THE STATE OF FACTORING ON QUANTUM COMPUTERS

arXiv:2410.14397

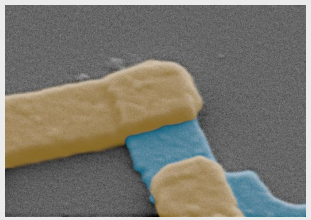
2024 I DR. DENNIS WILLSCH

supported by Bundesministerium für Bildung und Forschung
Ministerium für Kultur und Wissenschaft des Landes Nordrhein-Westfalen

RESEARCH OVERVIEW

Simulating QCs

JUQCS (HPC)
Comput. Phys. Commun. 176, 121 (2007)
Comput. Phys. Commun. 237, 47 (2019)
Comput. Phys. Commun. 278, 108411 (2022)



Superconducting Qubits (HPC)
Phys. Rev. A 96, 062302 (2017)
Phys. Rev. A 98, 052348 (2018)
PhD Thesis, arXiv:2008.13490 (2020)
Phys. Rev. A 101, 012327 (2020)
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Benchmarking QCs

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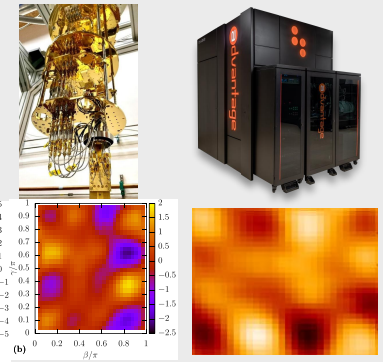

Algorithms on QCs

SVMs on Quantum Annealers
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Garden Optimization
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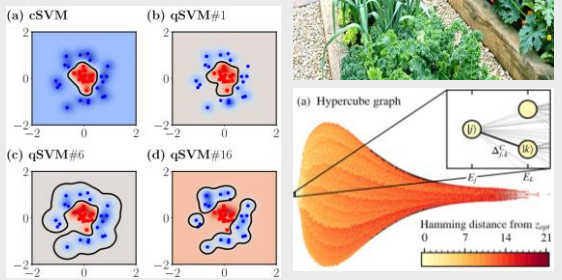

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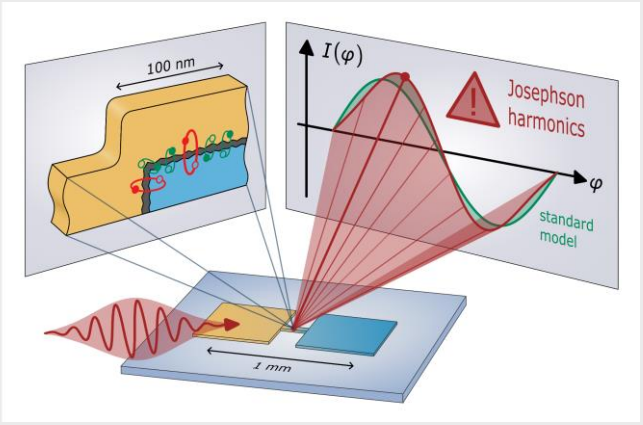

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Shor's Algorithm
Mathematics 11, 4222 (2023)

Quantum Factoring
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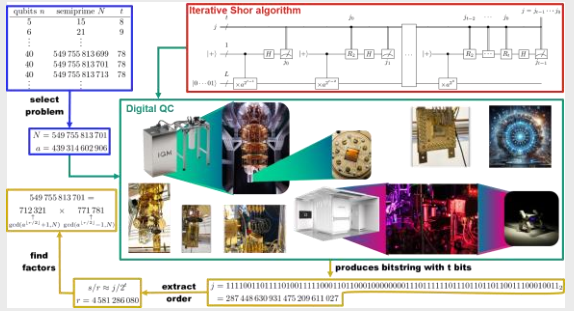

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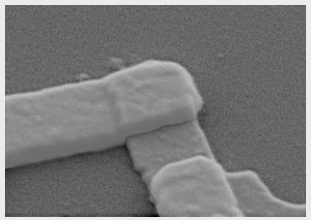

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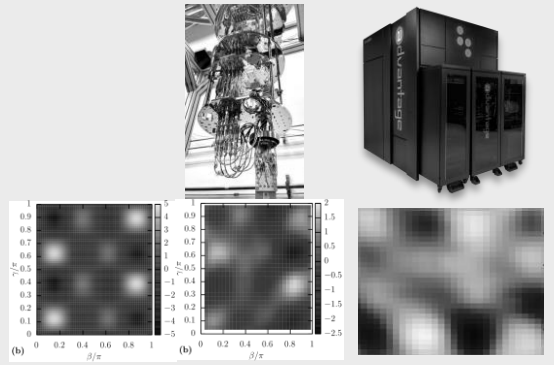

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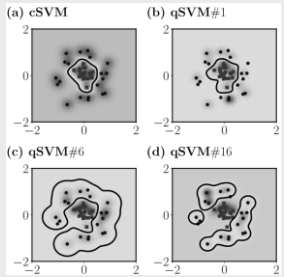

