MIXED INTERIOR TRANSMISSION EIGENVALUES

joint work with Jijun Liu

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Part I: Introduction & motivation
INTRODUCTION & MOTIVATION

A general physical configuration

- Obstacle $D$ is located in perfect conducting substrate $D_2$ with boundary $\Gamma_2 \subset \Gamma$.
- Remaining part of boundary $\Gamma_1 = \Gamma \setminus \Gamma_2$ contacts with surface of background dielectric medium $D_1$.
- We assume $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \neq \emptyset$, and $\Gamma_2 \neq \emptyset$.
- Scattering problem for isotropic inhomogeneous media (TE mode electromagnetic scattering) leads to ...
INTRODUCTION & MOTIVATION

A general physical configuration

- ... (acoustic) interior transmission problem with mixed boundary condition:

\[
\begin{align*}
\Delta u + k^2 u &= 0, \quad x \in D, \\
\Delta v + k^2 n v &= 0, \quad x \in D, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu}, \quad x \in \Gamma_1, \quad \text{(transmission condition)} \\
u &= v = 0, \quad x \in \Gamma_2. \quad \text{(hom. Dirichlet condition)}
\end{align*}
\] (1)

- Here, \( n \neq 1 \) is the real-valued index of refraction (constant).
- Find \( k \neq 0 \) and non-trivial \((u, v)\) such that (1) is satisfied.
- Such \( k \) will be called mixed interior transmission eigenvalues (MITEs).
INTRODUCTION & MOTIVATION

Goal

- This is a non-standard eigenvalue problem.
- It is neither elliptic nor self-adjoint.
- How to solve this problem numerically?
- No results for the computation of MITEs have yet been reported.
Part II: Some theory
A short review

- The set of MITEs is at most discrete.
- Does not accumulate at zero.
- There exists an infinite number of real MITEs.
- Only accumulation point is $\infty$.

- Nothing is known for complex-valued MITEs.
Part III: Boundary integral equations
Fundamental solution: \( \Phi_k(x, y) = \frac{iH_0^{(1)}(k|x - y|)}{4}, x \neq y \).  
Single- and double-layer potentials over \( \Gamma \) given for \( x \notin \Gamma \) by

\[
\begin{align*}
SL_k^\Gamma [\psi] (x) &= \int_\Gamma \Phi_k(x, y) \psi(y) \, ds(y), \\
DL_k^\Gamma [\psi] (x) &= \int_\Gamma \partial_\nu(y) \Phi_k(x, y) \psi(y) \, ds(y).
\end{align*}
\]

Green’s representation theorem:

\[
u(x) = SL_k^\Gamma [\partial_\nu u|_\Gamma] (x) - DL_k^\Gamma [u|_\Gamma] (x), \quad x \in D.
\]
BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- \( \Gamma \) is disjoint union of \( \Gamma_1 \) and \( \Gamma_2 \). Hence,

\[
\begin{align*}
    u(x) &= \text{SL}^{\Gamma_1}_k \left[ \partial_D u|_{\Gamma_1} \right] (x) + \text{SL}^{\Gamma_2}_k \left[ \partial_D u|_{\Gamma_2} \right] (x) \\
    &\quad - \text{DL}^{\Gamma_1}_k \left[ u|_{\Gamma_1} \right] (x) - \text{DL}^{\Gamma_2}_k \left[ u|_{\Gamma_2} \right] (x), \quad x \in D, \quad (2)
\end{align*}
\]

\[
\begin{align*}
    v(x) &= \text{SL}^{\Gamma_1}_{k\sqrt{n}} \left[ \partial_D v|_{\Gamma_1} \right] (x) + \text{SL}^{\Gamma_2}_{k\sqrt{n}} \left[ \partial_D v|_{\Gamma_2} \right] (x) \\
    &\quad - \text{DL}^{\Gamma_1}_{k\sqrt{n}} \left[ v|_{\Gamma_1} \right] (x) - \text{DL}^{\Gamma_2}_{k\sqrt{n}} \left[ v|_{\Gamma_2} \right] (x), \quad x \in D. \quad (3)
\end{align*}
\]

