

c_{SW} at One-Loop Order for Brillouin Fermions

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Wilson-like Dirac operators can be written in the form $D = \gamma_\mu \nabla_\mu - \frac{ar}{2} \Delta$. For Wilson fermions the standard two-point derivative $\nabla_\mu^{(\text{std})}$ and 9-point Laplacian $\Delta^{(\text{std})}$ are used. For Brillouin fermions these are replaced by improved discretizations $\nabla_\mu^{(\text{iso})}$ and $\Delta^{(\text{bri})}$ which have 54- and 81-point stencils respectively. We derive the Feynman rules in lattice perturbation theory for the Brillouin action and apply them to the calculation of the improvement coefficient c_{SW} , which, similar to the Wilson case, has a perturbative expansion of the form $c_{\text{SW}} = 1 + c_{\text{SW}}^{(1)} g_0^2 + \mathcal{O}(g_0^4)$. For $N_c = 3$ we find $c_{\text{SW}}^{(1)}_{\text{Brillouin}} = 0.12362580(1)$, compared to $c_{\text{SW}}^{(1)}_{\text{Wilson}} = 0.26858825(1)$, both for $r = 1$.

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1. Introduction

The massless Wilson Dirac Operator can be expressed in terms of the standard two-point derivative ∇_μ^{std} and the standard 9-point Laplacian Δ^{std}

$$D_W(x, y) = \sum_\mu \gamma_\mu \nabla_\mu^{\text{std}}(x, y) - \frac{ar}{2} \Delta^{\text{std}}(x, y). \quad (1)$$

The massless Brillouin Dirac operator is obtained by replacing the derivative and Laplacian by different, more complicated discretizations called ∇_μ^{iso} and Δ^{bri} [1][2]

$$D_B(x, y) = \sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(x, y) - \frac{ar}{2} \Delta^{\text{bri}}(x, y). \quad (2)$$

This way the Brillouin Dirac operator has an 81-point stencil containing all *off-axis* points that are 1-, 2-, 3-, and 4-hops away from x . The contributing points are weighted by the coefficients

$$(\rho_1, \rho_2, \rho_3, \rho_4) = \frac{1}{432} (64, 16, 4, 1) \quad (3)$$

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{r}{64} (-240, 8, 4, 2, 1), \quad (4)$$

such that

$$\begin{aligned} D_B(x, y) = & -\frac{\lambda_0}{2} \delta(x, y) \\ & + \sum_{\mu=\pm 1}^{\pm 4} \left(\rho_1 \gamma_\mu - \frac{\lambda_1}{2} \right) W_\mu(x) \delta(x + \hat{\mu}, y) \\ & + \sum_{\substack{\mu, \nu=\pm 1 \\ |\mu| \neq |\nu|}}^{\pm 4} \left(\rho_2 \gamma_\mu - \frac{\lambda_2}{4} \right) W_{\mu\nu}(x) \delta(x + \hat{\mu} + \hat{\nu}, y) \\ & + \sum_{\substack{\mu, \nu, \rho=\pm 1 \\ |\mu| \neq |\nu| \neq |\rho|}}^{\pm 4} \left(\frac{\rho_3}{2} \gamma_\mu - \frac{\lambda_3}{12} \right) W_{\mu\nu\rho}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho}, y) \\ & + \sum_{\substack{\mu, \nu, \rho, \sigma=\pm 1 \\ |\mu| \neq |\nu| \neq |\rho| \neq |\sigma|}}^{\pm 4} \left(\frac{\rho_4}{6} \gamma_\mu - \frac{\lambda_4}{48} \right) W_{\mu\nu\rho\sigma}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma}, y), \end{aligned} \quad (5)$$

where $|\mu| \neq |\nu| \neq \dots$ is used in a transitive way i.e. the sums are over indices with pairwise different absolute values. The W s are the average of the products of the link variables U along the paths:

$$W_\mu(x) = U_\mu(x) \quad (6)$$

$$W_{\mu\nu}(x) = \frac{1}{2} (U_\mu(x) U_\nu(x + \hat{\mu}) + \text{perm}) \quad (7)$$

$$W_{\mu\nu\rho}(x) = \frac{1}{6} (U_\mu(x) U_\nu(x + \hat{\mu}) U_\rho(x + \hat{\mu} + \hat{\nu}) + \text{perms}) \quad (8)$$

$$W_{\mu\nu\rho\sigma}(x) = \frac{1}{24} (U_\mu(x) U_\nu(x + \hat{\mu}) U_\rho(x + \hat{\mu} + \hat{\nu}) U_\sigma(x + \hat{\mu} + \hat{\nu} + \hat{\rho}) + \text{perms}) . \quad (9)$$

2. Feynman Rules of the Brillouin Action

We have derived the Feynman rules of the Brillouin action using a computer algebra system (Mathematica), supplemented by some analytical calculations by hand.

2.1 The Fermion Propagator

The Fourier transforms of the "free" derivative ∇_μ^{iso} and Laplace operator Δ^{bri} are [1]:

$$\nabla_\mu^{\text{iso}}(k) = \frac{i}{27} \sin(k_\mu) \prod_{\nu \neq \mu} (\cos(k_\nu) + 2) \quad (10)$$

$$\Delta^{\text{bri}}(k) = 4 \left(\cos^2(\tfrac{1}{2}k_1) \cos^2(\tfrac{1}{2}k_2) \cos^2(\tfrac{1}{2}k_3) \cos^2(\tfrac{1}{2}k_4) - 1 \right), \quad (11)$$

where k_μ is the fermion momentum in lattice units. Then the Fermion propagator is

$$S_{\text{Brillouin}}(k) = a \left(\sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k) \right)^{-1} = a \frac{-\sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k)}{\frac{r^2}{4} \Delta^{\text{bri}}(k)^2 - \left(\sum_\mu \nabla_\mu^{\text{iso}}(k) \right)^2}. \quad (12)$$

2.2 The Vertices

For the calculation of c_{SW} to one-loop order below the three vertices with one, two and three gluons coupling to a quark and anti-quark are needed (see Figure 1). We define the following notations to express the vertex Feynman rules:

$$s(k_\mu) = \sin\left(\tfrac{1}{2}k_\mu\right) \quad c(k_\mu) = \cos\left(\tfrac{1}{2}k_\mu\right) \quad (13)$$

$$\bar{s}(k_\mu) = \sin(k_\mu) \quad \bar{c}(k_\mu) = \cos(k_\mu) \quad (14)$$

$$K_{\mu\nu}^{(fg)}(p, q) = f(p_\mu + q_\mu) [\bar{g}(p_\nu) + \bar{g}(q_\nu)] \quad (15)$$

$$K_{\mu\nu\rho}^{(fgh)}(p, q) = f(p_\mu + q_\mu) \{ \bar{g}(p_\nu) \bar{h}(p_\rho) + \bar{g}(q_\nu) \bar{h}(q_\rho) + [\bar{g}(p_\nu) + \bar{g}(q_\nu)] [\bar{h}(p_\rho) + \bar{h}(q_\rho)] \} \quad (16)$$

$$K_{\mu\nu\rho\sigma}^{(fghj)}(p, q) = f(p_\mu + q_\mu) \{ 2 [\bar{g}(p_\nu) \bar{h}(p_\rho) \bar{j}(p_\sigma) + \bar{g}(q_\nu) \bar{h}(q_\rho) \bar{j}(q_\sigma)] + [\bar{g}(p_\nu) + \bar{g}(q_\nu)] [\bar{h}(p_\rho) + \bar{h}(q_\rho)] [\bar{j}(p_\sigma) + \bar{j}(q_\sigma)] \} \quad (17)$$

