

How (not) to renormalize integral equations with singular potentials in effective field theory

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(Dated: 11 January, 2018)

We discuss the connection between the perturbative and non-perturbative renormalization and related conceptual issues in the few-nucleon sector of the low-energy effective field theory of the strong interactions. General arguments are supported by examples from effective theories with and without pions as dynamical degrees of freedom. A quantum mechanical potential with explicitly specified short- and long-range parts is considered as an “underlying fundamental theory” and the corresponding effective field theory potential is constructed. Further, the problem of the effective field theoretical renormalization of the Skornyakov-Ter-Martirosian equation is revisited.

PACS numbers: 13.40.Gp, 11.10.Gh, 12.39.Fe, 13.75.Cs

I. INTRODUCTION

The underlying idea of the chiral effective field theory (EFT) of Quantum Chromodynamics (QCD) is that at low energies expressions of physical quantities given by QCD can be presented as perturbative expansions in powers of small masses and energy/momenta divided by some large scale(s). The aim of an EFT is to reproduce these perturbative series by applying an organising rule, the so-called power counting, based on the most general effective Lagrangian and requiring that the terms in the EFT Lagrangian up to given order generate contributions in physical quantities up to the corresponding order. In the few-nucleon sector of EFT, pioneered in Refs. [1, 2], the issue of renormalization turned out to be most problematic, leading to controversial approaches and considerations [3–54]. For recent reviews and references see e.g. Refs. [55–61]. In the few-nucleon sector one deals with non-perturbative expressions of physical quantities obtained by solving integral (or differential) equations plagued by ultraviolet (UV) divergences. The main source of disagreement seems to be a different understanding of the relation between perturbative and non-perturbative renormalization. The aim of the current paper is to bring some clarity to this issue.

Leaving out the whole complexity of the technical details our point of view can be summarized by saying that renormalization means expressing physical quantities in terms of other physical quantities instead of the bare parameters of the Lagrangian [62]. If all UV divergences disappear from the physical quantities after renormalization such a theory is called renormalizable. Renormalization is perturbative if it is applied to perturbative expressions and non-perturbative if applied to non-perturbative ones.

While there might exist quantum field theories (QFTs) in four space-time dimensions which are perturbatively non-renormalizable and non-perturbatively finite, a consistent EFT Lagrangian includes all of the infinite number of interactions allowed by the underlying symmetries, and therefore every ultraviolet divergence showing up in physical quantities is canceled by a corresponding counterterm [63].

In the few-nucleon sector of chiral EFT, an infinite number of Feynman diagrams contribute to the scattering amplitude already at leading order. Defining effective potentials as sums of irreducible diagrams one sums up these infinite sets of diagrams by solving the corresponding integral equations (or Schrödinger equation) [1, 2].

In EFT, un-subtracted *regularized* non-perturbative expressions obtained by solving integral equations, when expanded in powers of \hbar (corresponding to the loop expansion), reproduce the *regularized* perturbative series. Let us emphasize that this is *not* to say that non-perturbative expressions cannot contain *more* than perturbation theory can provide. In general, non-perturbative expressions of EFT may contain contributions which lead to trivial contributions in the perturbative series. A nice example is given by the instanton contribution in the action of QCD. It is given by $\exp(-A/g^2)$, with A some constant and g the strong coupling constant. This non-perturbative contribution has a perturbative expansion (in powers of g) of the form $0 + 0 + 0 + \dots$, i.e. it is trivial at any order. Notice, however, that an EFT is obtained by quantizing the corresponding classical theory, i.e. its whole construction is based on the assumption that the $\hbar \rightarrow 0$ limit coincides with the classical EFT. The authors are not aware of any result obtained

in EFT which indicates that regularized non-perturbative expressions of physical quantities if expanded around the $\hbar = 0$ limit do not reproduce the perturbative series of regularized EFT Feynman diagrams.

As mentioned above, EFTs are renormalizable QFTs. It is very well known how to renormalize Feynman diagrams. In self-consistent EFTs properly renormalized non-perturbative expressions, when expanded in \hbar , must reproduce the renormalized perturbative series. There is no reason to expect that this is not the case, although, except of some very special cases, in general it is not feasible to solve equations in closed forms and to carry out the non-perturbative renormalization explicitly.

Any EFT Lagrangian is written in terms of bare parameters and fields. To carry out the QFT renormalization one needs either to split the bare quantities in renormalized ones and counterterms or, equivalently, apply the BPHZ renormalization procedure (see, e.g., Ref. [64]). For few-body problems, the BPHZ procedure can be applied only in very special cases, therefore an essential issue is what are the power counting rules for the EFT Lagrangian meant to be applied to? To operators with bare quantities, or with the renormalized ones? We advocate the point of view that power counting should be applied to interaction terms of the EFT Lagrangian with renormalized coupling constants, *not* to bare ones, unless the Wilsonian approach is used, in which case the cutoff should be kept between the soft and the hard scales of the problem [49, 51, 65]. The power counting for the terms in the EFT Lagrangian with renormalized coupling constants translates into a corresponding power counting for physical quantities. On the other hand, there is no power counting for counterterms as they are divergent in the removed cutoff limit and only make sense in combination with corresponding contributions of loop diagrams.

In case of the pionless theory for the nucleon-nucleon (NN) interaction it is straightforward to implement the power counting applied to renormalized couplings by using BPHZ renormalization. In this case the equivalent counterterm formalism can also be worked out exactly. Within the BPHZ scheme, all loop diagrams are subtracted and the bare couplings are replaced by renormalized ones. In the language of counterterms this means that the power counting is applied to sums of unsubtracted loop diagrams and corresponding counterterm diagram(s), *but not separately to each of them*. The renormalization procedure becomes much more complicated when pions are included as dynamical degrees of freedom. Closed expressions cannot be obtained and, therefore, subtractive renormalization cannot be carried out except for some very special cases. Lessons learned from exactly solvable pionless EFT tell us, at least, what procedure should *not* be followed in order to carry out a self-consistent EFT renormalization and *not just to get rid off the ultraviolet divergences*.

Our paper is organized as follows. In section II we consider a very well known simple example of the S-wave NN scattering in pionless EFT. Section III addresses the case with the one-pion-exchange (OPE) potential. A non-singular potential considered as the “underlying theory” will be compared to the corresponding leading order (LO) EFT potential in Sec. IV. Sec. V considers the Skornyakov-Ter-Martirosian equation for the three particle system and we summarize our findings in Sec. VI.

II. CONTACT INTERACTION POTENTIAL

In this section we consider an exactly solvable EFT potential of contact interactions and recapitulate the problems encountered (and their solutions) in connection with the renormalization of non-perturbative expressions. This will prove to be useful for the considerations in the next sections where we deal with the cases in which no exact analytic solution can be obtained.

