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Can foreign exchange rates violate Bell inequalities?

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ABSTRACT

The analysis of empirical data through model-free inequalities leads to the conclusion that violations of Bell-type inequalities by empirical data cannot have any significance unless one believes that the universe operates according to the rules of a mathematical model.

1. Introduction

The issue of "violation of Bell inequalities" is at the center of many heated discussions about the foundations and interpretations of quantum physics. The huge interest in the subject is reflected by the award of the Nobel Prize in physics 2022 "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science". Remarkably and surprisingly, there is still no consensus about what such violations actually **imply** [1–50].

Recently, we presented new **model-free** inequalities that reduce to Bell-like inequalities[1] in a very special case [53]. These inequalities put constraints on certain linear combinations of correlations of the (two-valued) data and, very importantly, are independent of the model that one imagines to have produced the data. In this paper, we add a new model-free inequality to the family of model-free inequalities presented in Ref. [53] and use the new inequality to analyze foreign exchange data. The model-independent character of the inequalities that we derive, and not the Bell-type inequalities themselves, provide the appropriate background for discussing the "implications" (whatever they are) of violations of Bell-type inequalities **by empirical data**. An important point of these model-free inequalities is that their derivation only exploits elementary arithmetic properties of the data sets and do not refer to physics at all. They equally apply to any kind of discrete data [53].

Relating these model-free inequalities to physics involves making additional assumptions about the model that one imagines to have produced the data. We scrutinize the latter, subtle point by analyzing data of "non-physical" origin, namely, publicly available foreign exchange data [54], see Section 2.

In Section 3, we present a new model-free inequality involving only three correlations and in Section 4 we use the foreign exchange data to search for violations of the Bell-type inequality for **empirical data**, obtained as a special case of the model-free inequality. The main conclusion of this analysis is the following (see Section 4 for a detailed account). Computing correlations by considering the **whole** data set, elementary arithmetic dictates that there can be no violation of the Boole inequality [55], a

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[1] We use this term to refer to Bell's inequality involving three correlations [1], the Clauser–Horn–Shimony–Holt (CHSH) [51], Clauser–Horn [52], and all other inequalities that directly follow from Bell's model for the EPRB thought experiment.

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predecessor of the Bell inequality involving three correlations [1,56]. However, dividing (with respect to the time of the transactions) the data set in three equal parts, the Boole–Bell inequality can be violated (but the corresponding model-free inequality cannot). Clearly, the observed violation merely reflects **our choice** of selecting data rather than the "reality" which, in the present example, is the **complete** data set. The observation that, depending on the grouping of foreign exchange data, the analysis can lead to very different conclusions (e.g., violation versus no violation), is reminiscent of Simpson's paradox [57]. In this case, the word "paradox" does not mean that there is a contradiction involved but rather emphasizes that viewing the same data in different ways can lead to very different conclusions.

All the Einstein–Podolsky–Rosen–Bohm (EPRB) **laboratory** experiments reporting violations of Bell-type inequalities that we are aware of employ at least one mechanism for selecting the data from which the correlations are computed. Thus, potentially they all fall victim to Simpson's paradox. Most EPRB experiments [58–63] use time-coincidence windows to select pairs of photons. Other EPRB experiments [64–66] use voltage thresholds to classify a detection event as being the arrival of a photon or as something else. The sets of empirical data thus obtained can never violate the model-free inequalities [53]. Furthermore, unless **all** the cited experiments generate these data such that they can be reshuffled to form triples and quadruples (which is extremely unlikely), there is no mathematically sound argument why **these experimental data** should satisfy any of the Bell-type inequalities [53]. As in the case of the foreign exchange data, a violation of Bell-type inequalities by experimental data merely reflects the properties of the process, **chosen by the experimenter**, to select groups of data, not an intrinsic property of the data themselves. For instance, the violation of the CHSH inequality by the data of the EPRB experiment by Weihs et al. [62] smoothly changes into a non-violation by increasing the time coincidence window (see Fig. 5 in Ref. [53]), a clear case of conclusions depending on viewing the same data differently, recall Simpson's paradox. For a discussion of the physics aspects of this phenomenon, see section 9 in Ref. [53].

The preceding discussion is "model-free". Within this framework, one can only prove Bell-type inequalities if the data satisfies what Boole called "conditions of possible experience" [55], that is if the data derives, without reshuffling, from triples, quadruples [17], etc., conditions which are highly unlikely to be satisfied in any EPRB laboratory experiment (but easily satisfied in computer experiments). Clearly, in the absence of a proof that Bell-type inequalities exist for general **empirical data**, no conclusion can be drawn from a violation of one of them.

For violations of Bell-type inequalities by experimental data to have any relevance for physics, it is essential to introduce models that one imagines to be able to produce the data and derive inequalities from these models. The simplest but fairly general model for EPRB experiments is undoubtedly the one introduced by Bell [1,56]. Bell's model almost trivially yields Bell-type inequalities which are used to prove Bell's theorem [56], stating that Bell's model can never reproduce the full functional form of the correlation of a quantum system in the singlet state (see sections 5 and 6). As discussed in Section 5, Bell's theorem is very important for the foundations of quantum theory.

