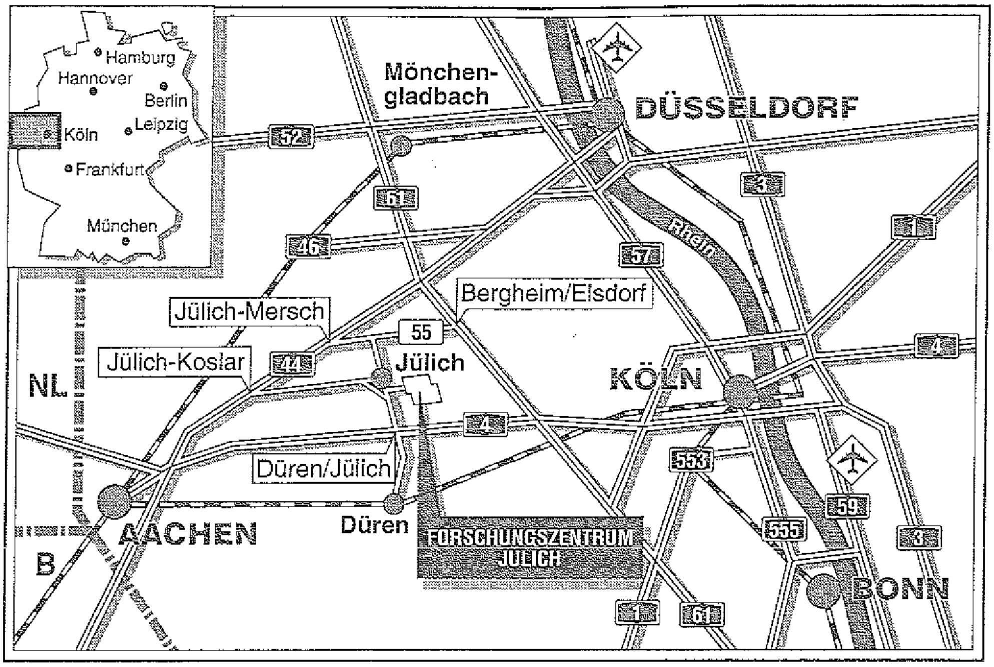
FORSCHUNGSZENTRUM JÜLICH GmbH

KFA

Institut für Plasmaphysik

Impurity Transport at the Plasma Edge: the Transition from 2D to 1D Models in Axisymmetric Toroidal Systems and the Role of Poloidal Source Distributions

H.A. Claaßen H. Gerhauser



Berichte des Forschungszentrums Jülich ; 2576
ISSN 0366-0885
Institut für Plasmaphysik Jül-2576

Zu beziehen durch: Forschungszentrum Jülich GmbH · Zentralbibliothek
Postfach 1913 · D-5170 Jülich · Bundesrepublik Deutschland
Telefon: 02461/61-6102 · Telefax: 02461/61-6103 · Telex: 833556-70 kfa d

Impurity Transport at the Plasma Edge: the Transition from 2D to 1D Models in Axisymmetric Toroidal Systems and the Role of Poloidal Source Distributions

H.A. Claaßen H. Gerhauser

Impurity Transport at the Plasma Edge: the Transition from 2D to 1D Models in axisymmetric Systems and the Role of poloidal Source Distributions

Summary:

Under due consideration of anomalous transport phenomena resulting from electromagnetic field fluctuations a 2D drift-kinetic equation for impurity ions is formulated in toroidal coordinates assuming the usual model for the helical magnetic field in axisymmetric tokamak configurations. A general solution for $T_{\parallel} \neq T_{\perp}$ (different kinetic temperatures for longitudinal and cross-field thermal motion) can be constructed in the form of a series expansion in Hermite polynomials with respect to $v_{\parallel}$ and in Laguerre polynomials with respect to $v_{\perp}$, which in the limit $T_{\parallel} = T_{\perp}$ is equivalent to a gyro-averaged version of Grad's expansion in tensorial Hermite polynomials. The lowest order approximations with respect to the Laguerre polynomials lead to kinetic equations for the reduced guiding-centre distribution functions $2\pi \int_0^\infty dv_{\perp} v_{\perp} (1,\ v_{\perp}^2) \zeta(\vec{x},\ v_{\parallel}, v_{\perp})$, which in the presence of anomalous transport can be more easily treated than the original drift-kinetic equation. The corresponding 2D balance equations in toroidal coordinates are averaged over the magnetic surfaces with appropriate weighting factors, leading to the usual 1D balance equations describing the netto fluxes in radial direction. In this derivation the role of parallel shear flows in the presence of anomalous viscosity and the effect of poloidally dependent source functions on the netto cross-field fluxes become apparent. It is shown as an example that a source concentration around the ALT-II limiter of TEXTOR yields an outward netto convection of impurity ions under normal operating conditions concerning the current and magnetic field direction.

Introduction:

Radial C^{z+} ($z \geq 4$) density profiles have been measured over a distance $r \geq 3o$ cm at the outer equatorial plane of TEXTOR using Li^{o} beam activated cx-spectroscopy /1/. This was done with the toroidal ALT II limiter in operation, which is located at $\theta = -\pi/4$ as seen from the outer midplane. Since this measurement is considered to be rather accurate and detailed, it should be appropriate for an impurity transport analysis in the r-θ plane assuming axisymmetry of the plasma. Unfortunately, such a 2D analysis is rather cumbersome, because it has to be combined with a numerical evaluation of the background hydrogen plasma /2/ with a feedback via the radiation losses and to some extent via the Coulomb collisions. Therefore, as a preliminary step, a 1D model, which averages the particle balance equation over the magnetic flux surfaces, was used for the interpretation of the measured C^{z+} density profiles taking experimental data for the background hydrogen plasma parameters.

In the following we discuss at some length the 2D drift-kinetic equations for impurity ions in an axisymmetric plasma (presupposing a toroidally symmetric limiter) and outline their solution by a specific moment approach, variations of which have already been reported and applied in previous work /3,4/. We then derive the 1D model by averaging the 2D balance equations over the magnetic surfaces, assumed to be concentric toroids.

The equations consider both anomalous diffusion and neoclassical drifts (ignoring the small effect of cross-field transport due to Coulomb collisions) and contain the relevant ionization, charge-exchange and recombination processes. In the derivation of the 1D-model the role of poloidal asymmetries in the source functions in modifying the Pfirsch-Schlüter transport is clarified. It is shown that for normal TEXTOR operation concerning the direction of the toroidal magnetic field and the induced tokamak current the ALT II limiter position at $\theta = -\pi/4$ leads to a net outward convection of the impurity ions. It turns out that consideration of an outward convection velocity yields, indeed, a better fit of the experimental data /5/, although other driving mechanisms cannot be ruled out and the interpretation of a poloidally localized measurement by a 1D-model based on a magnetic surface average of the balance equations must be considered with caution.

<u>2D-model:</u>

Using a toroidal coordinate system $\vec{x} = (r,\theta,\phi)$ with $\partial/\partial\phi = 0$ (axisymmetric case) and assuming concentric toroids for the magnetic surfaces with $R = R_o + r\cos\theta \equiv R_o h$ and $\vec{B} = (0,-\Theta,1)B_\phi^o/h$, where $\Theta(r) = r/(R_o q) \ll 1$ with $q(r)$ = safety factor, we have the following form of the drift-kinetic equation (neglecting the classical cross-field contributions from elastic collisions and calculating the drift velocities in the $\mathrm{rot}\vec{B} = 0$ approximation)/6/:

$$\left(\frac{1}{rR}\frac{\partial}{\partial r}(rR..) + \left(\frac{ZeE_r}{M} + \frac{v_{\parallel}^2\cos\theta}{R}\right)\frac{1}{v_{\perp}}\frac{\partial}{\partial v_{\perp}} - \frac{\cos\theta}{R}\left(1+ v_{\parallel}\frac{\partial}{\partial v_{\parallel}}\right)\right) D_{\perp}\frac{\partial}{\partial r}\frac{v_{\perp}}{2}\frac{\partial \overline{F}}{\partial v_{\perp}} +$$

$$+ v_r^D\frac{\partial \overline{F}}{\partial r} + (v_\theta^D - \Theta v_{\parallel})\frac{1}{r}\frac{\partial \overline{F}}{\partial \theta} - \frac{\Theta\sin\theta}{R}\frac{v_{\parallel}v_{\perp}^2}{2}\left(\frac{1}{v_{\perp}}\frac{\partial \overline{F}}{\partial v_{\perp}} - \frac{1}{v_{\parallel}}\frac{\partial \overline{F}}{\partial v_{\parallel}}\right) -$$

$$- \frac{E_z}{BR}\left(\frac{v_{\perp}}{2}\frac{\partial \overline{F}}{\partial v_{\perp}} + v_{\parallel}\frac{\partial \overline{F}}{\partial v_{\parallel}}\right) + \frac{ZeE_{\parallel}}{M}\frac{\partial \overline{F}}{\partial v_{\parallel}} = \frac{\partial_s \overline{F}}{\partial t} + \overline{Q} - \nu_I \overline{F}$$

$\overline{F}$, $\overline{Q}$ are the guiding-centre distribution and source functions of the impurity ions under consideration and depend on the space coordinates r,θ and the velocity components $v_{\parallel}$, $v_{\perp}$ parallel and perpendicular to the magnetic field.

$$v_\theta^D = -\frac{E_r}{B} - \frac{v_{\parallel}^2 + \frac{v_{\perp}^2}{2}}{\Omega R}\cos\theta \quad \text{and} \quad v_r^D = \frac{E_\theta}{B} - \frac{v_{\parallel}^2 + \frac{v_{\perp}^2}{2}}{\Omega R}\sin\theta$$

where $\Omega = ZeB/M$, are the drift velocities of the impurity ions due to electrostatic, centrifugal and gradB forces. The latter two strongly affect the vertical component of the electric field

$$E_z = E_r\sin\theta + E_\theta\cos\theta$$

creating a parallel electric field $E_\parallel = -\Theta E_\theta$, which drives the Pfirsch-Schlüter currents along $\vec{B}$. $\partial_s/\partial t|_{\vec{x}}$ denotes the Fokker-Planck collision operator for Coulomb interactions of the impurity ions. For interactions with hydrogen ions and electrons this operator adopts a differential form because of the low mass ratio $M_\pm/M \ll 1$ /7/:

$$\left.\frac{\partial_s \overline{F}}{\partial t}\right|_{\vec{x}} = \left.\frac{\partial_s \overline{F}}{\partial \tau_\parallel}\right|_{\vec{x}} + \left.\frac{\partial_s \overline{F}}{\partial \tau_\perp}\right|_{\vec{x}}$$

$$\left.\frac{\partial_s}{\partial \tau_\parallel}\right|_{\vec{x}} = \nu_c \left(v_T^2 \frac{\partial^2}{\partial v_\parallel^2} + (\alpha v_\parallel - \beta v_T) \frac{\partial}{\partial v_\parallel} + \alpha \right)$$

$$\left.\frac{\partial_s}{\partial \tau_\perp}\right|_{\vec{x}} = \nu_c \left(\frac{2\varepsilon v_T^2}{v_\perp} \frac{\partial}{\partial v_\perp} \left(v_\perp \frac{\partial}{\partial v_\perp}\right) + \gamma\left(v_\perp \frac{\partial}{\partial v_\perp} + 2\right) \right)$$

$$\nu_c = \frac{4\pi \ \ln\Lambda}{M^2} \left(\frac{ze^2}{4\pi\varepsilon_o}\right)^2 \frac{I_2}{v_T^2} \quad ,$$

$$\alpha = \frac{M}{M_\pm} v_T^2 \frac{I_4 - I_5}{I_2} \quad ; \quad \beta = \frac{M}{M_\pm} v_T \frac{I_3}{I_2} \quad ; \quad \gamma = \frac{M}{M_\pm} v_T^2 \frac{I_4}{I_2} \quad ; \quad \varepsilon = \frac{I_1}{2 I_2}$$