We start by considering the integral equations for the NN partial wave (PW) scattering amplitudes

$$T_{ll'}^{sj}(p, p', q) = V_{ll'}^{sj}(p, p') + \hbar \sum_{l''} \int_0^\infty \frac{dk k^2}{(2\pi)^3} V_{ll''}^{sj}(p, k) \frac{m}{q^2 - k^2 + i0^+} T_{l''l'}^{sj}(k, p', q), \quad (1)$$

which can be rewritten symbolically as

$$T = V + \hbar VGT. \quad (2)$$

Note that we have included the factor \hbar accompanying the loop integration as the expansion in \hbar corresponds to the standard QFT loop expansion. Below we revisit the 1S_0 PW in nucleon-nucleon scattering up to next-to-leading order (NLO) in pionless EFT considered in Ref. [66] and reiterate the considerations of Refs. [46, 50, 67, 68] but also adding some new insight. The starting NLO potential has the form

$$V_{\text{NLO}} = c + c_2 (p^2 + p'^2). \quad (3)$$

This potential is apparently perturbatively non-renormalizable, i.e. its iterations generate divergences, which can not be subtracted by redefining the available two parameters c and c_2 . On the other hand, in the framework of EFT all

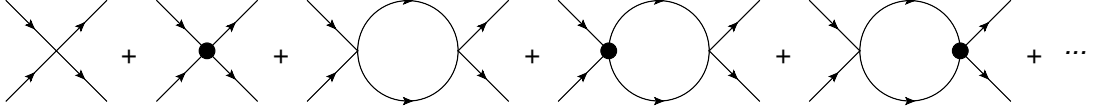


FIG. 1: Diagrams contributing to the NN scattering amplitude. The filled blob denotes the NLO interaction $\sim c_2$.

divergences generated by iterations of the potential of Eq. (3) are systematically removed by counter terms generated by interaction terms with higher derivatives. The on-shell amplitude corresponding to the potential of Eq. (3), i.e. the solution to Eq. (2), reads [66]:

$$T_{\text{NLO}}(q) = \frac{c_2 [\hbar c_2 (I_3 q^2 - I_5) - 2q^2] - c}{\hbar I(q^2) [c_2 (\hbar c_2 (I_5 - I_3 q^2) + 2q^2) + c] - (\hbar I_3 c_2 - 1)^2}, \quad (4)$$

where the integrals I_n are divergent and require regularization. In cutoff regularization these loop integrals are given by

$$\begin{aligned} I_n &= -m \int \frac{d^3 k}{(2\pi)^3} k^{n-3} \theta(\Lambda - k) = -\frac{m \Lambda^n}{2n \pi^2}, \\ I(p^2) &= m \int \frac{d^3 k}{(2\pi)^3} \frac{1}{p^2 - k^2 + i0^+} \theta(\Lambda - k) = -\frac{i p m_N}{4\pi} - \frac{m}{2\pi^2} \left[\Lambda + \frac{p}{2} \ln \frac{1 - \frac{p}{\Lambda}}{1 + \frac{p}{\Lambda}} \right] \\ &= -\frac{i p m}{4\pi} - \frac{m \Lambda}{2\pi^2} + \frac{m p^2}{2\pi^2 \Lambda} + O\left(\frac{1}{\Lambda^2}\right). \end{aligned} \quad (5)$$

Our aim is to renormalize the expression of Eq. (4) in a way consistent with the philosophy of EFT, i.e. to remove all divergences by absorbing them in the redefinition of the parameters of the effective Lagrangian. For any finite cutoff Λ the expansion of $T_{\text{NLO}}(q)$ in powers of \hbar is a convergent series for sufficiently small \hbar and exactly coincides with the perturbative series given by perturbative calculations of diagrams using the same EFT Lagrangian which has generated the potential of Eq. (3). In particular, the perturbative series

$$T_{\text{NLO}}(q) = c + 2c_2 q^2 + \hbar [c^2 I(q^2) + c_2^2 (3 I_3 q^2 + I_5 + 4 I(q^2) q^4) + 2c_2 c (I_3 + 2 I(q^2) q^2)] + \dots \quad (6)$$

is in one-to-one correspondence to the diagrams displayed in Fig. 1. Renormalization of any of the loop diagrams in Fig. 1 is done according to standard textbook procedures (see e.g. Ref. [64]). We can use subtractions by applying the BPHZ procedure, which is equivalent to the standard counterterm technique. For this particular case, the renormalized series can be easily summed up. This subtractive renormalization can be also achieved by replacing the loop integrals in Eq. (4) by subtracted ones, e.g., by discarding I_n entirely and subtracting $I(q^2)$ at $q^2 = -\mu^2$, and by simultaneously replacing the bare couplings c and c_2 with the subtraction-scale(s) dependent renormalized ones, denoted c_R and c_{2R} , respectively. The final result reads

$$T_{\text{NLO}}(q) = \frac{c_R + 2q^2 c_{2R}}{1 - \hbar [I(q^2) - I(-\mu^2)] (c_R + 2q^2 c_{2R})}. \quad (7)$$

Notice here that, as usual in any quantum field theory, the obtained renormalized result does not depend on the applied regularization scheme, e.g., using dimensional regularization and the same renormalization conditions leads to exactly the same result [67]. In agreement with our general expectations, the expansion of Eq. (7) in \hbar reproduces exactly the renormalized series of diagrams. Unlike the result of Ref. [66], the expression in Eq. (7) does *not* imply any restriction on the sign of the effective range imposed by the Wigner bound applicable to a zero-range potential. The Wigner bound does not apply to the expression of Eq. (7) because its expansion in \hbar generates an infinite series of renormalized diagrams, i.e. Eq. (7) already includes the contributions of an infinite number of counterterms (with powers of momentum/energy growing up to infinity) and thus the corresponding effective potential is not zero-ranged [67]. In particular, the bare potential with all those counterterms included has the form:

$$\begin{aligned} V_B &= \frac{c_R - \hbar c_{2R}^2 I(-\mu^2) (q^2 - p^2) (q^2 - p'^2) + c_{2R} [p^2 + p'^2 + \hbar c_{2R} (I_3 (p^2 + p'^2 - q^2) - I_5)]}{(\hbar I_3 c_{2R} + 1)^2 + \hbar I(-\mu^2) [c_R - c_{2R} (\hbar I_5 c_{2R} - q^2 (\hbar I_3 c_{2R} + 2))]} \\ &= c_R + c_{2R} (p^2 + p'^2) + \mathcal{O}(\hbar). \end{aligned} \quad (8)$$

This expression is consistent with the standard Weinberg power counting because the renormalized potential $c_R + c_{2R} (p^2 + p'^2)$ satisfies this power counting (provided that the scale μ is taken of the order of the hard scale of the problem

[69]) while all other terms in Eq. (8) are proportional to \hbar - they are counterterms with divergent coefficients which are not subject to the rules of the power counting. These counterterms contain higher orders of momenta/energy, but that does not cause a problem as we are interested in physical quantities and the power counting applies to subtracted loop integrals and *not* to the diverging loop integrals and diverging counter terms separately. What is mapped onto the power counting for the physical amplitude is the power counting for the potential with renormalized couplings. Notice that while, for the case of unnaturally large scattering length and natural effective range, the LO potential c must be iterated to obtain the LO EFT amplitude, the c_2 -term can be treated perturbatively, its first insertion generates the NLO contribution to the amplitude. The first two terms in the expansion of the amplitude $T_{\text{NLO}}(q)$ of Eq. (7) in powers of c_{2R} exactly reproduce the LO and NLO contributions to the renormalized EFT amplitude and the higher order terms are all small, as expected.

In case the NLO potential is treated perturbatively, all divergences are removed by the bare potential (expanded in c_{2R}), which does not contain counterterms with higher orders of momenta/energy:

$$V_{B,\text{expanded}} = \frac{c_R}{1 + \hbar c_R I(-\mu^2)} - \frac{2\hbar c_{2R} c_R I_3}{[1 + \hbar c_R I(-\mu^2)]^2} + \frac{c_{2R} [p^2 + p'^2 + \hbar c_R I(-\mu^2) (p^2 + p'^2 - 2q^2)]}{[1 + \hbar c_R I(-\mu^2)]^2}. \quad (9)$$

As it is seen from the above expressions it does not matter whether we treat the c_2 term perturbatively or non-perturbatively, the difference in the corresponding expressions for the scattering amplitude is indeed of higher order provided that we renormalize it properly, i.e. that we take into account contributions of all necessary counterterms, or equivalently, subtract UV divergences from *all* loop diagrams.