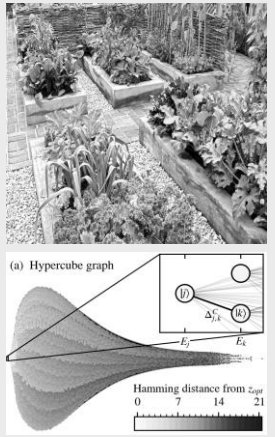

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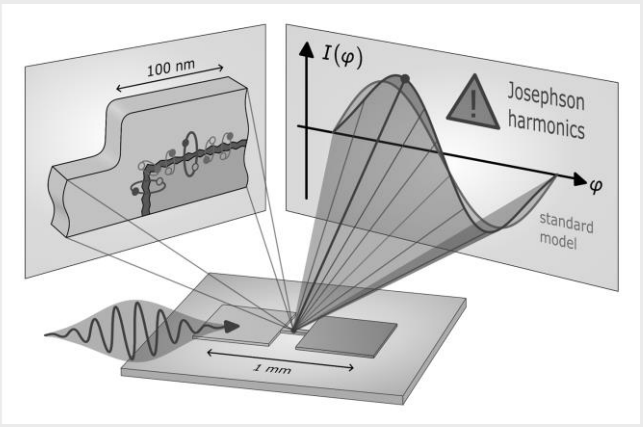

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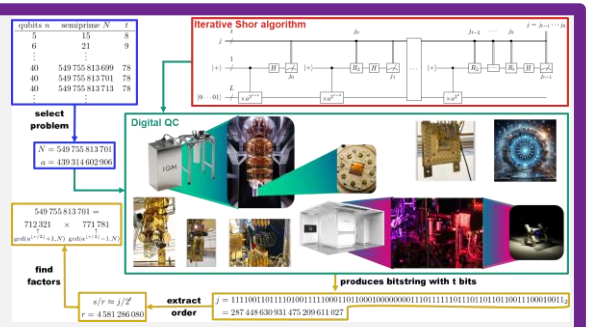

THE STATE OF FACTORING ON QUANTUM COMPUTERS

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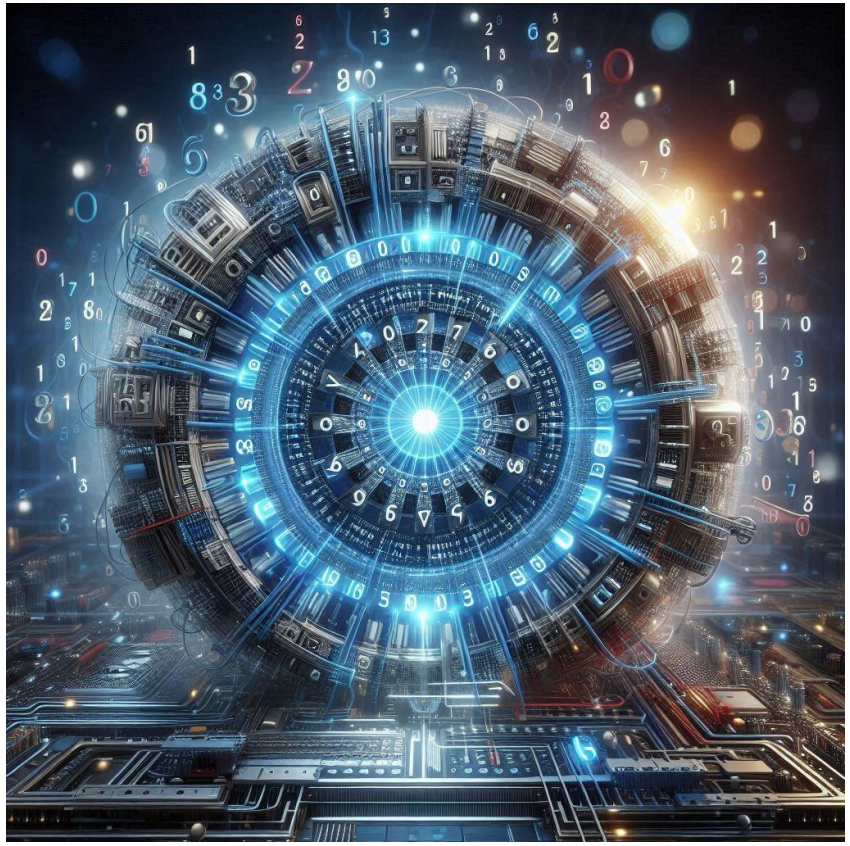

FACTORING PROBLEM

$$N = p \times q$$

Definition: Find a non-trivial factor of a composite L-bit integer N

Origin: One of the oldest mathematical problems with origins in ancient Greek times

Hardness: No classical polynomial-time algorithm known

Relevance: A lot of Internet security is based on this hardness

Public key cryptosystems like RSA used in TLS, SSH, …

FACTORING RECORDS

Digital QCs

Shor's Algorithm
N = 15, 21, (35) ~ 2^6

Martín-López et al., Nat. Photonics 6, 773 (2012)
Monz et al., Science 351, 1068 (2016)
Amico et al., Phys. Rev. A 100, 012305 (2019)

Careful: Oversimplified Shor's Algorithm
N ~ 2^{20000}

Smolin et al., Nature 499, 7457 (2013)
Gidney, algassert.com/post/2000 (2020)

Variational Quantum Factoring
N = 1099551473989 ~ 2^{41}

Anschuetz et al., arXiv:1808.08927 (2018)
Karamlou et al., npj Quantum Inf. 7, 156 (2021)

← Careful: preprocessed down to 3 qubits!