$\partial_s/\partial t|_{\vec{x}}$ is, of course, independent of the special choice of v_T. For further applications we will relate v_T to the temperature $T_\perp$ of the thermal ion motion across $\vec{B}$, i.e. we define $v_T = \sqrt{2kT_\perp/M}$. $I_1 \ldots I_5$ are integral coefficients given as velocity moments of the guiding-centre distribution function of the collision partner (+ hydrogen ions, - electrons).

The anomalous diffusion term $\sim D_\perp$ has been formulated with due consideration of curvature effects by the magnetic field. For simplification we consider here only the radial component of the anomalous diffusion flux, which is caused by fluctuations in the poloidal electric field and/or in the radial magnetic field with $D_\perp$ taken to be independent of $v_\parallel$ (a more general formulation is given in the Appendix). The consideration of anomalous diffusion in the drift-kinetic equation is complicated by the occurence of mixed derivatives with respect to $\vec{x}$ and $v_\perp$. To bypass this problem we expand $\bar{F}(\vec{x}, v_\parallel, v_\perp)$ in orthogonal polynamials with respect to $v_\perp$ generating a system of reduced kinetic equations for the expansion coefficients $a_n(\vec{x}, v_\parallel)$, which is easier to handle than the original drift-kinetic equation. A further expansion of $a_n(\vec{x}, v_\parallel)$ in orthogonal polynomials with respect to $v_\parallel$ ends up in a system of transport equations for the expansion coefficients $A_{mn}(\vec{x})$. In view of the eigenvalue properties of the Fokker-Planck collision operator for Coulomb interactions of impurity ions with hydrogen ions and electrons we choose an expansion in Laguerre and Hermite polynomials $L_n(\eta)$ and $H_m(\xi)$ respectively, where $\eta = v_\perp^2/v_T^2$ and

$\xi = \chi(v_\parallel - V_\parallel)$ are related to the parameters $v_\perp, \chi$ and $V_\parallel$ to be determined as part of the solution for the problem under consideration:

$$\bar{F}(\vec{x}, v_\parallel, v_\perp) = \sum_{n=0}^{\infty} \frac{a_n(\vec{x}, v_\parallel)}{\pi v_T^2} L_n(\eta) \exp(-\eta)$$

where $$a_n(\vec{x}, v_\parallel) = \frac{\chi}{\sqrt{\pi}} \sum_{m=0}^{\infty} A_{mn}(\vec{x}) H_m(\xi) \exp(-\xi^2)$$

χ is related to the kinetic temperature $T_\parallel$ for thermal ion motion along $\vec{B}$, i.e. $kT_\parallel = M/2\chi^2$, which in weakly collisional magnetized plasmas may strongly deviate from $T_\perp$. For $\chi = 1/v_T$ or $T_\parallel = T_\perp$ the above expansion corresponds to a gyro-averaged version of Grad's tensor polymial expansion /8/. In particular we find (after gyro-averaging)

within the 13-moment approximation

$$\left(\frac{\sqrt{\pi}^3 v_T^2}{\chi}\right) \bar{F}(\vec{x}, v_\parallel, v_\perp) \exp(\xi^2 + \eta) \simeq \sum_{m=0}^{3} A_{mo}(\vec{x}) H_m(\xi) L_o(\eta) + \sum_{m=0}^{1} A_{m1}(\vec{x}) H_m(\xi) L_1(\eta)$$

and $$\sum_{m=0}^{5} A_{mo}(\vec{x}) H_m(\xi) L_o(\eta) + \sum_{m=0}^{3} A_{m1}(\vec{x}) H_m(\xi) L_1(\eta) + \sum_{m=0}^{1} A_{m2}(\vec{x}) H_m(\xi) L_2(\eta)$$

within the 21-moment approximation

Due to the unified normalization of ξ and η (i.e. $\chi = 1/v_T$) the coefficients $A_{m,\alpha\geq 1}$ are related to A_{mo}. For the lowest-order coefficients the following relations hold:

$$A_{o1} = 2A_{2o} \;,\; A_{11} = -4A_{3o}$$

$$A_{o2} = 8A_{21} = -16A_{4o} \;,\; A_{12} = -4A_{31} = 32A_{5o}$$

connected to the velocity moments

$$A_{mo} = \frac{1}{2^m m!\chi} \int_{-\infty}^{+\infty} d\xi \; H_m(\xi) a_o(\vec{x}, v_\parallel)$$

The first three moments $A_{1o} \equiv \chi\Gamma_\parallel^\dagger$, $A_{2o} \equiv \chi^2\pi_{\parallel\parallel}/2M$ and $A_{3o} \equiv \chi^3 h_\parallel/5M$ are related to the longitudinal ("$\vec{B}$) diffusive particle flux with respect to $V_\parallel$, the dissipative (viscous) stress tensor component along $\vec{B}$ due to longitudinal gradients of $V_\parallel$, and the longitudinal heat flux of the impurity ions respectively. They are left (in addition to the density) from Grad's 13-moment approximation after gyro-averaging. The last two moments $A_{4o} \equiv \chi^4\sigma_{\parallel\parallel}/14M$ and $A_{5o} \equiv \chi^5 r_\parallel/70M$ appear in Grad's 21-moment approximation.

For the more general case $\chi \neq 1/v_T$ such simple relations between $A_{m,\alpha\geq 1}$ and A_{mo} do not exist and different sets of moment equations have to be formulated for A_{mn} with different index n /7/. In the following we use the guiding-centre distribution function $\overline{F}(\vec{x}, v_\parallel, v_\perp)$ and the gyro-averaged source function $\overline{Q}(\vec{x},v_\parallel,v_\perp)$ in zero-order approximation with respect to the Laguerre polynomial expansion taking n=0 with $L_o(\eta) \equiv 0$ (the index n=0 being omitted furtheron):

$$\begin{pmatrix} \overline{F} \\ \\ \overline{Q} \end{pmatrix}(\vec{x},v_\parallel,v_\perp) \approx \begin{pmatrix} a \\ \\ Q \end{pmatrix}(\vec{x},v_\parallel) \frac{\exp(-\eta)}{\pi v_T^2}$$

where $$\begin{pmatrix} a \\ \\ Q \end{pmatrix}(\vec{x},v_\parallel) \equiv 2\pi \int_{-\infty}^{+\infty} dv_\perp v_\perp \begin{pmatrix} \overline{F} \\ \\ \overline{Q} \end{pmatrix}(\vec{x},v_\parallel,v_\perp)$$

The distribution function $a(\vec{x},v_\parallel)$ satisfies the reduced kinetic equation

$$\hat{L}_o a = Q - \nu_I a$$

which is derived by taking the integral $2\pi \int_{-\infty}^{+\infty} dv_\perp v_\perp \ldots$ of the drift-kinetic equation. Whereas this equation includes the information about all parameters related to the parallel impurity ion motion, in particular $V_\parallel$ and χ, the parameter v_T related to the cross-field thermal motion of the impurity ions must be determined from

$$\hat{L}_1(v_T^2 a) = v_T^2(Q - \nu_I a) + 2(1-\gamma)\nu_c v_T^2 a$$

which follows from the integral $2\pi \int_{-\infty}^{+\infty} dv_\perp v_\perp^3 \ldots$ of the drift-kinetic equation. The second term on the rhs of this equation describes the collisional transfer of the cross-field thermal energy between impurity ions and hydrogen ions or electrons. $Q - \nu_I a$ denote the sources and losses due to inelastic collisions.

The operators $\hat{L}_o$ and $\hat{L}_1$ are given by the expressions

$$\hat{L}_o = \frac{1}{rR}\frac{\partial}{\partial r}\left(rR\left(-D_\perp \frac{\partial}{\partial r} + \hat{v}_r^D\right) .. \right) + \frac{1}{rR}\frac{\partial}{\partial \theta}\left(R\left(\hat{v}_\theta^D - \Theta v_\parallel\right) .. \right) +$$

$$\left(\frac{D_\perp \cos\theta}{R}\frac{\partial}{\partial r} - \frac{E_z}{BR}\right)\left(1 + v_\parallel \frac{\partial}{\partial v_\parallel}\right) + \left(\frac{ZeE_\parallel}{M} + \frac{v_T^2}{2}\frac{\Theta \sin\theta}{R}\right)\frac{\partial}{\partial v_\parallel} - \frac{\partial_s}{\partial \tau_\parallel}$$

$$\hat{L}_1 = \hat{L}_o^\dagger + \frac{E_z}{BR} + \frac{v_\parallel \Theta \sin\theta}{R} + \left(\frac{ZeE_r}{M} + \frac{v_\parallel^2 \cos\theta}{R}\right) 2D_\perp \frac{\partial}{\partial r}\left(\frac{1}{v_T^2} ..\right)$$

where $\hat{v}_r^D = \frac{E_\theta}{B} - \frac{v_\parallel^2 + \frac{v_T^2}{2}}{\Omega R}\sin\theta$ and $\hat{v}_\theta^D = -\frac{E_r}{B} - \frac{v_\parallel^2 + \frac{v_T^2}{2}}{\Omega R}\cos\theta$

$\hat{L}_o^\dagger$ results from $\hat{L}_o$ by substituting $D_\perp$ by $2D_\perp$ and $\frac{v_T^2}{2}$ by v_T^2 .