- Using \( u|_{\Gamma_2} = v|_{\Gamma_2} = 0 \), equations (2) and (3) can be simplified to

\[
\begin{align*}
    u(x) &= \text{SL}^{\Gamma_1}_k \left[ \partial_D u|_{\Gamma_1} \right] (x) + \text{SL}^{\Gamma_2}_k \left[ \partial_D u|_{\Gamma_2} \right] (x) - \text{DL}^{\Gamma_1}_k \left[ u|_{\Gamma_1} \right] (x), \quad x \in D, \quad (4)
\end{align*}
\]

\[
\begin{align*}
    v(x) &= \text{SL}^{\Gamma_1}_{k\sqrt{n}} \left[ \partial_D v|_{\Gamma_1} \right] (x) + \text{SL}^{\Gamma_2}_{k\sqrt{n}} \left[ \partial_D v|_{\Gamma_2} \right] (x) - \text{DL}^{\Gamma_1}_{k\sqrt{n}} \left[ v|_{\Gamma_1} \right] (x), \quad x \in D. \quad (5)
\end{align*}
\]
Boundary integral operators:

\[ S_{k}^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_j} \Phi_k(x, y) \psi(y) \, ds(y), \quad x \in \Gamma_j, \]

\[ K_{k}^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_j} \partial_{\nu_i(y)} \Phi_k(x, y) \psi(y) \, ds(y), \quad x \in \Gamma_j, \]

\[ K_{k}^{\top \Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_j} \partial_{\nu_j(x)} \Phi_k(x, y) \psi(y) \, ds(y), \quad x \in \Gamma_j, \]

\[ T_{k}^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \partial_{\nu_j(x)} \int_{\Gamma_i} \partial_{\nu_i(y)} \Phi_k(x, y) \psi(y) \, ds(y), \quad x \in \Gamma_j, \]

where \( i, j \in \{1, 2\}. \)
BOUNDARY INTEGRAL EQUATIONS

Derivation of first boundary integral equation

- \( D \ni x \rightarrow x \in \Gamma_1 \) in (4) and (5) and jump relations:

\[
\begin{align*}
u|_{\Gamma_1} &= S_{k \rightarrow \Gamma_1}^{\Gamma_1} \left[ \partial_\nu v|_{\Gamma_1} \right] + S_{k \rightarrow \Gamma_1}^{\Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - \left( K_{k \rightarrow \Gamma_1}^{\Gamma_1} \left[ u|_{\Gamma_1} \right] - \frac{1}{2} u|_{\Gamma_1} \right), \\
\end{align*}
\]

\( (6) \)

\[
\begin{align*}
u|_{\Gamma_1} &= S_{k \rightarrow \Gamma_1}^{\Gamma_1} \left[ \partial_\nu v|_{\Gamma_1} \right] + S_{k \rightarrow \Gamma_1}^{\Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - \left( K_{k \rightarrow \Gamma_1}^{\Gamma_1} \left[ v|_{\Gamma_1} \right] - \frac{1}{2} v|_{\Gamma_1} \right), \\
\end{align*}
\]

\( (7) \)

\[
\begin{align*}
\text{Difference of (6) and (7), } u|_{\Gamma_1} &= v|_{\Gamma_1} \text{ and } \partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}: \\
0 &= \left( S_{k \rightarrow \Gamma_1}^{\Gamma_1} - S_{k \rightarrow \Gamma_1}^{\Gamma_1} \right) \left[ \partial_\nu u|_{\Gamma_1} \right] + S_{k \rightarrow \Gamma_1}^{\Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] - S_{k \rightarrow \Gamma_1}^{\Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - \left( K_{k \rightarrow \Gamma_1}^{\Gamma_1} - K_{k \rightarrow \Gamma_1}^{\Gamma_1} \right) \left[ u|_{\Gamma_1} \right]. \\
\end{align*}
\]

\( (8) \)
BOUNDARY INTEGRAL EQUATIONS

Derivation of second boundary integral equation

- \( D \ni x \rightarrow x \in \Gamma_2 \) in (4) and (5):

\[
\begin{align*}
  u|_{\Gamma_2} &= S_{k}^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_1} \right] + S_{k}^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] - K_{k}^{\Gamma_1 \rightarrow \Gamma_2} \left[ u|_{\Gamma_1} \right], \\
  v|_{\Gamma_2} &= S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_1} \right] + S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \left[ v|_{\Gamma_1} \right].
\end{align*}
\]  

(9)  

(10)

- Difference of (9) and (10), \( u|_{\Gamma_2} = v|_{\Gamma_2} = 0, u|_{\Gamma_1} = v|_{\Gamma_1} \) and \( \partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1} \):

\[
\begin{align*}
  0 &= \left( S_{k}^{\Gamma_1 \rightarrow \Gamma_2} - S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \right) \left[ \partial_\nu u|_{\Gamma_1} \right] + S_{k}^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] \\
  &\quad - S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - \left( K_{k}^{\Gamma_1 \rightarrow \Gamma_2} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \right) \left[ u|_{\Gamma_1} \right].
\end{align*}
\]  