However, as explained in Section 7, these Bell-type inequalities are derived within a particular mathematical model (Bell's model) and therefore a violation of one or more of them **by empirical data** only implies that this model cannot serve as a description of the data, as in the case of our example of the foreign exchange data.

In summary, feeding empirical data from **any** experiment into a Bell-type inequality and observing a violation only implies that Bell's model does not apply to the case at hand. In particular, the conclusions must be that Bell's model fails to describe how the data of actual EPRB experiments are collected and analyzed and that it is necessary to develop other, better models. As a matter of fact, a straightforward extension of Bell's model which accounts for the data selection process can, in the appropriate limit, **exactly** reproduce the results of a quantum system in the singlet state (see Ref. [53], section 11.5 and references therein). Other conclusions than the two just mentioned constitute logical fallacies, see Section 8.

2. Analysis of foreign exchange rates

The reader may wonder why the publicly available foreign exchange data are going to be analyzed by the procedure described in this section. Comparing this procedure to the one used to analyze data obtained by performing EPRB experiments, see Ref. [53](section 3) for a detailed explanation, it becomes clear that these two procedures are the same. However, there is no need to be familiar with these experiments to understand this procedure and the conclusions drawn from it. All that is necessary to know now is that changes of the exchange rates have to be digitized (means mapped onto ± 1) and that the quantities of interest are the correlations between the two-valued representation of changes in different exchange rates.

The raw data set contains the exchange rates of the currencies of twenty-two different countries relative to the US Dollar, starting on 3 January 2000 and ending on 31 December 2019 [54]. On some days (e.g. Christmas) there is no trading and therefore also no data. After removing the "no data" records, the data set contains 5015 records of twenty-two foreign exchange rates.

In detail, the procedure to calculate correlations is as follows.

1. Read the raw ratio data from the file *Foreign_Exchange_Rates.csv* (downloadable here), skipping the "no data" records and store the floating point data in an array of dimensions (5015,22).
2. Compute the forward (in time, that is record-wise) differences of these ratios and store the floating point results in an array of dimensions (5014,22).
3. Store the sign of all these differences in an integer array of dimensions (5014,22) as values ± 1. These data will be referred to as foreign exchange data in the following.
4. Divide the ± 1 data set into three equal parts of $N = 1671$ records (dropping one record), and denote the parts by $\mathcal{D}_s$ where the subscript $s = 1, 2, 3$ labels the data set and the data contained in them.

5. Compute the correlations

$$C_s(A,B) = \frac{1}{N}\sum_{i=1}^{N} A_{s,i} B_{s,i} \;, \tag{1}$$

where A and B each symbolize one of the twenty-two currencies and $A_{s,i} = \pm 1$, $B_{s,i} = \pm 1$. Excluding the trivial correlation $C_s(A,A) = 1$ and noting that $C_s(A,B) = C_s(B,A)$ for $s = 1,2,3$, this procedure yields $3 \times 21 \times 22/2 = 693$ different correlations.

3. Model-free inequalities

The prime focus of this paper is on the conclusions that can be drawn from violations of inequalities on certain linear combinations of the correlations $C_s(A,B)$, $C_{s'}(A',B')$, etc. As explained next, without knowing the actual values of these correlations or without assuming a particular model for the process that generates the data, elementary arithmetic alone already yields nontrivial inequalities that can never be violated by data.

As the $C_s(A,B)$'s are averages of ± 1 values, it follows immediately that $-1 \le C_s(A,B) \le 1$ for $s = 1,2,3$ and all pairs (A,B) of currencies. Then, obviously, $-2 \le C_s(A,B) + C_{s'}(A',B') \le 2$ where (A,B) and (A',B') denote any two pairs of currencies and $s,s' = 1,2,3$. Then what about $C_1(A,B) + C_2(A',B') + C_3(A'',B'')$, for instance?

Let (x,y,x',y',x'',y'') stand for any of the sextuples $(A_{1,i}, B_{1,i}, A'_{2,i}, B'_{2,i}, A''_{3,i}, B''_{3,i})$. If $x,y,x',y',x'',y'' = \pm 1$ it follows immediately that $-3 \le xy + x'y' + x''y'' \le 3$ and therefore, $-3 \le C_1(A,B) + C_2(A',B') + C_3(A'',B'') \le 3$. But can one find stricter bounds?