Note that $$\frac{1}{rR}\left(\frac{\partial}{\partial r}(rR\hat{v}_r^D) + \frac{\partial}{\partial \theta}(R\hat{v}_\theta^D)\right) = -\left.\frac{\partial}{\partial z}\right|_R \left(\frac{\frac{2Ze\Phi}{M} + v_\parallel^2 + \frac{v_T^2}{2}}{\Omega R}\right)$$

where $\left.\frac{\partial}{\partial z}\right|_R = \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial r}$

is the derivative with respect to the vertical direction z (for which R = const.). Φ denotes the electrostatic potential.

The first line in the expression for $\hat{L}_o$ describes the flux divergence due to drift motion and anomalous diffusion in radial direction as well as drift motion and convection in poloidal direction. The second line accounts for the longitudinal ($\parallel\vec{B}$) ac(de)celeration of impurity ions due to the Coriolis force (first term), the electric field and the magnetic mirror force (second term) as well as the friction and thermal forces contained in $\partial_s/\partial\tau_\parallel$. In a reference frame moving with $V_\parallel$ we have

$$\frac{\partial_s}{\partial\tau_\parallel} = \nu_c\left(v_T^2\frac{\partial^2}{\partial v_\parallel^2} + \left[\alpha(v_\parallel - V_\parallel) - \beta^\dagger v_T\right]\frac{\partial}{\partial v_\parallel} + \alpha\right)$$

where the connection between the drag (i.e. friction and thermal) forces in the moving and the laboratory reference frame is given by the relation

$$\beta^\dagger = \beta - \frac{\alpha V_\parallel}{v_T}$$

For shifted Maxwellian distribution functions of hydrogen ions and electrons the collision frequency ν_c reads

$$\nu_c = \frac{8\sqrt{\pi}}{3}\frac{\ln\Lambda}{M^2 v_T^2}\left(\frac{Ze^2}{4\pi\varepsilon_o}\right)^2\frac{n_\pm}{v_{T\pm}}$$

Solution method for $a(\vec{x}, v_{\shortparallel})$:

$\hat{L}_o$ may be decomposed into $\hat{O} + \hat{R}$ such that in connection with the already mentioned series expansion

$$\begin{pmatrix} a \\ \\ Q \end{pmatrix} (\vec{x}, v_{\shortparallel}) = \sum_{m=0}^{\infty} \begin{pmatrix} A_m \\ \\ Q_m \end{pmatrix} (\vec{x}) \Psi_m(\vec{x}, \xi)$$

$$\Psi_m(\vec{x}, \xi) \equiv \frac{1}{\sqrt{\pi}} \chi^{m+1}(\vec{x}) H_m(\xi) \exp(-\xi^2) \quad , \quad A_m(\vec{x}) \equiv \frac{A_{mo}}{\chi^m}(\vec{x})$$

an equation of the form

$$\hat{O}\Psi_m(\vec{x}, \xi) = \lambda_m \nu_c(\vec{x}) \Psi_m(\vec{x}, \xi) + c_I(\vec{x}) \Psi_{m+1}(\vec{x}, \xi) + c_{II}(\vec{x}) \Psi_{m+2}(\vec{x}, \xi)$$

holds. By setting $c_I(\vec{x}) = 0 = c_{II}(\vec{x})$ we obtain an eigenvalue equation for $\Psi_m(\vec{x}, \xi)$ with the eigenvalues $\lambda_m \nu_c(\vec{x})$ and auxiliary conditions for the parameters of the variable ξ : for $V_{\shortparallel}(\vec{x})$, the velocity shift, from $c_I(\vec{x}) = 0$, and $M/2\chi^2(\vec{x})$, the kinetic temperature of the ion motion along $\vec{B}$, from $c_{II}(\vec{x}) = 0$. The eigenvalues and the auxiliary conditions depend, of course, on the choice of the operator $\hat{O}$. This choice is dictated by the requirement to approach the centre-of-mass velocity and the true kinetic temperature along $\vec{B}$ by the parameters of ξ as far as possible, which guarantees a good convergence of the series expansion (moment) method.

For example, the choice $\hat{O} = \hat{L}_o - \hat{R}$ with

$$\hat{R} = -(v_{\parallel} - V_{\parallel})\left(\frac{\Theta}{rR}\frac{\partial(R..)}{\partial\theta} + \frac{E_z}{BR}\frac{\partial}{\partial v_{\parallel}}\right) - \frac{v_{\parallel}^2 - V_{\parallel}^2}{\Omega R}\frac{\partial}{\partial z}\bigg|_R +$$

$$+ \frac{D_{\perp}\cos\theta}{R}\frac{\partial}{\partial r}v_{\parallel}\frac{\partial}{\partial v_{\parallel}} + \frac{D_{\perp}}{2\chi^2}\frac{\partial\ln\chi}{\partial r}\frac{\partial^2}{\partial v_{\parallel}^2}\left(\frac{\partial}{\partial r} - \frac{\partial V_{\parallel}}{\partial r}\frac{\partial}{\partial v_{\parallel}}\right)$$

would generate the eigenvalues

$$\lambda_m \nu_c = m\alpha\nu_c + \frac{\Theta V_{\parallel}\sin\theta}{R} - \frac{\partial}{\partial z}\bigg|_R\left(\frac{\frac{Ze\Phi}{M} + \frac{v_T^2}{2}}{\Omega R}\right)$$

and the auxiliary conditions (zero-order balance equations for parallel momentum and energy of impurity ions)

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}(rD_{\perp}\frac{\partial}{\partial r}) + \vec{V}^{nc}\vec{\nabla}\right)\begin{cases} V_{\parallel} = \dfrac{-E_z V_{\parallel}}{BR} + \left(\dfrac{ZeE_{\parallel}}{M} + \dfrac{v_T^2}{2}\dfrac{\Theta\sin\theta}{R} + \beta^{\dagger}\nu_c v_T\right) \\[2ex] \dfrac{1}{4\chi^2} = \left(\dfrac{-\alpha}{2\chi^2} + v_T^2\right)\nu_c + D_{\perp}\left(\dfrac{\partial V_{\parallel}}{\partial r}\right)^2 \end{cases}$$

where

$$\vec{V}^{nc}\vec{\nabla} \equiv V_r^D\frac{\partial}{\partial r} + \frac{V_\theta^D - \Theta V_{\parallel}}{r}\frac{\partial}{\partial\theta}$$

with

$$V_\theta^D = -\frac{E_r}{B} - \frac{V_\parallel^2 + \frac{v_T^2}{2}}{\Omega R}\cos\theta \quad \text{and} \quad V_r^D = \frac{E_\theta}{B} - \frac{V_\parallel^2 + \frac{v_T^2}{2}}{\Omega R}\sin\theta$$

Note that $\frac{E_z}{BR} = -\hat{z}(Ze\Phi/M)$ with $\hat{z} \equiv \frac{1}{\Omega R}\frac{\partial}{\partial z}\Big|_R$

and that the anomalous viscosity effect on $V_\parallel$ and χ appears in the limit of small aspect ratio (i.e. in cylindrical coordinates).

Under these conditions the expansion coefficients A_m of the reduced guiding-centre distribution function $a(\vec{x}, v_\parallel)$ satisfy the moment equations+

$$Q_m - \nu_I A_m + R_m = \frac{1}{rR}\frac{\partial}{\partial r}\left(rR\left(-D_\perp \frac{\partial A_m}{\partial r} + \frac{E_\theta}{B}A_m - \frac{1}{M\Omega r}\frac{\partial p_m}{\partial \theta}\right)\right) -$$

$$-\frac{1}{rR}\frac{\partial}{\partial \theta}\left(R\left(\frac{E_r}{B} + \Theta V_\parallel\right)A_m + R\Theta(m+1)A_{m+1} - \frac{R}{M\Omega}\frac{\partial p_m}{\partial r}\right) - m\frac{D_\perp\cos\theta}{R}\frac{\partial A_m}{\partial r} +$$

$$+ mA_m\left(\alpha\nu_c - \frac{\Theta}{r}\frac{\partial V_\parallel}{\partial \theta} + \frac{E_z}{BR} - \hat{z}(V_\parallel^2) + \frac{m-1}{2\chi^2}\hat{z}(\ln\chi)\right) - m(m+1)A_{m+1}\hat{z}(V_\parallel)$$

+ where $\frac{p_m}{M} = \left(V_\parallel^2 + \frac{v_T^2}{2} + \frac{2m+1}{2\chi^2}\right)\frac{A_m}{2} + (m+1)V_\parallel A_{m+1} + (m+1)(m+2)\frac{A_{m+2}}{2}$

with the specific recursion term (related to the above choice of the operator $\hat{R}$)

$$R_m = (1-\delta_{mo})\left[2D_\perp \frac{\partial V_\parallel}{\partial r}\frac{\partial A_{m-1}}{\partial r} + \frac{\chi^{m-1}}{rR}\frac{\partial}{\partial\theta}\left(\frac{R\Theta A_{m-1}}{2\chi^{m+1}}\right) + \chi^{m-1}\hat{z}\left(\frac{V_\parallel A_{m-1}}{\chi^{m+1}}\right) + \right.$$

$$\left. + \frac{2m-1}{2\chi^2} A_{m-1}\hat{z}(V_\parallel) + \frac{D_\perp\cos\theta}{R}\left(\frac{\partial}{\partial r}(V_\parallel A_{m-1}) + mA_{m-1}\frac{\partial V_\parallel}{\partial r}\right)\right] +$$

$$+ (1-\sum_{r=0}^{1}\delta_{mr})\left[\left(\frac{\Theta}{r}\frac{\partial V_\parallel}{\partial\theta} + \hat{z}\left(\frac{Ze\Phi}{M} + V_\parallel^2 + \frac{2m+1}{4\chi^2}\right) + \right.\right.$$

$$\left. + \frac{\chi^2 D_\perp\cos\theta}{R}\frac{\partial}{\partial r}\left(V_\parallel^2 + \frac{m+1}{2\chi^2}\right)\right)\frac{A_{m-2}}{2\chi^2} + \frac{\hat{z}(A_{m-2})}{4\chi^4} + \frac{D_\perp}{R}\frac{\partial}{\partial r}\left(\frac{R}{2\chi^2}\right)\frac{\partial A_{m-2}}{\partial r}\right] +$$

$$+ (1-\sum_{r=0}^{2}\delta_{mr})\left[- D_\perp\frac{\partial\chi}{\partial r}\frac{\partial V_\parallel}{\partial r} - \frac{\Theta}{4\chi r}\frac{\partial \ln\chi}{\partial\theta} + \frac{\chi}{4}\hat{z}\left(\frac{V_\parallel}{\chi^2}\right) + \right.$$

$$\left. + \frac{D_\perp\cos\theta}{R}\frac{\chi^2}{2}\frac{\partial}{\partial r}\left(\frac{V_\parallel}{\chi}\right)\right]\frac{A_{m-3}}{\chi^3} +$$

$$+ (1-\sum_{r=0}^{3}\delta_{mr})\left[D_\perp\left(\frac{\partial\ln\chi}{\partial r}\right)^2 - \frac{\hat{z}(\ln\chi)}{2\chi^2} - \frac{D_\perp\cos\theta}{R}\frac{\partial\ln\chi}{\partial r}\right]\frac{A_{m-4}}{4\chi^4}$$

Such a solution method does not account for the influence of impurity ion density gradients on $V_\parallel$ and χ. Only volume forces proportional to the ion density itself (which in the auxiliary conditions cancels out) are involved in the determination of $V_\parallel$ and χ. For $D_\perp = 0$ the auxiliary conditions formally correspond to the cold ion approximation $\chi \to \infty$ (presupposing also $T_{\pm\parallel} \to 0$), where $\Psi_o(\vec{x},\xi) \to A_{oo}\delta(v_\parallel - V_\parallel)$. For finite χ the method is adequate (with good convergence properties of the series expansion) for the description of neoclassical impurity ion transport either in plasmas with weak impurity density gradients along $\vec{B}$ (i.e. in the presence of rather homogeneous longitudinal source distributions) independent of the longitudinal collisionality (see ref./7/) or in collision-dominated plasmas, where strong impurity density gradients can also build-up along $\vec{B}$. The inclusion of anomalous cross-field transport in the presence of the always existing radial shell structure of the impurity charge state profiles may deteriorate the convergence properties of the series expansion method adopted here, since the influence of the anomalous ion flux on $V_\parallel$ and χ is neglected, although the anomalous viscosity effect is partly taken into account.

