



A PATH TO PROCESS GENERAL MATRIX FIELDS

joint work with Bernhard Burgeth

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Part I: Introduction & motivation

INTRODUCTION & MOTIVATION

Processing fields of rotation matrices

- Interpolation of rotation matrices?

$$\frac{1}{2} \cdot \text{[Diagram 1]} + \frac{1}{2} \cdot \text{[Diagram 2]} = ?$$

The diagram shows two ellipses representing fields of rotation matrices. The first ellipse (left) has a white center, a black arrow pointing up-right, and a blue arrow pointing down-right. The second ellipse (right) has a white center, a black arrow pointing up-left, and a blue arrow pointing down-left. The result is indicated by a question mark.

- Interpolation specific for rotation matrices (M. Moakher, SIAM, 2002).

$$\frac{1}{2} \odot \text{[Diagram 1]} \oplus \frac{1}{2} \odot \text{[Diagram 2]} = \text{[Diagram 3]}$$

The diagram shows the same two ellipses as above. The first ellipse (left) has a white center, a black arrow pointing up-right, and a blue arrow pointing down-right. The second ellipse (right) has a white center, a black arrow pointing up-left, and a blue arrow pointing down-left. The result is indicated by a question mark.

- What about further operations?
- What about other classes of matrices?

INTRODUCTION & MOTIVATION

Beyond symmetric matrices (B. Burgeth & A. K., Springer, 2017)

- Unifying concept for psup/pinf of rotations R_1, R_2 in $\mathbb{R}^d$?

$$\text{psup} \left(\begin{array}{c} \text{Diagram 1} \end{array}, \begin{array}{c} \text{Diagram 2} \end{array} \right) = ?$$

$$\text{pinf} \left(\begin{array}{c} \text{Diagram 1} \end{array}, \begin{array}{c} \text{Diagram 2} \end{array} \right) = ?$$

- What about other classes of non-symmetric matrices?
- Generalize $\text{Sym}(n)$.
- **Idea:** Complexification, Hermitian matrices, $\text{Her}(n)$.

Part II: Calculus for Hermitian matrices

CALCULUS FOR HERMITIAN MATRICES

Basic properties

- $\text{Her}(n) = \{\mathbf{H} \in \mathbb{C}^{n \times n} \mid \mathbf{H} = \mathbf{H}^*\}$ is $\mathbb{R}$ -vector space.
 - $*$ stands for transposition with complex conjugation.
- $\mathbf{H} = \text{Re}(\mathbf{H}) + \text{Im}(\mathbf{H})i$,
 - Symmetric real part $\text{Re}(\mathbf{H})$.
 - Skew-symmetric imaginary part $\text{Im}(\mathbf{H})$.
- $\mathbf{H}$ unitarily diagonalizable: $\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^*$,
 - $\mathbf{U}$ unitary: $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$.
 - $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$ diagonal matrix with real-valued $d_1 \geq \dots \geq d_n$.
- Loewner order: $\mathbf{H}_1 \geq \mathbf{H}_2 \iff \mathbf{H}_1 - \mathbf{H}_2$ positive semidefinite.

CALCULUS FOR HERMITIAN MATRICES

Dictionary for Hermitian matrices

Setting	Scalar-valued	Matrix-valued
Function	$f : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \mapsto f(x) \end{cases}$	$F : \begin{cases} \text{Her}(n) \longrightarrow \text{Her}(n) \\ \mathbf{H} \mapsto \mathbf{U} \text{diag}(f(d_1), \dots, f(d_n)) \mathbf{U}^* \end{cases}$
Partial derivatives	$\partial_\omega h,$ $\omega \in \{t, x_1, \dots, x_d\}$	$\bar{\partial}_\omega \mathbf{H} := (\partial_\omega h_{ij})_{ij},$ $\omega \in \{t, x_1, \dots, x_d\}$
Gradient	$\nabla h(x) := (\partial_{x_1} h(x), \dots, \partial_{x_d} h(x))^\top,$ $\nabla h(x) \in \mathbb{R}^d$	$\bar{\nabla} \mathbf{H}(x) := (\bar{\partial}_{x_1} \mathbf{H}(x), \dots, \bar{\partial}_{x_d} \mathbf{H}(x))^\top,$ $\bar{\nabla} \mathbf{H}(x) \in (\text{Her}(n))^d$

