta\theta \\
 \epsilon_4 = \gamma_{r\phi} &= \frac{1}{G_{r\phi}} \sigma_{r\phi} \\
 \epsilon_5 = \gamma_{rz} &= \frac{1}{G_{rz}} \sigma_{rz} \\
 \epsilon_6 = \gamma_{\phi z} &= \frac{1}{G_{\phi z}} \sigma_{\phi z} ;
 \end{aligned} \tag{6}$$

with

$$\begin{aligned}
 \frac{V_{\phi r}}{E_\phi} &= \frac{V_{r\phi}}{E_r} ; \\
 \frac{V_{z\phi}}{E_z} &= \frac{V_{\phi z}}{E_\phi} ; \\
 \frac{V_{rz}}{E_r} &= \frac{V_{zr}}{E_z} .
 \end{aligned} \tag{7}$$

E_r, E_ϕ, E_z are the Young's moduli, $v_{r\phi}$ etc. Poisson's ratios and $G_{r\phi}, G_{rz}, G_{\phi z}$ are the shear moduli. Equ. (6) can be solved for the σ 's:

$$\begin{bmatrix} \sigma_r \\ \sigma_\varphi \\ \sigma_z \\ \sigma_{r\varphi} \\ \sigma_{rz} \\ \sigma_{\varphi z} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ & D_{22} & D_{23} & & & \\ & & D_{33} & & & \\ & & & D_{44} & & \\ \text{symmetric} & & & & D_{55} & \\ & & & & & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} - \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta\theta \quad (8)$$

The relations between the D_{ij} , λ_i and the E_K , ν_{K1} , G_{K1} , α_K can be found in ref. /9/, p. 252 and ref. /10/, p.1016.

2. Establishing the shape functions

Now we turn back towards the considerations about equ. (1), and put the question about necessary conditions for $\delta\pi_p = 0$. The solid body is divided in N finite elements. One is shown in fig. 1 (ref. 11, p. 43). It may be regarded as a volume-element in cylinder coordinates r, ϕ, z represented in a cartesian x, y, z coordinate system. The area $P_1 P_2 P_6 P_5$ for instance is the coordinate-surface $r = r_1$. The points P_i have the following coordinates:

$$\begin{aligned}
 &P_1 (r_1, \varphi_1, z_2); \quad P_2 (r_1, \varphi_1, z_1); \quad P_3 (r_2, \varphi_1, z_2) \\
 &P_4 (r_2, \varphi_1, z_1); \quad P_5 (r_1, \varphi_2, z_2); \quad P_6 (r_1, \varphi_2, z_1) \\
 &P_7 (r_2, \varphi_2, z_2); \quad P_8 (r_2, \varphi_2, z_1)
 \end{aligned}$$

The points P_i are associated with the following shape-functions $p_i (r, \phi, z)$ which have the property:

$$p_i (P_j) = \delta_{ij} \quad (9)$$

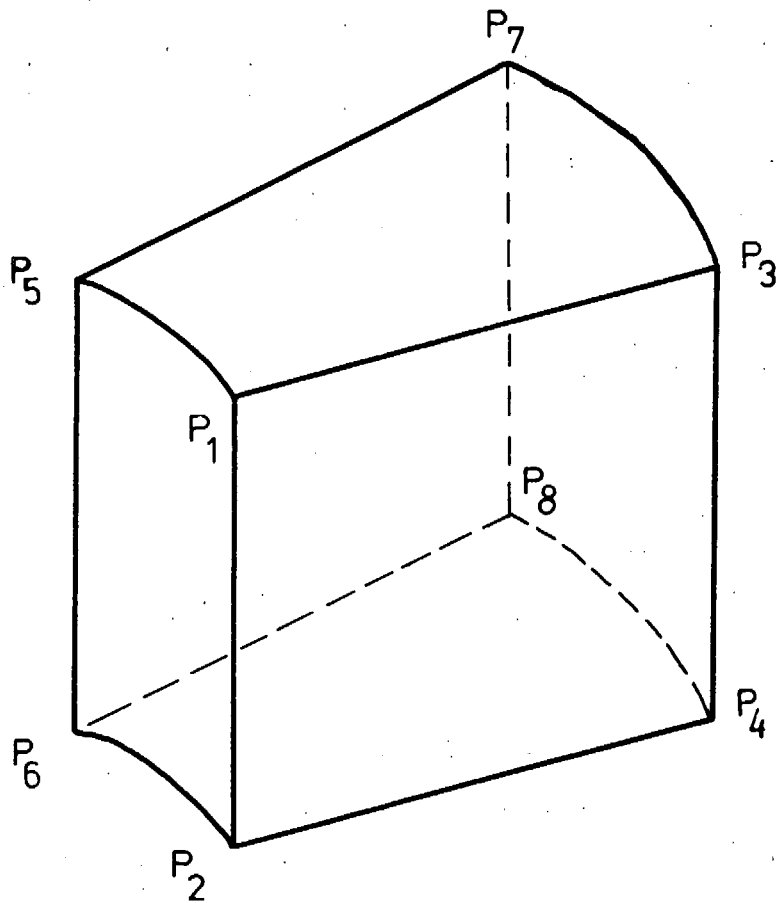


Fig. 1

We use the following abbreviations:

$$\Delta R = r_2 - r_1 ; \quad \Delta \varphi = \varphi_2 - \varphi_1 ; \quad \Delta z = z_2 - z_1 ;$$

$$r^* = \frac{r - r_1}{\Delta R} ; \quad \varphi^* = \frac{\varphi - \varphi_1}{\Delta \varphi} ; \quad z^* = \frac{z - z_1}{\Delta z}$$