A more general procedure, which is not bound to the above mentioned restrictions, but ceases to have the advantage of calculating $V_\parallel$ and χ independent of A_o, incorporates the equation $\hat{O}\Psi_m = \lambda_m \nu_c \Psi_m + c_I \Psi_{m+1} + c_{II}\Psi_{m+2}$ into

$$\sum_{m=0}^{\infty} \hat{L}_o (A_m \Psi_m) = \sum_{m=0}^{\infty} (Q_m - \nu_I A_m)\Psi_m$$

without setting $c_I = 0 = c_{II}$. Such a procedure adds the terms $-(1-\delta_{mo})c_I A_{m-1} - (1-\sum_{r=0}^{1}\delta_{mr})c_{II}A_{m-2}$ to the recursion formula R_m and produces more general equations for $V_\parallel$ and χ, if the conditions $A_o c_I - R_1 = 0 = A_o c_{II} - R_2$ are imposed. In contrast to $c_I = 0 = c_{II}$ these conditions do not depend on the special decomposition of $\hat{L}_o$ into $\hat{O} + \hat{R}$ and follow in a more conventional way from the complete set of particle, parallel momentum and energy balance equations (outlined on the following pages) by setting $A_{m\geq 1} = 0$ and $Q_{m\geq 1} = 0$. The more general equations for $V_\parallel$ and χ are interrelated and have to be solved simultaneously with the particle balance.

Balance equations:

The 2D neoclassical balance equations (with additive terms describing anomalous transport) for particle density, parallel momentum and total kinetic energy (with respect to cross-field and parallel motion in the laboratory system) of the impurity ions are easily derived by integrating the drift-kinetic equation for $\bar{F}(\vec{x},v_{\parallel},v_{\perp})$ over the velocity space $(v_{\parallel},v_{\perp})$ with the weighting factors 1, $Mv_{\parallel}$ and $M(v_{\parallel}^2+v_{\perp}^2)/2$ respectively.[+] In zero-order approximation with respect to the Laguerre polynomial expansion of $\bar{F}(\vec{x},v_{\parallel},v_{\perp})$ the moment equations for A_m with $m \leq 2$ define the corresponding balance equations with respect to the parallel ion motion in a reference frame moving with $V_{\parallel}$. The parallel momentum and energy equations can be transformed from one reference system to the other by due consideration of the auxiliary conditions for $V_{\parallel}$ and χ.

particle balance equation: $\vec{\nabla}\vec{\Gamma} = \delta S$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[rR\left(-D_{\perp}\frac{\partial n}{\partial r}+\frac{E_{\theta}}{B}n\right)-\frac{R}{\Omega}\frac{\partial}{\partial\theta}\left(\frac{p_{\parallel}^{T}+p_{\perp}}{2M}\right)\right]-$$

$$-\frac{1}{rR}\frac{\partial}{\partial\theta}\left[R\left(\Theta\Gamma_{\parallel}+\frac{E_r}{B}n\right)-\frac{R}{\Omega}\frac{\partial}{\partial r}\left(\frac{p_{\parallel}^{T}+p_{\perp}}{2M}\right)\right]=S-\nu_I n\equiv\delta S$$

where $S \equiv \int_{-\infty}^{+\infty} dv_{\parallel} Q(\vec{x},v_{\parallel})$, $\Gamma_{\parallel} \equiv \int_{-\infty}^{+\infty} dv_{\parallel} v_{\parallel} a(\vec{x},v_{\parallel})$

and $p_{\parallel}^{T}+p_{\perp} \equiv 2\pi\int_{0}^{+\infty} dv_{\perp} v_{\perp}\int_{-\infty}^{+\infty} dv_{\parallel} M\left(v_{\parallel}^2+\frac{v_{\perp}^2}{2}\right)\bar{F}(\vec{x},v_{\parallel},v_{\perp})$

+ for the purely neoclassical transport see also ref./9/

The zero-order approximation with respect to the Laguerre polynamial expansion of $\vec{\vec{F}}(\vec{x},v_{\parallel},v_{\perp})$ generates the following relations between physical quantities and the coefficients A_m:

$n = A_o$ (particle density), $\Gamma_{\parallel} = V_{\parallel}A_o + A_1$ (total, i.e. convective plus conductive particle flux), $p_{\parallel} = M(A_o/2\chi^2 + 2A_2)$ (diagonal element $\parallel\vec{B}$ of the pressure tensor in a reference frame moving with $V_{\parallel}$), $p_{\perp} \simeq A_o M v_T^2/2$ (pressure of cross-field thermal motion of the ions) and $p_{\parallel}^T = p_{\parallel} + MV_{\parallel}^2 A_o + 2MV_{\parallel}A_1$ (diagonal element $\parallel\vec{B}$ of the total pressure tensor). Note that $p_{\perp} = (n - A_{o1})kT_{\perp}$, but

$$\frac{p_{\parallel}^T + p_{\perp}}{2M} \rightarrow \frac{nk(T_{\parallel}+T_{\perp})}{2M} + \frac{nV_{\parallel}^2}{2} + (V_{\parallel}A_1 + A_2) \quad \text{for } \vec{\vec{F}}(\vec{x},v_{\parallel},v_{\perp}) = \frac{a(\vec{x},v_{\parallel})}{\pi v_T^2}\exp(-$$

where A_2 is the dissipative (viscous) part of the diagonal stress tensor element $\parallel\vec{B}$.

parallel momentum balance: $(\vec{\nabla}\vec{\vec{T}})_{\parallel} = J_{\parallel}$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[rR\left(-D_{\perp}\frac{\partial\Gamma_{\parallel}}{\partial r} + \frac{E_{\theta}\Gamma_{\parallel}}{B}\right) - \frac{R^2}{2M}\frac{\partial P_{\parallel\perp}}{\partial\theta}\right] - \frac{D_{\perp}\cos\theta}{R}\frac{\partial\Gamma_{\parallel}}{\partial r} -$$

$$-\frac{1}{rR}\frac{\partial}{\partial\theta}\left[R\left(\frac{\Theta p_{\parallel}^T}{M} + \frac{E_r\Gamma_{\parallel}}{B}\right) - \frac{R^2}{2M}\frac{\partial P_{\parallel\perp}}{\partial r}\right] + \frac{E_z\Gamma_{\parallel}}{BR} -$$

$$-\left(\frac{ZeE_{\parallel}}{M} + \frac{v_T^2}{2}\frac{\Theta\sin\theta}{R}\right)n = \frac{\delta I_{\parallel}}{M} \equiv \int_{-\infty}^{+\infty} dv_{\parallel} v_{\parallel}\left(\frac{\partial_s a}{\partial\tau_{\parallel}} + Q - \nu_I a\right)$$

$$\text{where}\quad P_{\parallel\perp} = 2\pi\int_0^{+\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel M v_\parallel \frac{v_\parallel^2 + \frac{v_\perp^2}{2}}{\Omega R}\,\overline{F}(\vec{x}, v_\parallel, v_\perp)$$

$$\frac{2M}{\Omega R}\left(v_\parallel^2 A_1 + 3(V_\parallel A_2 + A_3)\right) + (MV_\parallel^2 + kT_\perp + 3kT_\parallel)\frac{\Gamma_\parallel}{\Omega R} \quad \text{for}\ \overline{F}(\vec{x}, v_\parallel, v_\perp) = \frac{a(\vec{x}, v_\parallel)}{\pi v_T^2}\exp(-\eta)$$

is the parallel momentum transported by the vertical drift due to centrifugal and gradB forces. The parallel momentum balance equation can be written in a more compact form by introducing the stress tensor components $T_{\phi r}$ and $T_{\phi\theta}$ in toroidal coordinates (note that $T_{\phi r,\theta} \simeq T_{\parallel r,\theta}$ and $J_\phi \simeq J_\parallel$)

$$\frac{1}{rR}\left[\frac{\partial}{\partial r}(rRT_{\phi r}) + \frac{\partial}{\partial\theta}(RT_{\phi\theta})\right] + \frac{1}{R}(\cos\theta\ T_{\phi r} - \sin\theta\ T_{\phi\theta}) =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(rT_{\phi r}) + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial\theta} + \frac{2}{R}(\cos\theta\ T_{\phi r} - \sin\theta\ T_{\phi\theta}) = J_\phi$$

where ($V_r^E \equiv E_\theta/B$ and $V_\theta^E \equiv -E_r/B$)

$$T_{\parallel r}^{nc} = 2\pi\int_0^{+\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel M v_\parallel v_r^D\,\overline{F}(\vec{x}, v_\parallel, v_\perp) = M\Gamma_\parallel V_r^E - P_{\parallel\perp}\sin\theta$$

and

$$T_{\parallel\theta}^{nc} = 2\pi\int_0^{+\infty} dv_\perp v_\perp \int_{-\infty}^{+\infty} dv_\parallel M v_\parallel (v_\theta^D - \Theta v_\parallel)\overline{F}(\vec{x}, v_\parallel, v_\perp) = M\Gamma_\parallel V_\theta^E - P_{\parallel\perp}\cos\theta - \Theta p_\parallel^T$$

are the neoclassical parts of the stress tensor components, and $$T^{a}_{\parallel r} = - \eta^{a}_{\perp} \frac{\partial}{\partial r}\left(\frac{\Gamma_{\parallel}}{n}\right) + M\Gamma_{\parallel} V^{a}_{r}$$

with $\eta^{a}_{\perp} = MnD_{\perp}$ and $nV^{a}_{r} = -D_{\perp}\frac{\partial n}{\partial r}$

denotes the anomalous part. $\eta^{a}_{\perp}$ and nV^{a}_{r} are respectively the anomalous viscosity and the anomalous diffusive particle flux in radial direction. An anomalous contribution to $T_{\parallel\theta}$ would also appear, if diffusion due to radial electric field fluctuations would be taken into account.

