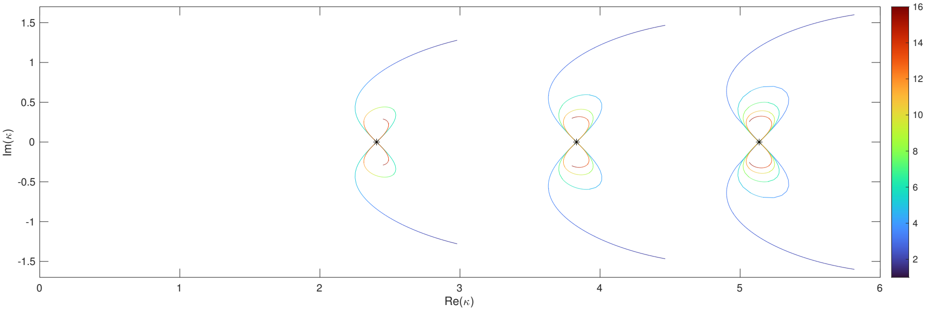


INTERIOR TRANSMISSION EIGENVALUE TRAJECTORIES

(joint work with Dr. Lukas Pieronek, Karlsruhe Institute of Technology)

SIAM CSE 2023 Amsterdam | February 27, 2023 | Andreas Kleefeld | Jülich Supercomputing Centre

INTRODUCTION

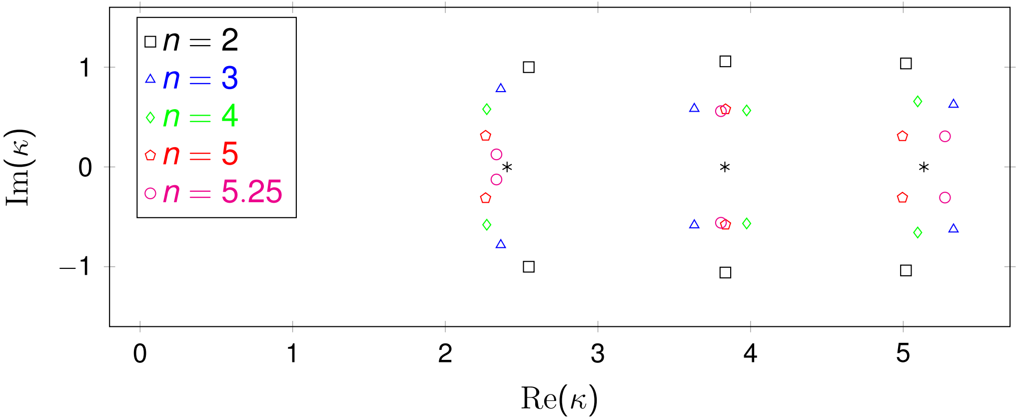
Motivation

- Consider non-linear non-self-adjoint eigenvalue (EV) problem for given bounded planar domain.
- Existence of discrete set of real-valued EVs is known.
- Existence of complex-valued EVs only proven for special geometries.
- Proof for general domains is still open.
- But complex-valued EVs can be computed numerically for general domains.

- **Can a certain structure/behavior of those be conjectured?**

INTRODUCTION

Motivation



Complex-valued EVs within $[0, 5.5] \times [-1.5, 1.5]\mathrm{i}$ for unit disk for the parameters $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n = 5.25$. Real-valued EVs are not shown.

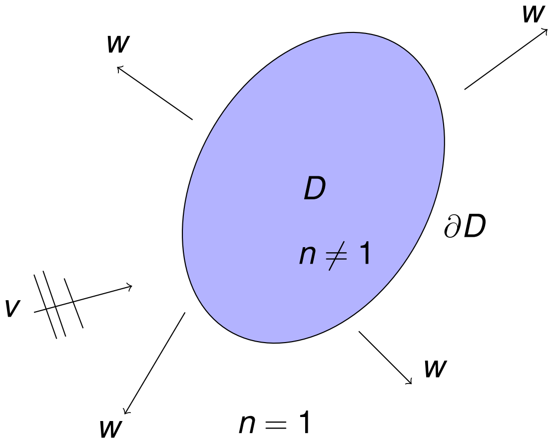
INTRODUCTION

Problem setup

Interior transmission problem (ITP):

$$\begin{aligned} \Delta v + \kappa^2 v &= 0 \quad \text{in } D\,, \\ \Delta w + n\kappa^2 w &= 0 \quad \text{in } D\,, \\ v &= w \quad \text{on } \partial D\,, \\ \partial_\nu v &= \partial_\nu w \quad \text{on } \partial D\,. \end{aligned}$$

Given $n \in L^\infty(D)$, $\kappa \in \mathbb{C}\backslash\{0\}$ is called interior transmission eigenvalue (ITE) if the ITP is solved for non-trivial $v, w \in L^2(D)$ such that $(v - w) \in H_0^2(D)$.