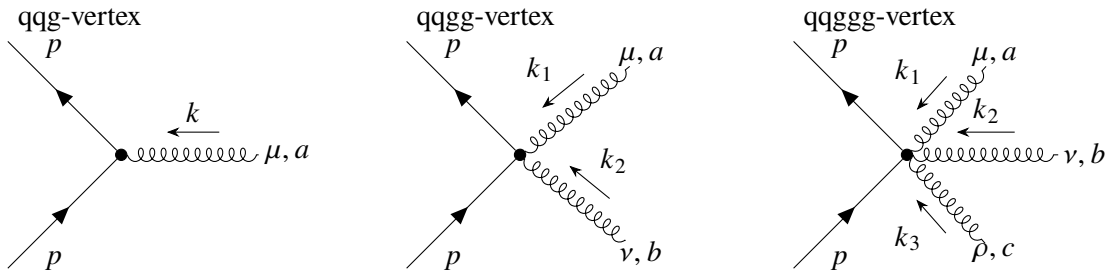


Figure 1: Momentum assignments for the vertices with one, two and three gluons.

$$L_{\mu\nu}^{(fg)}(p, q, k) = f(p_\mu + q_\mu + k_\mu)g(2p_\nu + k_\nu) \quad (18)$$

$$L_{\mu\nu\rho}^{(fgh)}(p, q, k) = f(p_\mu + q_\mu + k_\mu)g(2p_\nu + k_\nu) [\bar{h}(p_\rho) + \bar{h}(p_\rho + k_\rho) + \bar{h}(q_\rho)] \quad (19)$$

$$\begin{aligned} L_{\mu\nu\rho\sigma}^{(fghj)}(p, q, k) &= f(p_\mu + q_\mu + k_\mu)g(2p_\nu + k_\nu) \\ &\times \{ \bar{h}(p_\rho)\bar{j}(p_\sigma) + \bar{h}(p_\rho + k_\rho)\bar{j}(p_\sigma + k_\sigma) + \bar{h}(q_\rho)\bar{j}(q_\sigma) \\ &+ [\bar{h}(p_\rho) + \bar{h}(p_\rho + k_\rho) + \bar{h}(q_\rho)][\bar{j}(p_\sigma) + \bar{j}(p_\sigma + k_\sigma) + \bar{j}(q_\sigma)] \} \end{aligned} \quad (20)$$

$$M_{\mu\nu\rho}^{(fgh)}(p, q, k_1, k_2) = f(p_\mu + q_\mu + k_{1\mu} + k_{2\mu})g(2p_\nu + k_{1\nu} + 2k_{2\nu})h(2p_\rho + k_{2\rho}) \quad (21)$$

$$\begin{aligned} M_{\mu\nu\rho\sigma}^{(fghj)}(p, q, k_1, k_2) &= f(p_\mu + q_\mu + k_{1\mu} + k_{2\mu})g(2p_\nu + k_{1\nu} + 2k_{2\nu})h(2p_\rho + k_{2\rho}) \\ &\times \{ \bar{j}(p_\sigma) + \bar{j}(p_\sigma + k_{2\sigma}) + \bar{j}(p_\sigma + k_{1\sigma} + k_{2\sigma}) + \bar{j}(q_\sigma) \} \end{aligned} \quad (22)$$

with $f, g, h, j \in \{s, c\}$.

The $q\bar{q}g$ -vertex:

$$\begin{aligned} V_{1\mu}^a(p, q) &= -g_0 T^a \left[\lambda_1 s(p_\mu + q_\mu) + 2i\rho_1 c(p_\mu + q_\mu)\gamma_\mu \right. \\ &+ \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^4 \left\{ \lambda_2 K_{\mu\nu}^{(sc)}(p, q) + 2i\rho_2 \left(K_{\mu\nu}^{(cc)}(p, q)\gamma_\mu - K_{\mu\nu}^{(ss)}(p, q)\gamma_\nu \right) \right\} \\ &+ \frac{1}{3} \sum_{\substack{\nu, \rho=1 \\ \neq (\nu, \rho; \mu)}}^4 \left\{ \lambda_3 K_{\mu\nu\rho}^{(scc)}(p, q) + 2i\rho_3 \left(K_{\mu\nu\rho}^{(ccc)}(p, q)\gamma_\mu - 2K_{\mu\nu\rho}^{(ssc)}(p, q)\gamma_\nu \right) \right\} \\ &+ \frac{1}{9} \sum_{\substack{\nu, \rho, \sigma=1 \\ \neq (\nu, \rho, \sigma; \mu)}}^4 \left\{ \lambda_4 K_{\mu\nu\rho\sigma}^{(sccc)}(p, q) + 2i\rho_4 \left(K_{\mu\nu\rho\sigma}^{(cccc)}(p, q)\gamma_\mu - 3K_{\mu\nu\rho\sigma}^{(sscc)}(p, q)\gamma_\nu \right) \right\} \left. \right] \quad (23) \end{aligned}$$

The $q\bar{q}gg$ -vertex:

$$\begin{aligned} V_{2\mu\nu}^{ab}(p, q, k_1, k_2) &= ag_0^2 T^a T^b \left\{ -\frac{1}{2} \lambda_1 \delta_{\mu\nu} c(p_\mu + q_\mu) + i\rho_1 \delta_{\mu\nu} s(p_\mu + q_\mu)\gamma_\mu \right. \\ &+ \lambda_2 \left((1 - \delta_{\mu\nu}) L_{\mu\nu}^{(ss)}(p, q, k_2) - \frac{1}{2} \delta_{\mu\nu} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 K_{\mu\alpha}^{(cc)}(p, q) \right) \\ &+ \lambda_3 \left(\frac{2}{3} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho=1 \\ \neq (\rho, \mu, \nu)}}^4 L_{\mu\nu\rho}^{(ssc)}(p, q, k_2) - \frac{1}{6} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho=1 \\ \neq (\alpha, \rho, \mu)}}^4 K_{\mu\alpha\rho}^{(ccc)}(p, q) \right) \\ &+ \lambda_4 \left(\frac{1}{6} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho, \sigma=1 \\ \neq (\rho, \sigma, \mu, \nu)}}^4 L_{\mu\nu\rho\sigma}^{(sscc)}(p, q, k_2) - \frac{1}{18} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho, \sigma=1 \\ \neq (\alpha, \rho, \sigma, \mu)}}^4 K_{\mu\alpha\rho\sigma}^{(cccc)}(p, q) \right) \end{aligned}$$

$$\begin{aligned}
& +i\rho_2 \left(2(1 - \delta_{\mu\nu}) \left[L_{\mu\nu}^{(cs)}(p, q, k_2) \gamma_\mu + L_{\mu\nu}^{(sc)}(p, q, k_2) \gamma_\nu \right] \right. \\
& \quad \left. + \delta_{\mu\nu} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 \left[K_{\mu\alpha}^{(sc)}(p, q) \gamma_\mu + K_{\mu\alpha}^{(cs)}(p, q) \gamma_\alpha \right] \right) \\
& +i\rho_3 \left(\frac{4}{3}(1 - \delta_{\mu\nu}) \sum_{\substack{\rho=1 \\ \neq(\rho;\mu,\nu)}}^4 \left[L_{\mu\nu\rho}^{(csc)}(p, q, k_2) \gamma_\mu + L_{\mu\nu\rho}^{(scc)}(p, q, k_2) \gamma_\nu - L_{\mu\nu\rho}^{(sss)}(p, q, k_2) \gamma_\rho \right] \right. \\
& \quad \left. + \frac{1}{3} \delta_{\mu\nu} \sum_{\substack{\alpha,\rho=1 \\ \neq(\alpha,\rho;\mu)}}^4 \left[K_{\mu\alpha\rho}^{(scc)}(p, q) \gamma_\mu + 2K_{\mu\alpha\rho}^{(ccs)}(p, q) \gamma_\rho \right] \right) \\
& +i\rho_4 \left(\frac{1}{3}(1 - \delta_{\mu\nu}) \sum_{\substack{\rho,\sigma=1 \\ \neq(\rho,\sigma,\mu,\nu)}}^4 \left[L_{\mu\nu\rho\sigma}^{(csc)}(p, q, k_2) \gamma_\mu + L_{\mu\nu\rho\sigma}^{(sccc)}(p, q, k_2) \gamma_\nu - 2L_{\mu\nu\rho\sigma}^{(sssc)}(p, q, k_2) \gamma_\rho \right] \right. \\
& \quad \left. + \frac{1}{9} \delta_{\mu\nu} \sum_{\substack{\alpha,\rho,\sigma=1 \\ \neq(\alpha,\rho,\sigma,\mu)}}^4 \left[K_{\mu\alpha\rho\sigma}^{(sccc)}(p, q) \gamma_\mu + 3K_{\mu\alpha\rho\sigma}^{(ccsc)}(p, q) \gamma_\rho \right] \right) \Bigg\} \quad (24)
\end{aligned}$$