Schnorr's Algorithm
N = 261980999226229 ~ 2^{48}

Schnorr, cryptoeprint:2021/933 (2021)
Hegade and Solano, arXiv:2301.1105 (2023)

← seems not scalable? →

Ducas, github.com/lducas/SchnorrGate (2021)
Grebnev et al., IEEE Access 11, 134760 (2023)
Khattar and Yosri, arXiv:2307.09651 (2023)
Tesoro et al., arXiv:2410.16355 (2024)

Analog QCs

Direct Method
N = 1042441 ~ 2^{20}

Peng et al., Phys. Rev. Lett. 101, 220405 (2008)
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Multiplier Circuit
N = 8219999 ~ 2^{23}

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Simulated Digital QCs

Shor's Algorithm
N = 549755813701 ~ 2^{39}

Willsch et al., Mathematics 11, 4222 (2023)

Classical (Super)Computers

Number Field Sieve
N = RSA-250 ~ 10^{250} ~ 2^{829}

Boudot et al., CRYPTO 2020, 62 (2020)

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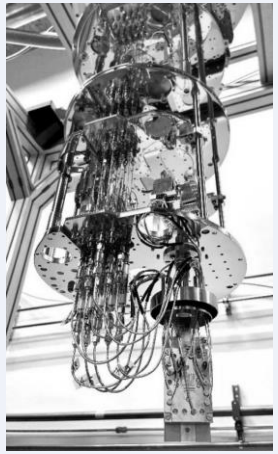

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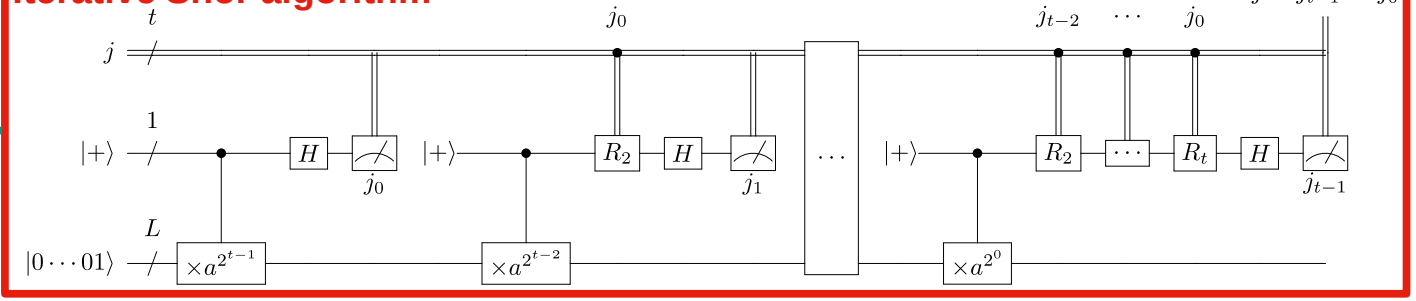
DIGITAL QCS: SHOR'S FACTORING ALGORITHM

qubits n	semiprime N	t
5	15	8
6	21	9
⋮	⋮	
40	549 755 813 699	78
40	549 755 813 701	78
40	549 755 813 713	78
⋮	⋮	

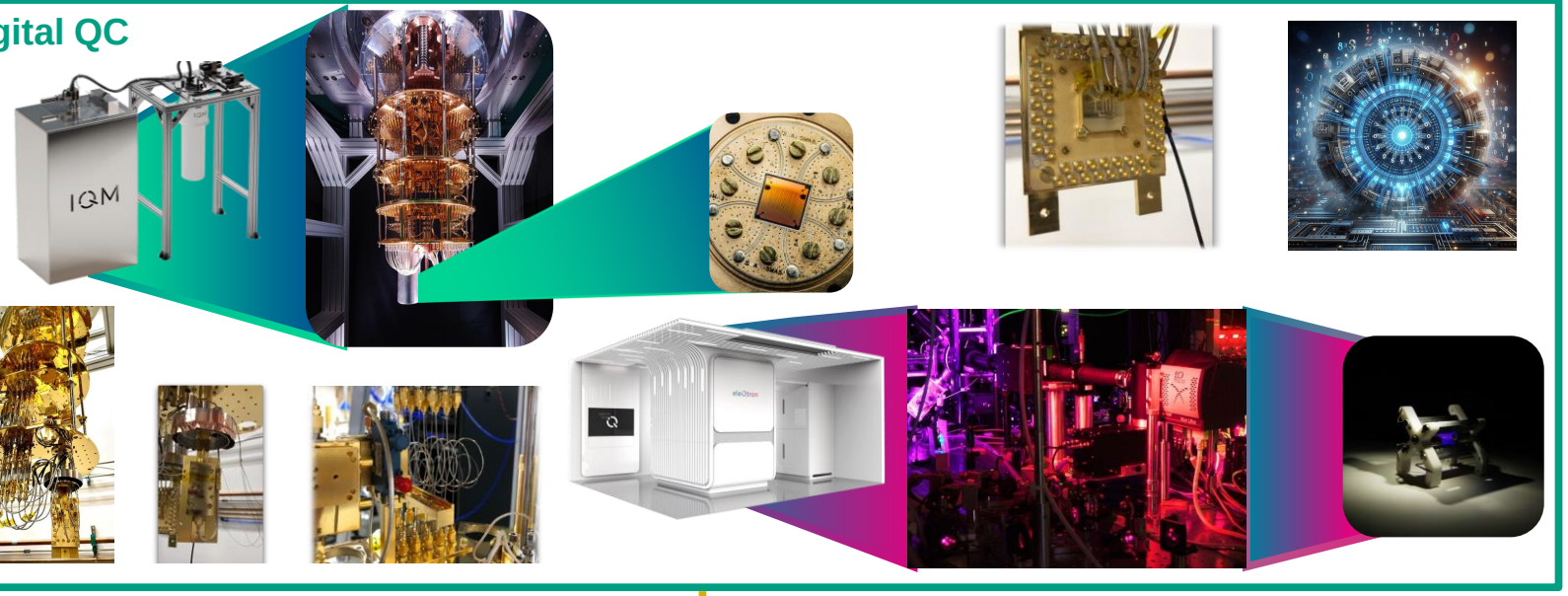
select problem

$N = 549\,755\,813\,701$
$a = 439\,314\,602\,906$

Iterative Shor algorithm



Digital QC



produces bitstring with t bits

$j = 11110011011110100111110001101100010000000011101111101110110110110011100010011_2$
$= 287\,448\,630\,931\,475\,209\,611\,027$