INTERIOR TRANSMISSION PROBLEM

Results known so far (list not complete)

- **Introduction of ITP:**
  Kirsch (1986) and Colton & Monk (1988).
- **Discreteness of real ITEs:**
  Colton & Kirsch & Päivärinta (1989), Rynne & Sleeman (1991), Cakoni & Haddar (2007), Colton & Päivärinta & Sylvester (2007), Kirsch (2009), Cakoni & Haddar (2009), and Hickmann (2012).
- **Existence of real ITEs:**
  Päivärinta & Sylvester (2009), Kirsch (2009), Cakoni & Gintides & Haddar (2011), Cakoni & Haddar (2011), Cakoni & Kirsch (2011), Bellis & Cakoni & Guzina (2011), and Cossonnière (2011).
- **Existence of complex-valued ITEs:**
  Colton & Leung & Meng (2015) D is a disc/ball and n is spherically stratified.

INTERIOR TRANSMISSION PROBLEM

Recent work (partial list) on numerical methods

- **Boundary integral equations:** Cossonnière (2011), Cossonnière & Haddar (2013), Kleefeld (2013), Zeng & Sun & Xu (2016), Cakoni & Kress (2017).
- **Inside-outside-duality method:** Kirsch & Lechleiter (2013), Lechleiter & Peters (2014), Lechleiter & Rennoch (2015), Peters & Kleefeld (2016).
- **Finite-element-method:** Monk & Sun (2012), Ji & Sun (2013), Sun & Xu (2013), Ji & Sun & Xie (2014), Li & Huang & Lin & Liu (2015), Sun & Zhou (2017), Li & Yang (2018), Bi & Han & Yang (2019), Yang & Zhang & Bi (2020), Liu & Sun (2021), Gong & Sun & Turner & Zheng (2022).
- **Method of fundamental solutions:** Kleefeld & Pieronek (2018), (2019).

- More researchers will follow.

ITE TRAJECTORIES

The considered domains

- Study ITE trajectories for homogeneous media ($n = \mathrm{const}$).
- At first, the unit disk is considered.
- Then also other planar bounded domains such as an ellipse, a triangle, a square, a deformed ellipse, and a clover are investigated.

ITE TRAJECTORIES

The unit disk

ITP eigenfunctions are given by Bessel functions

$$v_n(r,\phi) = J_p(\kappa_n r)\big(\alpha\cos(p\phi) + \beta\sin(p\phi)\big)$$
$$w_n(r,\phi) = \gamma_n J_p(\sqrt{n}\kappa_n r)\big(\alpha\cos(p\phi) + \beta\sin(p\phi)\big)$$

$$\Longrightarrow \kappa_n \text{ is an ITE} \quad \Leftrightarrow \quad \underbrace{\kappa_n J_p'(\kappa_n)J_p(\kappa_n\sqrt{n}) - \kappa_n\sqrt{n}J_p(\kappa_n)J_p'(\kappa_n\sqrt{n})}_{=:F_p(n,\kappa_n)} = 0\,.$$

By implicit function theorem:
$n \mapsto \kappa_n$ is continuously differentiable as long as $\partial_\kappa F_p(n,\kappa)|_{(n,\kappa_n)} \neq 0$.

ITE TRAJECTORIES

The unit disk

For ITP eigenfunctions v_n, w_n it holds that

$$\int_D |v_n|^2 - n|w_n|^2 \,\mathrm{d}x = \begin{cases} 0 & \text{for } \kappa_n \in \mathbb{C}\backslash\mathbb{R} \\ \frac{(1-n)(\alpha^2+\beta^2)}{2} J_p(\kappa_n)^2 \int_0^{2\pi} \cos(p\phi)^2 \,\mathrm{d}\phi & \text{for } \kappa_n \in \mathbb{R} \end{cases}$$

$\Longrightarrow$ Intersection points of complex-valued ITE trajectories with real axis are Dirichlet eigenvalues (DELs) of the (negative) Laplacian.

$\Longrightarrow$ 1-1 correspondence: For all DELs κ^* of the unit disk, there exists a unique complex-valued ITE trajectory $n \mapsto \kappa_n$ such that $\lim_{n\to\infty} \kappa_n = \kappa^*$.