The $q\bar{q}gggg$ -vertex:

$$\begin{aligned}
V_{3\nu\rho}^{abc}(p, q, k_1, k_2, k_3) &= a^2 g_0^3 T^a T^b T^c \left\{ \frac{1}{6} \delta_{\mu\nu} \delta_{\mu\rho} (\lambda_1 s(p_\mu + q_\mu) + 2i\rho_1 c(p_\mu + q_\mu) \gamma_\mu) \right. \\
& + \lambda_2 \left(\frac{1}{6} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 K_{\mu\alpha}^{(sc)}(p, q) \right. \\
& \quad \left. + \frac{1}{2} \delta_{\mu\nu} (1 - \delta_{\mu\rho}) L_{\mu\rho}^{(cs)}(p, q, k_3) + \frac{1}{2} \delta_{\nu\rho} (1 - \delta_{\mu\nu}) L_{\mu\nu}^{(sc)}(p, q, k_2 + k_3) \right) \\
& + \lambda_3 \left(-\frac{2}{3} (1 - \delta_{\mu\nu}) (1 - \delta_{\mu\rho}) (1 - \delta_{\nu\rho}) M_{\mu\nu\rho}^{(sss)}(p, q, k_2, k_3) + \frac{1}{18} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha,\beta=1 \\ \neq(\alpha,\beta,\mu)}}^4 K_{\mu\alpha\beta}^{(scc)}(p, q) \right. \\
& \quad \left. + \frac{1}{3} \delta_{\mu\nu} (1 - \delta_{\mu\rho}) \sum_{\substack{\alpha=1 \\ \neq(\alpha,\mu,\rho)}}^4 L_{\mu\rho\alpha}^{(csc)}(p, q, k_3) + \frac{1}{3} \delta_{\nu\rho} (1 - \delta_{\mu\nu}) \sum_{\substack{\alpha=1 \\ \neq(\alpha,\mu,\nu)}}^4 L_{\mu\nu\alpha}^{(scc)}(p, q, k_2 + k_3) \right) \\
& + \lambda_4 \left(-\frac{1}{3} (1 - \delta_{\mu\nu}) (1 - \delta_{\mu\rho}) (1 - \delta_{\nu\rho}) \sum_{\substack{\sigma=1 \\ \neq(\sigma,\mu,\nu,\rho)}}^4 M_{\mu\nu\rho\sigma}^{(sssc)}(p, q, k_2, k_3) \right. \\
& \quad \left. + \frac{1}{54} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha,\beta,\sigma=1 \\ \neq(\alpha,\beta,\sigma,\mu)}}^4 K_{\mu\alpha\beta\sigma}^{(sccc)}(p, q) \right. \\
& \quad \left. + \frac{1}{12} \delta_{\mu\nu} (1 - \delta_{\nu\rho}) \sum_{\substack{\alpha,\sigma=1 \\ \neq(\alpha,\sigma;\mu,\rho)}}^4 L_{\mu\rho\alpha\sigma}^{(csc)}(p, q, k_3) + \frac{1}{12} \delta_{\nu\rho} (1 - \delta_{\mu\nu}) \sum_{\substack{\alpha,\sigma=1 \\ \neq(\alpha,\sigma;\mu,\nu)}}^4 L_{\mu\nu\alpha\sigma}^{(sccc)}(p, q, k_2 + k_3) \right) \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& +i\rho_2 \left(\frac{1}{3} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 \left[K_{\mu\alpha}^{(cc)}(p, q) \gamma_\mu - K_{\mu\alpha}^{(ss)}(p, q) \gamma_\alpha \right] \right. \\
& \quad + \delta_{\mu\nu} (1 - \delta_{\mu\rho}) \left[L_{\mu\rho}^{(cc)}(p, q, k_3) \gamma_\rho - L_{\mu\rho}^{(ss)}(p, q, k_3) \gamma_\mu \right] \\
& \quad \left. + \delta_{\nu\rho} (1 - \delta_{\mu\nu}) \left[L_{\mu\rho}^{(cc)}(p, q, k_2 + k_3) \gamma_\mu - L_{\mu\rho}^{(ss)}(p, q, k_2 + k_3) \gamma_\nu \right] \right) \\
& +i\rho_3 \left(-\frac{4}{3} (1 - \delta_{\mu\nu}) (1 - \delta_{\mu\rho}) (1 - \delta_{\nu\rho}) \right. \\
& \quad \times \left[M_{\mu\nu\rho}^{(css)}(p, q, k_2, k_3) \gamma_\mu + M_{\mu\nu\rho}^{(scs)}(p, q, k_2, k_3) \gamma_\nu + M_{\mu\nu\rho}^{(ssc)}(p, q, k_2, k_3) \gamma_\rho \right] \\