$$p_1(r, \varphi, z) = (1 - r^*)(1 - \varphi^*)z^*$$

$$p_2(r, \varphi, z) = (1 - r^*)(1 - \varphi^*)(1 - z^*)$$

$$p_3(r, \varphi, z) = r^*(1 - \varphi^*)z^*$$

(10)

$$p_4(r, \varphi, z) = r^*(1 - \varphi^*)(1 - z^*)$$

$$p_5(r, \varphi, z) = (1 - r^*)\varphi^*z^*$$

$$p_6(r, \varphi, z) = (1 - r^*)\varphi^*(1 - z^*)$$

$$p_7(r, \varphi, z) = r^*\varphi^*z^*$$

$$p_8(r, \varphi, z) = r^*\varphi^*(1 - z^*)$$

We approximate u, v, w the components of the displacement vector $\vec{v}$ in one element by:

$$\begin{aligned} u &= \sum_{i=1}^8 p_i(r, \varphi, z) u_i = \{p\}^T \{u\} \\ v &= \sum_{i=1}^8 p_i(r, \varphi, z) v_i = \{p\}^T \{v\} \\ w &= \sum_{i=1}^8 p_i(r, \varphi, z) w_i = \{p\}^T \{w\} \end{aligned} \quad (11)$$

Where u_i, v_i, w_i are the unknown values of u, v, w at the point P_i ; $\{p\}^T$ denotes a row, $\{u\}$ a column. This approximation can be done in each element, so that u, v, w is assumed to be defined 'piecewise' in the whole solid (/2/, p. 74 ff, p.133).

3. Element matrix and system matrix

Having in mind equ.'s (5) and (8) we insert (11) into (1). π_p is a function of u_K, v_K, w_K . Now the necessary condition for $\delta\pi_p = 0$ can be written:

$$\begin{aligned} \frac{\partial \pi_p}{\partial u_K} &= \sum_e \frac{\partial \pi_p^e}{\partial u_K} = 0 \\ \frac{\partial \pi_p}{\partial v_K} &= \sum_e \frac{\partial \pi_p^e}{\partial v_K} = 0 \\ \frac{\partial \pi_p}{\partial w_K} &= \sum_e \frac{\partial \pi_p^e}{\partial w_K} = 0 \end{aligned} \quad (12)$$

The index K denotes those points P_K of the solid which were generated when the division of the solid in finite elements was

carried out. The summation over e concerns all elements which have a common P_K . If we write down the equations (12) for all K we get equations which are linear in u_K, v_K, w_K . Expressed in matrix-form this is known as 'system matrix equation' (ref./2/, p. 99).

Now we come to the 'element-matrices': We define the column (vector) $\{U^e\}$ as:

$$\{U^e\} = \begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ \vdots \\ u_B \\ v_1 \\ \vdots \\ v_B \\ w_1 \\ \vdots \\ w_B \end{Bmatrix}, \quad (13)$$

where $u_1, \dots, w_B$ are the unknown values of (11).
We write:

$$\begin{aligned} \delta \pi_p^e &= \frac{\partial \pi_p^e}{\partial u_1} \delta u_1 + \dots + \frac{\partial \pi_p^e}{\partial w_B} \delta w_B \\ &= \{\delta U^e\}^T \frac{\partial \pi_p^e}{\partial \{U^e\}}, \end{aligned} \quad (14)$$

where:

$$\frac{\partial \pi_p^e}{\partial \{U^e\}} = \begin{Bmatrix} \frac{\partial \pi_p^e}{\partial u_1} \\ \vdots \\ \frac{\partial \pi_p^e}{\partial w_B} \end{Bmatrix}. \quad (15)$$

In equ. (1) we see that there are 4 terms in $\delta\pi_p^e$ so that we can write:

$$\frac{\partial \pi_p^e}{\partial \{U^e\}} = [K^e] \{U^e\} + \{F^e\}, \quad (16)$$

with

$$\{F^e\} = -\{F_F^e\} - \{F_{\Delta\theta}^e\} - \{F_T^e\}. \quad (17)$$

Equ. (16) is called the 'expanded element equation' (ref. /2/, p.59), $[K^e]$, $\{F_F^e\}$, $\{F_{\Delta\theta}^e\}$ and $\{F_T^e\}$ are our 'element-matrices'. They contain only constants which are not functions of $u_1, \dots, w_8$. We will not explain here how the element-equations (16) are assembled to the system-matrix-equation which results from (12); and how Dirichlet boundary conditions can be inserted (ref./2/, p.144, p.179).

4. The element matrices

After some notation-conventions we will give now explicit expressions for the element matrices.

4.1 Initial remarks

Looking at (1) we see that there are volume-integrals and surface-integrals.

a) Volume-Integrals

In the case of our finite element a standard integration is possible: $dV = r dr d\phi dz$. The integration over r , ϕ , z can be done independently. As indicated in (10) we envisage only those finite elements which don't contain the z -axis, therefore:

$$0 < r_1 \leq r \leq r_2 \quad ; \quad \varphi_1 \leq \varphi \leq \varphi_2 \quad ; \quad z_1 \leq z \leq z_2$$

Now we treat 2 integral-types which we need for the r -integration:

$$I_1(a_1, a_2, a_3) = \int_0^1 (a_1 t + b_1)(a_2 t + b_2)(a_3 t + b_3) dt$$

$$= \frac{1}{4} a_1 a_2 a_3 + \frac{1}{3} (a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3) + \frac{1}{2} (b_1 b_2 a_3 + b_1 a_2 b_3 + a_1 b_2 b_3) + b_1 b_2 b_3$$

$$I_2(m_1, m_2) = \int_{r_1}^{r_2} \frac{1}{r} (m_1 t + n_1)(m_2 t + n_2) dt$$

$$= \frac{1}{2} m_1 m_2 \Delta(R^2) + (m_1 n_2 + m_2 n_1) \Delta R + n_1 n_2 \ln \frac{r_2}{r_1},$$

whereby

$$\Delta(R^2) = r_2^2 - r_1^2.$$

There is a definite relation between the a_i 's and the b_i 's and between the m_i 's and n_i 's in such a way that when the a_i 's or m_i 's are given the b_i 's or n_i 's are fixed:

$$a_i = \begin{cases} 0 \\ 1 \\ -1 \\ \Delta R \end{cases} \Rightarrow b_i = \begin{cases} 1 \\ 0 \\ 1 \\ r_1 \end{cases} ; \quad m_i = \begin{cases} 0 \\ 1 \\ -1 \end{cases} \Rightarrow n_i = \begin{cases} 1 \\ -r_1 \\ r_2 \end{cases}$$

b) Surface Integrals

The six surfaces of our finite element are coordinate-surfaces (/ 3 /, p. 20); the integration is standard.