$$J_{\parallel} \equiv ZenE_{\parallel} + (p^{T}_{\parallel} + p_{\perp})\frac{\Theta\sin\theta}{R} + \delta I_{\parallel}$$

is the force component $\parallel\vec{B}$. It comprises electrostatic, gradB and centrifugal force components as well as the collisional momentum transfer $\parallel\vec{B}$ due to elastic collisions (friction and thermal forces) and inelastic processes. By definition of $T^{nc}_{\parallel r,\theta}$, $T^{a}_{\parallel r}$ and $J_{\parallel}$ we have

$$J_{\parallel} - \frac{2}{R}(\cos\theta\ T_{\parallel r} - \sin\theta\ T_{\parallel\theta}) =$$

$$-2M\left(\frac{E_z}{B} - D_{\perp}\cos\theta\frac{\partial\ln\Gamma_{\parallel}}{\partial r}\right)\frac{\Gamma_{\parallel}}{R} - p^{T}_{\parallel}\frac{\Theta\sin\theta}{R} + ZenE_{\parallel} + p_{\perp}\frac{\Theta\sin\theta}{R} + \delta I_{\parallel}$$

where the first two terms on the rhs denote the negative

Coriolis force and the centripetal force. In a (r,θ)-coordinate system rotating with the angular frequency $\Gamma_\phi/nR(\Gamma_\phi \simeq \Gamma_\parallel)$ around the major axis R = O of the toroidal coordinate system these forces are partly canceled.+

energy balance: $\vec{\nabla}\vec{q} - Ze\vec{E}\vec{\Gamma} = \delta Q$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left[rR\left(-D_\perp\frac{\partial}{\partial r}\left(\frac{p_\parallel^T}{2}+2p_\perp\right)+\frac{E_\theta}{B}\left(\frac{p_\parallel^T}{2}+p_\perp\right)\right)-\frac{R^2}{2}\frac{\partial q_\perp}{\partial\theta}\right]-$$

$$-\frac{1}{rR}\frac{\partial}{\partial\theta}\left[R\left(\theta q_\parallel^T+\frac{E_r}{B}\left(\frac{p_\parallel^T}{2}+p_\perp\right)\right)-\frac{R^2}{2}\frac{\partial q_\perp}{\partial r}\right]+\frac{E_z(p_\parallel^T+p_\perp)}{BR}-$$

$$-Ze\left(E_\parallel\Gamma_\parallel - E_rD_\perp\frac{\partial n}{\partial r}\right)=2\pi\int_0^{+\infty}dv_\perp v_\perp\int_{-\infty}^{+\infty}dv_\parallel\frac{M}{2}(v_\parallel^2+v_\perp^2)\left(\frac{\partial_s\bar{F}}{\partial t}+\bar{Q}-\nu_I\bar{F}\right)\equiv\delta Q$$

where the energy fluxes are defined by the integrals

$$\begin{pmatrix} q_\parallel^T \\ \\ q_\perp \end{pmatrix}\equiv 2\pi\int_0^{+\infty}dv_\perp v_\perp\int_{-\infty}^{+\infty}dv_\parallel\frac{M}{2}(v_\parallel^2+v_\perp^2)\begin{pmatrix} v_\parallel \\ \\ \dfrac{v_\parallel^2+v_\perp^2/2}{\Omega R}\end{pmatrix}\bar{F}(\vec{x},v_\parallel,v_\perp)$$

We note that $-D_\perp\partial/\partial r(p_\parallel^T/2+2p_\perp)$ may be decomposed into a conductive part $-\lambda_\perp^a\partial(0.5kT_\parallel+2kT_\perp)/\partial r$ with $\lambda_\perp^a = nD_\perp$

+) $E_z/B - D_\perp\cos\theta\,\partial\ln\Gamma_\parallel/\partial r$ may be considered as an effective velocity (directed along R) relative to the rotating coordinate system.

and a corresponding convective part $\left(\frac{kT_{\parallel}}{2} + 2kT_{\perp}\right) nV_r^a$

Moreover+ $q_{\parallel}^T \to \frac{P_{\parallel\perp}}{2}\Omega R + \frac{kT_{\perp}}{2}\Gamma_{\parallel}$ for $\bar{F}(\vec{x},v_{\parallel},v_{\perp}) = \frac{a(\vec{x},v_{\parallel})}{\pi v_T^2}\exp(-\eta)$

and $\frac{E_z(p_{\parallel}^T + p_{\perp})}{BR} = -Ze(E_r\Gamma_r^{C,\vec{\nabla}B} + E_{\theta}\Gamma_{\theta}^{C,\vec{\nabla}B})$, where

$$\begin{pmatrix}\Gamma_r \\ \\ \Gamma_{\theta}\end{pmatrix}^{C,\vec{\nabla}B} \equiv -2\pi\int_0^{+\infty} dv_{\perp} v_{\perp} \int_{-\infty}^{+\infty} dv_{\parallel} \frac{v_{\parallel}^2 + \frac{v_{\perp}^2}{2}}{\Omega R} \bar{F}(\vec{x},v_{\parallel},v_{\perp}) \begin{pmatrix}\sin\theta \\ \\ \cos\theta\end{pmatrix}$$

are the cross-field particle fluxes due to centrifugal and gradB drifts. We mention that the divergence terms appearing in the balance equations can be put in one of the following forms

$$\frac{1}{rR}\left(\frac{\partial}{\partial r}\left(-\frac{R^2}{2}\frac{\partial G}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{R^2}{2}\frac{\partial G}{\partial r}\right)\right) =$$

$$-\frac{1}{rR}\left(\frac{\partial}{\partial r}(rRG\sin\theta) + \frac{\partial}{\partial \theta}(RG\cos\theta)\right) = -\left.\frac{\partial G}{\partial z}\right|_R$$

with $G = \frac{p_{\parallel}^T + p_{\perp}}{M\Omega R}$, $P_{\parallel\perp}$ and $q_{\perp}$ respectively.

+) exact formulae for $P_{\parallel\perp}$ and $q_{\parallel}^T$ would result from $kT_{\perp}\Gamma_{\parallel} \to kT_{\perp}(\Gamma_{\parallel} - \frac{A_{11}}{X} - V_{\parallel}A_{01})$

Transition to a θ-averaged 1D-model:

By averaging the particle balance equation over the magnetic surfaces of the standard magnetic field configuration (which are concentric toroids) with the weighting factor $h = 1+r\cos\theta/R_o \equiv R/R_o$ we obtain the 1D equation

$$\frac{1}{r}\frac{\partial}{\partial r} r\int_0^{2\pi}\frac{d\theta}{2\pi}\left[-hD_\perp\frac{\partial n}{\partial r}+h^2\left(\frac{E_\theta}{B_o}n-\frac{1}{\Omega_o r}\frac{\partial}{\partial\theta}\left(\frac{p_\parallel^T+p_\perp}{2M}\right)\right)\right]=\int_0^{2\pi}\frac{d\theta}{2\pi}h(S-\nu_I n)$$

The averaged radial electric and diamagnetic drift terms can be expressed by the $\parallel\vec{B}$ momentum transfer via elastic and inelastic collisions and via the anomalous viscosity in sheared $\parallel\vec{B}$ flows, if we neglect the momentum transport due to the drift motion. To this end we average the parallel momentum equation with respect to θ using the weighting factor h^2 and assuming

$$\int_0^{2\pi}\frac{d\theta}{2\pi}h\left[h\frac{E_z\Gamma_\parallel}{BR}+\frac{1}{r}\frac{\partial}{\partial r}\left(rh\left(v_r^E\Gamma_\parallel-\frac{P_{\parallel\perp}}{M}\sin\theta\right)\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(h\left(v_\theta^E\Gamma_\parallel-\frac{P_{\parallel\perp}}{M}\cos\theta\right)\right)\right]$$

$$<<\int_0^{2\pi}\frac{d\theta}{2\pi}\left(h^2\frac{\delta I_\parallel}{M}+\frac{1}{r}\frac{\partial}{r}\left(rh^2D_\perp\frac{\partial\Gamma_\parallel}{\partial r}\right)\right)\approx -\Theta\Omega_o\int^{\pi}\frac{d\theta}{2\pi}h^2\left(\frac{E_\theta}{B_o}n-\frac{1}{\Omega_o r}\frac{\partial}{\partial\theta}\left(\frac{p_\parallel^T+p_\perp}{2M}\right)\right)$$

the pressure terms being integrated by parts and $E_\parallel$ set equal to $-\Theta E_\theta$. Inserting this result into the averaged particle balance equation yields

$$\frac{1}{r}\frac{\partial}{\partial r}\, r \int_0^{2\pi}\frac{d\theta}{2\pi}\left[-hD_\perp\frac{\partial n}{\partial r}+\frac{1}{\Theta\Omega_o}\left(h^2\frac{\delta I_\parallel}{M}+\frac{1}{r}\frac{\partial}{\partial r}(rh^2D_\perp\frac{\partial\Gamma_\parallel}{\partial r})\right)\right]\simeq$$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\, h(S-\nu_I n)$$

It follows from this equation that – in addition to asymmetries from inelastic collision processes in the presence of radial electron density and temperature gradients – sheared $\parallel\vec{B}$ flows in the presence of cross-field viscosity (assumed to be anomalous) can also lead to asymmetries around the maxima of the radial charge state profiles of the impurities.

The traditional way to perform the averaging process in the above 1D particle balance equation is to ignore shear-flow terms $\sim\partial\Gamma_\parallel/\partial r$ and to neglect the θ-dependence of densities, temperatures and electrostatic potential (compared to their r-dependence) in the determination of the poloidal variation of the parallel particle and heat fluxes $\Gamma_\parallel$ and $h_\parallel$ within a magnetic surface and therewith in the determination of the average value of $\delta I_\parallel$. To demonstrate the procedure we restrict the consideration to the collision-dominated Pfirsch-Schlüter regime, in which $p_\parallel - p_\perp \simeq 0$.