Recall that a basic property of a sum of numbers is that the order in which the numbers are added is irrelevant. This elementary arithmetic fact and this fact alone can reduce the values of contributions to $C_1(A,B) + C_2(A',B') + C_3(A'',B'')$. To see this, use the freedom to reshuffle the contributions to $C_2(A',B')$ and $C_3(A'',B'')$ and consider the expression $A_{1,i}\, B_{1,i} + A'_{2,j}\, B'_{2,j} + A''_{3,k}\, B''_{3,k}$. Further assume that for a given i, it is possible to find at least one pair (j,k) such that $A'_{2,j} = A_{1,i}$, $B'_{2,j} = B''_{3,k}$ and $A''_{3,k} = B_{1,i}$. In this particular case, the sextuple $(A_{1,i}, B_{1,i}, A'_{2,j}, B'_{2,k}, A''_{3,k}, B''_{3,k})$ derives from the triple $(x = A_{1,i}, y = B_{1,i}, z = B'_{2,j})$ and its contribution to $C_1(A,B) + C_2(A',B') + C_3(A'',B'')$ is given by $xy + xz + yz$. It is easy to verify that $-1 \le xy + xz + yz \le 3$ by considering all eight combinations of $x,y,z = \pm 1$. Thus, if by reshuffling the data it becomes possible to reduce sextuples to triples, the contributions of sextuples is bounded from below by minus one instead of minus three. In other words, reshuffling can reduce the contributions to linear combinations of the three correlations.

By a straightforward extension of the proof given in Ref. [53](appendix B), we can prove that in general

$$|C_1(A,B) \pm C_2(A',B')| \le 3 - 2\Gamma \pm C_3(A'',B'') \;, \tag{2}$$

or, equivalently

$$|C_1(A,B) \pm C_2(A',B')| \mp C_3(A'',B'') - 1 \le 2(1-\Gamma) \;, \tag{3}$$

where Γ denotes the maximum fraction of triples that one can identify by reshuffling the data in $\mathcal{D}_s$ with $s = 2,3$. Γ can be computed by a slightly modified version of the procedure described in Ref. [53](appendix B). Note that Eq. (2) is equivalent to $|C_1(A,B) \pm C_3(A'',B'')| \le 3 - 2\Gamma \pm C_2(A',B')$ or $|C_3(A'',B'') \pm C_2(A',B')| \le 3 - 2\Gamma \pm C_1(A,B)$, which follow directly from Eq. (N.6) in Ref. [53].

If and only if all the N pairs of records in the three data sets $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ can be reshuffled to create N triples, the fraction of triples $\Gamma = 1$. In this particular case, Eq. (2) takes the form of the Boole inequality [55]

$$|C_1(A,B) \pm C_2(A,B')| \le 1 \pm C_3(B,B') \;, \tag{4}$$

written here in a different but equivalent form than Boole did.

The inequality Eq. (2) (for $0 \le \Gamma \le 1$) holds for any set of sextuples $\mathcal{D} = \{(A_{1,i}, B_{1,i}, A'_{2,i}, B'_{2,i}, A''_{3,i}, B''_{3,i}) \mid i = 1,\ldots,N\}$, irrespective of how the data was obtained or generated. It is therefore **model free**, meaning that inequality Eq. (2) holds for data, regardless of any imaginary model that is believed to have produced these data. Model-free inequalities, involving four correlations and reducing to the Clauser–Horn–Shimony–Holt [52,56] and Clauser–Horn [51] inequalities in the exceptional case that all octuples of data can be reshuffled to form quadruples, are given in Ref. [53].

4. Application of the model-free inequalities to the foreign exchange data

It is clear that Eq. (2), being the result of basic arithmetic only, can never be violated if Γ is chosen as defined. With regard to Bell-type inequalities, to be discussed in Section 7, the main question of interest is "can certain combinations of the foreign exchange data violate Eq. (2) with $\Gamma = 1$?" Note that if these data were uniformly random, $\Gamma = 1$ up to statistical fluctuations. The proof of this fact is similar to the one given in Ref. [53](appendix B).

Computing the pairwise correlations of all foreign exchange rates according to the procedure outlined in Section 2, there are 85 out of 18480 possible combinations of triples of foreign currencies that violate at least one of Bell-like inequalities

$$|C_1(A,B) \pm C_2(A,C)| \mp C_3(B,C) - 1 \le 0 \,. \tag{5}$$

The maximum value of $|C_1(A,B) + C_2(A,C)| - C_3(B,C) - 1 = 0.17$ with A representing the Euro, B the Swiss Franc, and C the Danish Krone. The average values of the A's, B's, and C's are at most 0.012, and $C_1(A,B) = 0.80$, $C_2(A,C) = 0.96$, and $C_3(B,C) = 0.59$.

Calculating the maximum number of triples by solving the minimization problem by a slightly modified version of the procedure described in Ref. [53](appendix B) yields $2(1-\Gamma)=0.17$, not only in agreement with Eq. (2) (with the plus sign) but also signaling that the left hand side of Eq. (3) is equal to the bound $2(1-\Gamma)$.

The maximum value of $|C_1(A,B)-C_2(A,C)|+C_3(B,C)-1=0.23$ where in this case, A represents the Mexican Peso, B the Euro, and C the Danish Krone. The average values of the A's, B's, and C's are at most 0.021, $C_1(A,B)=-0.039$, $C_2(A,C)=0.25$, and $C_3(B,C)=0.94$. Also in this case, the value of the left hand side of Eq. (3) (with the minus sign) is the same as $2(1-\Gamma)=0.23$.