To summarize, the expression of Eq. (4) reproduces the perturbative diagrams if expanded around $\hbar = 0$. That perturbative series is convergent for arbitrarily large (but finite) cutoff and sufficiently small \hbar . The corresponding renormalized series is also convergent, and it converges to the expression of Eq. (7). Thus, the non-perturbative expression, when expanded in \hbar , reproduces the perturbative series and the corresponding renormalized non-perturbative expression, when expanded in \hbar , reproduces the renormalized perturbative series.

Next, we discuss a procedure often referred to as “non-perturbative renormalization” which makes the non-perturbative expression of Eq. (4) finite, but does not qualify as a renormalization of EFT as it does not match the well established quantum field theoretical perturbative renormalization in the region where perturbation theory is applicable.

In Ref. [66] the two parameters c and c_2 have been fixed by demanding that the amplitude of Eq. (4) reproduces the scattering length and the effective range. The obtained amplitude has the form

$$T(q) = \frac{-4i\pi a [4a\hbar\Lambda + \pi(aq^2 r_e + 2)]}{m [\pi(a^2 q^3 r_e + 2aq - 2i) + 2a\hbar\Lambda(aq(2 + iq r_e) - 2i)]} = \frac{-4\pi/m}{-\frac{1}{a} + \frac{q^2 r_e}{2} - iq} - \frac{\pi^2 a q^4 r_e^2 / m}{\hbar\Lambda \left(-\frac{1}{a} + \frac{q^2 r_e}{2} - iq\right)^2} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right). \quad (10)$$

This expression happens to be finite in the limit $\Lambda \rightarrow \infty$ leading to

$$T(q) = -\frac{4\pi/m}{-\frac{1}{a} + \frac{q^2 r_e}{2} - iq}. \quad (11)$$

Notice, however, that the expansion of the expression of Eq. (10) in powers of \hbar leads to:

$$T(q) = \frac{2\pi a(aq^2 r_e + 2)}{m} + \hbar \left[\frac{2a^4 \Lambda q^4 r_e^2}{m} - \frac{i\pi a^2 q(aq^2 r_e + 2)^2}{m} \right] + \dots, \quad (12)$$

which is, again, a convergent series for arbitrarily large but finite Λ and sufficiently small \hbar . In this expression the term of order \hbar as well as all higher-order terms contain positive powers of Λ . Thus, terms in Eq. (12) correspond to partially renormalized diagrams, that means that some positive powers of the cutoff are removed while others are not. We remark that standard quantum field theoretical renormalization applied to Feynman diagrams of the perturbative series removes all positive powers and logarithms of the cutoff parameter. Thus the perturbative expansion of the “nonperturbatively-renormalized” expression of Eq. (10) does not reproduce the renormalized perturbative series.

Notice that the closed non-perturbative expression of Eq. (10) satisfies the *necessary* condition imposed on a properly renormalised expression that for large values of the cutoff the cutoff-dependent part is suppressed by inverse powers of Λ . However, it is *not* an EFT renormalized expression unless one defines EFT renormalization in such a way that there is a clear mismatch between the renormalization of the perturbative series and a convergent sum of this series (for small \hbar).

The expression of Eq. (10) is affected by the Wigner bound, i.e. the cutoff cannot be taken very large unless the effective range is non-positive (otherwise the bare parameters c and c_2 become complex). However, this restriction has nothing to do with EFT because Eq. (10) is *not* renormalized from the EFT point of view.

One might argue that we are addressing an irrelevant issue here as the c_2 -term is not of LO for the NN system and therefore we do not have to include it non-perturbatively. However, there might exist systems (at least theoretically) for which the c_2 -term needs to be iterated, and most importantly we aim at using the pionless EFT as a “theoretical laboratory” where we can learn how to deal with problems for which we do not have exact expressions.

According to Weinberg’s power counting in an EFT with pions treated as explicit degrees of freedom, the OPE potential is of LO. However, from the point of view of the renormalization, if compared to the above problem of the pionless EFT, it acts *not* like the c -term but rather like the c_2 -term, i.e. it is perturbatively non-renormalizable, see e.g. Ref. [52].

Next let us have a closer look at the issue of renormalization for the OPE potential making use of the lessons learned in this section.

III. INCLUDING THE ONE-PION-EXCHANGE

According to Ref. [1] in EFT with pions and nucleons as dynamical degrees of freedom, the LO NN potential is given by energy- and momentum-independent contact interaction terms plus the OPE potential. If this power counting rule is applied to the bare quantities, then the full LO potential has the form

$$V_{\text{LO}} = V_C + V_\pi, \quad (13)$$

where V_π is the OPE potential and the contact interaction part V_C contributes only in S-waves. However, following the approach of Ref. [33] let us apply “non-perturbative renormalization” by including an additional single contact interaction term in each attractive triplet PW and fixing them by minimizing the cutoff dependence of the solutions of the integral equation for the corresponding PW amplitudes. Below we argue that such a “non-perturbative renormalization” is an *ad hoc* procedure, inconsistent with the quantum field theoretical renormalization of EFT.

To calculate the scattering amplitude for our LO potential we need to solve the PW integral equations (2) to which we apply the cutoff regularization. For arbitrarily large but finite cutoff Λ and for sufficiently small \hbar , the iterations of Eq. (2) generate a perturbative series which converges to the solution of the equation. That is, for arbitrarily large but finite cutoff, the solution of Eq. (2) can be expanded in a convergent series of \hbar around $\hbar = 0$. This series exactly reproduces the results for diagrams obtained by iterating the OPE potential and the contact interaction (if the latter is non-vanishing for a given PW). Each term in this convergent (for sufficiently small \hbar) series of diagrams can be renormalized by either using BPHZ subtractions or the counterterm technique. Unlike the analogous case of the pionless EFT, we do not know how to sum up the resulting renormalized series. However, a simple counting of orders of UV divergences of perturbative loop diagrams makes it clear that the single contact interaction term included in the potential in each spin-triplet attractive channel cannot generate all those subtractions of loop diagrams (see e.g. Ref. [52]). That is, for a large (but finite) cutoff, independently from the choice of the available single contact interaction term, which is tuned (as a function of the cutoff) to minimize the cutoff dependence in the non-perturbative solution of the corresponding PW integral equation [33], the expansion of the solution to Eq. (2) in powers of \hbar for sufficiently small \hbar generates a convergent series of terms which are only partially renormalized, they contain positive powers of the cutoff. Therefore, these diagrams do not coincide with perturbatively renormalized diagrams. That is, analogously to the contact interaction potential of the previous section the “non-perturbatively renormalized” expressions, when expanded, do not reproduce the perturbative series renormalized according to the rules of quantum field theory. We conclude that also for the OPE potential, the “non-perturbative renormalization” is inconsistent with proper EFT renormalization.