4.2 Evaluation of the stiffness matrix

Inserting (11) into (5) we can write:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \{\varepsilon\} = \begin{bmatrix} \frac{\partial \{p\}^T}{\partial r} & 0 & 0 \\ \frac{\{p\}^T}{r} & \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} & 0 \\ 0 & 0 & \frac{\partial \{p\}^T}{\partial z} \\ \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} & \frac{\partial \{p\}^T}{\partial r} - \frac{\{p\}^T}{r} & 0 \\ \frac{\partial \{p\}^T}{\partial z} & 0 & \frac{\partial \{p\}^T}{\partial r} \\ 0 & \frac{\partial \{p\}^T}{\partial z} & \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} \end{bmatrix} \{U^e\} = [B] \{U^e\} \quad (18)$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \{\sigma\} = [E] [B] \{U^e\} \quad (19)$$

Where $[E]$ is the matrix of the D_{ij} from equ. (8). As we have indicated this matrix is symmetric. The first term in (1) can now be written:

$$\begin{aligned} & \iiint_{V^e} \left[\{\delta U^e\}^T ([B]^T [E] [B]) \{U^e\} \right] dV \quad \text{or} \\ & [K^e] = \iiint_{V^e} [B]^T [E] [B] dV \\ & = \iiint_{V^e} \begin{bmatrix} K_{11}^* & K_{12}^* & K_{13}^* \\ \text{symm.} & K_{22}^* & K_{23}^* \\ & & K_{33}^* \end{bmatrix} dV = \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} \\ & \tilde{K}_{22} & \tilde{K}_{23} \\ \text{symm.} & & \tilde{K}_{33} \end{bmatrix}. \end{aligned} \quad (20)$$

The matrix $[B]^T [E] [B]$ is also symmetric. It is easy to verify that:

$$\begin{aligned}
 K_{11}^* &= \left(D_{11} \frac{\partial \{p\}}{\partial r} + D_{21} \frac{\{p\}}{r} \right) \frac{\partial \{p\}^T}{\partial r} + \left(D_{12} \frac{\partial \{p\}}{\partial r} + D_{22} \frac{\{p\}}{r} \right) \frac{\{p\}^T}{r} + D_{44} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} + \\
 &\quad + D_{55} \frac{\partial \{p\}}{\partial z} \frac{\partial \{p\}^T}{\partial z} \\
 K_{12}^* &= \left(D_{12} \frac{\partial \{p\}}{\partial r} + D_{22} \frac{\{p\}}{r} \right) \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} + D_{44} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \left(\frac{\partial \{p\}^T}{\partial r} - \frac{\{p\}^T}{r} \right) \\
 K_{13}^* &= \left(D_{13} \frac{\partial \{p\}}{\partial r} + D_{23} \frac{\{p\}}{r} \right) \frac{\partial \{p\}^T}{\partial z} + D_{55} \frac{\partial \{p\}}{\partial z} \frac{\partial \{p\}^T}{\partial r} \\
 K_{22}^* &= D_{22} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} + D_{44} \left(\frac{\partial \{p\}}{\partial r} - \frac{\{p\}}{r} \right) \left(\frac{\partial \{p\}^T}{\partial r} - \frac{\{p\}^T}{r} \right) + D_{66} \frac{\partial \{p\}}{\partial z} \frac{\partial \{p\}^T}{\partial z} \\
 K_{23}^* &= D_{23} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{\partial \{p\}^T}{\partial z} + D_{66} \frac{\partial \{p\}}{\partial z} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} \\
 K_{33}^* &= D_{33} \frac{\partial \{p\}}{\partial z} \frac{\partial \{p\}^T}{\partial z} + D_{55} \frac{\partial \{p\}}{\partial r} \frac{\partial \{p\}^T}{\partial r} + D_{66} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} .
 \end{aligned} \tag{21}$$

The matrices $K_{11}^*, \dots, K_{33}^*$ contain all together 24 terms (submatrices). Only 10 of them must be integrated, the remaining submatrices can be easily obtained (from the 10 integrated submatrices) by transposing. That can be derived from the fact;

$$\{x\} \{y\}^T = \left(\{y\} \{x\}^T \right)^T, \tag{22}$$

where $\{x\}$ and $\{y\}^T$ are columns and rows respectively of the same length. The 10 integrated submatrices are:

$$K_1 = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ -1 & -\frac{1}{2} & 1 & \\ -\frac{1}{2} & -1 & \frac{1}{2} & 1 \end{bmatrix}, \quad \text{symm.}$$

$$\iiint_{V_e} \frac{\partial \{p\}}{\partial r} \frac{\partial \{p\}^T}{\partial r} dV = \begin{bmatrix} K_1 & \text{symm.} \\ \frac{1}{2} K_1 & K_1 \end{bmatrix} \frac{r_1 + r_2}{18 \Delta R} \Delta \varphi \Delta z ;$$

$$K_{21} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -K_{21} & -K_{21} \\ K_{21} & K_{21} \end{bmatrix}$$