extract order

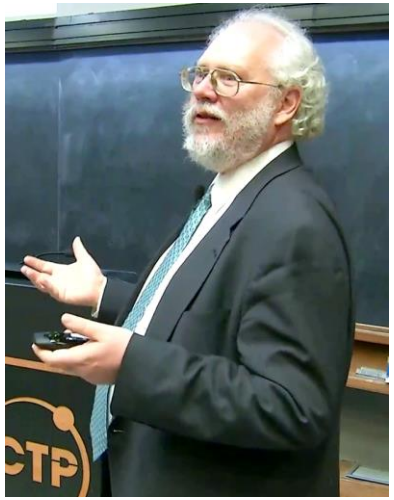
$s/r \approx j/2^t$
$r = 4\,581\,286\,080$

find factors

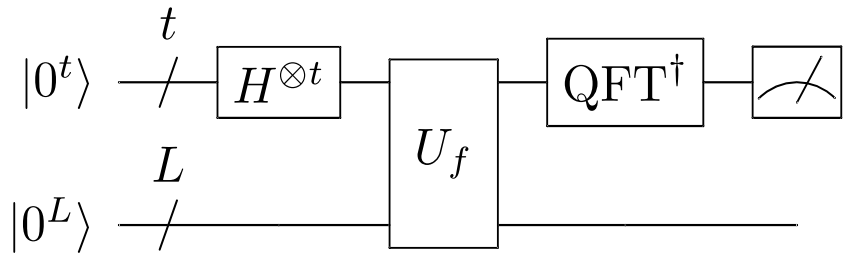
$549\,755\,813\,701 =$
$712\,321 \times 771\,781$
$\gcd(a^{\lfloor r/2 \rfloor}+1, N)$ $\gcd(a^{\lfloor r/2 \rfloor}-1, N)$

SHOR'S FACTORING ALGORITHM

The quantum part



- **Conventional** Shor algorithm needs 3L qubits

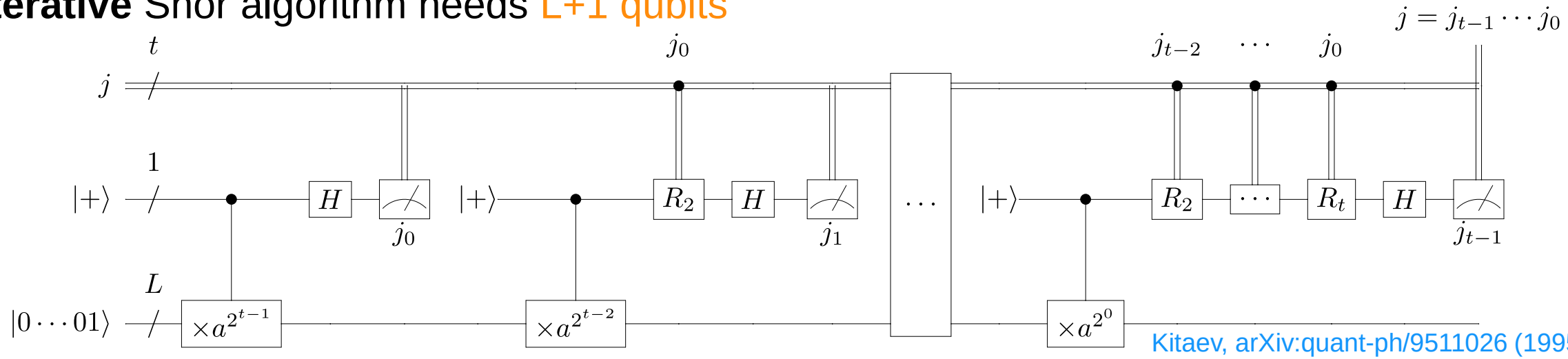


$$t \approx 2L$$

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus (a^x \bmod N)\rangle$$

Shor, Proc. Found. Comp. Sci. 35, 124 (1994)
Shor, SIAM J. Comput. 26, 1484 (1997)

- **Iterative** Shor algorithm needs L+1 qubits



Kitaev, arXiv:quant-ph/9511026 (1995)
Griffiths and Niu, PRL 76, 3228 (1996)