& \quad + \frac{1}{9} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha, \beta=1 \\ \neq (\alpha, \beta, \mu)}}^4 \left[K_{\mu\alpha\beta}^{(ccc)}(p, q) \gamma_\mu - K_{\mu\alpha\beta}^{(ssc)}(p, q) \gamma_\alpha - K_{\mu\alpha\beta}^{(scs)}(p, q) \gamma_\beta \right] \\
& \quad - \frac{2}{3} \delta_{\mu\nu} (1 - \delta_{\mu\rho}) \sum_{\substack{\alpha=1 \\ \neq (\alpha, \mu, \rho)}}^4 \left[L_{\mu\rho\alpha}^{(ssc)}(p, q, k_3) \gamma_\mu - L_{\mu\rho\alpha}^{(ccc)}(p, q, k_3) \gamma_\rho + L_{\mu\rho\alpha}^{(css)}(p, q, k_3) \gamma_\alpha \right] \\
& \quad - \frac{2}{3} \delta_{\nu\rho} (1 - \delta_{\mu\nu}) \sum_{\substack{\alpha=1 \\ \neq (\alpha, \mu, \nu)}}^4 \left[-L_{\mu\nu\alpha}^{(ccc)}(p, q, k_2 + k_3) \gamma_\mu + L_{\mu\nu\alpha}^{(ssc)}(p, q, k_2 + k_3) \gamma_\nu \right. \\
& \quad \left. + L_{\mu\nu\alpha}^{(scs)}(p, q, k_2 + k_3) \gamma_\alpha \right] \Big) \\
& +i\rho_4 \left(-\frac{2}{3} (1 - \delta_{\mu\nu}) (1 - \delta_{\mu\rho}) (1 - \delta_{\nu\rho}) \sum_{\substack{\sigma=1 \\ \neq (\sigma, \mu, \nu, \rho)}}^4 \left[M_{\mu\nu\rho\sigma}^{(cssc)}(p, q, k_2, k_3) \gamma_\mu + M_{\mu\nu\rho\sigma}^{(scsc)}(p, q, k_2, k_3) \gamma_\nu \right. \right. \\
& \quad \left. + M_{\mu\nu\rho\sigma}^{(sscc)}(p, q, k_2, k_3) \gamma_\rho - M_{\mu\nu\rho\sigma}^{(ssss)}(p, q, k_2, k_3) \gamma_\sigma \right] \\
& \quad + \frac{1}{27} \delta_{\mu\nu} \delta_{\mu\rho} \sum_{\substack{\alpha, \beta, \sigma=1 \\ \neq (\alpha, \beta, \sigma, \mu)}}^4 \left[K_{\mu\alpha\beta\sigma}^{(cccc)}(p, q) \gamma_\mu - K_{\mu\alpha\beta\sigma}^{(sscc)}(p, q) \gamma_\alpha \right. \\
& \quad \left. - K_{\mu\alpha\beta\sigma}^{(scsc)}(p, q) \gamma_\beta - K_{\mu\alpha\beta\sigma}^{(sccs)}(p, q) \gamma_\sigma \right] \\
& \quad - \frac{1}{6} \delta_{\mu\nu} (1 - \delta_{\mu\rho}) \sum_{\substack{\alpha, \sigma=1 \\ \neq (\alpha, \sigma, \mu, \rho)}}^4 \left[L_{\mu\rho\alpha\sigma}^{(sscc)}(p, q, k_3) \gamma_\mu - L_{\mu\rho\alpha\sigma}^{(cccc)}(p, q, k_3) \gamma_\rho \right. \\
& \quad \left. + L_{\mu\rho\alpha\sigma}^{(cssc)}(p, q, k_3) \gamma_\alpha + L_{\mu\rho\alpha\sigma}^{(cscs)}(p, q, k_3) \gamma_\sigma \right] \\
& \quad + \frac{1}{6} \delta_{\nu\rho} (1 - \delta_{\mu\nu}) \sum_{\substack{\alpha, \sigma=1 \\ \neq (\alpha, \sigma, \mu, \nu)}}^4 \left[L_{\mu\nu\alpha\sigma}^{(cccc)}(p, q, k_2 + k_3) \gamma_\mu - L_{\mu\nu\alpha\sigma}^{(sscc)}(p, q, k_2 + k_3) \gamma_\nu \right. \\
& \quad \left. + L_{\mu\nu\alpha\sigma}^{(scsc)}(p, q, k_2 + k_3) \gamma_\alpha + L_{\mu\nu\alpha\sigma}^{(sccs)}(p, q, k_2 + k_3) \gamma_\sigma \right] \Big) \Big\} \quad (25)
\end{aligned}$$