$$\iiint_{V^e} \frac{\{p\}}{r} \frac{\partial \{p\}^T}{\partial r} dV = \begin{bmatrix} K_2 & \frac{1}{2}K_2 \\ \frac{1}{2}K_2 & K_2 \end{bmatrix} \frac{\Delta \varphi \Delta z}{18};$$

$$K_3 = \begin{bmatrix} I_2(-1, -1) & I_2(-1, -1) & I_2(1, 1) & I_2(1, 1) \\ \frac{1}{2}I_2(-1, -1) & \frac{1}{2}I_2(1, -1) & \frac{1}{2}I_2(1, 1) & I_2(1, 1) \\ I_2(1, -1) & I_2(1, -1) & I_2(1, 1) & I_2(1, 1) \\ \frac{1}{2}I_2(1, -1) & I_2(1, -1) & \frac{1}{2}I_2(1, 1) & I_2(1, 1) \end{bmatrix}$$

symmetric

$$\iiint_{V^e} \frac{\{p\}}{r} \frac{\{p\}^T}{r} dV = \begin{bmatrix} K_3 & \text{symm.} \\ \frac{1}{2}K_3 & K_3 \end{bmatrix} \frac{\Delta \varphi \Delta z}{9(\Delta R)^2};$$

$$\iiint_{V^e} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} dV = \begin{bmatrix} K_3 & \text{symm.} \\ -K_3 & K_3 \end{bmatrix} \frac{\Delta z}{3\Delta \varphi (\Delta R)^2};$$

$$K_4 = \begin{bmatrix} I_1(-1, -1, \Delta R) & I_1(-1, -1, \Delta R) & I_1(1, 1, \Delta R) & I_1(1, 1, \Delta R) \\ -I_1(-1, -1, \Delta R) & -I_1(1, -1, \Delta R) & -I_1(1, 1, \Delta R) & I_1(1, 1, \Delta R) \\ I_1(1, -1, \Delta R) & I_1(1, -1, \Delta R) & -I_1(1, 1, \Delta R) & I_1(1, 1, \Delta R) \\ -I_1(1, -1, \Delta R) & I_1(1, -1, \Delta R) & -I_1(1, 1, \Delta R) & I_1(1, 1, \Delta R) \end{bmatrix}$$

symmetric

$$\iiint_{V^e} \frac{\partial \{p\}}{\partial z} \frac{\partial \{p\}^T}{\partial z} dV = \begin{bmatrix} K_4 & \text{symm.} \\ \frac{1}{2}K_4 & K_4 \end{bmatrix} \frac{\Delta \varphi \Delta R}{3\Delta z};$$

$$\iiint_{V^e} \frac{\partial \{p\}}{\partial r} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} dV = \begin{bmatrix} -K_2 & K_2 \\ -K_2 & K_2 \end{bmatrix} \frac{\Delta z}{12};$$

$$\iiint_{V^e} \frac{\{p\}}{r} \frac{1}{r} \frac{\partial \{p\}^T}{\partial \varphi} dV = \begin{bmatrix} -K_3 & K_3 \\ -K_3 & K_3 \end{bmatrix} \frac{\Delta z}{6(\Delta R)^2};$$

$$K_{51} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} K_{51} & \frac{1}{2}K_{51} \\ \frac{1}{2}K_{51} & K_{51} \end{bmatrix},$$

$$\iiint_{V^e} \frac{\{p\}}{r} \frac{\partial \{p\}^T}{\partial z} dV = \begin{bmatrix} K_5 & \frac{1}{2}K_5 \\ \frac{1}{2}K_5 & K_5 \end{bmatrix} \frac{\Delta R \Delta \varphi}{18};$$

$$K_{61} = \begin{bmatrix} -I_1(0, -1, \Delta R) & I_1(0, -1, \Delta R) \\ -I_1(0, -1, \Delta R) & I_1(0, -1, \Delta R) \end{bmatrix}, \quad K_{62} = \begin{bmatrix} -I_1(0, 1, \Delta R) & I_1(0, 1, \Delta R) \\ -I_1(0, 1, \Delta R) & I_1(0, 1, \Delta R) \end{bmatrix},$$

$$K_6 = \begin{bmatrix} K_{61} & K_{62} \\ -K_{61} & -K_{62} \end{bmatrix}, \quad \iiint_{V^e} \frac{\partial \{p\}}{\partial r} \frac{\partial \{p\}^T}{\partial z} dV = \begin{bmatrix} K_6 & \frac{1}{2}K_6 \\ \frac{1}{2}K_6 & K_6 \end{bmatrix} \frac{\Delta \varphi}{6};$$

$$K_{71} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$K_7 = \begin{bmatrix} K_{71} & \frac{1}{2}K_{71} \\ \frac{1}{2}K_{71} & K_{71} \end{bmatrix},$$

$$\iiint_{V^e} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \frac{\partial \{p\}^T}{\partial z} dV = \begin{bmatrix} K_7 & K_7 \\ -K_7 & -K_7 \end{bmatrix} \frac{\Delta R}{12};$$

Multiplying these 10 submatrices and the corresponding transposed matrices by appropriate D_{ij} and adding them according equ. (21) we get the matrices $\tilde{K}_{ij}$ and with them the stiffness matrix $[K^e]$.

4.3 Evaluation of the body-force-matrix

The components f_r, f_ϕ, f_z of the body-force-vector $\vec{F}$ are assumed to be constants and not functions of the space-coordinates r, ϕ, z throughout one finite element. The second term in (1) can therefore be written:

$$\{\delta U^e\}^T \iiint_{V^e} \begin{Bmatrix} f_r \{p\} \\ f_\phi \{p\} \\ f_z \{p\} \end{Bmatrix} dV.$$

According equ. (14), (16) we have:

$$\{\bar{F}_F^e\} = \iiint_{V^e} \begin{Bmatrix} f_r \{p\} \\ f_\phi \{p\} \\ f_z \{p\} \end{Bmatrix} dV. \quad (23)$$

Noting that:

$$\iiint_{V^e} \{p\} dV = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \frac{\Delta R}{8} \Delta \phi \Delta z, \quad (24)$$

we are able to compose (23).