CALCULUS FOR HERMITIAN MATRICES

Dictionary for Hermitian matrices

Setting	Scalar-valued	Matrix-valued
Length	$\ w\ _p := \sqrt[p]{ w_1 ^p + \dots + w_d ^p},$ $\ w\ _p \in [0, +\infty[$	$ \mathbf{W} _p := \sqrt[p]{ \mathbf{W}_1 ^p + \dots + \mathbf{W}_d ^p},$ $ \mathbf{W} _p \in \text{Her}^+(n)$
Supremum	$\sup(a, b)$	$\text{psup}(\mathbf{A}, \mathbf{B}) = \frac{1}{2}(\mathbf{A} + \mathbf{B} + \mathbf{A} - \mathbf{B})$
Infimum	$\inf(a, b)$	$\text{pinf}(\mathbf{A}, \mathbf{B}) = \frac{1}{2}(\mathbf{A} + \mathbf{B} - \mathbf{A} - \mathbf{B})$

Image processing tools for symmetric matrices carry over to Hermitian matrices.

CALCULUS FOR HERMITIAN MATRICES

Embedding $M_{\mathbb{R}}(n)$ into $\text{Her}(n)$

- Linear mapping $\Xi : M_{\mathbb{R}}(n) \longrightarrow \text{Her}(n)$

$$\Xi : \mathbf{M} \longmapsto \frac{1}{2}(\mathbf{M} + \mathbf{M}^{\top}) + \frac{i}{2}(\mathbf{M} - \mathbf{M}^{\top})$$

- Inverse mapping $\Xi^{-1} : \text{Her}(n) \longrightarrow M_{\mathbb{R}}(n)$

$$\Xi^{-1} : \mathbf{H} \longmapsto \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\top}) - \frac{i}{2}(\mathbf{H} - \mathbf{H}^{\top})$$

- Processing strategy:

$$\begin{array}{ccc} \text{Her}(n) & \xrightarrow{\mathcal{IO}} & \text{Her}(n) \\ \Xi \uparrow & & \downarrow \Xi^{-1} \\ M_{\mathbb{R}}(n) & \xrightarrow{\Xi^{-1} \circ \mathcal{IO} \circ \Xi} & M_{\mathbb{R}}(n) \end{array}$$

- Operations on Hermitian matrices via operator $\mathcal{IO}$.
- $\mathcal{IO}$ represents averaging, psup, pinf, time-step in numerical scheme, etc.

Part III: Processing orthogonal matrices

PROCESSING ORTHOGONAL MATRICES

Processing orthogonal matrices, $\mathbf{Q} \in O(n)$

- $O(n) \subset M_{\mathbb{R}}(n)$
- There is a problem.
 - **Before** processing: $\mathbf{Q} \in O(n)$.
 - **After** processing: $(\Xi^{-1} \circ \mathcal{IO} \circ \Xi)(\mathbf{Q}) \notin O(n)$.
- There is a remedy.
 - Projection from $M_{\mathbb{R}}(n)$ back to $O(n)$ via best Frobenius norm approximation $\tilde{\mathbf{Q}} \in O(n)$
$$\|(\Xi^{-1} \circ \mathcal{O} \circ \Xi)(\mathbf{Q}) - \tilde{\mathbf{Q}}\|_{\text{F}}^2 \longrightarrow \min .$$
- This **nearest matrix problem** allows for explicit solution:
 - Orthogonal factor in polar decomposition of $\mathbf{M}$.
 - $\tilde{\mathbf{Q}} = \mathcal{P}_{O(n)}(\mathbf{M}) = \mathbf{M} (\mathbf{M}^{\top} \mathbf{M})^{-1/2}$.