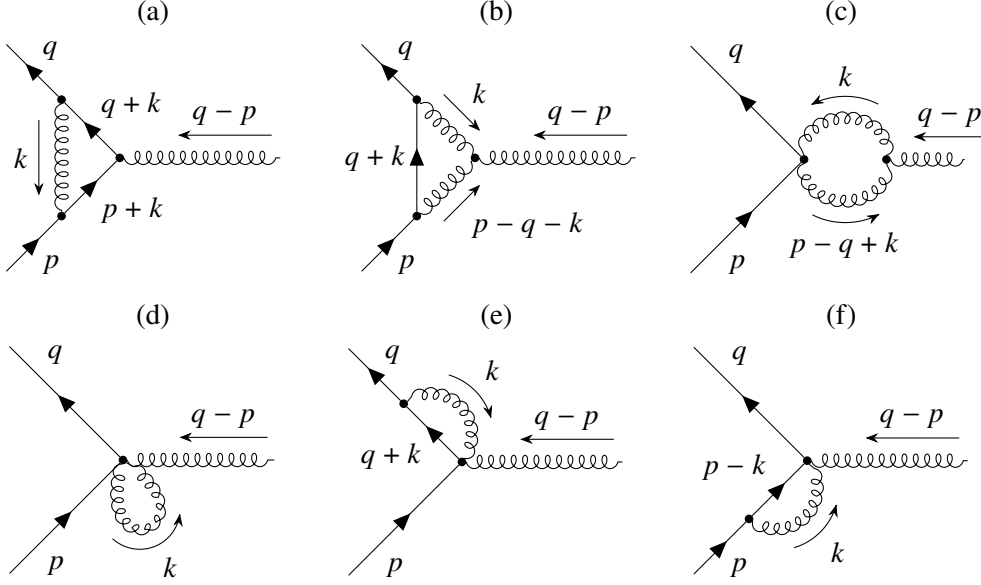


Figure 2: The six one-loop diagrams contributing to the vertex function in lattice perturbation theory.