4.4 Evaluation of the temperature-matrix

It is assumed that the temperature θ or the temperature difference $\Delta\theta = \theta - \theta_r$ is known all over the solid especially at the points P_i ($i = 1, 2, \dots, 8$) of one finite element. Introducing the matrices

$$\{\lambda\} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \{\Delta\theta\} = \begin{Bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \vdots \\ \Delta\theta_8 \end{Bmatrix} \quad (25)$$

the third term in (1) can be written:

$$\{\delta U^e\}^T \left[\iiint_{V^e} [B]^T \{\lambda\} \{p\}^T dV \right] \{\Delta\theta\}.$$

According to equ. (14) and (17) we have

$$\{F_{\Delta\theta}^e\} = \left[\iiint_{V^e} \begin{Bmatrix} \lambda_1 L_1^* \\ \lambda_2 L_2^* \\ \lambda_3 L_3^* \end{Bmatrix} dV \right] \{\Delta\theta\} = \begin{Bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \end{Bmatrix} \{\Delta\theta\}, \quad (26)$$

with

$$\begin{aligned} L_1^* &= \frac{\partial \{p\}}{\partial r} \{p\}^T \\ L_2^* &= \left[\frac{\{p\}}{r} + \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \right] \{p\}^T \\ L_3^* &= \frac{\partial \{p\}}{\partial z} \{p\}^T \end{aligned} \quad (27)$$

Integrating (27) we receive:

$$L_{11} = \begin{bmatrix} I_1(0, -1, \Delta R) & I_1(0, -1, \Delta R) \\ I_1(0, -1, \Delta R) & I_1(0, -1, \Delta R) \end{bmatrix}, \quad L_{12} = \begin{bmatrix} I_1(0, 1, \Delta R) & I_1(0, 1, \Delta R) \\ I_1(0, 1, \Delta R) & I_1(0, 1, \Delta R) \end{bmatrix},$$

$$L_1 = \begin{bmatrix} -L_{11} & -L_{12} \\ L_{11} & L_{12} \end{bmatrix},$$

$$\iiint_{V^e} \frac{\partial \{p\}}{\partial r} \{p\}^T dV = \begin{bmatrix} L_1 & \frac{1}{2}L_1 \\ \frac{1}{2}L_1 & L_1 \end{bmatrix} \frac{\Delta \varphi \Delta z}{9};$$

$$L_{21} = \begin{bmatrix} 1 & & & \\ \frac{1}{2} & & & \\ \frac{1}{2} & 1 & & \\ \frac{1}{4} & \frac{1}{4} & 1 & \\ & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \quad \iiint_{V^e} \frac{\{p\}}{r} \{p\}^T dV = \begin{bmatrix} L_{21} & \text{symm.} \\ \frac{1}{2}L_{21} & L_{21} \end{bmatrix} \frac{\Delta R \Delta \varphi \Delta z}{27};$$

$$L_{22} = \begin{bmatrix} -K_{21} & -\frac{1}{2}K_{21} \\ -\frac{1}{2}K_{21} & -K_{21} \end{bmatrix}, \quad \iiint_{V^e} \frac{1}{r} \frac{\partial \{p\}}{\partial \varphi} \{p\}^T dV = \begin{bmatrix} L_{22} & L_{22} \\ -L_{22} & -L_{22} \end{bmatrix} \frac{\Delta R \Delta z}{18};$$

$$L_{31} = \begin{bmatrix} I_1(-1, -1, \Delta R) & I_1(-1, -1, \Delta R) \\ -I_1(-1, -1, \Delta R) & -I_1(-1, -1, \Delta R) \end{bmatrix}, \quad L_{32} = \begin{bmatrix} I_1(1, -1, \Delta R) & I_1(1, -1, \Delta R) \\ -I_1(1, -1, \Delta R) & -I_1(1, -1, \Delta R) \end{bmatrix},$$

$$L_{33} = \begin{bmatrix} I_1(1, 1, \Delta R) & I_1(1, 1, \Delta R) \\ -I_1(1, 1, \Delta R) & -I_1(1, 1, \Delta R) \end{bmatrix}, \quad L_3 = \begin{bmatrix} L_{31} & L_{32} \\ L_{32} & L_{33} \end{bmatrix}$$

$$\iiint_{V^e} \frac{\partial \{p\}}{\partial z} \{p\}^T dV = \begin{bmatrix} L_3 & \frac{1}{2}L_3 \\ \frac{1}{2}L_3 & L_3 \end{bmatrix} \frac{\Delta \varphi \Delta R}{6};$$

Inserting these matrices into (26) we obtain $\{F_{\Delta\theta}^e\}$.