NUMERICAL RESULTS

The unit disk

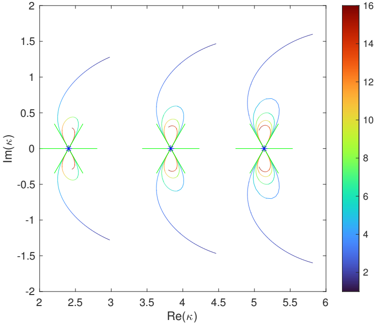


Figure: ITE trajectories for unit disk

- Complex-valued ITE trajectories show recurrent DEL-behavior as $n \to \infty$ and do not approach other DELs.
- There is no complex-valued ITE trajectories which are not linked to any DEL.

NUMERICAL RESULTS

The unit disk

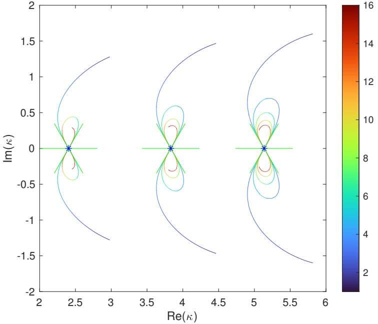


Figure: ITE trajectories for unit disk

- $\lim_{n\to n^*} |\kappa_n'| = \infty$ (where n^* is such that κ_{n^*} is an DEL).
- $\lim_{n\to n^*} \arg(\kappa_n') = \pm\pi/3$.

1-1 correspondence between DELs and complex-valued ITE trajectories also for other domains?

NUMERICAL RESULTS

The ellipse with half-axis 0.95 and 1

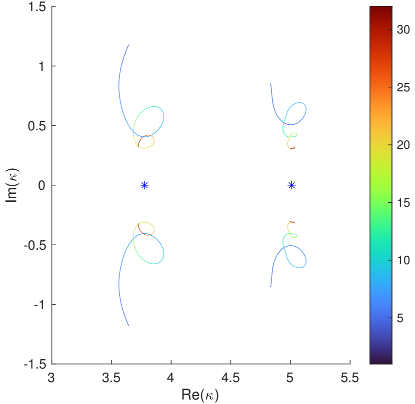


Figure: ITE trajectories for ellipse

- No recurrent DEL behavior, but still $\lim_{n\to\infty} \kappa_n = \kappa^*$ has been proven.
- A 1-1 correspondence is observed.

NUMERICAL RESULTS

For polygons (unit edges)

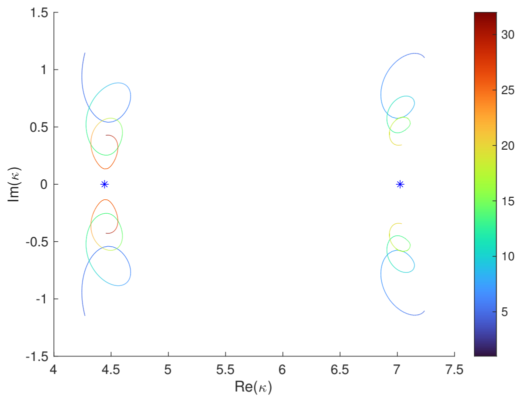


Figure: ITE trajectories for square

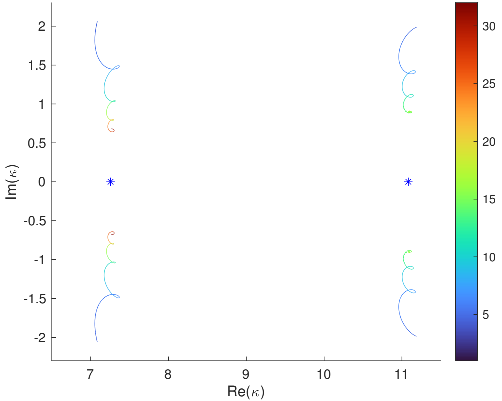


Figure: ITE trajectories for triangle

A 1-1 correspondence and $\lim_{n\to\infty} \kappa_n = \kappa^*$ for square and triangle is observed.