If, for simplicity, it is assumed that the standard deviation on the correlations is approximately given by $1/2\sqrt{N}=1/2\sqrt{1671}\approx 0.012$, the foreign exchange data shows violations of more than 10 standard deviations in the two cases mentioned earlier.

Finally, it should be mentioned that using $C(A,B)=(3N)^{-1}\sum_{s=1}^{3}\sum_{i=1}^{N}A_{s,i}B_{s,i}$ to compute the correlations (that is without breaking up each of the whole data sets in three parts), and repeating the analysis never produces violations of $|C(A,B)\pm C(A,C)|\mp C(B,C)-1\le 0$. This is to be expected because in this case, the data for the A's, B's, and C's form triples ($\Gamma=1$) and then, as already shown by Boole [55], there can be no violation.

5. Mathematical models: importance of Bell's theorem

In brief, Bell proposed to model the correlations of an EPRB thought experiment by [56,67]

$$C(\mathbf{a},\mathbf{b})=\int A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)\,\mu(\lambda)\,d\lambda\ ,\ |A(\mathbf{a},\lambda)|\le 1\ ,\ |B(\mathbf{b},\lambda)|\le 1\ ,\ 0\le\mu(\lambda)\ ,\ \int\mu(\lambda)\,d\lambda=1\ , \tag{6}$$

where $A(\mathbf{a},\lambda)$ and $B(\mathbf{b},\lambda)$ are mathematical functions of the conditions $\mathbf{a}$ and $\mathbf{b}$, respectively, and the common variable λ denoting an arbitrary set of "hidden" variables. Bell gave a proof that $C(\mathbf{a},\mathbf{b})$ cannot arbitrarily closely approximate the correlation $-\mathbf{a}\cdot\mathbf{b}$ *for all* unit vectors $\mathbf{a}$ and $\mathbf{b}$ [1]. According to Bell himself (see Ref. [56] (p.65)), **this is the theorem.**

Within the universe of mathematical models, Bell's theorem is of great importance. Apparently not well-known seems to be the fact that the theorem excludes a probabilistic description of the Stern–Gerlach experiment with spin-1/2 particles in terms of the model Eq. (6) [53]. In this particular case, the quantum-theoretical description goes in terms of a single particle. Therefore all implications pertaining to physics, other than the one just mentioned, drawn from a violation of a Bell-type inequalities become void. Very well-known is the fact that the theorem excludes all models of the type Eq. (6) as possible candidates for describing a quantum system of two spin-1/2 objects for which in a certain case $C(\mathbf{a},\mathbf{b})=-\mathbf{a}\cdot\mathbf{b}$. By far the most important general consequence of Bell's theorem is that there is no hope for recovering **all** the results of quantum theory by expressions of the kind Eq. (6), that is by averaging (the integration over λ with probability density $\mu(\lambda)$) over an ensemble of "classical" physics models formulated in terms of scalar functions with values in the interval $[-1,+1]$.

Eq. (6) generalizes the idea of separation of variables, that is the idea that a scalar function of two variables ($\mathbf{a}$ and $\mathbf{b}$) can, in particular cases, be written as (the integral over) a product of scalar functions, each of which depend on one of these variables only. **Within the universe of mathematical models**, there is absolutely no valid argument for restricting the search for models yielding correlations $C(\mathbf{a},\mathbf{b})=-\mathbf{a}\cdot\mathbf{b}$ to the domain of models formulated in terms of scalar functions (as in Eq. (6)). In fact, as is well-known, quantum theory provides such a separated model in terms of Pauli spin matrices [53]. Moreover, with a suitable definition of the notion of locality, the quantum-theoretical model can also be argued to exhibit locality [68–71]. Another possibility is to resort to non-Diophantine arithmetics [72].

6. Mathematical models: implications of violating Bell-type inequalities

From Eq. (6), $|ac\pm ac|\le 1\pm bc$ for $-1\le a,b,c\le 1$, and the application of the triangle inequality, it follows directly that

$$|C(\mathbf{a},\mathbf{b})\pm C(\mathbf{a},\mathbf{c})|\le 1\pm C'(\mathbf{b},\mathbf{c})\ , \tag{7}$$

where

$$C'(\mathbf{b},\mathbf{c})=\int B(\mathbf{b},\lambda)B(\mathbf{c},\lambda)\,\mu(\lambda)\,d\lambda\ . \tag{8}$$

Inequality (7) will be referred to as a Boole–Bell inequality. Key arguments in Bell's original proof of his theorem are the assumption of perfect anticorrelation (meaning $A(\mathbf{x},\lambda)=-B(\mathbf{x},\lambda)$ for all $\mathbf{x}$) and that $C(\mathbf{a},\mathbf{b})=\pm\mathbf{a}\cdot\mathbf{b}$. Bell then shows that Eq. (7) can be violated [1,56]. For instance, with the choice $\mathbf{a}=(1,0,0)$, $\mathbf{b}=(1,1,0)/\sqrt{2}$, and $\mathbf{c}=\pm(1,-1,0)/\sqrt{2}$, Eq. (7) becomes $\sqrt{2}\le 1$, a clear violation.