4.5 Evaluation of the surface-traction-matrix

As mentioned above the surfaces of the finite element are coordinate surfaces. Looking at fig. 1 we characterize these 6 surfaces $F_1, \dots, F_6$ as follows:

$$\begin{aligned}
 F_1 : & \quad P_1 P_2 P_5 P_6 & R_1 d\varphi dz & \quad r = R_1 & \quad 0 \leq \varphi^* \leq 1 & \quad 0 \leq z^* \leq 1 \\
 F_2 : & \quad P_3 P_4 P_7 P_8 & R_2 d\varphi dz & \quad r = R_2 & \quad " & \quad " \\
 F_3 : & \quad P_2 P_4 P_6 P_8 & r dr d\varphi & \quad 0 \leq r^* \leq 1 & \quad " & \quad z = z_1 \quad (28) \\
 F_4 : & \quad P_1 P_3 P_5 P_7 & " & \quad " & \quad " & \quad z = z_2 \\
 F_5 : & \quad P_1 P_2 P_3 P_4 & dr dz & \quad " & \quad \varphi = \varphi_1 & \quad 0 \leq z^* \leq 1 \\
 F_6 : & \quad P_5 P_6 P_7 P_8 & " & \quad " & \quad \varphi = \varphi_2 & \quad "
 \end{aligned}$$

On each surface 4 shape-functions are zero:

$$\begin{aligned}
 \text{on } F_1 : & \quad p_3(r, \varphi, z) = p_4(r, \varphi, z) = p_7(r, \varphi, z) = p_8(r, \varphi, z) = 0 \\
 " \quad F_2 : & \quad p_1 = p_2 = p_5 = p_6 = 0 \\
 " \quad F_3 : & \quad p_1 = p_3 = p_5 = p_7 = 0 \quad (29) \\
 " \quad F_4 : & \quad p_2 = p_4 = p_6 = p_8 = 0 \\
 " \quad F_5 : & \quad p_5 = p_6 = p_7 = p_8 = 0 \\
 " \quad F_6 : & \quad p_1 = p_2 = p_3 = p_4 = 0
 \end{aligned}$$

Similar to chapter 4.3 we assume that the components t_r , t_ϕ , t_z of the vector $\vec{T}$ of surface tractions are constants and not functions of the space-variables r , ϕ , z throughout one finite element. The fourth term in (1) can therefore be written:

$$\{\delta U^e\}^T \sum_{i=1}^6 \iint_{F_i} \begin{Bmatrix} t_r \{p\} \\ t_\phi \{p\} \\ t_z \{p\} \end{Bmatrix} dF_i.$$

According to equ. (14) and (16) we have:

$$\{F_F^e\} = \begin{Bmatrix} t_r \{\tilde{T}\} \\ t_\phi \{\tilde{T}\} \\ t_z \{\tilde{T}\} \end{Bmatrix} \quad (30)$$

with:

$$\{\tilde{T}\} = \sum_{i=1}^6 \{\tilde{T}_i\} = \sum_{i=1}^6 \iint_{F_i} \{p\} dF_i \quad (31)$$

$$\{\tilde{T}_1\} = \iint_{F_1} \{p\} R_1 d\phi dz = \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \frac{R_1 \Delta z \Delta \phi}{4};$$

$$\{\tilde{T}_2\} = \iint_{F_2} \{p\} R_2 d\phi dz = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{Bmatrix} \frac{R_2 \Delta \phi \Delta z}{4};$$

$$\{\tilde{T}_3\} = \iint_{F_3} \{p\} r dr d\varphi = \begin{Bmatrix} 0 \\ I_1(0, -1, \Delta R) \\ 0 \\ I_1(0, 1, \Delta R) \\ 0 \\ I_1(0, -1, \Delta R) \\ 0 \\ I_1(0, 1, \Delta R) \end{Bmatrix} \frac{\Delta R \Delta \varphi}{2} ;$$

$$\{\tilde{T}_4\} = \iint_{F_4} \{p\} r dr d\varphi = \begin{Bmatrix} I_1(0, -1, \Delta R) \\ 0 \\ I_1(0, 1, \Delta R) \\ 0 \\ I_1(0, -1, \Delta R) \\ 0 \\ I_1(0, 1, \Delta R) \\ 0 \end{Bmatrix} \frac{\Delta R \Delta \varphi}{2} ;$$

$$\{\tilde{T}_5\} = \iint_{F_5} \{p\} dr dz = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \frac{\Delta R \Delta z}{4} ;$$

$$\{\tilde{T}_6\} = \iint_{F_6} \{p\} dr dz = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \frac{\Delta R \Delta z}{4} .$$

Inserting this in (31) we get (30). The correspondence between (29) and the zero's in $\{\tilde{T}_1\}$ is remarkable.

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Appendix 3

About the coherency of the orthotropic properties

Examinations were executed concerning the properties of the epoxy layers. As mentioned above the oscillations sometimes occurring in the stresses were suspected to depend on the discrepancies in the given properties. This suspicion was emphasized by the computer message that the large stiffness matrix were not positive definite. We had to use a special solver for semi definite matrices, because of negative values on the matrix diagonal.

1. Stress-strain relations

We understand this if looking at the stress-strain relations, equation 8, page 57, in appendix 2. The D's, explicitly written and confined to the first three diagonal expressions, are

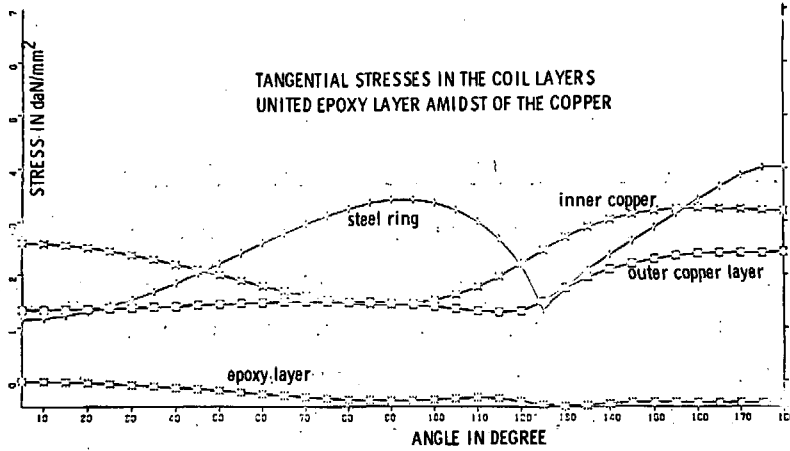
$$D_{11} = \frac{1 - \nu_{\phi z} \nu_{z\phi}}{F} E_r \quad (1)$$