PROCESSING ORTHOGONAL MATRICES

Projection into $O(n)$

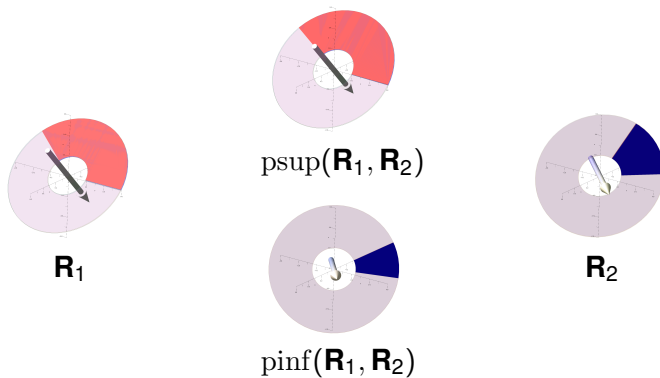
- Augmented processing strategy

$$\begin{array}{ccccc} \text{Her}(n) & \xrightarrow{\quad \mathcal{IO} \quad} & \text{Her}(n) & & \\ \uparrow \Xi & & \downarrow \Xi^{-1} & & \\ O(n) \subset M_{\mathbb{R}}(n) & \xrightarrow[\Xi^{-1} \circ \mathcal{IO} \circ \Xi]{} & M_{\mathbb{R}}(n) & \xrightarrow[\mathcal{P}_{O(n)}]{} & O(n) \end{array}$$

- General strategy allows for processing of
 - any square real matrix $\in M_{\mathbb{R}}(n)$.
 - any matrices from an “interesting” subset $S \subset M_{\mathbb{R}}(n)$.
- But $\mathcal{P}_S$ needs to be calculated.

PROCESSING ORTHOGONAL MATRICES

Experiments in $SO(3)$



Pseudo-supremum and pseudo-infimum of two rotations.

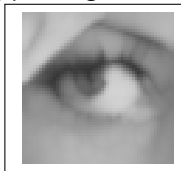
PROCESSING ORTHOGONAL MATRICES

Experiments in $SO(2)$ and $SO(3)$, B. Burgeth & A. K., LNCS 10225, 2017

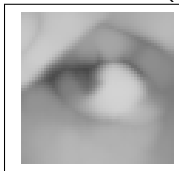
- Scalar ($\rightarrow$ angle ϕ) image as rotations in $SO(2)$.



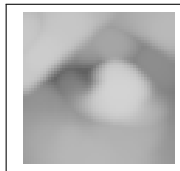
Original



Dilation, $T = 2$



Dilation, $T = 4$

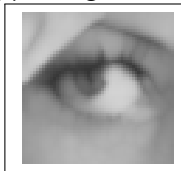


Dilation, $T = 6$

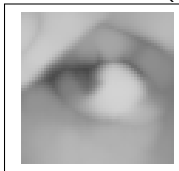
- Scalar ($\rightarrow$ angle ϕ) image as rotations in $SO(3)$, **single axis**.



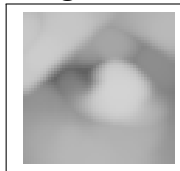
Original



Dilation, $T = 2$



Dilation, $T = 4$

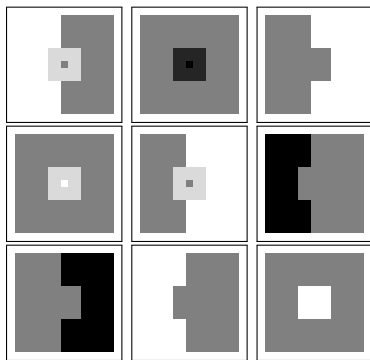


Dilation, $T = 6$

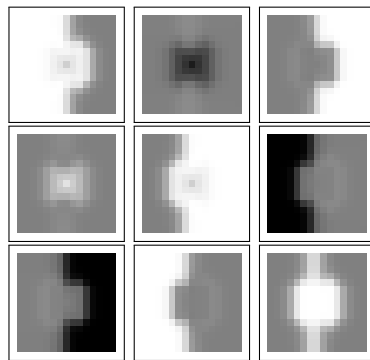
PROCESSING ORTHOGONAL MATRICES

Experiment in $SO(3)$

- Tiled view: original 15×15 -field of $SO(3)$ -matrices, its dilation with $T = 1$.