One might be tempted to declare this mismatch between “non-perturbative renormalization” and renormalized perturbative series, which is even more obvious for the case of a repulsive singular interaction, as irrelevant by taking the cutoff to infinity, in which case the solution to the equation becomes a non-analytic function of the coupling constant of the OPE potential (see, e.g., Ref. [70]). That is, the resulting non-perturbative amplitude cannot be expanded perturbatively in powers of the coupling constant, i.e. it cannot be expanded in powers of \hbar either. One could claim that non-perturbative expressions have nothing to do with perturbation theory because it is a solution in an intrinsically non-perturbative regime. Such an argument is not valid here because the non-analyticity in the coupling constant of the OPE potential originates from the singular $1/r^3$ behaviour of OPE potential for $r \rightarrow 0$ [70]. The OPE potential of chiral EFT is obtained in the low-energy region and its singular $1/r^3$ behaviour for $r \rightarrow 0$ has

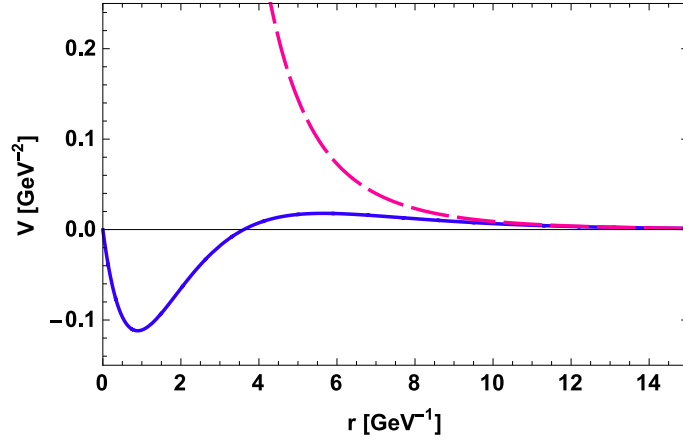


FIG. 2: Exact and approximate potentials as discussed in the text. The solid (blue) and the long-dashed (magenta) lines corresponds to the exact and the approximate potentials, respectively.

nothing to do with either EFT or the real world and the underlying fundamental theory, QCD,¹ which is believed to describe the real world where the only bound state in the NN system is the deuteron, while the singular $1/r^3$ behaviour inevitably entails deeply bound states. Notice further that if we could sum up the properly renormalized perturbative series for sufficiently small \hbar , the existence of its unique continuation to the physical value of \hbar would depend on the choice of the renormalization condition.

To bring further insight into the above discussion, we consider below a quantum mechanical non-singular potential with the long range part behaving like $1/r^3$ if extended to small values of r while dropping the short range parts.

IV. NON-SINGULAR POTENTIAL OF THE “UNDERLYING THEORY”

Let us consider the potential

$$V(r) = \frac{\alpha(e^{-m_1 r} - e^{-Mr})}{r^3} + \frac{\alpha(m_1 - M)e^{-m_1 r}}{r^2} + \frac{\alpha(M - m_1)^2 e^{-m_2 r}}{2r} - \frac{1}{6}\alpha(2m_1 - 3m_2 + M)(M - m_1)^2 e^{-m_1 r}, \quad (14)$$

where the light mass M is the small scale and the heavy masses m_1, m_2 are the large scales. Our choice of parameters is $\alpha = -36 \text{ GeV}^{-2}$, $M = 0.1385 \text{ GeV}$, $m_1 = 0.75 \text{ GeV}$ and $m_2 = 1.15 \text{ GeV}$. The factor α sets the strength of the interaction. This strength is taken equal for all terms, so that the potential $V(r)$ vanishes for $r \rightarrow 0$ and it behaves as $-\alpha e^{-Mr}/r^3$ for large r . In Fig. 2 we show the full potential and its long range part extended to small values of r .

The Fourier transform of the potential $V(r)$ has the form

$$\begin{aligned} V(q) &= \int d^3\mathbf{r} e^{i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}) \\ &= \frac{2}{3}\pi\alpha(M - m_1)^2 \left(\frac{3}{m_2^2 + q^2} - \frac{2m_1(2m_1 - 3m_2 + M)}{(m_1^2 + q^2)^2} \right) + 4\pi\alpha \ln \frac{q + iM}{q + im_1} \\ &\quad + 2\pi\alpha \left(1 - \frac{iM}{q} \right) \left[\ln \frac{M + iq}{M - iq} - \ln \frac{m_1 + iq}{m_1 - iq} \right]. \end{aligned} \quad (15)$$

We consider the LS equation for the scattering amplitude in the center-of-mass frame of two particles with unit masses interacting via the potential of Eq. (15)

$$T_E(\mathbf{p}, \mathbf{p}') = V(|\mathbf{p} - \mathbf{p}'|) + \int \frac{d^3\mathbf{q}}{(2\pi)^3} V(|\mathbf{p} - \mathbf{q}|) \frac{1}{E - q^2 + i\epsilon} T_E(\mathbf{q}, \mathbf{p}'), \quad (16)$$

¹ Thus, while derived from the chiral EFT and therefore relevant for large distances, the $\sim 1/r^3$ part of the OPE potential has nothing to do with the nucleon-nucleon interaction at shorter distances, where its singular behaviour shows up.

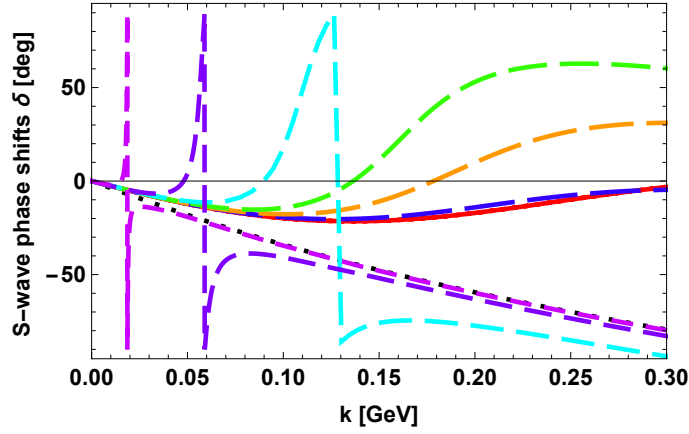


FIG. 3: S-wave phase shifts versus the particle momentum in the center-of-mass frame. The solid (red) line corresponds to the underlying potential and the dashed lines with decreasing length of dashes are phase shifts for the cutoff $\Lambda = 0.6, 0.8, 1.0, 1.4, 2.0$ and 3.0 GeV, respectively. The constant contact interaction term is fitted to reproduce the scattering length. The dotted (black) line represents the phase shifts corresponding to the singular long range potential in the infinite cutoff limit.

with $E = k^2/1$ GeV the total energy and \mathbf{p} and \mathbf{p}' the relative momenta of the incoming and the outgoing particles, respectively. We concentrate on the S-wave integral equation which has the form

$$t_E(p, p') = v(p, p') + \int_0^\infty \frac{dq q^2}{2\pi^2} v(p, q) \frac{1}{E - q^2 + i\epsilon} t_E(q, p'), \quad (17)$$

where $t_E(p, p') = (1/2) \int T_E(\mathbf{p}, \mathbf{p}') d\cos\theta_{\mathbf{p}\mathbf{p}'}$ and the S-wave potential $v(p, q) = (1/2) \int V(|\mathbf{p} - \mathbf{q}|) d\cos\theta_{\mathbf{p}\mathbf{q}}$ is given by

$$\begin{aligned} v(p, q) = & -\frac{\pi\alpha(p-q)^2 \ln \frac{|p-q|+iM}{|p-q|+im_1}}{pq} + \frac{\pi\alpha((p-q)^2 - 2iM|p-q|) \ln \frac{m_1+i|p-q|}{m_1-i|p-q|}}{2pq} \\ & - \frac{\pi\alpha((p-q)^2 - 2iM|p-q|) \ln \frac{M+i|p-q|}{M-i|p-q|}}{2pq} - \frac{4\pi\alpha m_1(M-m_1)^2(2m_1-3m_2+M)}{3(m_1^2+(p-q)^2)(m_1^2+(p+q)^2)} \\ & - \frac{\pi\alpha(M-m_1)^2 \ln \frac{m_2^2+(p-q)^2}{m_2^2+(p+q)^2}}{2pq} + \frac{\pi\alpha(p+q)^2 \ln \frac{iM+p+q}{im_1+p+q}}{pq} - \frac{\pi\alpha(p+q)(p+q-2iM) \ln \frac{m_1+i(p+q)}{m_1-i(p+q)}}{2pq} \\ & + \frac{\pi\alpha m_1(m_1-2M) \ln \frac{m_2^2+(p-q)^2}{m_2^2+(p+q)^2}}{2pq} + \frac{\pi\alpha M^2 \ln \frac{M^2+(p-q)^2}{M^2+(p+q)^2}}{2pq} + \frac{\pi\alpha(p+q)(p+q-2iM) \ln \frac{M+i(p+q)}{M-i(p+q)}}{2pq}. \quad (18) \end{aligned}$$

