# Quantum annealing : Sampling efficiency for 2-SAT problems with multiple solutions

V. Mehta[1,2], F. Jin[1], K. Michielsen[1,2] and H. De Raedt[1,3]

[1] *Institute for Advanced Simulation, Jülich Supercomputing Centre, Forschungszentrum Jülich, D-52425 Jülich, Germany*

[2] *RWTH Aachen University, D-52056 Aachen, Germany*

[3] *Zernike Institute for Advanced Materials, University of Groningen, NL-9747AG Groningen, The Netherlands*

## 2-Satisfiability (SAT) problems

A 2-SAT problem is specified by $N$ binary variables $x_i$ and a conjunction of $M$ clauses defining a binary-valued cost function

$$C(x_1,\dots,x_N) = (L_{1,1} \vee L_{1,2}) \wedge (L_{2,1} \vee L_{2,2}) \wedge \dots \wedge (L_{M,1} \vee L_{M,2})$$

where the literal $L_{\alpha,j}$ stands for either $x_{i(\alpha,j)}$ or its negation $\bar{x}_{i(\alpha,j)}$ for $\alpha = 1,\dots,M$ and $j = 1,2$.

The corresponding formulation of Ising Hamiltonian is given by [1]

$$H_{2SAT} = \sum_{\alpha=1}^{M} h_{2SAT}(\varepsilon_{\alpha,1} S_{i(\alpha,1)}, \varepsilon_{\alpha,2} S_{i(\alpha,2)})$$

where $\varepsilon_{\alpha,j} = +1(-1)$ if $L_{\alpha,j}$ stands for $x_i$ ($\bar{x}_i$) and $h_{2SAT}(S_l, S_m) = (S_l - 1)(S_m - 1)$.

**Chosen problem Hamiltonians** :

1) Have four degenerate ground states.
2) Highly degenerated first excited states.
3) 1000 problems in each set with 6 ≤ N ≤ 20.

## Methods

### Standard quantum annealing :

System starts in the uniform superposition state and is annealed towards the Hamiltonian encoding the problem to be solved.

### Reverse annealing :

System is initialized in one of the low-lying classical states and is annealed backwards up to a reversal distance $s_r$. From there the system is swept back to the problem Hamiltonian, after an optional wait of $T_W$.

### Resources :

Simulations [2] and D-Wave quantum annealers [3]

## Standard quantum annealing

### Simulations

Result for N=14 problems with four ground states ($|\psi_0^i\rangle$ where i=1,2,3,4)

Example 1

| State | $T_A$ = 10 | $T_A$ = 1000 |
|---|---|---|
| $\psi_0^1$ | 0.0357 | 0.2497 |
| $\psi_0^2$ | 0.0320 | 0.2490 |
| $\psi_0^3$ | 0.0419 | 0. 2510 |
| $\psi_0^4$ | 0.0384 | 0.2502 |
| Total | 0.1471 | 0.9999 |

Example 2

| State | $T_A$ = 10 | $T_A$ = 1000 |
|---|---|---|
| $\psi_0^1$ | 0.1233 | 0.4986 |
| $\psi_0^2$ | 0.0742 | 0.2507 |
| $\psi_0^3$ | 0.0648 | $9.56\times10^{-10}$ |
| $\psi_0^4$ | 0.0589 | 0.2506 |
| Total | 0.3212 | 0.9909 |

**Perturbation theory [4]** : $V_{i,j} = \langle \psi_0^i | H_I | \psi_0^j \rangle$

$$V = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$|\nu_1\rangle$ = 1/2(1, 1, 1, 1)
p = 0.25, 0.25, 0.25, 0.25

$$V = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$|\nu_1\rangle$ = 1/2(√2, 1, 0, 1)
p = 0.5, 0.25, 0, 0.25

**➔Annealing results agree with perturbation theory predictions for long $T_A$**

### D-Wave Advantage_5.1 annealer

Almost equal sampling probabilities for all ground states for almost all problems

**➔ Temperature effects and noise play a significant role**

## Reverse annealing

### Simulation results

- Different annealing times
  $s_r = 0.7, T_W = 0$,
  initial state = $|\psi_0^1\rangle$

| State | $T_A$ = 10 | $T_A$ = 90 | $T_A$ = 100 |
|---|---|---|---|
| $\psi_0^1$ | 0.1891 | 0.9011 | 0.0296 |
| $\psi_0^2$ | 0.4053 | 0.0495 | 0.4051 |
| $\psi_0^3$ | 0.0001 | 0.000003 | 0.000005 |
| $\psi_0^4$ | 0.4031 | 0.0493 | 0.4852 |
| Total | 0.9976 | 1.000 | 1.0000 |

Success probabilities corresponding to different annealing times for the problem posed by example 2 and $s_r = 0.7$

