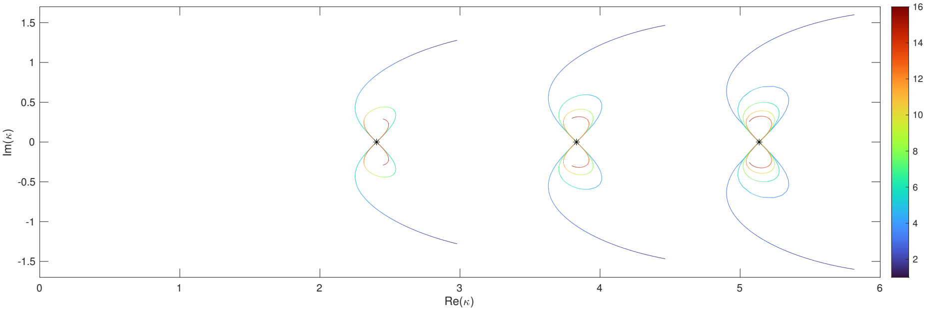

INTERIOR TRANSMISSION EIGENVALUE TRAJECTORIES

(joint work with Dr. Lukas Pieronek, Karlsruhe Institute of Technology)

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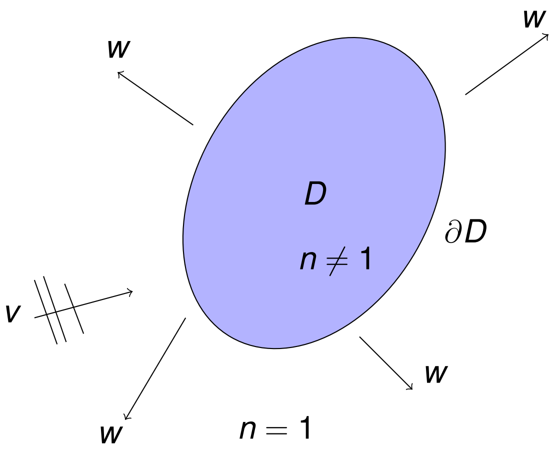
INTRODUCTION

Problem setup (first introduced by Kirsch (1986) and Colton & Monk (1988))

Interior transmission problem (ITP):

$$
\begin{aligned}
\Delta v + \kappa^2 v = 0 \quad & \text{in } D\,, \\
\Delta w + n\kappa^2 w = 0 \quad & \text{in } D\,, \\
v = w \quad & \text{on } \partial D\,, \\
\partial_\nu v = \partial_\nu w \quad & \text{on } \partial D\,.
\end{aligned}
$$

Given $n \in L^\infty(D)$, $\kappa \in \mathbb{C}\backslash\{0\}$ is called interior transmission eigenvalue (ITE) if the ITP is solved for non-trivial $v, w \in L^2(D)$ such that $(v - w) \in H_0^2(D)$.



INTRODUCTION

Motivation

- ITP is non-linear & non-self-adjoint eigenvalue problem.
- Real-valued ITEs are discrete.

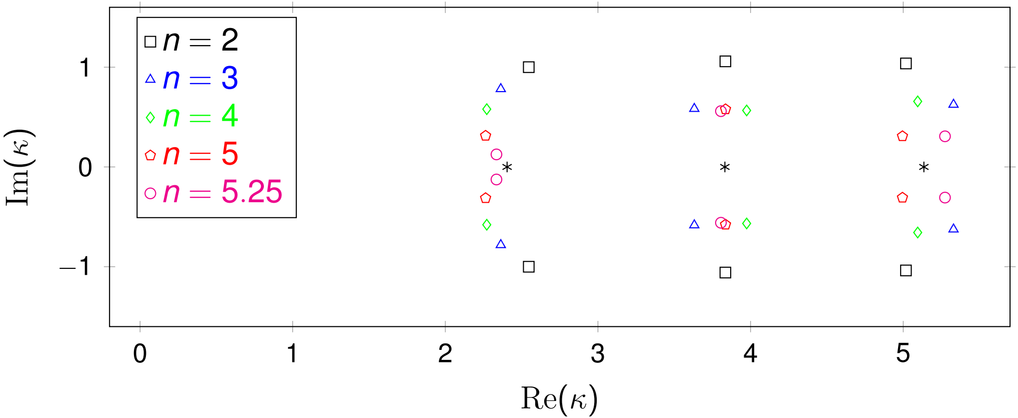
  Colton & Kirsch & Päivärinta (1989), Rynne & Sleeman (1991), Cakoni & Haddar (2007), Colton & Päivärinta & Sylvester (2007), Kirsch (2009), Cakoni & Haddar (2009), and Hickmann (2012), ...
- Existence of real-valued ITEs is known.