Always within the realm of mathematical models, the only logically correct conclusion that one can draw from a violation of Eq. (7) is that the model Eq. (6) cannot describe the correlation $\pm\mathbf{a}\cdot\mathbf{b}$ for all unit vectors $\mathbf{a}$ and $\mathbf{b}$, which is Bell's theorem. However, that does not imply that only the model Eq. (6) is excluded.

By virtue of Fine's theorem [6,7] (for an alternative proof, see Appendix), for a fixed triple of conditions $(\mathbf{a},\mathbf{b},\mathbf{c})$, a violation of Eq. (7) also excludes the existence of any description in terms of a normalized nonnegative distribution $f(x,y,z)$ ($0\le f(x,y,z)\le 1$, $\sum_{x,y,z=\pm 1}f(x,y,z)=1$) with the property that

$$C(\mathbf{a},\mathbf{b})=\sum_{x,y,z=\pm 1}xy\,f(x,y,z)\ ,\ C(\mathbf{a},\mathbf{c})=\sum_{x,y,z=\pm 1}xz\,f(x,y,z)\ ,\ C(\mathbf{b},\mathbf{c})=\sum_{x,y,z=\pm 1}yz\,f(x,y,z)\ . \tag{9}$$

As a matter of fact, Fine's theorem shows much more, namely that the existence of a normalized nonnegative distribution $f(x,y,z)$ is equivalent to the Boole–Bell inequalities being satisfied (for a technically precise statement, see Appendix).

From a violation of Eq. (7) one can, if one wishes to do so, draw the conclusion that the correlations cannot be obtained from the model Eq. (6) but Fine's theorem leaves no room for interpretations of violations of a Boole–Bell inequality other than the mathematical equivalence.

It is important to mention here that Fine's theorem has no implications for Bell's theorem. Furthermore, the existence of a normalized nonnegative distribution $f(x,y,z)$ **for a fixed triple of conditions** $(\mathbf{a},\mathbf{b},\mathbf{c})$ does not imply that there exists a probabilistic description of the EPRB experiment in terms of a trivariate, simply because one cannot perform one EPRB experiment conditioned on the three settings $(\mathbf{a},\mathbf{b},\mathbf{c})$. In order to do so, one would needs to perform an extended EPRB experiment [53,73]. In this case, it is impossible to violate any of the Bell-type inequalities [53,73].

7. Empirical data: irrelevance of violating Bell-type inequalities

As shown in Section 2 correlations computed from two-valued data must **always** satisfy the inequality Eq. (2) with Γ being the value of the fraction of triples. The Boole–Bell inequality **for empirical data** Eq. (4) is recovered in the exceptional case that $\Gamma = 1$, the **only** case in which one can prove the Boole–Bell inequality for empirical data. The same is true for the Clauser–Horn–Shimony–Holt [52,56] and Clauser–Horn [51] inequalities [53]. For empirical data, the mathematical proof of these two inequalities can only be carried out if the empirical data can be reshuffled to form quadruples [53].

The violation of Bell-type inequalities by data from EPRB experiments and the conclusions drawn from it are regarded as a landmark in the development of modern quantum technologies. But what then with the violation of Boole–Bell inequalities by the foreign exchange rate data? The idea that this would be a consequence of "quantum or non-classical physics at work" sounds strange, to put it mildly. Clearly, the logic, arguments and concepts (such as locality as encapsulated by Eq. (6)) that are being used in contemporary quantum physics to interpret violations of Bell-type inequalities need to be scrutinized further.

8. Conclusion

The discussion that follows uses the three-correlations case as an example. The same arguments and conclusions almost trivially extend to the four-correlations case as well [53]. The results presented in the earlier sections can be summarized as follows:

1. Correlations of empirical data cannot violate the model-free inequality Eq. (2).
2. Correlations of empirical data can violate inequality Eq. (4) when not all sextuples of data can be rearranged to form triples (i.e., $\Gamma < 1$). Such a violation has no meaning (other than that not all the triples can form sextuples) because empirical data should comply with Eq. (2), not with Eq. (4), as exemplified by the analysis of the foreign exchange data.
3. The proof of inequality Eq. (7) assumes that the correlation is given by model Eq. (6). If a properly discretized version of model Eq. (6) is used as a (computer) model for producing synthetic data, this model generates data that can be reshuffled into a set of triples [53], that is $\Gamma = 1$ [53].
4. A violation of the Bell-type inequality Eq. (4) by empirical data rules out the model Eq. (6) as a potential candidate for **describing** the data. If the empirical data cannot be described by Bell's model Eq. (6), the model has to be rejected or extended.
5. Unless one subscribes to the idea that empirical data are generated by "mathematical rules governing the universe", the role of any mathematical model is limited to **describing** empirical data. If the model description fails (e.g., because a model-dependent inequality is violated), the proper action, practiced in most fields of science, would be to try improving the model, not to philosophize about the premises that went into its formulation.