Original $SO(3)$ field

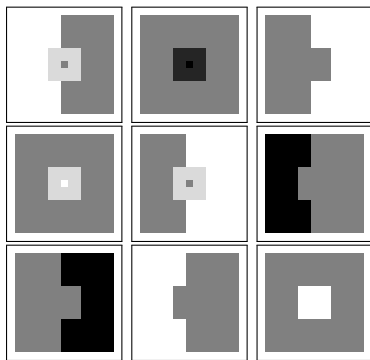


Dilated $SO(3)$ field, $T = 1$

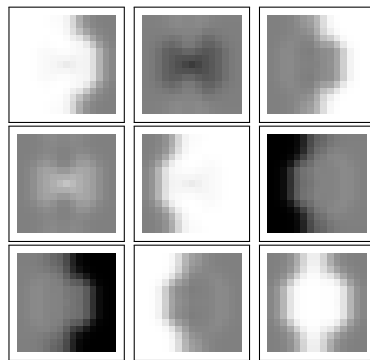
PROCESSING ORTHOGONAL MATRICES

Experiment in $SO(3)$

- Tiled view: original 15×15 -field of $SO(3)$ -matrices, its dilation with $T = 2$.



Original $SO(3)$ field



Dilated $SO(3)$ field, $T = 2$

Part IV: Processing Moebius transformations

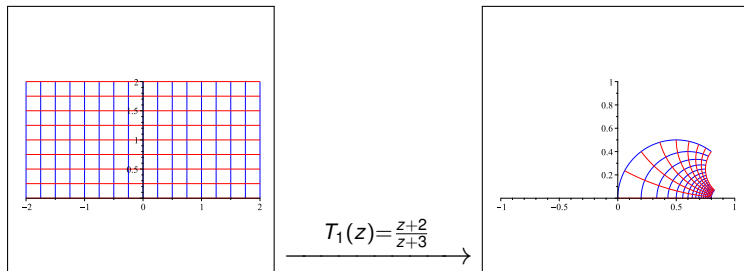
PROCESSING MOEBIUS TRANSFORMATIONS

Setup and illustrative example

- Möbius transformations are given by

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

- Play an important role in hyperbolic geometry.
- They represent motions in the Poincaré half-space model.



PROCESSING MOEBIUS TRANSFORMATIONS

Setup and illustrative example

- Algebraic description via matrices of the form

$$\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$.

- Matrices are elements of $M_{\mathbb{R}}(2)$ with determinant $\neq 0$.
- Hence, $\mathbf{T} \in \text{Gl}_2(\mathbb{R})$.
- Poincaré half model

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y > 0\}$$

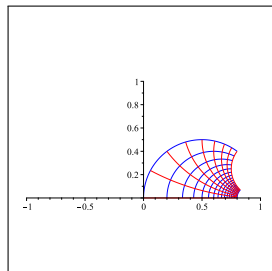
is invariant under Moebius transformations.

PROCESSING MOEBIUS TRANSFORMATIONS

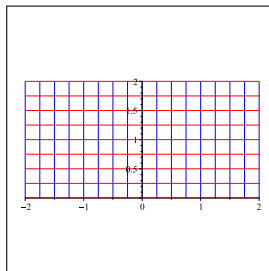
Example

$$T_1(z) = \frac{z+2}{z+3}, \quad \mathbf{T}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

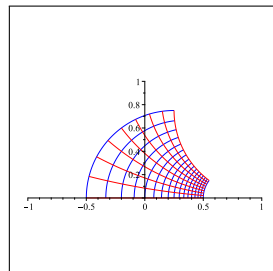
$$T_2(z) = \frac{z+1}{z+4}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$



$\xleftarrow{T_1(z)}$



$\xrightarrow{T_2(z)}$



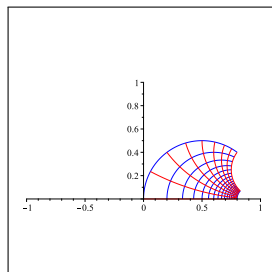
What is the pseudo supremum and infimum of two Moebius transformations?