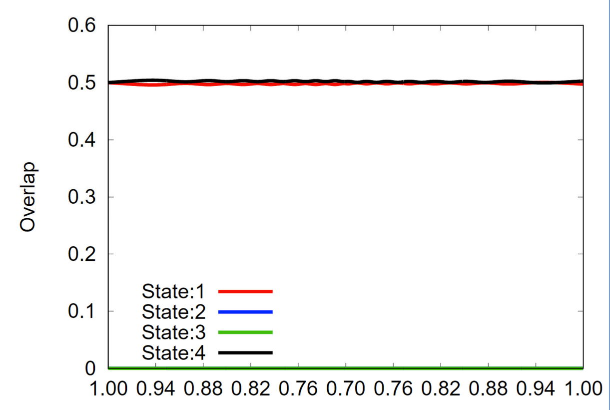


Fig. 1 : Overlap of the state of the system with the lowest four states of the instantaneous Hamiltonian for $T_A$=100 and $s_r = 0.7$

- Different reversal distance
  $T_A = 1000, T_W = 0$,
  initial state = $|\psi_0^1\rangle$

| State | $s_r$ = 0.4 | $s_r$ = 0.6 | $s_r$ = 0.8 |
|---|---|---|---|
| $\psi_0^1$ | 0.0703 | 0.8310 | 0.6147 |
| $\psi_0^2$ | 0.2874 | 0.0566 | 0.1926 |
| $\psi_0^3$ | 0.0012 | 0.000002 | 0 |
| $\psi_0^4$ | 0.2739 | 0.0579 | 0.1926 |
| Total | 0.6396 | 0.9454 | 1.0000 |

Success probabilities corresponding to different reversal distances for the problem posed by example 2

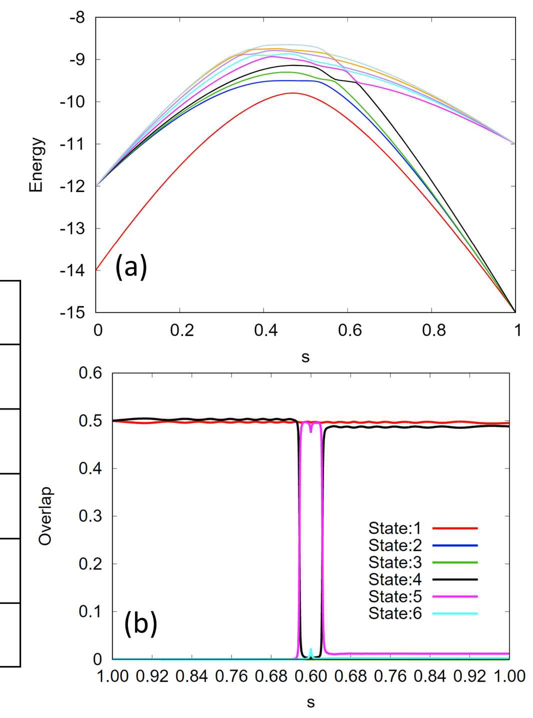


Fig. 2 : (a) Energy spectrum and (b) overlap of the state with the low-lying states of the instantaneous Hamiltonian

### D-Wave results

- Different annealing times
  $s_r = 0.7,\ T_W = 0$, initial state = $|\psi_0^1\rangle$

| State | $T_A$ = 0.5μs | $T_A$ = 50μs | $T_A$ = 200μs |
|---|---|---|---|
| $\psi_0^1$ | 0.9991 | 0.6709 | 0.5405 |
| $\psi_0^2$ | 0.0006 | 0.1470 | 0.2490 |
| $\psi_0^3$ | 0 | 0 | 0 |
| $\psi_0^4$ | 0.0003 | 0.0830 | 0.2105 |
| Total | 1.000 | 0.9009 | 1.0000 |

- Different reversal distances
  $T_A = 20\ \mu s, T_W = 0$, initial state = $|\psi_0^1\rangle$

| State | $s_r$ = 0.4 | $s_r$ = 0.6 | $s_r$ = 0.8 |
|---|---|---|---|
| $\psi_0^1$ | 0.4080 | 0.3120 | 1.000 |
| $\psi_0^2$ | 0.3235 | 0.4075 | 0 |
| $\psi_0^3$ | 0.0065 | 0 | 0 |
| $\psi_0^4$ | 0.2556 | 0.2751 | 0 |
| Total | 0.9936 | 0.9946 | 1.0000 |

## Summary and outlook

Our study focused on efficiently sampling all the degenerate ground states of a problem using quantum annealing. We observe that using the standard quantum annealing Hamiltonian, the numerically obtained sampling probabilities, even if not fair, can, in the case of long annealing times, be explained using perturbation theory. The sampling probabilities from the D-Wave quantum annealer remain comparable for almost all the problems. On the other hand, the sampling probabilities for the reverse annealing protocol depends greatly on the choice of annealing times, reversal distance, waiting time, and the initial state for both simulations and D-Wave results.

References:

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