  Päivärinta & Sylvester (2009), Kirsch (2009), Cakoni & Gintides & Haddar (2011), Cakoni & Haddar (2011), Cakoni & Kirsch (2011), Bellis & Cakoni & Guzina (2011), and Cossonnière (2011), ...
- Existence of complex-valued ITEs only proven for special geometries.

  Colton & Leung & Meng (2015)
- Proof for general domains is still open.
- But complex-valued ITEs can be computed numerically for general domains.

- **Can a certain pattern of those be conjectured?**

INTRODUCTION
Motivation



Complex-valued ITEs of unit disk within $[0, 5.5] \times [-1.5, 1.5]\mathrm{i}$ for the parameters $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n = 5.25$. Real-valued ITEs are not shown.

ITE TRAJECTORIES

The considered domains

- Study ITE trajectories $n \mapsto \kappa_n$ for homogeneous media ($n = \mathrm{const}$).
- At first, consider $D =$ unit disk.
- Then, other planar bounded domains D such as
  - an ellipse,
  - a triangle,
  - a square,
  - a deformed ellipse, &
  - a clover

  are investigated numerically.

ITE TRAJECTORIES

The unit disk

ITP eigenfunctions are explicitly given by Bessel functions

$$v_n(r,\phi) = J_p(\kappa_n r)\big(\alpha\cos(p\phi) + \beta\sin(p\phi)\big)$$
$$w_n(r,\phi) = \gamma_n J_p(\sqrt{n}\kappa_n r)\big(\alpha\cos(p\phi) + \beta\sin(p\phi)\big)$$

$$\overset{\text{BC}}{\Longrightarrow} \kappa_n \text{ is an ITE} \quad \Leftrightarrow \quad \underbrace{\kappa_n J_p'(\kappa_n) J_p(\kappa_n\sqrt{n}) - \kappa_n\sqrt{n} J_p(\kappa_n) J_p'(\kappa_n\sqrt{n})}_{=:F_p(n,\kappa_n)} = 0\,.$$

By implicit function theorem:
Locally, $n \mapsto \kappa_n$ is continuously differentiable as long as $\underbrace{\partial_\kappa F_p(n,\kappa)|_{(n,\kappa_n)} \neq 0}_{\text{holds for } \textbf{all } \kappa_n \in \mathbb{C}\setminus\mathbb{R}}$.

Can show:
ITE trajectories $n \mapsto \kappa_n$ are even globally defined and continuous.

ITE TRAJECTORIES

Main results

Theorem (1-1 correspondence for D = unit disk)

- *Each **complex-valued** ITE trajectory $n \mapsto \kappa_n$ converges to some Dirichlet eigenvalue of the Laplacian (DEL) κ^* as $n \to \infty$.*
- *For each DEL κ^* there exists a **unique complex-valued** ITE trajectory $n \mapsto \kappa_n$ (modulo complex conjugation) such that $\lim_{n\to\infty} \kappa_n = \kappa^*$.*

Note: For real-valued ITE trajectories $n \mapsto \kappa_n$ it holds that $\lim_{n\to\infty} \kappa_n = 0$.

Theorem (Limit points for arbitrary D)

***If** a complex-valued ITE trajectory $n \mapsto \kappa_n$ is convergent as $n \to \infty$, then the limit point is a DEL.*

Recall: Existence of complex-valued ITEs for arbitrary D is open.

INTERIOR TRANSMISSION PROBLEM

Recent work (partial list) on numerical methods

- Boundary integral equations:

  Cossonnière (2011), Cossonnière & Haddar (2013), Kleefeld (2013), Zeng & Sun & Xu (2016), Cakoni & Kress (2017).

- Inside-outside-duality method:

  Kirsch & Lechleiter (2013), Lechleiter & Peters (2014), Lechleiter & Rennoch (2015), Peters & Kleefeld (2016)

- Finite-element-method:

  Monk & Sun (2012), Ji & Sun (2013), Sun & Xu (2013), Ji & Sun & Xie (2014), Li & Huang & Lin & Liu (2015), Sun & Zhou (2017), Li & Yang (2018), Bi & Han & Yang (2019), Yang & Zhang & Bi (2020), Liu & Sun (2021), Gong & Sun & Turner & Zheng (2022)

- Method of fundamental solutions:

  Kleefeld & Pieronek (2018), (2019)

- More researchers will follow.