PROCESSING MOEBIUS TRANSFORMATIONS

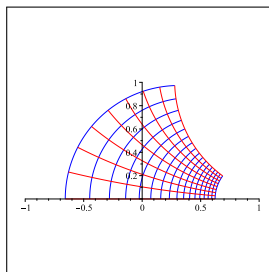
Example pseudo supremum

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},$$

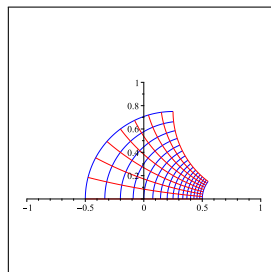
$$\mathbf{T}_{\text{psup}} = \text{psup}(\mathbf{T}_1, \mathbf{T}_2) \approx \begin{pmatrix} 1.2887 & 1.2113 \\ 1.0000 & 4.0774 \end{pmatrix}$$



$\mathbf{T}_1$



$\mathbf{T}_{\text{psub}}$



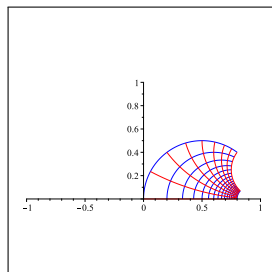
$\mathbf{T}_2$

PROCESSING MOEBIUS TRANSFORMATIONS

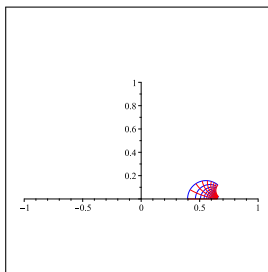
Example pseudo infimum

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix},$$

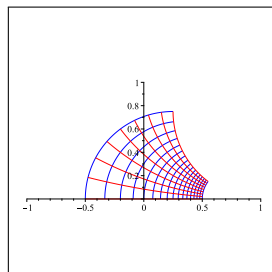
$$\mathbf{T}_{\text{pinf}} = \text{pinf}(\mathbf{T}_1, \mathbf{T}_2) \approx \begin{pmatrix} 0.7113 & 1.7887 \\ 1.0000 & 2.9226 \end{pmatrix}$$



$\mathbf{T}_1$



$\mathbf{T}_{\text{pinf}}$



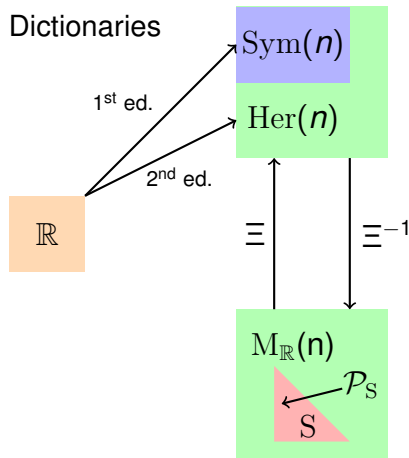
$\mathbf{T}_2$

Part V: Summary & outlook

SUMMARY & OUTLOOK

Summary

- Transition from scalar calculus to calculus for symmetric matrices.
- Proposed an extension to Hermitian matrices.
- 1-to-1 link to general square matrices.
- Specialization to “interesting” matrix subsets possible, for example $S = O(n)$.



SUMMARY & OUTLOOK

Outlook

- Extending the “dictionary”.
- Considering other interesting classes of matrices.
- Solving (numerically) nearest matrix problems.
- Looking for interesting fields of applications:
 - Material science (crack formation), problem size: $10^3 \times 10^3 \times 10^3$ -grid, 10 matrix entries, 10^3 -iterations.
 - Processing of correlation matrices within “big data”.
 - High resolution 10^7 , multispectral $(10^2)^2$ images, 10^3 -iterations.
- Visualization is a problem.
- Increasing the efficiency of computations.
- HPC for real applications are necessary.

REFERENCES

Partial list



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