Notice here that despite its appearance, $v(p, q)$ is a real function of its arguments. Eq. (17) has a unique well-defined solution and the corresponding phase shifts are represented by the solid (red) line in Fig. 3.

Considering the expression of Eq. (14) as an “underlying fundamental” potential, we can construct the corresponding EFT. The LO EFT potential consists of a constant contact interaction corresponding to the delta potential in coordinate space and the long range part $-\alpha e^{-Mr}/r^3$, which is singular if extended to the small r region. The coupling constant $\alpha = -36 \text{ GeV}^{-2} \approx -1/(0.167 \text{ GeV})^2$ is chosen such that the full LO potential as well as its long range part are non-perturbative for the momenta $k \sim M = 0.1385 \text{ GeV}$. A simple UV analysis shows that the LO potential is perturbatively non-renormalizable, i.e. to remove the divergences from its iterations one needs to introduce counterterms of higher order (in momentum and energy) which themselves generate new divergences etc. up to infinity. On the other hand, the solution to the regularized integral equation has a well defined removed cutoff limit. However, this “non-perturbatively renormalized” expression is not only conceptually incorrect from the point of view of EFT but also does not describe the data as will be seen below. Analogously to Ref. [71] we regularize the LO potential by multiplying its singular long-range part with a local regulator function $f(r/R)$ which we choose for convenience of the following form

$$f\left(\frac{r}{R}\right) = \left(1 - e^{-\frac{r}{R}}\right)^2, \quad (19)$$

and the constant contact interaction is regularized in momentum space by multiplying with

$$e^{-\frac{p^2+q^2}{\Lambda^2}}. \quad (20)$$

We choose $R = 2/\Lambda$ and fix the constant contact interaction term by demanding that the scattering length of the underlying potential is reproduced. As expected from general consideration of cutoff EFT [34, 68, 72, 73] for the cutoffs around the optimal value ~ 0.6 GeV (which is of the order of the large scale of the problem) the phase shifts are reasonably well described by the LO EFT potential. For increasing cutoff, the region where the phase shifts are well described at LO decreases eventually vanishing in the removed cutoff limit. Note further that the phase shifts corresponding to a repulsive long-range singular potential without adding a strong attractive contact interaction have a well defined removed cutoff limit which strongly deviates from the data as seen in Fig. 3. One might be tempted to try to reproduce the data by treating the higher order contact interactions perturbatively. However, and on top to the conceptual problems discussed above, such a perturbative treatment would be questionable due to the large discrepancy between the data and the LO phase shifts, see also Ref. [74] for a related discussion. Thus, as mentioned above, taking the cutoff beyond the large scale of the problem for the considered perturbatively non-renormalizable LO potential (without including contributions of an infinite number of counter terms) is not only conceptually incorrect from the point of view of EFT [46], but does not allow one to describe the data either.

V. RENORMALIZING THE SKORNYAKOV-TER-MARTIROSYAN EQUATION IN EFT

The arguments of the previous sections also apply to the doublet channel neutron-deuteron scattering in pionless EFT as well as the three-body problem of self-interacting scalar particles as the corresponding LO integral equations have characteristic features of a singular $1/r^2$ potential.

Consider the integral equation for the half off-shell amplitude of the bound state of two scalar particles scattering off another scalar particle

$$\begin{aligned} t(p, k) &= M(p, k) + V_3(\Lambda, E, p, k) + \hbar \frac{2}{\pi} \int_0^\infty \frac{dq q^2 t(q, k)}{-\frac{1}{a_2} + \sqrt{3q^2/4 - mE}} F(k, q, \Lambda) [M(p, q) + V_3(\Lambda, E, p, q)], \\ M(k, p) &= \frac{1}{pk} \ln \frac{k^2 + p^2 + kp - mE}{k^2 + p^2 - kp - mE}. \end{aligned} \quad (21)$$

Here, $E = 3k^2/4m - B_2$ is the energy of the three-body system, $B = 1/(ma_2^2)$ is the binding energy of two scalars with a_2 the two-body scattering length, and $F(k, q, \Lambda)$ is an ultraviolet regulator, with Λ the pertinent cutoff parameter. We included a more general three-body interaction V_3 than considered in Ref. [75], where this equation has been obtained using a non-relativistic EFT of self-interacting scalar particles. Eq. (21) for $V_3 = 0$ is equivalent to the Skorniyakov-Ter-Martirosyan (S-TM) equation [76]. For $V_3 = 0$ and $F(k, p, \Lambda) \equiv 1$ it does not have a unique solution [77]. By considering the cutoff-regularized S-TM equation one obtains a unique solution, however, the limit of the cutoff going to infinity does not exist [78]. This problem has been solved by considering a momentum- and energy-independent three-body force $V_3(k, p, \Lambda) = H(\Lambda)/\Lambda^2$ and by choosing $H(\Lambda)$ as a function of the cutoff so that the strong cutoff dependence of the scattering amplitude is canceled. The obtained three-body force $H(\Lambda)/\Lambda^2$ exhibits a limit-cycle behaviour as a function of the cutoff [75, 78].

As detailed in Ref. [79] a proper EFT renormalization of the solution to Eq. (21) requires the inclusion of contributions of an infinite number of local three-body counterterms. This disagrees with the widely accepted conclusions of Refs. [75, 78] that a single constant three-body interaction term is sufficient. Note that a simple power counting demonstrates that iterations of Eq. (21) with a constant V_3 generate overall divergences which cannot be removed by renormalizing the available constant three-body term but rather require the inclusion of contributions of higher-order (up to infinity) counterterms. Analogously to the previous sections, the non-perturbative renormalization utilising a single counterterm is not consistent with the perturbative renormalization. It can be easily seen by considering the subset of iterated diagrams involving only a constant three-body term that the perturbative expansion of the solution to the equation for small \hbar does not reproduce the perturbatively renormalized series. Indeed, for $V_3(k, p, \Lambda) = v_3(\Lambda)$, all these diagrams are generated by expanding the following expression in powers of \hbar :

$$\begin{aligned} t_3(E, \Lambda) &= \frac{v_3(\Lambda)}{1 - \hbar \frac{2}{\pi} v_3(\Lambda) \int_0^\infty \frac{dq q^2}{-\frac{1}{a_2} + \sqrt{3q^2/4 - mE}} F(k, q, \Lambda)} \\ &\equiv \frac{1}{1/v_3(\Lambda) - \hbar \frac{2}{\pi} \left(\frac{\Lambda^2}{\sqrt{3}} + \frac{4\Lambda}{3a_2} - \frac{4(ma_2^2 E + 2) \ln(a_2 \Lambda)}{3\sqrt{3}a_2^2} + \text{finite} \right)} = \frac{1}{1/v_3^R + \hbar \frac{2}{\pi} \left(\frac{4ma_2^2 E \ln(a_2 \Lambda)}{3\sqrt{3}a_2^2} + \text{finite} \right)}, \end{aligned} \quad (22)$$

where we have used a sharp cutoff. Divergences with energy-independent coefficients are absorbed in the redefinition of the constant three-body force, however, the logarithmic divergence with an energy-dependent coefficient remains.