From (1–5), it follows that it is only by adopting the view that a mathematical model such as Eq. (6) **is** a reality existing in the world in which we live that it may become possible to interpret the premises that led to the formulation of Eq. (6) as genuine properties of "nature". As it is unknown whether "mathematics rules the universe", it is perhaps more apt to analyze data without relying on the premise that it does.

CRediT authorship contribution statement

Hans De Raedt: Conceptualization, Formal analysis, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing. **Mikhail I. Katsnelson:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing, Methodology, Validation. **Manpreet S. Jattana:** Investigation, Methodology, Writing – original draft, Writing – review & editing, Formal analysis, Validation. **Vrinda Mehta:** Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing, Formal analysis. **Madita Willsch:** Formal analysis, Investigation, Methodology, Software, Writing – original draft, Writing – review & editing, Validation. **Dennis Willsch:** Formal analysis, Investigation, Methodology, Software, Validation, Writing – original draft, Writing – review & editing. **Kristel Michielsen:** Investigation, Methodology, Validation, Writing – review & editing. **Fengping Jin:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix. Trivariates of two-valued variables

Without loss of generality, any real-valued, normalized function $f(x_1,x_2,x_3)$ of the two-valued variables $x_1=\pm1$, $x_2=\pm1$, and $x_3=\pm1$ can be written as

$$f(x_1,x_2,x_3)=\frac{1+K_1\,x_1+K_2\,x_2+K_3\,x_3+K_{12}\,x_1x_2+K_{13}\,x_1x_3+K_{23}\,x_2x_3+K_{123}\,x_1x_2x_3}{8}\;. \tag{A.1}$$

From Eq. (A.1) it follows that

$$1 \;=\; \sum_{x_1=\pm1}\sum_{x_2=\pm1}\sum_{x_3=\pm1} f(x_1,x_2,x_3)\;, \tag{A.2a}$$

$$K_i \;=\; \sum_{x_1=\pm1}\sum_{x_2=\pm1}\sum_{x_3=\pm1} x_i f(x_1,x_2,x_3)\;,\; i\in\{1,2,3\}\;, \tag{A.2b}$$

$$K_{ij} \;=\; \sum_{x_1=\pm1}\sum_{x_2=\pm1}\sum_{x_3=\pm1} x_ix_j f(x_1,x_2,x_3)\;,\; (i,j)\in\{(1,2),(1,3),(2,3)\}\;, \tag{A.2c}$$

$$K_{123} \;=\; \sum_{x_1=\pm1}\sum_{x_2=\pm1}\sum_{x_3=\pm1} x_1x_2x_3 f(x_1,x_2,x_3)\;, \tag{A.2d}$$

where the K's are the moments of $f(x_1,x_2,x_3)$ and Eq. (A.2a) is a restatement of the normalization of $f(x_1,x_2,x_3)$.

If $f(x_1,x_2,x_3)$ is going to be used as a model for empirical frequencies, it must satisfy $0\le f(x_1,x_2,x_3)\le 1$ and $\sum_{x_1,x_2,x_3=\pm1} f(x_1,x_2,x_3)=1$. From $0\le f(x_1,x_2,x_3)\le 1$, it follows immediately that all the K's in Eq. (A.2) are smaller than one in absolute value. Furthermore the marginals $f_3(x_1,x_2)=\sum_{x_3=\pm1} f(x_1,x_2,x_3)$, $f_2(x_1,x_3)=\sum_{x_2=\pm1} f(x_1,x_2,x_3)$, and $f_1(x_2,x_3)=\sum_{x_1=\pm1} f(x_1,x_2,x_3)$ are real-valued, normalized and nonnegative bivariates that is $0\le f_3(x_1,x_2)\le 1$, $\sum_{x_1,x_2=\pm1} f_3(x_1,x_2)=1$, etc. From the nonnegativity of these marginals, it follows that $|K_i\pm K_j|\le 1\pm K_{ij}$ for $(i,j)\in\{(1,2),(1,3),(2,3)\}$ [53]. Other inequalities involving moments follow by making linear combinations of the inequalities $f(x_1,x_2,x_3)\ge 0$ for different values of (x_1,x_2,x_3). For instance, from $4[f(+1,+1,+1)+f(-1,-1,-1)]=1+K_{12}+K_{13}+K_{23}\ge 0$ and $4[f(-1,+1,+1)+f(+1,-1,-1)]=1-K_{12}-K_{13}+K_{23}\ge 0$ it follows that $|K_{12}+K_{13}|\le 1+K_{23}$, one instance the Boole–Bell inequality. Recall that the latter implies that the inequalities $|K_{12}\pm K_{23}|\le 1\pm K_{13}$ and $|K_{13}\pm K_{23}|\le 1\pm K_{12}$ are also satisfied see Eq. (N.6) in Ref. [53].