$$D_{22} = \frac{1 - \nu_{rz} \nu_{zr}}{F} E_{\phi} \quad (2)$$

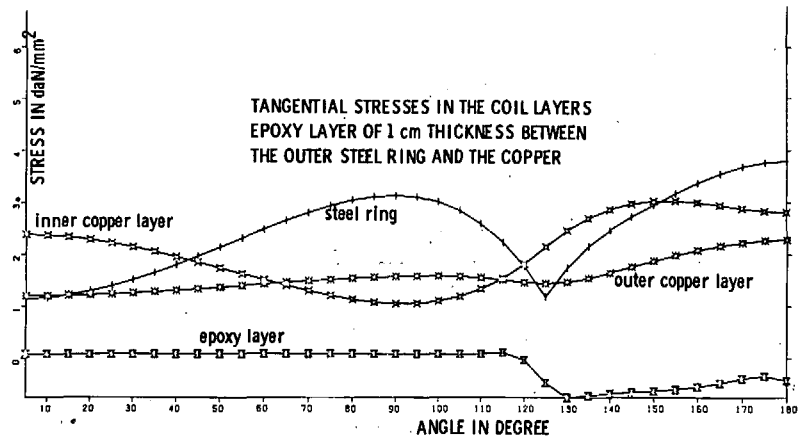
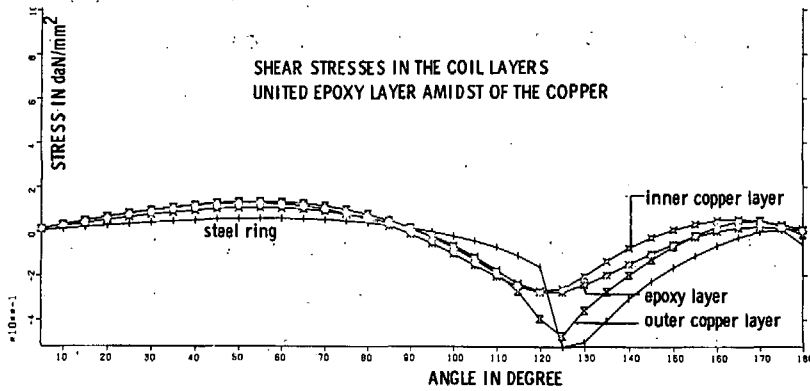
$$D_{33} = \frac{1 - \nu_{r\phi} \nu_{\phi r}}{F} E_z, \quad (3)$$

$$\text{with } F = 1 - \nu_{r\phi} \nu_{\phi r} - \nu_{rz} \nu_{zr} - \nu_{\phi z} \nu_{z\phi} - \nu_{r\phi} \nu_{\phi z} \nu_{zr} + \nu_{\phi r} \nu_{rz} \nu_{z\phi} \quad (4)$$

The matrix D is only then positive definite if all diagonal elements are positive.



model 1



model 2

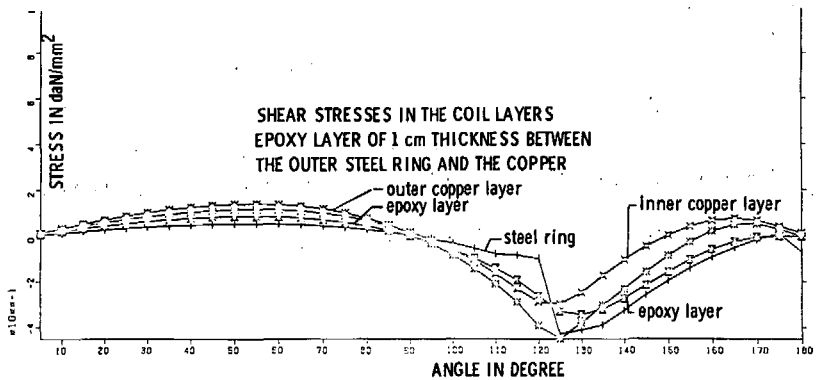


FIG.1 STRESS FIELDS IN THE TWO CALCULATION MODELS IF USING COHERENT STIFFNESS PROPERTIES IN THE EPOXY LAYER.

2. Necessary conditions

For that purpose the D's must be positive expressions. This restriction has its physical background in the fact that the strain energy stored in a structure must be positive, see ref. /1/, p.142, and ref. /2/, p.14.

It means, because of the positive amounts of the E's, the product of a pair of ν 's in equ. 1 to 3 has to be less than 1, and F, equ. 4, must be positive.

Three ν 's can be prescribed. The three remnant ν 's are then calculated according to the relations given in equ. 7, p.56, of appendix 2.

3. Calculations testing certain properties

Those necessary conditions were not satisfied in the already described calculations. The three given Poisson's ratios were

$$\nu_{r\phi} = \nu_{rz} = \nu_{\phi z} = 0.3 ,$$

wherefrom follows if we use $E_{\phi} = E_z = 2500 \text{ daN/mm}^2$ and $E_r = 350 \text{ daN/mm}^2$:

$$\nu_{\phi r} = \nu_{zr} = 2.143$$

$$\nu_{z\phi} = 0.3 .$$

F, D_{11} , D_{22} and D_{33} become less than zero. The D-matrix is semidefinite.