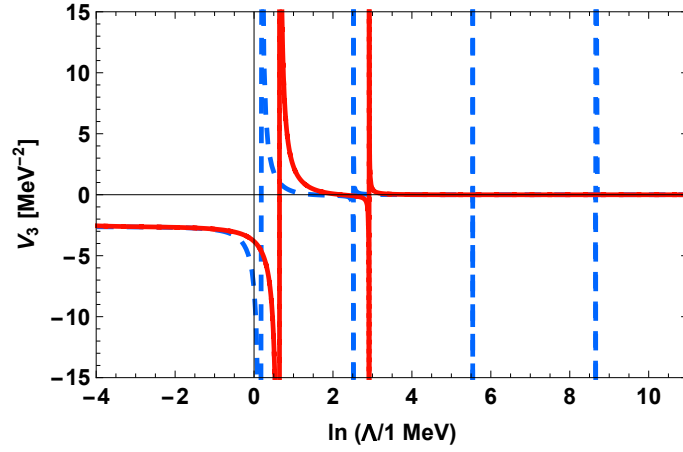


FIG. 4: Three-body forces as function of the cutoff. The solid and dashed lines correspond to the exact potential and the constant contact interaction, respectively.

Expansion of the resulting partially renormalized expression in \hbar generates partially renormalized loop diagrams, i.e. does not reproduce the renormalized expressions of diagrams involving only a constant three-body term.

On the other hand, using the RG equation we are able to calculate the exact three-body force which removes the full cutoff dependence. It is no surprise that no limit cycle is observed in the exact three-body force for large values of the cutoff. Notice that we obtain the limit cycle corresponding to the Efimov physics by taking larger and larger values for the two-body scattering length a_2 , in agreement with Refs. [80, 81]. Calculation of the exact three-body force which removes the full cutoff dependence can be performed as demonstrated below.

To perform a RG analysis of the S-TM equation (21), it is convenient to employ the regularized Green's function²

$$G_\Lambda(q, k) = \hbar \frac{2}{\pi} \frac{q^2}{-\frac{1}{a_2} + \sqrt{3q^2/4 - mE}} \left(\frac{k^2 + \Lambda^2}{q^2 + \Lambda^2} \right)^3. \quad (23)$$

By demanding cutoff-independence of the scattering amplitude at threshold, we obtain the cutoff dependence of the constant contact interaction term $V_3 = H(\Lambda)/\Lambda^2$. We fix the single parameter $H(\Lambda)$ such that $\cot \delta(E) = -51.6267$ for $E = -0.00106486$ MeV. There is no particular reason for choosing these numbers except that they correspond to an exactly vanishing three-body force for $\Lambda = \Lambda_0 \equiv 100$ MeV within the given regularization scheme. In all calculations, we take $m = 939$ MeV and $a_2 = 1$ MeV⁻¹, again for no particular reason. As seen from our results plotted in Fig. 4, in full agreement with Refs. [75, 78] the constant three-body force shows a limit cycle behaviour for the cutoff getting large. Notice here that we plot $H(\Lambda)/\Lambda^2$, not $H(\Lambda)$ as is done in those papers.

Next, by solving the exact RG equation we obtain $V^\Lambda(E, p, k) = M(p, k) + V_3(\Lambda, E, p, k)$, which leads to an exactly cutoff independent solution of Eq. (21). We fix the renormalization conditions of the three-body force by demanding that $V^{\Lambda_0}(E, p, k) = M(p, k)$, i.e. the three-body force vanishes for $\Lambda = \Lambda_0$. The exact RG equation for $V^\Lambda(E, p, k)$ satisfying the above renormalization condition has the form

$$V^\Lambda = M + M(G_{\Lambda_0} - G_\Lambda)V^\Lambda. \quad (24)$$

Exact cutoff independence of the scattering amplitude corresponding to the solution of Eq. (24) can easily be seen by re-writing Eq. (21) formally as

$$\frac{1}{V^\Lambda} = \frac{1}{t} + G_\Lambda, \quad (25)$$

and subtracting its analogue for $\Lambda = \Lambda_0$.

The three-body force, i.e. $V_3(\Lambda, E, p, k) = V^\Lambda(E, p, k) - M(p, k)$, is obtained by solving numerically Eq. (24) for on-shell kinematics with $E = -0.00106486$ MeV and is plotted in Fig. 4. As seen from the figure no limit cycle is obtained for large values of the cutoff.

² Notice here that the results of Refs. [75, 78] do not depend on the particular form of the regulator.

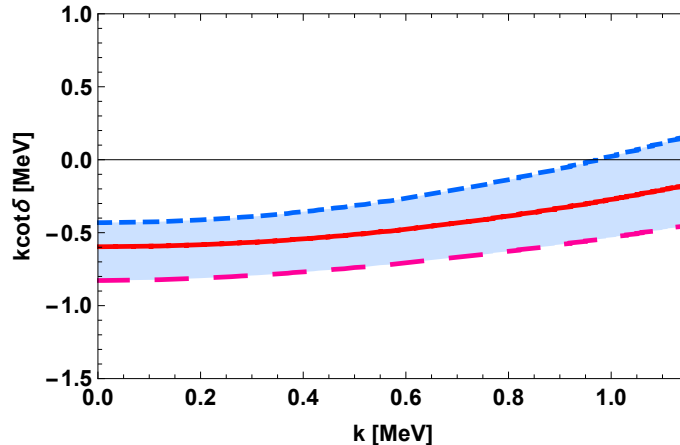


FIG. 5: $k \cot \delta$ as the function of k calculated without including the three-body force. The short-dashed, solid and long dashed lines correspond to $\Lambda = 75, 100$, and 125 MeV, respectively.

A self-consistent and practically applicable solution to the problem of non-perturbative renormalization is provided by the cutoff EFT. For the considered unphysical case the optimal choice is per construction $\Lambda \sim 100$ MeV. We plot $k \cot \delta$ as the function of k in Fig. 5 for Λ varying between 75 and 125 MeV without including the three-body force. As seen from this figure, a 25 % deviation from the optimal value of the cutoff leads to about a 30 % change in the physical quantity which is acceptable. One could reduce the cutoff dependence by promoting the constant three-body force to the LO as done in Refs. [75, 78], however, one still must keep the cutoff in the vicinity of the optimal value \sim hard scale of the problem to arrive at a self-consistent EFT framework.