In this case ($T_\parallel \simeq T_\perp$ or $\chi \simeq 1/v_T$ assuming a common ion temperature T_+) also Coulomb collisions between the various charge states of the impurity ions may be involved ($p_\parallel \neq p_\perp$ is more likely to occur for trace impurity levels). Assuming $F = F^{Maxw}(1+\Psi)$ with $|\Psi|<<1$ for all plasma components we start with the linearized Boltzmann collision integral to get a unified formulation for friction and thermal forces. Within Grad's 13-moment approximation of the distribution functions F (being sufficient for the present application/1o/) we obtain the representation (using the index z to distinguish the impurity charge state under consideration from the other charge states z')

$$\frac{\delta I_{\parallel z}}{M_z} \equiv \int_{-\infty}^{+\infty} dv_\parallel v_\parallel \left.\frac{\partial_s a}{\partial \tau_\parallel}\right|_z \simeq \sum_{i=+,z'} \frac{n_z \mu_{zi} \nu_{zi}}{M_z} \left(\left(\frac{\Gamma_{\parallel i}}{n_i} - \frac{\Gamma_{\parallel z}}{n_z}\right) - \frac{3}{5}\frac{\mu_{zi}}{kT_+}\left(\frac{h_{\parallel i}}{\rho_i} - \frac{h_{\parallel z}}{\rho_z}\right)\right) - \frac{3}{5}\frac{M_-}{M_z}\nu_{z-}\frac{n_z}{n_-}\frac{h_{\parallel -}}{kT_-}$$

where the electron contribution to $I_\parallel$ reduces to the thermal force (last term on the rhs of the equation).

By definition

$\rho \equiv nM$ and $h_\parallel \equiv q_\parallel - \frac{5}{2}\Gamma_\parallel^\dagger$, where $q_\parallel$ and $\Gamma_\parallel^\dagger$ are now related to the centre-of-mass velocity $U_\parallel$ of the whole plasma.

We note that $\left| \frac{h_{\parallel z} n_+ M_+}{h_{\parallel +} n_z M_z} \right| \leq \max \left(\frac{M_+}{z^2 M_z} , \frac{M_+^2}{M_z^2} \right)$, which usually is small. For the trace impurity limit and $M_+/M_z \ll 1$ (where $U_\parallel \simeq V_{\parallel +}$) the momentum integral $\delta I_\parallel$ of the Boltzmann collision operator is, of course, identical with the corresponding integral of the Fokker-Planck collision operator

$$\int_{-\infty}^{+\infty} dv_\parallel v_\parallel \left. \frac{\partial_s a}{\partial \tau_\parallel} \right|_z = (\beta n_z v_{Tz} - \alpha \Gamma_{\parallel z}) \nu_c \simeq (\beta^\dagger \nu_c)_{z+,-} n_z v_{Tz} + (\alpha \nu_c)_{z+} (n_z V_{\parallel +} - \Gamma_{\parallel z})$$

where $(\beta^\dagger \nu_c)_{z+,-} \equiv -\frac{3}{5} \sum_{c=+,-} \frac{M_c}{M_z} \nu_{zc} \frac{h_{\parallel c}}{kT_c} \frac{n_z}{n_c}$ and $(\alpha \nu_c)_{z+} \equiv \frac{M_+}{M_z} \nu_{z+}$

In these formulae $\nu_{zi} = \frac{z^2 z_i^2 n_i}{\sqrt{\mu_{zi}}} Q_+^\dagger$ and $\nu_{z-} = \frac{z^2 n_-}{\sqrt{M_-}} Q_-^\dagger$

where $Q_\pm^\dagger \equiv \frac{4\sqrt{2\pi}}{3} \frac{\ln\Lambda}{\sqrt{kT_\pm}^3} \left(\frac{e^2}{4\pi\varepsilon_o} \right)^2$ and $\mu_{zi} \equiv \frac{M_i M_z}{M_i + M_z}$

The poloidal variation of $\Gamma_\parallel$ and $h_\parallel/\rho$ within a magnetic surface is calculated from the particle and heat balance equations by application of the operator $(1 - \int_0^{2\pi} d\theta\ h/2\pi)$. Besides neglecting the poloidal gradients of densities, temperatures and electrostatic potential compared to their radial gradients, which may be acceptable even for the plasma edge within the last closed magnetic surface, the traditional averaging process also neglects any poloidal source variations and any centre-of-mass velocities of the plasma as a whole. This, however, cannot be justified at the plasma edge because of the source concentration near the limiters and because of the plasma convection along $\vec{B}$ and the plasma rotation across $\vec{B}$ with strong radial gradients and associated shear flows. To focus the attention on the role of poloidal source variations in modifying the averaged radial fluxes, we ignore in the following the additional influence of $U_{\parallel}$ and plasma rotation across $\vec{B}$, i.e. we assume a collision-dominated plasma with $T_{\parallel} \simeq T_{\perp} \rightarrow T$, $V_{\parallel} \simeq U_{\parallel} \rightarrow 0$ and $\oint \Gamma_{\parallel} d\theta = 0$. As a consequence of these assumptions we operate with the limiting values

$$q_{\parallel}^{T} \rightarrow q_{\parallel} \equiv h_{\parallel} + \frac{5}{2} kT\Gamma_{\parallel}^{\dagger}$$

$$p_{\parallel} \simeq p_{\perp} >> M \left| \frac{A_4}{v_T^2}, A_2 \right| \quad \text{and} \quad q_{\perp} \rightarrow \frac{5}{2} p v_T^2$$

Adding the assumptions

$$\left| \frac{1}{r} \frac{\partial}{\partial \theta} (n,p,\Phi) \right| << \left| \frac{\partial}{\partial r} (n,p,\Phi) \right| , \quad \frac{r}{R_o} << 1 \quad \text{and} \quad \frac{\partial D_{\perp}}{\partial \theta} \simeq 0$$

we finally obtain the θ-dependence of $\Gamma_{\parallel}^{\dagger}$ and $h_{\parallel}$ in the approximate form

$$\Gamma_{\parallel}^{\dagger} \simeq -\frac{r}{\Theta}\left(\frac{D_{\perp}\sin\theta}{R_o}\frac{\partial n}{\partial r} + \frac{2\cos\theta}{M\Omega_o R_o}\left(ZenE_r - \frac{\partial p}{\partial r}\right) + \left(1-\int_0^{2\pi}\frac{d\theta}{2\pi}\right)\int^{\theta} d\theta'\left(1-\int_0^{2\pi}\frac{d\theta'}{2\pi}h'\right)(S-\nu_I n)'\right)$$

$$h_{\parallel} \simeq -\frac{r}{\Theta}\left(\frac{D_{\perp}\sin\theta}{R_o}\, n\,\frac{\partial}{\partial r}\left(\frac{5}{2}kT\right) - \frac{2\cos\theta}{M\Omega_o R_o}\, p\,\frac{\partial}{\partial r}\left(\frac{5}{2}kT\right) + \right.$$

$$\left. + kT\left(1-\int_0^{2\pi}\frac{d\theta}{2\pi}\right)\int^{\theta} d\theta'\left(1-\int_0^{2\pi}\frac{d\theta'}{2\pi}h'\right)(W+\nu_I n)'\right) \quad ; \quad W \equiv 2\pi\int_{-\infty}^{+\infty} dv_{\parallel}\int_0^{+\infty} dv_{\perp} v_{\perp}\left(\frac{v_{\parallel}^2+v_{\perp}^2}{v_T^2} - \frac{5}{2}\right)\bar{\bar{Q}}$$

where the integration constants have been chosen in such a way as to guarantee $\oint d\theta(\Gamma_{\parallel}^{\dagger},h_{\parallel}) = 0$. Note that dashed quantities depend on θ' and that n, T, Φ have been taken constant in the θ-integration process. A proper inclusion (via a Fourier analysis) of the θ- dependence of n, T, Φ (induced by the poloidal source variations) in the final calculation of $\int_0^{2\pi} d\theta h^2 I_{\parallel}/2\pi$ would amount to take (up to terms of the order r^2/R_o^2) the θ- averaged values in all but the source terms. In addition to the parallel flux contributions $\sim \cos\theta$ resulting from the ion drift motion there are contributions $\sim \sin\theta$ due to the anomalous diffusion in the toroidal plasma configuration, taking into account that within the above assumptions

$$\int^{\theta} d\theta'\left(1-\int_0^{2\pi}\frac{d\theta'}{2\pi}h'\right)\left(\frac{1}{rR}\frac{\partial}{\partial r}\left(-rRD_{\perp}\frac{\partial n}{\partial r}\right)\right) \simeq -\frac{D_{\perp}\sin\theta}{R_o}\frac{\partial n}{\partial r}$$

producing an inboard-outboard transport along $\vec{B}$ due to $R(\theta)$ at constant (θ- independent) radial flux density $-D_\perp \partial n/\partial r$, superimposed on the "$\vec{B}$ flux compensating the vertical drift motion.

The θ- integration in $\Gamma_\parallel^\dagger$ and $h_\parallel$ are performed after expanding the source functions in Fourier series (for a more general discussion concerning the neoclassical effect of poloidally varying source functions see ref./11/):

$$\begin{pmatrix} S-\nu_I n \\ \\ W+\nu_I n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S_o \\ \\ W_o \end{pmatrix} + \sum_{k=1}^{\infty}\begin{pmatrix} S_k^c \\ \\ W_k^c \end{pmatrix}\cos(k\theta) + \sum_{k=1}^{\infty}\begin{pmatrix} S_k^s \\ \\ W_k^s \end{pmatrix}\sin(k\theta)$$

with the result that within the order $r/R_o \ll 1$

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\, h^2\left(1-\int_0^{2\pi}\frac{d\theta}{2\pi}\right)\int_0^{\theta} d\theta'\left(1-\int_0^{2\pi}\frac{d\theta'}{2\pi}\, h'\right)\begin{pmatrix} S-\nu_I n \\ \\ W+\nu_I n \end{pmatrix}' \simeq -\frac{r}{R_o}\begin{pmatrix} S_1^s \\ \\ W_1^s \end{pmatrix}$$

With these expressions we obtain in the limit $M_+/M_z \ll 1$:

$$\frac{1}{\theta\Omega_{zo}}\int_0^{2\pi}\frac{d\theta}{2\pi}\, h^2\int_{-\infty}^{+\infty} dv_\parallel v_\parallel \frac{\partial_s a}{\partial\tau_\parallel}\Bigg|_{r,\theta}^{z} \simeq \frac{2q^2}{\Omega_{zo}^2}\,\frac{p_z}{M_z}\left[\frac{M_+\nu_{z+}}{M_z}\left(-\frac{\partial \ln n_z}{\partial r} + \right.\right.$$

$$+ Z \frac{\partial \ln n_+}{\partial r} - \left(1+\frac{Z}{2}\right)\frac{\partial \ln T}{\partial r}\Bigg) + \sum_{z' \neq zz} \frac{\nu_{zz'}}{2}\left(-\frac{\partial \ln n_z}{\partial r} + \frac{Z}{Z'}\frac{\partial \ln n_{z'}}{\partial r} - \frac{Z'-Z}{4Z'}\frac{\partial \ln T}{\partial r}\right)\Bigg] +$$

$$+ \frac{q^2 R_o}{\Omega_{zo}}\left[\frac{M_+ \nu_{z+}}{M_z}\left(\frac{n_z}{n_+}\left(S^s_{1+} - \frac{3}{5} W^s_{1+}\right) - S^s_{1z}\right) + \sum_{z' \neq z} \frac{\nu_{zz'}}{2}\left(\frac{n_z}{n_{z'}}\left(S^s_{1z'} - \frac{3}{1o} W^s_{1z'}\right) - \left(S^s_{1z} - \frac{3}{1o} W^s_{1z}\right)\right)\right]$$