(11)
BOUNDARY INTEGRAL EQUATIONS

Derivation of third boundary integral equation

- Normal derivative of (4) and (5), \( D \ni x \rightarrow x \in \Gamma_1 \), and jump relations:

\[
\partial_{\nu} u|_{\Gamma_1} = K_k^{\Gamma_1 \rightarrow \Gamma_1} \left[ \partial_{\nu} u|_{\Gamma_1} \right] + \frac{1}{2} \partial_{\nu} u|_{\Gamma_1} + K_k^{\Gamma_2 \rightarrow \Gamma_1} \left[ \partial_{\nu} u|_{\Gamma_2} \right] - T_k^{\Gamma_1 \rightarrow \Gamma_1} \left[ u|_{\Gamma_1} \right], \quad (12)
\]

\[
\partial_{\nu} v|_{\Gamma_1} = K_k^{\Gamma_1 \rightarrow \Gamma_1} \left[ \partial_{\nu} v|_{\Gamma_1} \right] + \frac{1}{2} \partial_{\nu} v|_{\Gamma_1} + K_k^{\Gamma_2 \rightarrow \Gamma_1} \left[ \partial_{\nu} v|_{\Gamma_2} \right] - T_k^{\Gamma_1 \rightarrow \Gamma_1} \left[ v|_{\Gamma_1} \right]. \quad (13)
\]

- Difference of (12) and (13), \( u|_{\Gamma_1} = v|_{\Gamma_1} \) and \( \partial_{\nu} u|_{\Gamma_1} = \partial_{\nu} v|_{\Gamma_1} \):

\[
0 = \left( K_k^{\Gamma_1 \rightarrow \Gamma_1} - K_k^{\Gamma_2 \rightarrow \Gamma_1} \right) \left[ \partial_{\nu} u|_{\Gamma_1} \right] + K_k^{\Gamma_2 \rightarrow \Gamma_1} \left[ \partial_{\nu} u|_{\Gamma_2} \right] - K_k^{\Gamma_2 \rightarrow \Gamma_1} \left[ \partial_{\nu} v|_{\Gamma_2} \right] - \left( T_k^{\Gamma_1 \rightarrow \Gamma_1} - T_k^{\Gamma_1 \rightarrow \Gamma_1} \right) \left[ u|_{\Gamma_1} \right]. \quad (14)
\]
Normal derivative of (4) and (5), $D \ni x \rightarrow x \in \Gamma_2$, and jump relations:

\[
\partial_\nu u|_{\Gamma_2} = K_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_1} \right] + K_k^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] + \frac{1}{2} \partial_\nu u|_{\Gamma_2} - T_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ u|_{\Gamma_1} \right], \quad (15)
\]

\[
\partial_\nu v|_{\Gamma_2} = K_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_1} \right] + K_k^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] + \frac{1}{2} \partial_\nu v|_{\Gamma_2} - T_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ v|_{\Gamma_1} \right]. \quad (16)
\]

Equations (15) and (16) can be rewritten as

\[
0 = K_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_1} \right] + K_k^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] - T_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ u|_{\Gamma_1} \right] - \frac{1}{2} \partial_\nu u|_{\Gamma_2}, \quad (17)
\]

\[
0 = K_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_1} \right] + K_k^{\Gamma_2 \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - T_k^{\Gamma_1 \rightarrow \Gamma_2} \left[ v|_{\Gamma_1} \right] - \frac{1}{2} \partial_\nu v|_{\Gamma_2}. \quad (18)
\]
Difference of (17) and (18), \( u|_{\Gamma_1} = v|_{\Gamma_1} \) and \( \partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1} \):

\[
0 = \left( K^\top_{k \rightarrow \Gamma_2} - K^\top_{k \sqrt{n} \rightarrow \Gamma_2} \right) \left[ \partial_\nu u|_{\Gamma_1} \right] + K^\top_{k \rightarrow \Gamma_2} \left[ \partial_\nu u|_{\Gamma_2} \right] \\
- K^\top_{k \sqrt{n} \rightarrow \Gamma_2} \left[ \partial_\nu v|_{\Gamma_2} \right] - \left( T^\Gamma_{k \rightarrow \Gamma_2} - T^\Gamma_{k \sqrt{n} \rightarrow \Gamma_2} \right) \left[ u|_{\Gamma_1} \right] - \frac{1}{2} \partial_\nu u|_{\Gamma_2} \\
+ \frac{1}{2} \partial_\nu v|_{\Gamma_2}.
\] (19)
Four equations (8), (11), (14), and (19) can be written as

\[ Z(k)g = 0 \]

with

\[
Z(k) = \begin{pmatrix}
S_{1 \rightarrow 1} - S_{1 \rightarrow 1} & S_{1 \rightarrow 2} - S_{1 \rightarrow 2} & S_{2 \rightarrow 1} - S_{2 \rightarrow 1} & S_{2 \rightarrow 2} - S_{2 \rightarrow 2} \\
S_{1 \rightarrow 1} & S - S_{1 \rightarrow 1} & S_{2 \rightarrow 1} & S_{2 \rightarrow 2} \\
S_{1 \rightarrow 2} & S_{2 \rightarrow 2} & \frac{S}{k \sqrt{n}} & \frac{S}{k \sqrt{n}} \\
S_{1 \rightarrow 1} & S_{1 \rightarrow 2} & \frac{S}{k \sqrt{n}} & \frac{S}{k \sqrt{n}} \\
\end{pmatrix}
\]