VI. SUMMARY

In this work we discussed the connection between the perturbative and non-perturbative renormalization in the few-nucleon sector of the low-energy effective field theory of the strong interactions. The corresponding integral equations for the scattering amplitude are obtained by means of summing up an infinite number of ladder diagrams. Solutions to regularized equations reproduce perturbative series of Feynman diagrams when expanded in powers of \hbar , which corresponds to the loop expansion. These series are convergent for a finite cutoff and sufficiently small \hbar . We argued that properly renormalized non-perturbative physical quantities, expanded in powers of \hbar , must reproduce the corresponding renormalized series of standard perturbation theory. We considered once again the renormalization of the NLO 1S_0 nucleon-nucleon potential in pionless EFT. For this exactly solvable model we carried out renormalization explicitly and demonstrated the above mentioned connection between non-perturbative and perturbative expressions for physical quantities. We have also identified, once again, problems of the so called “non-perturbative renormalization”. Next we applied the knowledge gained from this “theoretical laboratory” to the LO contact interaction and the OPE potential in the pionful EFT. We argued that the “non-perturbative renormalization” of the solution to Lippmann-Schwinger equation performed by eliminating/minimizing the cutoff dependence of partial wave amplitudes for the values of the cutoff much larger than the hard scale of the problem by adjusting a finite number of counterterms is not consistent with QFT renormalization. To demonstrate our point of view with a further example we considered a quantum mechanical non-singular potential with explicitly specified short- and long-range parts as an “underlying theory” and addressed the problem of renormalization in the corresponding low-energy effective theory. We also considered the Skornyakov-Ter-Martirosyan equation describing the scattering of three scalar particles. We argued that fixing the non-uniqueness of the solution of this equation by introducing a single constant three-body force is incompatible with the EFT formalism. This is because the iterations of the three-body force generate divergences whose removal requires the introduction of an infinite number of other momentum- (and energy-) dependent three-body terms. We compared the Wilsonian renormalization group behaviour of the cutoff regulated exact three-body force to the cutoff-dependence of the constant three-body force. The cutoff dependence of the latter was obtained by demanding cutoff-independence of the scattering amplitude at threshold. We observe that the constant three-body force as a function of the cutoff shows a qualitatively different behaviour compared to the cutoff-dependent exact potential for the cutoff much larger than the hard scale of the problem. In particular, the constant contact interaction exhibits a limit cycle behaviour which is absent in the exact potential. The examples considered here support once again the arguments of Refs. [34, 68, 72, 73] that in the absence of a possibility of subtracting all UV divergences in

the solutions of the integral equations of the non-relativistic EFT, the solution of the problem of renormalization is provided by cutoff regularized EFT with the cutoff kept of the order of the hard scale of the problem.

Acknowledgments

This work was supported in part by BMBF (contract No. 05P2015 - NUSTAR R&D), by DFG and NSFC through funds provided to the Sino-German CRC 110 “Symmetries and the Emergence of Structure in QCD” (NSFC Grant No. 11621131001, DFG Grant No. TRR110), by the Georgian Shota Rustaveli National Science Foundation (grant FR/417/6-100/14), by VolkswagenStiftung (grant no. 93562) and by the CAS President’s International Fellowship Initiative (PIFI) (Grant No. 2018DM0034).

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- [1] S. Weinberg, Phys. Lett. B **251**, 288 (1990).
 - [2] S. Weinberg, Nucl. Phys. **B363**, 3 (1991).
 - [3] C. Ordóñez and U. van Kolck, Phys. Lett. B **291**, 459 (1992).
 - [4] N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A **625**, 758 (1997), [arXiv:nucl-th/9706045].
 - [5] N. Kaiser, Phys. Rev. C **61**, 014003 (2000), [arXiv:nucl-th/9910044].
 - [6] P. F. Bedaque, H. W. Hammer and U. van Kolck, Nucl. Phys. A **676**, 357 (2000), [arXiv:nucl-th/9906032].
 - [7] P. F. Bedaque, G. Rupak, H. W. Griesshammer and H. W. Hammer, Nucl. Phys. A **714**, 589 (2003), [arXiv:nucl-th/0207034].
 - [8] H. W. Griesshammer, Nucl. Phys. A **760**, 110 (2005), [arXiv:nucl-th/0502039].
 - [9] N. Kaiser, Phys. Rev. C **64**, 057001 (2001), [arXiv:nucl-th/0107064].
 - [10] M. P. Valderrama and E. Ruiz Arriola, Phys. Lett. B **580**, 149 (2004), [arXiv:nucl-th/0306069].
 - [11] M. P. Valderrama and E. Ruiz Arriola, Phys. Rev. C **72**, 054002 (2005), [arXiv:nucl-th/0504067].
 - [12] M. P. Valderrama and E. Ruiz Arriola, Phys. Rev. C **74**, 064004 (2006), [arXiv:nucl-th/0507075].
 - [13] M. C. Birse and J. A. McGovern, Phys. Rev. C **70**, 054002 (2004), [arXiv:nucl-th/0307050].
 - [14] M. C. Birse, Phys. Rev. C **74**, 014003 (2006), [arXiv:nucl-th/0507077].
 - [15] J. V. Steele and R. J. Furnstahl, Nucl. Phys. A **645** (1999) 439, [arXiv:nucl-th/9808022].
 - [16] M. Lutz, Nucl. Phys. A **677**, 241 (2000), [arXiv:nucl-th/9906028].
 - [17] R. Higa and M. R. Robilotta, Phys. Rev. C **68**, 024004 (2003), [arXiv:nucl-th/0304025].
 - [18] R. Higa, M. R. Robilotta, C. A. da Rocha, Phys. Rev. C **69**, 034009 (2004), [arXiv:nucl-th/0310011].
 - [19] M.D. Cozma, O. Scholten, R.G. Timmermans, J.A. Tjon, Phys. Rev. C **75**, 014006 (2007).
 - [20] L. Girlanda, M. Viviani and W. H. Klink, Phys. Rev. C **76**, 044002 (2007), [nucl-th/0702024 [NUCL-TH]].
 - [21] E. Epelbaum, U.-G. Meißner, W. Glöckle, Nucl. Phys. **A714**, 535 (2003), nucl-th/0207089.
 - [22] E. Epelbaum *et al.*, Phys. Rev. Lett. **86**, 4787 (2001), nucl-th/0007057.
 - [23] E. Epelbaum *et al.*, Phys. Rev. **C66**, 064001 (2002), nucl-th/0208023.
 - [24] E. Epelbaum, W. Glöckle, U.-G. Meißner, Nucl. Phys. **A637**, 107 (1998), nucl-th/9801064.
 - [25] E. Epelbaum, W. Glöckle, U.-G. Meißner, Nucl. Phys. A **671**, 295 (2000), [arXiv:nucl-th/9910064].
 - [26] D. B. Kaplan, M. J. Savage, M. B. Wise, Nucl. Phys. **B478**, 629 (1996), [arXiv:nucl-th/9605002].
 - [27] D. B. Kaplan, M. J. Savage, M. B. Wise, Phys. Lett. B **424**, 390 (1998), [arXiv:nucl-th/9801034].
 - [28] D. B. Kaplan, M. J. Savage, M. B. Wise, Nucl. Phys. **B534**, 329 (1998), [arXiv:nucl-th/9802075].
 - [29] T. D. Cohen and J. M. Hansen, Phys. Rev. C **59**, 3047 (1999), [arXiv:nucl-th/9901065].
 - [30] S. Fleming, T. Mehen, and I. W. Stewart, Nucl. Phys. **A677**, 313 (2000), [arXiv:nucl-th/9911001].
 - [31] S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. van Kolck, Phys. Rev. A **64**, 042103 (2001), [quant-ph/0010073].
 - [32] S. R. Beane, P. F. Bedaque, M. J. Savage, U. van Kolck, Nucl. Phys. **A700**, 377 (2002), [arXiv:nucl-th/0104030].
 - [33] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005), [arXiv:nucl-th/0506005].
 - [34] G. P. Lepage, arXiv:nucl-th/9706029.
 - [35] T. S. Park, K. Kubodera, D. P. Min, and M. Rho, Nucl. Phys. **A646**, 83 (1999), [arXiv:nucl-th/9807054].
 - [36] G. P. Lepage, *Conference summary*, Prepared for INT Workshop on Nuclear Physics with EFT, Seattle, Washington, 25-26 Feb 1999.
 - [37] T. Frederico, V. S. Timoteo and L. Tomio, Nucl. Phys. A **653**, 209 (1999), [nucl-th/9902052].
 - [38] V. S. Timoteo, T. Frederico, A. Delfino and L. Tomio, Phys. Lett. B **621**, 109 (2005), [nucl-th/0508006].
 - [39] V. S. Timoteo, T. Frederico, A. Delfino and L. Tomio, Phys. Rev. C **83**, 064005 (2011), [arXiv:1006.1942 [nucl-th]].
 - [40] C. -J. Yang, C. Elster and D. R. Phillips, Phys. Rev. C **80**, 044002 (2009), [arXiv:0905.4943 [nucl-th]].
 - [41] M. C. Birse, Phil. Trans. Roy. Soc. Lond. A **369**, 2662 (2011), [arXiv:1012.4914 [nucl-th]].
 - [42] M. P. Valderrama, Phys. Rev. C **83**, 024003 (2011), [arXiv:0912.0699 [nucl-th]].
 - [43] M. Pavon Valderrama, Phys. Rev. C **84**, 064002 (2011), [arXiv:1108.0872 [nucl-th]].
 - [44] B. Long and C. J. Yang, Phys. Rev. C **85**, 034002 (2012), [arXiv:1111.3993 [nucl-th]].