Summarizing: if the data can be modeled by a nonnegative, normalized trivariate $f(x_1,x_2,x_3)$, all inequalities

$$|K_1|\le 1\;,\;|K_2|\le 1\;,\;|K_3|\le 1\;,|K_{12}|\le 1\;,|K_{13}|\le 1\;,\;|K_{23}|\le 1\;, \tag{A.3a}$$

$$|K_1\pm K_2|\le 1\pm K_{12}\;,\;|K_1\pm K_3|\le 1\pm K_{13}\;,\;|K_2\pm K_3|\le 1\pm K_{23}\;, \tag{A.3b}$$

$$|K_{12}\pm K_{13}|\le 1\pm K_{23}\;, \tag{A.3c}$$

which include the Boole–Bell inequalities are satisfied. Thus, the inequalities Eq. (A.3) do not only derive from Bell's model Eq. (6) but also from a much more general model defined by the trivariate Eq. (A.1).

A remarkable fact, first shown by Fine [6,7] by a different approach than the one taken here, is that if all inequalities Eq. (A.3) are satisfied, it is possible to construct a real-valued, normalized trivariate $0\le f(x_1,x_2,x_3)\le 1$ of the two-valued variables $x_1=\pm1$, $x_2=\pm1$, and $x_3=\pm1$, yielding all the moments that appear in Eq. (A.3). The text that follows replaces the corresponding part and theorem in Ref. [53], which are not correct.

First note that if Eqs. (A.3a) and (A.3b) hold, the existence of the three normalized bivariates $0\le f_3(x_1,x_2)\le 1$, $0\le f_2(x_1,x_3)\le 1$, and $0\le f_1(x_2,x_3)\le 1$ is guaranteed [53]. Indeed, for instance, if $|K_1|\le 1$, $|K_2|\le 1$, $|K_{12}|\le 1$, and $|K_1\pm K_2|\le 1\pm K_{12}$ it follows immediately that $0\le f_3(x_1,x_2)=(1+K_1\,x_1+K_2\,x_2+K_{12}\,x_1x_2)/4\le 1$ is the desired normalized bivariate. Therefore, what remains to be proven is the existence of a real-valued trivariate $g(x_1,x_2,x_3)$ that (i) takes values in the interval $[0,1]$ and (ii) yields the three named bivariates with their respective moments $K_1,\dots,K_{23}$ that appear in Eqs. (A.3a) and (A.3b) as marginals.

Second, note that without loss of generality, any real-valued trivariate $g(x_1,x_2,x_3)$ of the two-valued variables $x_1=\pm1$, $x_2=\pm1$, and $x_3=\pm1$ can be written as

$$g(x_1,x_2,x_3)=\frac{K'_0+K'_1\,x_1+K'_2\,x_2+K'_3\,x_3+K'_{12}\,x_1x_2+K'_{13}\,x_1x_3+K'_{23}\,x_2x_3+K'_{123}\,x_1x_2x_3}{8}\;. \tag{A.4}$$

Imposing requirement (ii) immediately yields $K'_0=1$, $K'_1=K_1$, $K'_2=K_2$, $K'_3=K_3$, $K'_{12}=K_{12}$, $K'_{13}=K_{13}$, and $K'_{23}=K_{23}$, leaving only K'_{123} as unknown.

Third, the requirement that $0\le g(x_1,x_2,x_3)$ is used to derive conditions for the unknown K'_{123} in terms of all the moments that appear in Eq. (A.3). This can be accomplished as follows. The eight inequalities $g(x_1,x_2,x_3)\ge0$ are rewritten as bounds on K'_{123}. For instance $g(+1,+1,+1)\ge0 \iff -1-K_1-K_2-K_3-K_{12}-K_{13}-K_{23}\le K'_{123}$ and $g(-1,-1,-1)\ge0 \iff K'_{123}\le 1-K_1-K_2-K_3+K_{12}+K_{13}+K_{23}$, and so on. The resulting eight inequalities can be summarized as

$$\begin{aligned}&\max\big[-1+K_1-K_2-K_3-K_{12}-K_{13}+K_{23},-1-K_1+K_2-K_3-K_{12}+K_{13}-K_{23},\\&\quad-1-K_1-K_2+K_3+K_{12}-K_{13}-K_{23},-1+K_1+K_2+K_3+K_{12}+K_{13}+K_{23}\big]\\&\quad\le K_{123}\le\min\big[1-K_1-K_2-K_3+K_{12}+K_{13}+K_{23},1+K_1+K_2-K_3+K_{12}-K_{13}-K_{23},\\&\qquad\qquad 1+K_1-K_2+K_3-K_{12}+K_{13}-K_{23},1-K_1+K_2+K_3-K_{12}-K_{13}+K_{23}\big]\;.\end{aligned} \tag{A.5}$$

Using the inequalities $\max(a,b,c,d)\ge(a+b+c+d)/4$ and $\min(a,b,c,d)\le(a+b+c+d)/4$ it immediately follows from Eq. (A.5) that $-1\le K'_{123}\le1$, which together with Eq. (A.3a), guarantees that $g(x_1,x_2,x_3)\le1$.