For getting a positive definite matrix we only need to change some Poisson's ratios. We may assume

$$\nu_{r\phi} = \nu_{rz} = 0.2$$

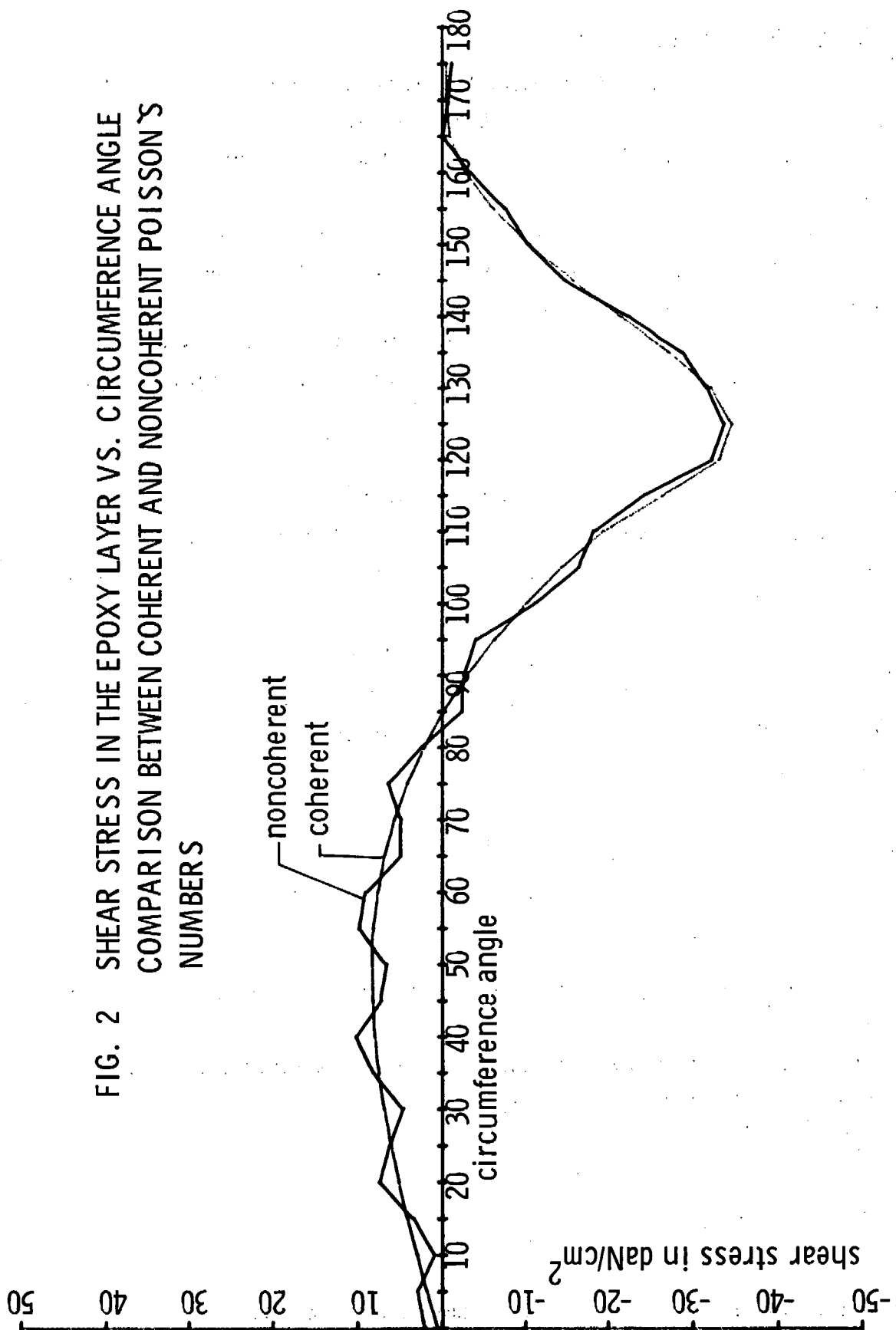
$$\nu_{\phi z} = 0.3 .$$

Then we obtain with the same E-moduli as above

$$\nu_{\phi r} = \nu_{zr} = 1.429$$

$$\nu_{z\phi} = 0.3 .$$

FIG. 2 SHEAR STRESS IN THE EPOXY LAYER VS. CIRCUMFERENCE ANGLE
COMPARISON BETWEEN COHERENT AND NONCOHERENT POISSON'S
NUMBERS



It is easy to check that the units F and D_{ii} now become positive.

Results of stress distributions gained with the new v 's are to see in fig. 1. Again we considered the former two models, where in the first case the epoxy formed an united orthotropic layer amidst of the copper. In the second case we have an epoxy layer of 1 cm thickness between the steel ring and the copper. Both computer runs concern the unheated state. We observe that all oscillations have vanished. Also there was no computer message complaining about semi definite matrices.

Comparing these results with the former ones we see that, fortunately, the oscillations did not deteriorate the former results in a way that they are useless. The average values agree quite well with the results gained with the purged properties, as can be seen in fig.2. Here the shear stresses in the epoxy of the 2nd model (epoxy between steel and copper) are plotted. The stress oscillations of the noncoherent material frame the monotonous contour of the epoxy with coherent properties.

We see further that the influence is small of changing a little the Poisson's ratios if we only consider the average shear stresses. But, of course, it does not satisfy completely to use on the one side given but not reasonable v 's, on the other side coherent but selfmade values, though the results seem to be logical. Measurements for finding the exact amounts are therefore necessary.

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Completions

Please attach on p.17 below:

It is to consider that these axial deflections will not occur in the real coil because they are caused only by our half coil model. The real whole coil carries lateral loads in its lower part which are equal but opposite directed compared with the upper lateral loads thus compensating the axial deformation. Only twisting moments remain.

Please attach on p.43 below:

A subsequent remark

After finishing the described optimization procedures we made a repetition but now with coherent orthotropic properties according to appendix 3. We learned that the somewhat disappointing result displayed in fig.23 was only a consequence of the noncoherency of the properties. Their destructive influence was even amplified by the iteration steps.

Using coherent properties delivers excellent results. Also the supposed goal of vanishing of the shearing stresses in the epoxy is to attain. We shall show that in a future report.