SHOR'S FACTORING ALGORITHM — SIMULATION

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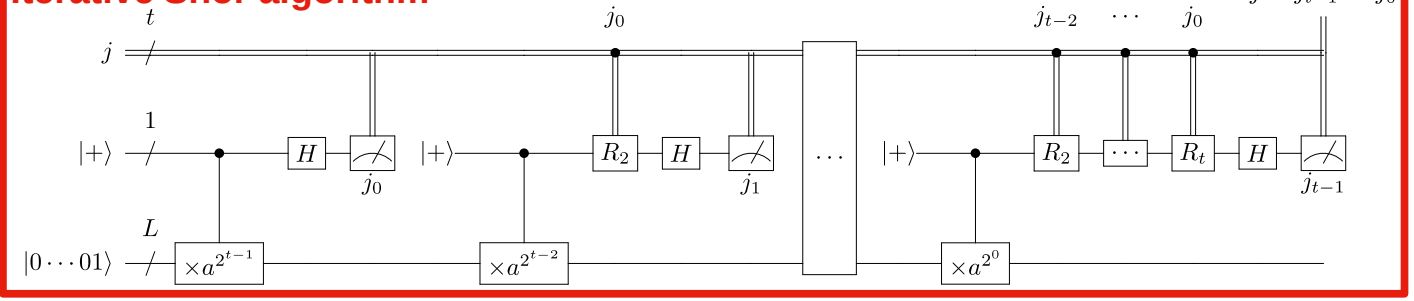
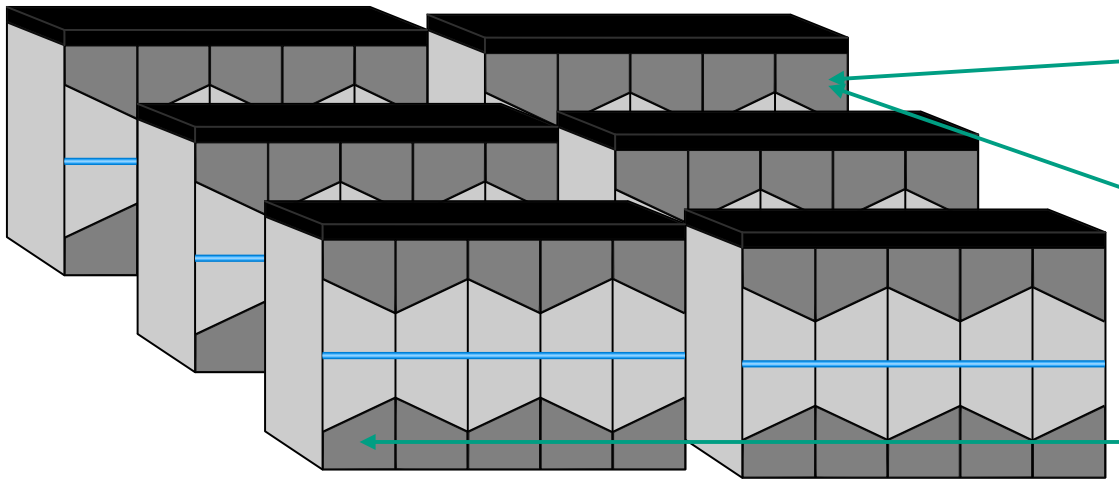


Simulation on 2048 GPUs



GPU #0

$$\begin{pmatrix} \psi_{00000000000\,0\cdots0} \\ \vdots \\ \psi_{00000000000\,1\cdots1} \end{pmatrix}$$

GPU #1

$$\begin{pmatrix} \psi_{00000000001\,0\cdots0} \\ \vdots \\ \psi_{00000000001\,1\cdots1} \end{pmatrix}$$

GPU #2047

$$\begin{pmatrix} \psi_{11111111111\,0\cdots0} \\ \vdots \\ \psi_{11111111111\,1\cdots1} \end{pmatrix}$$

Large-Scale Simulation of Shor's Quantum Factoring Algorithm, **Mathematics** 11, 4222 (2023)

produces bitstring with t bits

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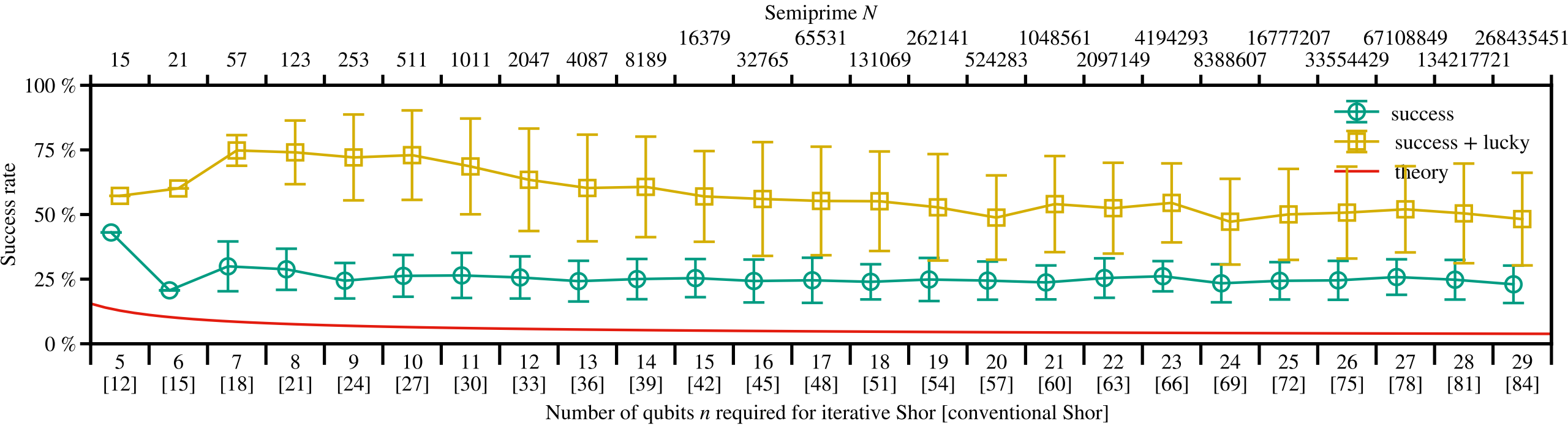
SHOR'S FACTORING ALGORITHM

Simulation Results



- **Average shows:** Performance much better than predicted by theory
- Success rate approaches ~ 50 % with so-called "lucky" cases
- **More information:**
  Large-Scale Simulation of Shor's Quantum Factoring Algorithm, **Mathematics** 11, 4222 (2023)

"lucky":
Conditions not satisfied—
But factoring works anyway!