- [45] B. Long and U. van Kolck, *Annals Phys.* **323**, 1304 (2008), [arXiv:0707.4325 [quant-ph]].
- [46] E. Epelbaum and J. Gegelia, *Eur. Phys. J. A* **41**, 341 (2009), [arXiv:0906.3822 [nucl-th]].
- [47] D. Djukanovic, J. Gegelia, S. Scherer and M. R. Schindler, *Few Body Syst.* **41**, 141 (2007), [arXiv:nucl-th/0609055].
- [48] J. Gegelia and G. Japaridze, *Phys. Lett. B* **517**, 476 (2001), [arXiv:nucl-th/0108005].
- [49] M. C. Birse, J. A. McGovern and K. G. Richardson, *Phys. Lett. B* **464**, 169 (1999), [hep-ph/9807302].
- [50] J. Gegelia, In **Adelaide 1998, Nonperturbative methods in quantum field theory** 30-35 [nucl-th/9802038].
- [51] K. Harada and H. Kubo, *Nucl. Phys. B* **758**, 304 (2006), [nucl-th/0605004].
- [52] M. J. Savage, In **Pasadena 1998, Nuclear physics with effective field theory** 247-267 [nucl-th/9804034].
- [53] X. L. Ren, K. W. Li, L. S. Geng, B. W. Long, P. Ring and J. Meng, *Chinese Physics C Vol. 42*, No. 1 (2018) 014103, [arXiv:1611.08475 [nucl-th]].
- [54] M. Pavón Valderrama, M. Sánchez Sánchez, C. J. Yang, B. Long, J. Carbonell and U. van Kolck, *Phys. Rev. C* **95**, no. 5, 054001 (2017), [arXiv:1611.10175 [nucl-th]].
- [55] R. Machleidt and D. R. Entem, *Phys. Rept.* **503**, 1 (2011), [arXiv:1105.2919 [nucl-th]].
- [56] E. Epelbaum, *Prog. Part. Nucl. Phys.* **57**, 654 (2006), [arXiv:nucl-th/0509032].
- [57] E. Epelbaum and U.-G. Meißner, *Ann. Rev. Nucl. Part. Sci.* **62**, 159 (2012), [arXiv:1201.2136 [nucl-th]].
- [58] P. F. Bedaque and U. van Kolck, *Ann. Rev. Nucl. Part. Sci.* **52**, 339 (2002), [nucl-th/0203055].
- [59] E. Epelbaum, H. -W. Hammer and U.-G. Meißner, *Rev. Mod. Phys.* **81**, 1773 (2009), [arXiv:0811.1338 [nucl-th]].
- [60] M. C. Birse, *PoS CD* **09**, 078 (2009) [arXiv:0909.4641 [nucl-th]].
- [61] M. P. Valderrama, *Int. J. Mod. Phys. E* **25**, no. 05, 1641007 (2016), [arXiv:1604.01332 [nucl-th]].
- [62] J. Gasser and H. Leutwyler, *Phys. Rept.* **87**, 77 (1982).
- [63] S. Weinberg, “The Quantum Theory Of Fields. Vol. 1,2: Foundations, Modern applications” *Cambridge, UK: Univ. Pr. (1995)*.
- [64] J. C. Collins, “*Renormalization. An introduction to renormalization, the renormalization group, and the operator product expansion,*” *Cambridge, Uk: Univ. Pr. (1984) 380p*.
- [65] E. Epelbaum, J. Gegelia and U.-G. Meißner, arXiv:1710.04178 [nucl-th].
- [66] S. R. Beane, T. D. Cohen and D. R. Phillips, *Nucl. Phys. A* **632**, 445 (1998), [nucl-th/9709062].
- [67] J. Gegelia, *Phys. Lett. B* **429**, 227 (1998).
- [68] J. Gegelia, *J. Phys. G* **25**, 1681 (1999), [nucl-th/9805008].
- [69] E. Epelbaum, J. Gegelia and U.-G. Meißner, *Nucl. Phys. B* **925**, 161 (2017), [arXiv:1705.02524 [nucl-th]].
- [70] W. Frank, D. J. Land and R. M. Spector, *Rev. Mod. Phys.* **43**, 36 (1971).
- [71] E. Epelbaum, H. Krebs and U.-G. Meißner, *Eur. Phys. J. A* **51**, no. 5, 53 (2015), [arXiv:1412.0142 [nucl-th]].
- [72] J. Gegelia and S. Scherer, *Int. J. Mod. Phys. A* **21**, 1079 (2006), [nucl-th/0403052].
- [73] E. Epelbaum and U.-G. Meißner, *Few Body Syst.* **54**, 2175 (2013), [nucl-th/0609037].
- [74] E. Epelbaum, A. M. Gasparyan, J. Gegelia and H. Krebs, *Eur. Phys. J. A* **51**, no. 6, 71 (2015), [arXiv:1501.01191 [nucl-th]].
- [75] P. F. Bedaque, H. W. Hammer and U. van Kolck, *Nucl. Phys. A* **646**, 444 (1999), [nucl-th/9811046].
- [76] G. V. Skornyyakov and Ter-Martirosyan, *Sov. Phys. JETP* **4**, 648 (1957) [*Zh. Eksp. Teor. Fiz.* **31**, 775 (1956)].
- [77] G.S. Danilov, *Sov. Phys. JETP* **13**, 349 (1961).
- [78] P. F. Bedaque, H. W. Hammer and U. van Kolck, *Phys. Rev. Lett.* **82**, 463 (1999), [nucl-th/9809025].
- [79] E. Epelbaum, J. Gegelia, U.-G. Meißner and D. L. Yao, *Eur. Phys. J. A* **53**, no. 5, 98 (2017), [arXiv:1611.06040 [nucl-th]].
- [80] S. Moroz, S. Floerchinger, R. Schmidt and C. Wetterich, *Phys. Rev. A* **79**, 042705 (2009), [arXiv:0812.0528 [cond-mat.stat-mech]].
- [81] K. Harada, H. Kubo and I. Yoshimoto, *Phys. Rev. D* **87**, no. 8, 085006 (2013), [arXiv:1208.0719 [hep-th]].