To prove that there exists at least one value of K'_{123} satisfying Eq. (A.5) if all inequalities in Eq. (A.3) are satisfied, it is sufficient to show that the difference $\min(\dots)-\max(\dots)$ cannot be negative. One straighforward way to do this is to use the sixteen inequalities Eqs. (A.3b) and Eq. (A.3c) to show that all sixteen possible differences deriving from Eq. (A.5) are nonnegative. Instead, it is more elegant to rewrite the arguments of $\max(\dots)$ and $\min(\dots)$ in Eq. (A.5) as

$$K'_{123}\ge\max(-1-K_3-K_{12}+|K_1+K_2+K_{13}+K_{23}|,-1+K_3+K_{12}+|K_1-K_2-K_{13}+K_{23}|)\qquad=:\ \text{LHS}\;, \tag{A.6a}$$

$$K'_{123}\le\min(1-K_3+K_{12}-|K_1+K_2-K_{13}-K_{23}|,1+K_3-K_{12}-|K_1-K_2+K_{13}-K_{23}|)\qquad=:\ \text{RHS}\;. \tag{A.6b}$$

The final step is then to prove that $\text{RHS}-\text{LHS}\ge0$. Combining Eqs. (A.6a) and (A.6b) and using $\min(a,b)+\min(c,d)=\min(a+c,a+d,b+c,b+d)$ yields

$$\begin{aligned}\text{RHS}-\text{LHS}=\min(&2+2K_{12}-|K_1+K_2-K_{13}-K_{23}|-|K_1+K_2+K_{13}+K_{23}|\;,\\&2-2K_{12}-|K_1-K_2+K_{13}-K_{23}|-|K_1-K_2-K_{13}+K_{23}|\;,\\&2-2K_3-|K_1+K_2-K_{13}-K_{23}|-|K_1-K_2-K_{13}+K_{23}|\;,\\&2+2K_3-|K_1-K_2+K_{13}-K_{23}|-|K_1+K_2+K_{13}+K_{23}|)\;.\end{aligned} \tag{A.7}$$

Using the inequalities Eq. (A.3) and the identity $-|a-b|-|a+b|=\min(-a+b,-b+a)+\min(-a-b,a+b)=2\min(-|a|,-|b|)$, it can be shown that each of the four arguments of $\min(.)$ in Eq. (A.7) is nonnegative. In detail

$$\begin{aligned}2+2K_{12}-|K_1+K_2-(K_{13}+K_{23})|-|K_1+K_2+K_{13}+K_{23}|&=2+2K_{12}+2\min(-|K_1+K_2|,-|K_{13}+K_{23}|)\\&\ge2+2K_{12}+2(-1-K_{12})=0\;,\end{aligned} \tag{A.8a}$$

$$\begin{aligned}2-2K_{12}-|K_1-K_2+K_{13}-K_{23}|-|K_1-K_2-(K_{13}-K_{23})|&=2-2K_{12}+2\min(-|K_1-K_2|,-|K_{13}-K_{23}|)\\&\ge2-K_{12}+2(-1+K_{12})=0\;,\end{aligned} \tag{A.8b}$$

$$\begin{aligned}2-2K_3-|K_1-K_{13}+K_2-K_{23}|-|K_1-K_{13}-(K_2-K_{23})|&=2-2K_3+2\min(-|K_1-K_{13}|,-|K_2-K_{23}|)\\&\ge2-2K_3+2(-1+K_3)=0\;,\end{aligned} \tag{A.8c}$$

$$\begin{aligned}2+2K_3-|K_1+K_{13}-(K_2+K_{23})|-|K_1+K_{13}+K_2+K_{23}|&=2+2K_3+2\min(-|K_1+K_{13}|,-|K_2+K_{23}|)\\&\ge2+2K_3+2(-1-K_3)=0.\end{aligned} \tag{A.8d}$$

This completes the proof that if all inequalities Eq. (A.3) are satisfied, there exists a normalized, nonnegative trivariate $0\le g(x_1,x_2,x_3)\le1$ given by Eq. (A.4) with $K'_{123}\in[\text{LHS},\text{RHS}]$. Summarizing we have proven

Theorem: Given a real-valued, normalized function $0\le f(x_1,x_2,x_3)\le1$ of two-valued variables, its moments Eq. (A.2a)–(A.2c) satisfy all the inequalities Eq. (A.3). Conversely, given the values of the moments Eq. (A.2a)–(A.2c) satisfying all the inequalities Eq. (A.3), it is always possible to choose K_{123} in the range $[\text{LHS},\text{RHS}]$ (defined by the right hand sides of Eqs. (A.6a) and (A.6b), respectively), and construct a real-valued, normalized function $0\le f(x_1,x_2,x_3)\le1$ of two-valued variables which yields the specified values of the moments Eq. (A.2a)–(A.2c).

A different strategy was implemented in Mathematica®, providing an independent proof of the theorem. The explicit form of $f(x_1,x_2,x_3)$ in terms of its moments is given by Eq. (A.1).

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