NUMERICAL RESULTS

The deformed ellipse (d-ellipse)

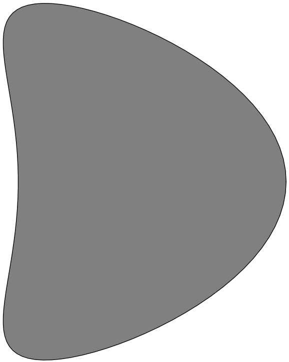


Figure: D-ellipse domain

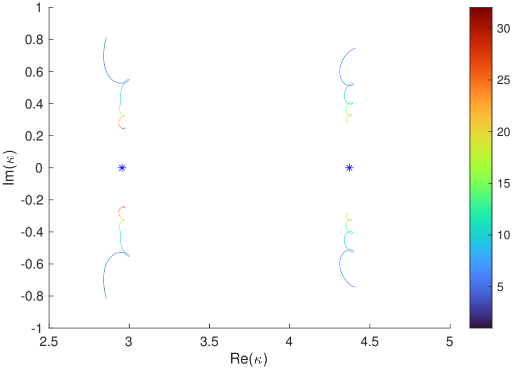


Figure: ITE trajectories for d-ellipse

A 1-1 correspondence and $\lim_{n\to\infty} \kappa_n = \kappa^*$ for deformed ellipse is observed.

NUMERICAL RESULTS

The clover

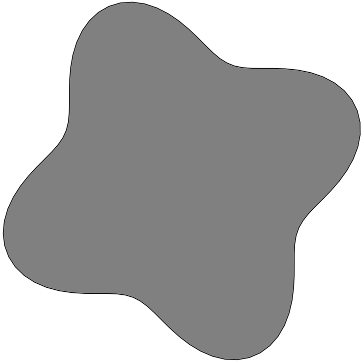


Figure: Clover domain

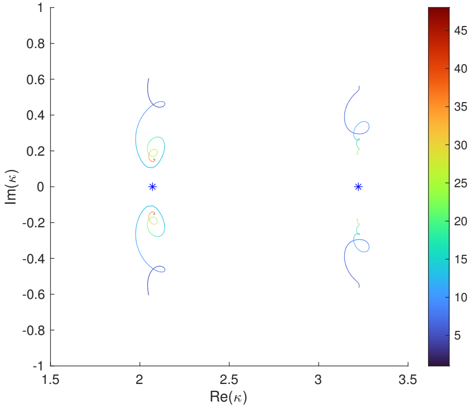


Figure: ITE trajectories for clover

A 1-1 correspondence and $\lim_{n\to\infty} \kappa_n = \kappa^*$ for clover is observed.

CONJECTURE

by Lukas Pieronek and Andreas Kleefeld

Conjecture:

For simply-connected scatterers in 2D and homogeneous media (index of refraction is constant), there is a 1-1 correspondence between DELs and complex-valued ITE trajectories determined by their limiting behavior as the index of refraction goes to infinity.

NUMERICAL RESULTS

Other interesting observation

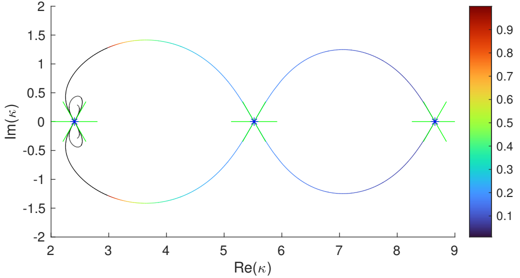


Figure: ITE trajectory for unit disk, $0 < n < 1$

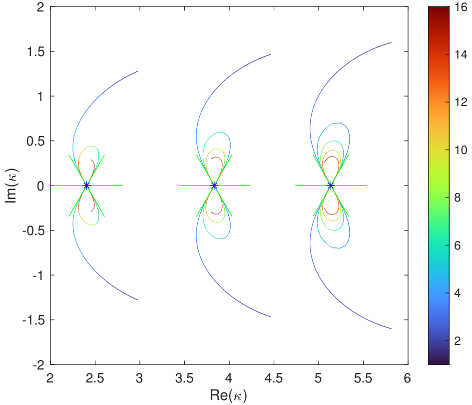


Figure: ITE trajectories for unit disk $n > 1$

- Case $0 < n < 1$ can be continuously extended to the case $n > 1$.
- Computed values for $n = 1$ are $2.9804 \pm 1.2796\mathrm{i}$, $4.4663 \pm 1.4675\mathrm{i}$, and $5.8169 \pm 1.6000\mathrm{i}$.

OUTLOOK

- Turn conjecture into theorem.
- Need to elaborate on infeasibility of non-simply-connected scatterers.
- If conjecture is true, does it extend to inhomogeneous media?
- Can new findings be used for a general existence proof of complex-valued ITEs?
- **Looking forward to your ideas, comments, and inspiring conversations.**

REFERENCE

LUKAS PIERONEK AND ANDREAS KLEEFELD, *On trajectories of complex-valued interior transmission eigenvalues*, arXiv:2205.11596 (2022).

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