NUMERICAL RESULTS

The unit disk

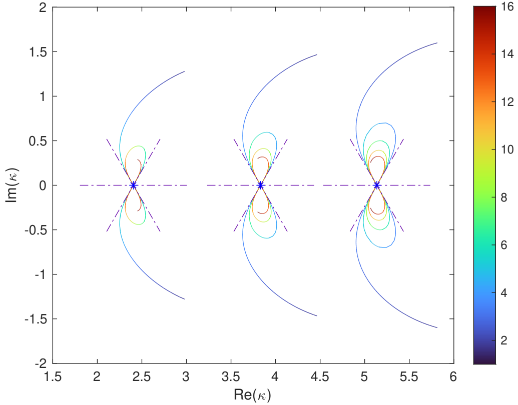


Figure: ITE trajectories for unit disk

$\Longrightarrow$ 1:1 correspondence is validated.

Further interesting observations:

- Complex-valued κ_n intersect DEL recurrently as $n \to \infty$.
- $\lim_{n\to n^*} |\kappa_n'| = \infty$ (where n^* is such that κ_{n^*} is a DEL).
- $\lim_{n\to n^*} \arg(\kappa_n') = \pm\pi/3$.

Does **1-1 correspondence** between DELs and complex-valued ITE trajectories also hold for other domains D?

NUMERICAL RESULTS

The ellipse with half-axis 0.95 and 1

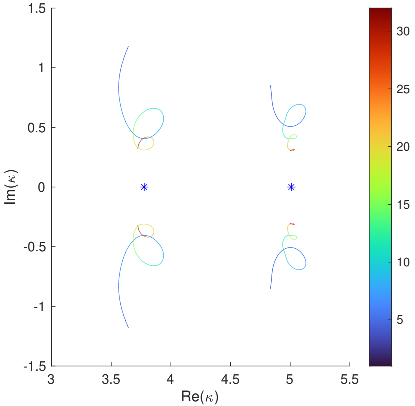


Figure: ITE trajectories for ellipse

Numerically, 1-1 correspondence between complex-valued ITE trajectories and DELs can again be observed for the ellipse.

NUMERICAL RESULTS

For polygons (unit edges)

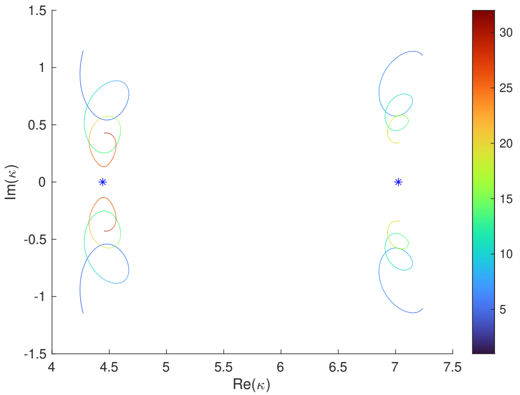


Figure: ITE trajectories for square

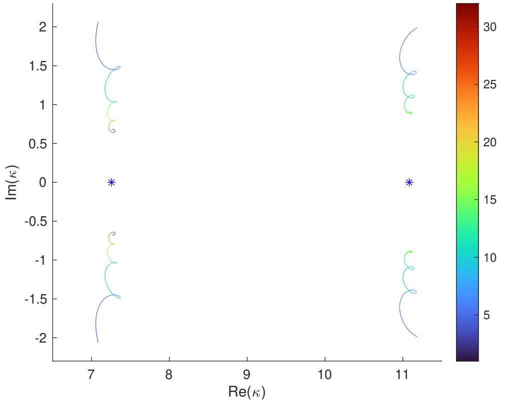


Figure: ITE trajectories for triangle

Also, a 1-1 correspondence between complex-valued ITE trajectories and DELs can be observed numerically for a triangle and a square.

NUMERICAL RESULTS

The deformed ellipse (d-ellipse)

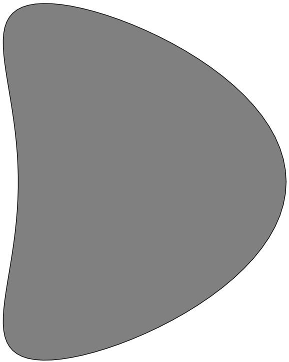


Figure: D-ellipse domain

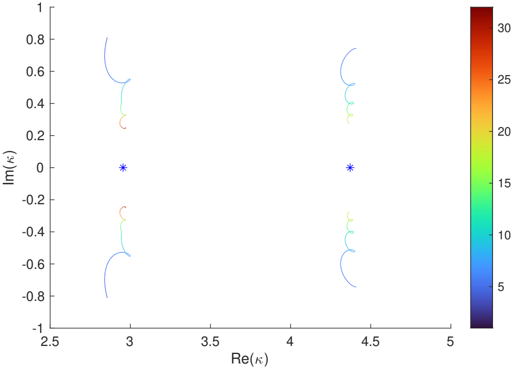


Figure: ITE trajectories for d-ellipse

Likewise, a 1-1 correspondence between complex-valued ITE trajectories and DELs can be observed numerically for a deformed ellipse.