Purely gluonic Feynman rules (gluon propagator and ggg -vertex) as well as contributions to the vertex Feynman rules coming from the clover term containing the improvement coefficient c_{SW} are unaffected by the fermion formulations and thus identical to the Wilson case (see Ref. [3]).

3. Perturbative Determination of c_{SW}

Adding the usual clover-term to the Brillouin action we obtain the $O(a)$ -improved Brillouin action (with chromo-hermitean field strength $F_{\mu\nu}$ for fixed $\mu < \nu$):

$$\mathcal{S}_{\text{Brillouin}}^{\text{Clover}}[\bar{\psi}, \psi] = \mathcal{S}_{\text{Brillouin}}[\bar{\psi}, \psi] + c_{\text{SW}} \cdot \sum_x \sum_{\mu < \nu} \bar{\psi}(x) \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x). \quad (26)$$

The improvement coefficient has a perturbative expansion $c_{\text{SW}} = c_{\text{SW}}^{(0)} + g_0^2 c_{\text{SW}}^{(1)} + O(g_0^4)$ and at tree level in both the Wilson and the Brillouin case:

$$c_{\text{SW}}^{(0)} = 1 (= r) \quad (27)$$

At one-loop level $c_{\text{SW}}^{(1)}$ is calculated from the one-loop vertex function $\Lambda_\mu^{a(1)}$ comprised of six one-loop Feynman diagrams (see Figure 2). The sum of all diagrams results in a finite integral, while diagrams (a), (b), (c), (e), and (f) on their own lead to IR-divergent integrals. To see the cancellation of the divergencies explicitly, we used a small gluon mass μ as a regulator (following Ref. [3]) and split off an analytically solvable divergent integral from the diagrams. Table 1 shows the results for each diagram for $N_c = 3$ and $r = 1$, comparing the Wilson and the Brillouin actions.

Diagram	Divergent part	Constant part Wilson	Constant part Brillouin
(a)	$-L/3$	0.004569626(1)	0.0047576939(1)
(b)	$-9L/2$	0.083078349(1)	0.055554134(1)
(c)	$+9L/2$	-0.081307544(1)	-0.057741323(1)
(d)	0	0.297394534(1)	0.142461144(1)
(e)	$L/6$	-0.017573359(1)	-0.010702925(1)
(f)	$L/6$	-0.017573359(1)	-0.010702925(1)
Sum	0	0.26858825(1)	0.12362580(1)

Table 1: Divergent and constant contributions from each diagram for $N_c = 3$ and $r = 1$. The logarithmic divergence is encoded in $L := \frac{1}{16\pi^2} \ln\left(\frac{\pi^2}{\mu^2}\right)$.

The sum of all diagrams finally results in the following value

$$c_{\text{SW}_{\text{Brillouin}}}^{(1)} = 0.045785517(3)N_c - 0.041192255(3)\frac{1}{N_c} = 0.12362580(1), \quad (28)$$

where we have set $N_c = 3$ in the last step. Compare to

$$c_{\text{SW}_{\text{Wilson}}}^{(1)} = 0.0988424712(4)N_c - 0.083817496(3)\frac{1}{N_c} = 0.26858825(1) \quad (29)$$

from Ref. [3], one sees that $c_{\text{SW}_{\text{Brillouin}}}^{(1)}$ is about half of $c_{\text{SW}_{\text{Wilson}}}^{(1)}$ for any value of N_c .

References

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