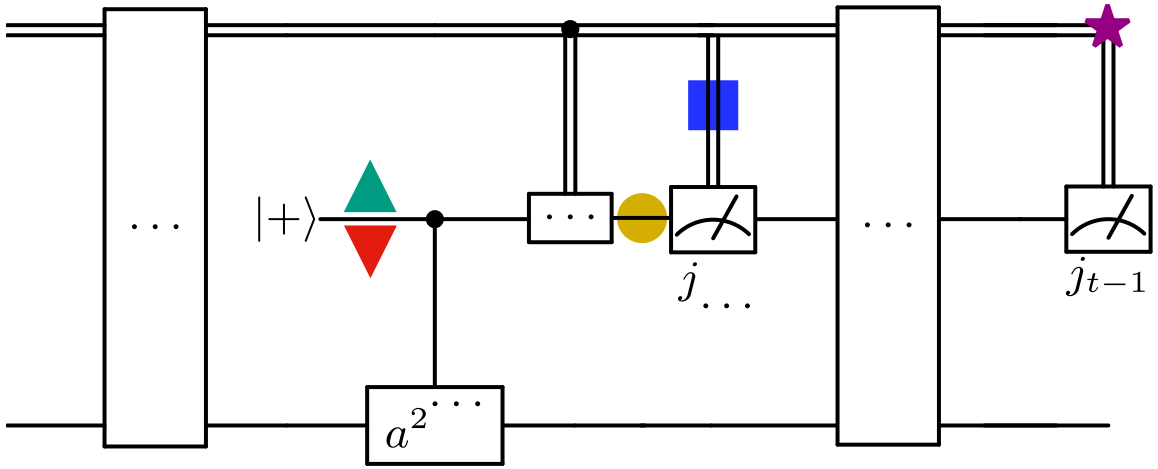
SHOR'S FACTORING ALGORITHM

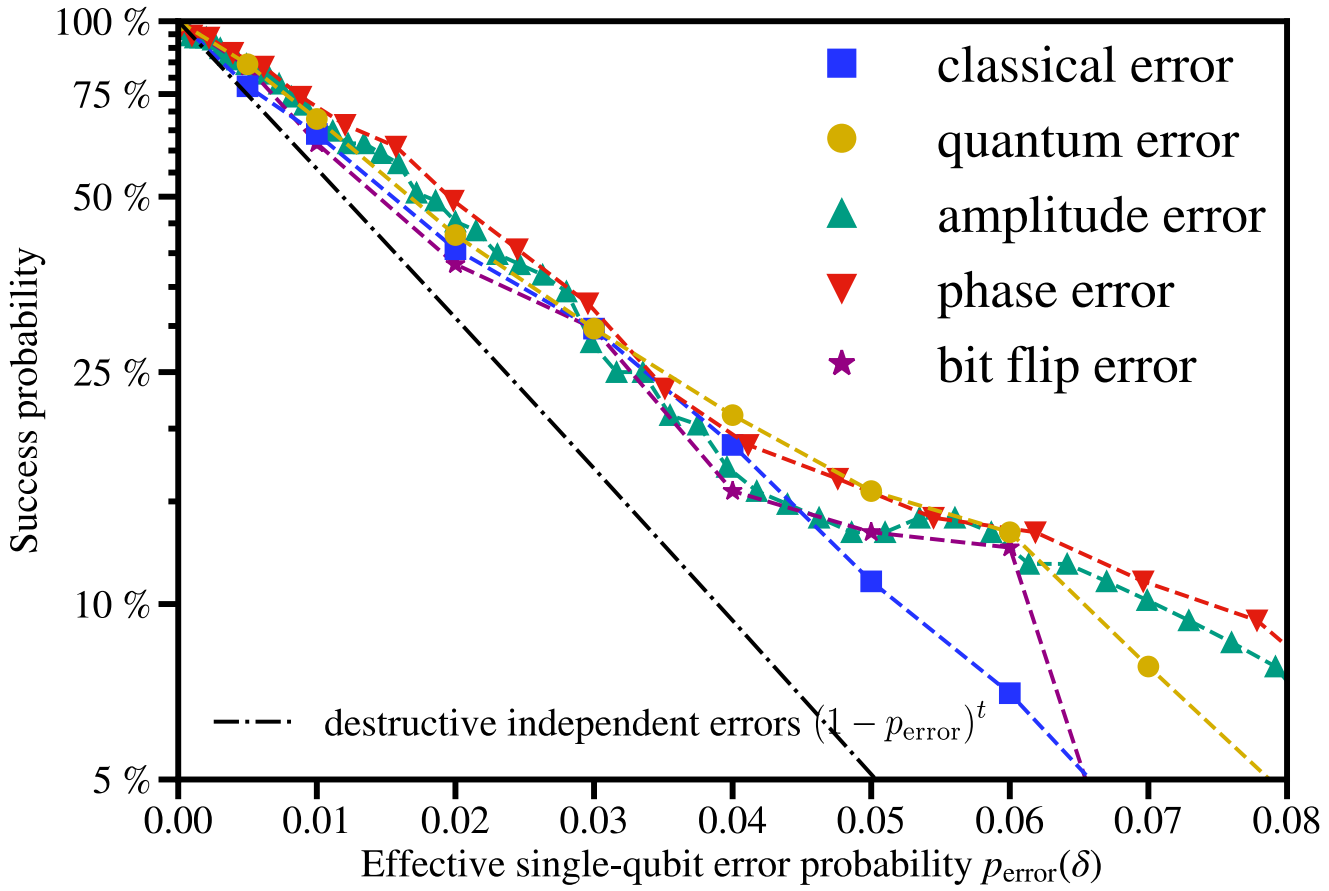
Performance under errors

- **Efficient post-processing** can bring success arbitrarily close to 100%

  Ekerå, Quantum Inf. Process. 20, 205 (2021)
  Ekerå, ACM Trans. Quantum Comput. 5, 11 (2024)