Note that because of $\sqrt{M_-/M_+} \ll 1$ the electron contribution is neglected in this expression. The term $\sim (S^s_{1+} - 3W^s_{1+}/5)$ results from the poloidal asymmetry of the hydrogen ion source due to poloidally localized hydrogen recycling. The other inelastic terms refer to the poloidal asymmetry of the impurity ion source. Concentrating on the term $\sim S^s_{1z}$, we see that in the case of a toroidal limiter like ALT-II of TEXTOR we would have (setting $S^\dagger \equiv S - \nu_I n$)

$$S^s_{1z} = \frac{1}{\pi}\int_0^{2\pi} d\theta \; S^\dagger(\theta)\sin\theta \;\begin{cases} < 0 \\ > 0 \end{cases} \text{for an ALT-II position at } \theta = \mp \frac{\pi}{4}$$

assuming $\quad S^\dagger(\theta) = S_{max}\exp\left(-\left(\theta \pm \frac{\pi}{4}\right)^2\right)$

This means that for a limiter position at $\theta = -\pi/4$ there will be an outward convection of the impurity ions, whereas for a mirrored position at $\theta = +\pi/4$ an inward convection will occur.

Adding to the average radial transport due to Coulomb collisions the inelastic part $\int_{-\infty}^{+\infty} dv_{\parallel} v_{\parallel} (Q - \nu_I a) = Q_{\parallel} - \nu_I \Gamma_{\parallel}$ we would get the additional term

$$\frac{1}{\Theta \Omega_{zo}} \int_0^{2\pi} \frac{d\theta}{2\pi} h^2 (Q_{\parallel} - \nu_I \Gamma_{\parallel})_z = \frac{2q^2 \nu_I}{M_z \Omega_{zo}^2} \left(Ze n_z E_r - \frac{\partial p_z}{\partial r} \right) -$$

$$- \frac{q^2 \nu_I R_o}{\Omega_{zo}} S^s_{1z} + \frac{q}{\Omega_{zo}} Q^c_{\parallel 1z} \quad \text{assuming} \quad Q_{\parallel oz} = \frac{1}{\pi} \int_0^{2\pi} d\theta \, Q_{\parallel z}(\theta) = 0$$

$Q_{\parallel}$ being again Fourier analysed. We see that any outboard limiter position causes

$$Q^c_{\parallel 1z} = \frac{1}{\pi} \int_0^{2\pi} d\theta \, Q_{\parallel z}(\theta) \cos\theta > 0$$

due to net creation of $\cos\theta$ weighted parallel momentum by ionization in the outboard region. Thus $Q^c_{\parallel 1z}$ enforces the effect of S^s_{1z} for a limiter position at the lower outboard region. In contrast to elastic momentum transfer depending on the relative velocity of the colliding particles and thus eliminating the electric drift velocity, the parallel momentum production or loss due to ionization or recombination is influenced by the radial electric field. The collisional part of the effective radial particle flux due to friction and thermal forces satisfies automatically the condition of local ambipolarity because of momentum conservation in the collision process, which can easily be seen from Grad's form of the collision integral. The validity of local ambipolarity in the presence of inelastic collisions rests, however, on the condition that a poloidally varying electric field along $\vec{B}$ is generated in such a way as to compensate for the current induced by the different thermal velocities involved in the ionization of neutral and charged particles.

References:

/1/ R.P.Schorn et al., Appl. Physics B, 52, 71 (1991)

/2/ H.Gerhauser, H.A.Claaßen, Contr. Plasma Physics 3o(1), 89 (199o)

/3/ H.A.Claaßen, H.Gerhauser, D.Reiter, 16th European Conf. on CFPP, Venice, pt.III, 975 (1989)

/4/ H.A.Claaßen, H.Gerhauser, D.Reiter, Contr. Plasma Physics 3o(1), 83 (199o)

/5/ H.A.Claaßen et al., J. Nucl. Mat. 176&177, 398 (199o)

/6/ H.A.Claaßen, H.Gerhauser, Rep. KFA JÜL-1971 (1985)
A.V.Gurevich et al., Nucl. Fusion 27(3), 453 (1987)

/7/ H.A.Claaßen, H.Repp, Nucl. Fusion 21(5), 589 (1981)

/8/ H.Grad, J. Pure & Appl. Math. 2, 324 (1949)

/9/ H.A.Claaßen, H.Repp, Rep. 79-o2-o5o (1979) SFB Plasmaphysik Bochum/Jülich

/1o/ R.Balescu, Transport Processes in Plasmas, Vol.2 Neoclassical Transport, p.535 (1988)

/11/ W.M.Stacey Jr., Nucl. Fusion 27(8), 1213 (1987)

Appendix: We briefly summarize here the different steps leading to a drift-kinetic equation, which includes the effect of low-frequency, long-wavelength fluctuations of the electromagnetic field:

- decomposition of electromagnetic fields and distribution functions

$$\begin{array}{lll} \text{quasi-stationary} & B_j + b_j^{\perp} & \\ \text{(equilibrium)} & E_j + e_j^{\perp} & \text{fluctuating parts, i.e. } \langle b_j, e_j, f\rangle = 0 \\ \text{parts} & F + f & \end{array}$$

$\langle \ldots \rangle$ = time average over fluctuations

assumptions: $b_j^{\perp} B_j = 0$, $e_j^{\perp} B_j = 0$, $E_j = -\nabla_j \phi$ and $|F| \gg f$

low frequency modes $\omega \ll \Omega = ZeB/M$ = gyrofrequency

comprising both electrostatic and electromagnetic fluctuations: $e_j = e_j^{\perp} + v_{\parallel}\, \varepsilon_{jkl} n_k b_l^{\perp}$ $\quad (n_k = B_k/B)$

general procedure to derive the drift-kinetic equation:

- splitting of the kinetic equation into equilibrium (fluctuation time averaged) and perturbation (time-dependent) part

$$\hat{L}F = -\frac{Ze}{M} \langle e_j \frac{\partial f}{\partial v_j} \rangle + C(F,F^{\dagger}) + Q - \nu_I F \qquad \text{equilibrium}$$

$$\hat{L}f = -\frac{Ze}{M}\, e_j \frac{\partial F}{\partial v_j} + C(f,F^{\dagger}) + C(F,f^{\dagger}) + q - \nu_I f \qquad \text{perturbation}$$

in quasilinear approximation

$$\hat{L} \equiv \left.\frac{\partial}{\partial t}\right|_{\vec{x},\vec{v}} + v_j \left.\frac{\partial}{\partial x_j}\right|_{\vec{v},t} + \frac{Ze}{M}\left(E_j + \varepsilon_{jkl} v_k B_l\right) \left.\frac{\partial}{\partial v_j}\right|_{\vec{x},t}$$

- splitting into gyrotropic and gyro-dependent parts

$$\hat{L} = \overline{L} + \tilde{L}$$

$$(F,f) = (\overline{F},\overline{f}) + (\tilde{F},\tilde{f})$$

- introduction of the velocity components

$$v_k = (v_{\parallel}, v_{\perp}, \phi) = v_{\parallel} n_k + v_{\perp} s_k^o$$

$$s^o = p_k \cos\phi + b_k \sin\phi \quad , \quad \phi = \text{gyroangle}$$

n_k = unit vector along B_k, p_k = normal vector, b_k = binormal vector

in these velocity components the operators read:

slow time variations

$$\overline{L} = v_{\parallel} n_j \left(\frac{\partial}{\partial x_j} - \frac{Ze}{M} \frac{\partial \Phi}{\partial x_j} \frac{1}{v_{\parallel}} \frac{\partial}{\partial v_{\parallel}} \right) + \frac{v_{\parallel} v_{\perp}^2}{2} n_j \frac{\partial \ln B}{\partial x_j} \left(\frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} - \frac{1}{v_{\parallel}} \frac{\partial}{\partial v_{\parallel}} \right) + \frac{\partial}{\partial t}$$

describing the magnetic mirroring (trapping) effect due to $\nabla_{\parallel} \ln B$

$$\tilde{L} = \tilde{L}_o - \Omega \frac{\partial}{\partial \phi}$$

$$\tilde{L}_o = v_{\perp} s_j^o \left(\frac{\partial}{\partial x_j} - \frac{Ze}{M} \frac{\partial \Phi}{\partial x_j} \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} \right) - v_{\parallel} v_{\perp} \left(v_j s_k^o \frac{\partial n_k}{\partial x_j} + \frac{v_{\perp}}{2} n_j \frac{\partial \ln B}{\partial x_j} \right) \left(\frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} - \frac{1}{v_{\parallel}} \frac{\partial}{\partial v_{\parallel}} \right)$$

$$- \frac{\varepsilon_{krs} n_r s_s^o}{v_{\perp}} \left(v_j \left(v_{\parallel} \frac{\partial n_k}{\partial x_j} + v_{\perp} \frac{\partial s_k^o}{\partial x_j} \right) + \frac{Ze}{M} \frac{\partial \Phi}{\partial x_k} \right) \frac{\partial}{\partial \phi}$$

- drift ordering $\delta \equiv v_T/\Omega l << 1$; $\delta^\dagger \equiv v_T/\Omega\lambda_\perp \sim k_\perp \rho_L$

  l = characteristic length for spatial changes in the guiding-centre distribution function

  $k_\perp$= characteristic wavenumber of the fluctuations considered assuming $\Omega\tau >> 1$ (τ = fluctuation time scale)

$$\underset{\delta}{(L} + \underset{\delta}{\tilde{L}_o} - \underset{1}{\Omega \frac{\partial}{\partial\phi}}) F = - \underset{\delta}{< \frac{e_j}{B} \frac{\partial f}{\partial v_j} >} + C(F,F^\dagger) + Q - \underset{\nu/\Omega < \delta}{\nu_I F}$$

$$\underset{\delta}{(L} + \underset{\delta^\dagger}{\tilde{L}_o} - \underset{1}{\Omega \frac{\partial}{\partial\phi}}) f = - \underset{1}{\frac{e_j}{B} \frac{\partial F}{\partial v_j}} + C(f,F^\dagger) + C(F,f^\dagger) + q - \underset{\nu/\Omega < \delta}{\nu_I f}$$