NUMERICAL RESULTS

The clover

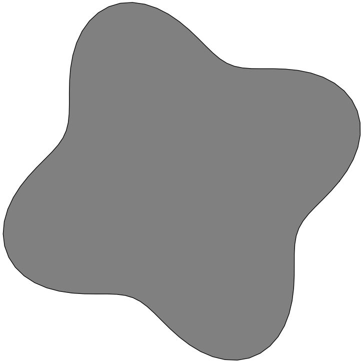


Figure: Clover domain

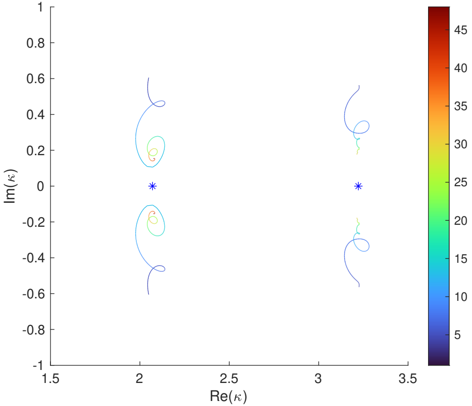


Figure: ITE trajectories for clover

And still, a 1-1 correspondence between complex-valued ITE trajectories and DELs can be observed numerically for a clover.

CONJECTURE

by Lukas Pieronek and Andreas Kleefeld

Conjecture:

For bounded simply-connected Lipschitz scatterers in 2D and homogeneous media (index of refraction is constant), there is a 1-1 correspondence between DELs and complex-valued ITE trajectories determined by their limiting behavior as the index of refraction goes to infinity.

NUMERICAL RESULTS

Other interesting observation

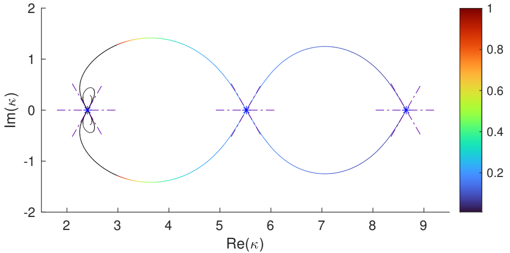


Figure: ITE trajectory for unit disk, $n > 0$

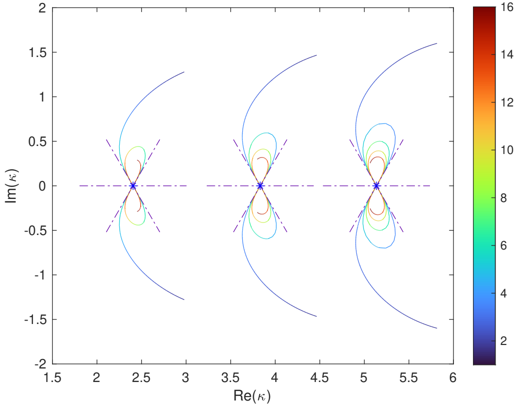


Figure: ITE trajectories for unit disk $n > 1$

- Case $0 < n < 1$ can be continuously extended to the case $n > 1$.
- Computed values for $n = 1$ are $2.9804 \pm 1.2796\mathrm{i}$, $4.4663 \pm 1.4675\mathrm{i}$, and $5.8169 \pm 1.6000\mathrm{i}$.

OUTLOOK

- Turn conjecture into theorem.
- Need to elaborate on infeasibility of non-simply-connected scatterers.
- If conjecture is true, does it extend to inhomogeneous media?
- Can new findings be used for a general existence proof of complex-valued ITEs?
- **Looking forward to your ideas, comments, and inspiring conversations.**

REFERENCES

- LUKAS PIERONEK & ANDREAS KLEEFELD, *On trajectories of complex-valued interior transmission eigenvalues*, arXiv:2205.11596 (2022).
- LUKAS PIERONEK & ANDREAS KLEEFELD, *On trajectories of complex-valued interior transmission eigenvalues*, Inverse Problems and Imaging, accepted.

- Data & source code are available at: https://github.com/kleefeld80/ITEtrajectory

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