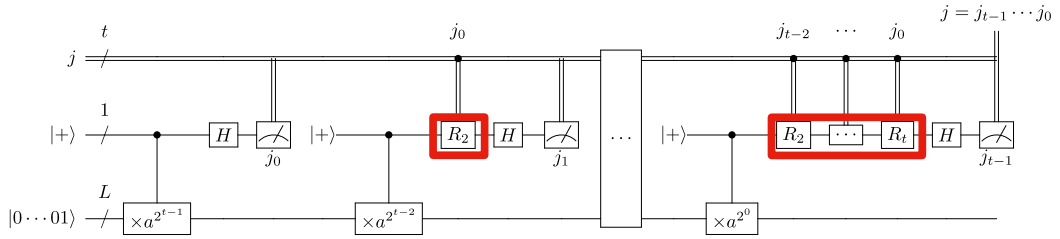
- But what happens under **errors**?





SHOR'S FACTORING ALGORITHM

A particularly interesting error



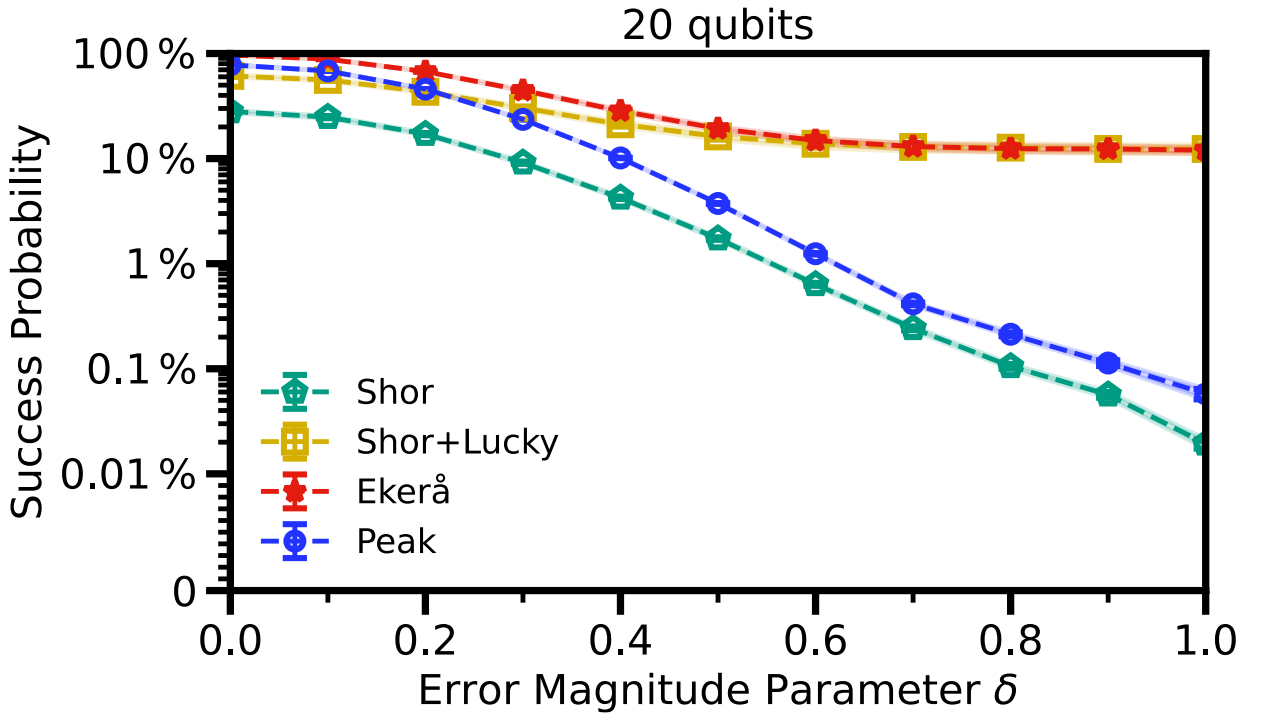
- Gaussian noise on rotation gates:

$$\widetilde{R_k} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(1+\delta r)/2^k} \end{pmatrix}$$

- Cai can **formally prove** that Shor fails!

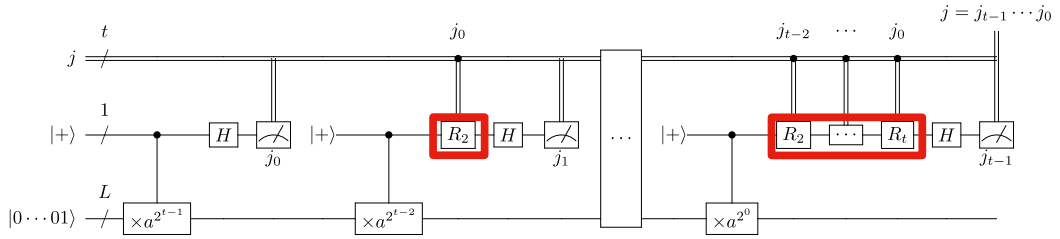
  Cai, Sci. China Inf. Sci. 67, 1 (2024)

- **Question:** Does that also affect the Ekerå-post-processed and the "lucky" cases?