(20)

\[
g = \begin{pmatrix}
\alpha & -\beta & \gamma & -\delta
\end{pmatrix}^T,
\]

where we used the notation

\[
\alpha = \partial_\nu u|_{\Gamma_1}, \quad \beta = u|_{\Gamma_1}, \quad \gamma = \partial_\nu u|_{\Gamma_2}, \quad \text{and} \quad \delta = \partial_\nu v|_{\Gamma_2}.
\]

(21)
Part IV: Numerical results
Discretize the resulting boundary integral operator via boundary element collocation method.

Curved boundary is approximated by lines.

Collocation nodes are the midpoints having $m$ collocation points in total.

Unknown function is approximated by constant interpolation at each midpoint.

Hence, we can regard (20) as non-linear eigenvalue problem of the form $Z(k)\tilde{g} = 0$ with $Z(k) \in \mathbb{C}^{m \times m}$ and $\tilde{g}$ the discretized version of $g$ given by (21).

Solved with Beyn’s algorithm (based on complex-valued contour integration of the resolvent).
NUMERICAL RESULTS

Unit circle, $n = 4$

Figure: Absolute value of $u$ (first row) and $v$ (second row) for the first four real-valued MITEs. The MITEs are 1.6818, 2.3185, 2.9533, 3.0791.
NUMERICAL RESULTS

Ellipse, major semi-axis $1$, minor semi-axis $4/5$, $n = 4$

Figure: Absolute value of $u$ (first row) and $v$ (second row) for the first four real-valued MITEs. The MITEs are $1.9111$, $2.4973$, $3.1282$, $3.4609$. 
**NUMERICAL RESULTS**

More on ellipses, major semi-axis

Table: **MITE**s for ellipses with various minor semi-axis using $n = 4$.

<table>
<thead>
<tr>
<th>Minor semi-axis</th>
<th>1\textsuperscript{st} MITE</th>
<th>2\textsuperscript{nd} MITE</th>
<th>3\textsuperscript{rd} MITE</th>
<th>4\textsuperscript{th} MITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.6818</td>
<td>2.3185</td>
<td>2.9533</td>
<td>3.0791</td>
</tr>
<tr>
<td>4/5</td>
<td>1.9111</td>
<td>2.4973</td>
<td>3.1282</td>
<td>3.4609</td>
</tr>
<tr>
<td>1/2</td>
<td>2.7709</td>
<td>3.1764</td>
<td>3.7892</td>
<td>4.3916</td>
</tr>
</tbody>
</table>

Table: **MITE**s for ellipses with various minor semi-axis using $n = 1/2$.

<table>
<thead>
<tr>
<th>Minor semi-axis</th>
<th>1\textsuperscript{st} MITE</th>
<th>2\textsuperscript{nd} MITE</th>
<th>3\textsuperscript{rd} MITE</th>
<th>4\textsuperscript{th} MITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.1620</td>
<td>4.5193</td>
<td>4.6482</td>
<td>5.8022</td>
</tr>
<tr>
<td>4/5</td>
<td>3.5798</td>
<td>4.8518</td>
<td>5.5187</td>
<td>6.2683</td>
</tr>
<tr>
<td>1/2</td>
<td>5.1115</td>
<td>6.1186</td>
<td>7.3248</td>
<td>8.4891</td>
</tr>
</tbody>
</table>
NUMERICAL RESULTS

Unit square, $n = 4$

Figure: Absolute value of $u$ (first row) and $v$ (second row) for the first three real-valued MITEs. The MITEs are 3.0503, 4.2622, 5.1805.
Figure: Absolute value of $u$ (first row) and $v$ (second row) for the first three real-valued MITEs. The MITEs are 2.6717, 3.6662, 4.8367.
NUMERICAL RESULTS

Unit square, $n = 4$

Figure: Absolute value of $u$ (first row) and $v$ (second row) for the first three real-valued MITEs. The MITEs are 4.0802, 5.2285, 5.7030.
Part V: Summary & outlook
SUMMARY & OUTLOOK

- Reviewed existence and discreteness of MITEs for real-valued constant $n$.
- Derived a system of boundary integral equations.
- Showed how to solve it.
- Provided extensive numerical results for a variety of 2D scatterers.

- Study behavior of the MITE eigenfunctions at corners.
- Investigate inside-outside-duality method both theoretically and practically.
REFERENCES