$\delta^\dagger << 1$ long wavelength case, $\delta^\dagger = O(1)$ short wavelength case

<u>equilibrium (quasi-stationary) part</u>

$$\overline{LF} + \overline{\tilde{L}\tilde{F}} = - \Omega < \frac{e_j}{B} \overline{\frac{\partial \tilde{f}}{\partial v_j}} > + \overline{C}(\overline{F},\overline{F}^\dagger) + \overline{Q} - \nu_I \overline{F}$$

describing classical transport "$\vec{B}$

due to drift ordering solution of gyro-dependent part by successive approximations:

$$\tilde{F}^{o} = 0 \quad , \quad \tilde{F}^{1} = \frac{1}{\Omega} \int^{\phi} (\tilde{LF} + \Omega < \frac{e_j}{B} \frac{\partial \overline{f}}{\partial v_j} >) \, d\phi'$$

without consideration of collisions (the latter lead to classical cross-field transport)

$$\rightarrow \quad \overline{LF} + \overline{\tilde{LF}^{1}} = - \Omega < \frac{e_j}{B} \overline{\frac{\partial \tilde{f}}{\partial v_j}} > + \overline{C}(\overline{F},\overline{F}^{\dagger}) + \overline{Q} - \nu_I \overline{F}$$

$\uparrow \qquad \uparrow$

produce anomalous cross-field ($e_j^{\perp} n_j = 0$) diffusion in $\vec{x}$-space and $v_{\perp}$-space

<u>perturbation part</u> for $\delta^{\dagger} << 1$, $\nu\tau << 1$, long wavelengths and negligible collisions

$$\overline{Lf} + \overline{\tilde{Lf}} = - \Omega \frac{e_j}{B} \overline{\frac{\partial \tilde{F}}{\partial v_j}}$$

$$\tilde{f}^{o} = - \frac{e_j}{B} \varepsilon_{jkl} n_k s_l^{o} \frac{\partial \overline{F}}{\partial v_{\perp}}$$

$$\rightarrow \quad \overline{L\overline{f}} = - \overline{\tilde{Lf}^{o}} - \Omega \frac{e_j}{B} \overline{\frac{\partial \tilde{F}^{1}}{\partial v_j}}$$

solution by integration along unperturbed orbits, described by the characteristics:

$$dt = \frac{d\zeta}{v_{\parallel}} \quad , \quad d\zeta = \frac{dv_{\perp}}{\frac{v_{\perp}}{2} \frac{\partial \ln B}{\partial \zeta}} \quad \rightarrow \quad \frac{\partial}{\partial \zeta}\left(\ln \frac{v_{\perp}^{2}}{B} \right) d\zeta = 0$$

$$\frac{d\zeta}{v_\parallel} = -\frac{dv_\parallel}{\frac{Ze}{M}\frac{\partial\Phi}{\partial\zeta} + \frac{v_\perp^2}{2}\frac{\partial \ln B}{\partial\zeta}} \rightarrow \frac{\partial}{\partial\zeta}\left(\frac{Ze\Phi}{M} + \frac{v^2}{2}\right)d\zeta = 0\,, \quad v^2 = v_\parallel^2 + v_\perp^2$$

showing the invariance of $\mu = \frac{Mv_\perp^2}{2B}$ (adiabatic) and $Ze^{\Phi} + \frac{Mv^2}{2}$ along $\vec{B}$

$$\bar{f} = -\int_{-\infty}^{t} \frac{\varepsilon_j}{B}\left(\vec{x}(\tau), v_\parallel(\tau), \tau\right)\frac{\partial\bar{F}}{\partial x_j}\left(\vec{x}(\tau), v_\parallel(\tau), v_\perp(\tau), \tau\right) d\tau$$

assuming $\frac{\partial\varepsilon_j}{\partial x_j} = 0$

inserting $\tilde{f}^{o}$ and the corresponding value for $\bar{f}$ into the first order drift-kinetic equation for the equilibrium part $\bar{F}$ one gets the following anomalous diffusion terms

$$\rightarrow \quad -\Omega < \frac{e_j}{B}\overline{\frac{\partial\tilde{f}^{o}}{\partial v_j}} > \; = \Omega < \frac{e_j e_k}{B^2} > \varepsilon_{jkl} n_l \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(\frac{v_\perp}{2}\frac{\partial\bar{F}}{\partial v_\perp}\right) \qquad \text{(I)}$$

where $\quad \varepsilon_j = \varepsilon_{jkl} e_k^{\perp} n_l + v_\parallel b_j^{\perp}$

$$\rightarrow \quad \overline{\tilde{IF}^1}\Big|_{an} = \left(\frac{\partial}{\partial x_j} + n_l \frac{\partial n_j}{\partial x_l}\left(1 + v_\parallel \frac{\partial}{\partial v_\parallel} - \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\right) - \frac{Ze}{M}\frac{\partial}{\partial x_j}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\right) G_j \qquad \text{(II)}$$

with $\quad G_j \equiv D_{jk}^{\perp}\frac{\partial}{\partial x_k}\left(\frac{v_\perp}{2}\frac{\partial\bar{F}}{\partial v_\perp}\right); \quad D_{jk}^{\perp}(\vec{x}, v_\parallel) = \int_{-\infty}^{t} < \frac{\varepsilon_j}{B}(\vec{x}, v_\parallel, t)\,\frac{\varepsilon_k}{B}(\vec{x}(\tau), v_\parallel(\tau), \tau) > d\tau$

assuming $\left|\frac{\partial \ln\bar{F}}{\partial\zeta}\right| << 1/(\text{correlation length } "\vec{R})$

this is added to the neoclassical part (formulated in the $\mathrm{rot}\vec{B} = 0$ approximation and neglecting the collisional part of $\tilde{F}^1$, which would add the classical cross-field transport terms)

$$\left.\overline{\widetilde{LF}^1}\right|_{nc} = v_j^D \frac{\partial \bar{F}}{\partial x_j} + (v_\parallel n_j + v_j^D) \frac{\partial \ln B}{\partial x_j} \frac{v_\perp^2}{2} \left(\frac{1}{v_\perp} \frac{\partial \bar{F}}{\partial v_\perp} - \frac{1}{v_\parallel} \frac{\partial \bar{F}}{\partial v_\parallel} \right) + \frac{Ze}{M} \frac{v_j E_j}{v_\parallel} \frac{\partial \bar{F}}{\partial v_\parallel}$$

where $v_j^D = \frac{\varepsilon_{jkl} n_k}{2} \left(\underset{\uparrow}{\frac{v_\perp^2}{2} \frac{\partial \ln B}{\partial x_l}} + \underset{\uparrow}{v_\parallel^2 n_r \frac{\partial n_l}{\partial x_r}} + \underset{\uparrow}{\frac{Ze}{M} \frac{\partial \Phi}{\partial x_l}} \right)$ is the drift

drift velocity due to gradB, centrifugal, electrostatic forces

- specification to a toroidal coordinate system $\vec{x} = (r, \theta, \phi)$ assuming axisymmetry $(\partial/\partial\phi = 0)$

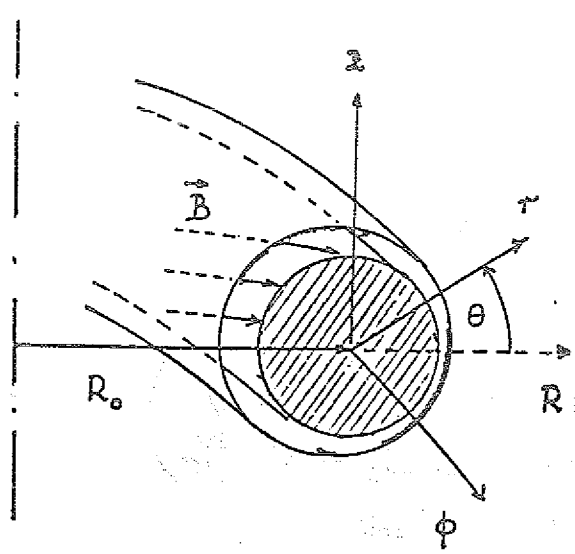
concentric magnetic flux surfaces

$R = R_0 + r \cos\theta$

$B = \left(0, -\Theta(r), 1 \right) B_\phi$

$B_\phi = B_\phi^0 / h \quad ; \; h = 1 + \frac{r}{R_0} \cos\theta$

$\rightarrow \; BR \simeq B_\phi R = \mathrm{const.}$



In the $\mathrm{rot}\vec{B} = 0$ approximation we have

$$n_l \frac{\partial n_j}{x_l} = \frac{\partial \ln B}{\partial x_j^{\perp}} \qquad \text{where} \qquad \frac{\partial}{\partial x_j^{\perp}} \equiv (\delta_{jk} - n_j n_k) \frac{\partial}{\partial x_k}$$

Neglecting the cross-correlations of the electromagnetic field components $\varepsilon_j^{\perp}$ we get

$$D_{jk}^{\perp} = \begin{pmatrix} D_{rr}^{\perp} & 0 & 0 \\ 0 & D_{\theta\theta}^{\perp} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{such that with} \quad \frac{\partial \ln B}{\partial x_j^{\perp}} = \frac{1}{R} \begin{pmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$\overline{\tilde{w}}_1 \Big|_{an}^{LF} = \left(\frac{1}{rR} \frac{\partial}{\partial r}(rR\ldots) + \left(\frac{ZeE_r}{M} + \frac{v_\parallel^2 \cos\theta}{R}\right) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} - \frac{\cos\theta}{R} \left(1 + v_\parallel \frac{\partial}{\partial v_\parallel}\right) \right) D_{rr}^{\perp} \frac{\partial}{\partial r} \frac{v_\perp}{2} \frac{\partial \overline{F}}{\partial v_\perp} +$$

$$+ \left(\frac{1}{rR} \frac{\partial}{\partial \theta}(R\ldots) + \left(\frac{ZeE_\theta}{M} - \frac{v_\parallel^2 \sin\theta}{R}\right) \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} + \frac{\sin\theta}{R} \left(1 + v_\parallel \frac{\partial}{\partial v_\parallel}\right) \right) \frac{D_{\theta\theta}^{\perp}}{r} \frac{\partial}{\partial \theta} \frac{v_\perp}{2} \frac{\partial \overline{F}}{\partial v_\perp}$$

The expression (I) on p.36 is of Fokker-Planck type and may be combined with the corresponding collisional part $\partial_s \overline{F}/\partial \tau_\perp$ on p.4 to describe an effective ion diffusion in $v_\perp$- space.

FORSCHUNGSZENTRUM JÜLICH GmbH KFA

Jül-2576
January 1992
ISSN 0366-0885