SHOR'S FACTORING ALGORITHM

A Particular "Software" Error



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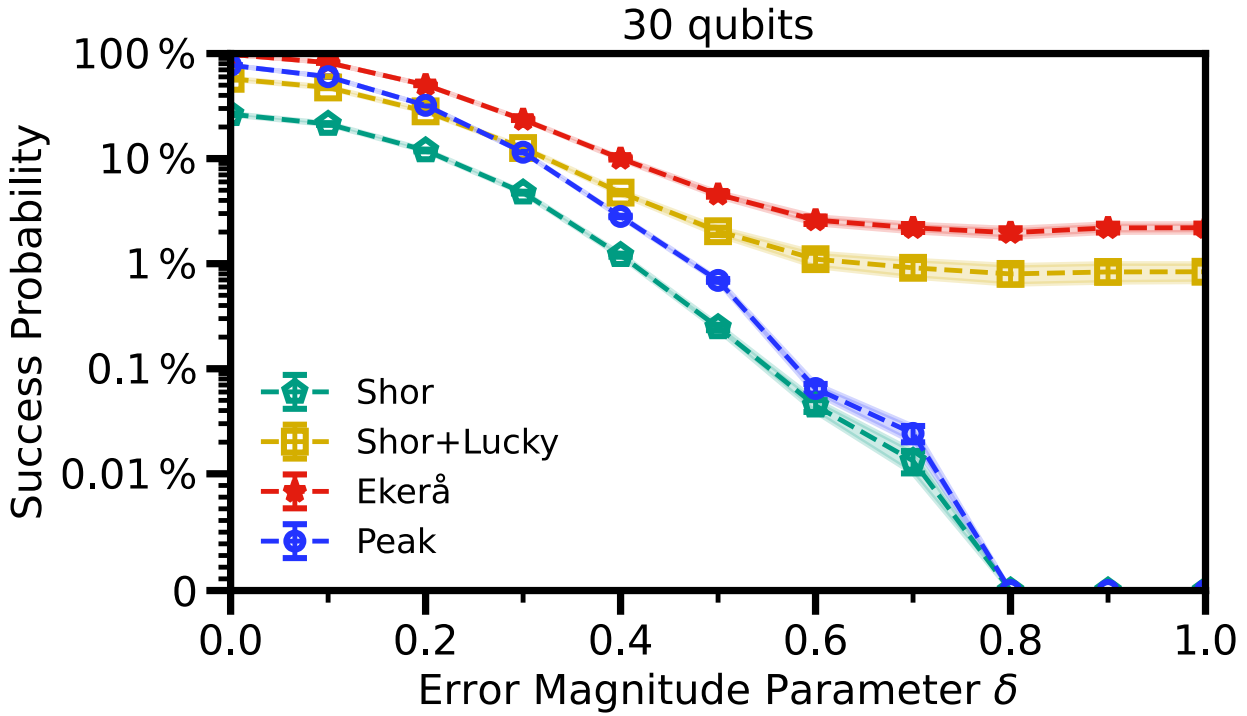


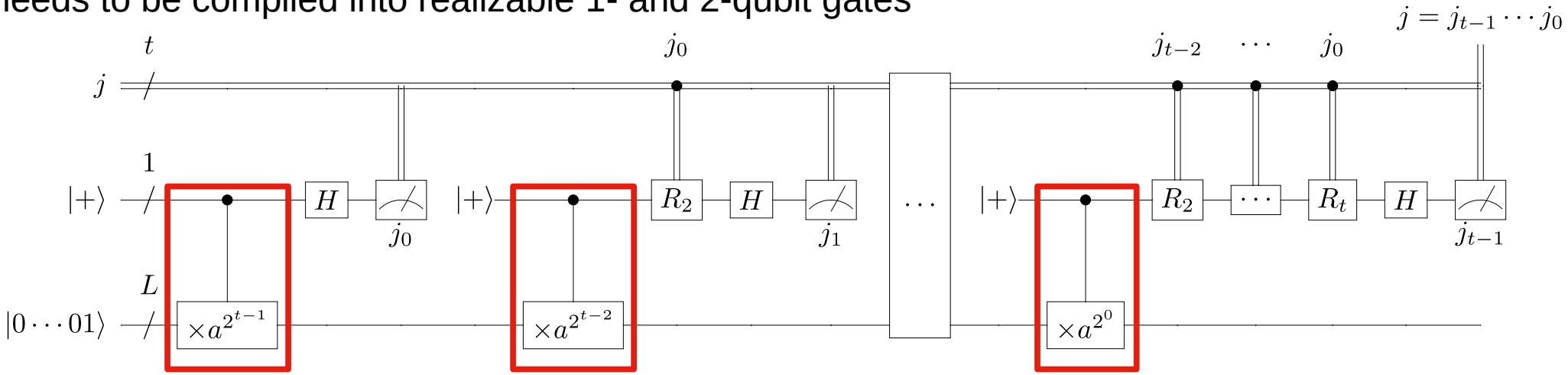
- **Question:** Does that also affect the Ekerå-post-processed and the "lucky" cases?
- **Conclusion:** Not as severely!
- **But:** General consensus is that quantum error correction is required

arXiv:2410.14397

SHOR'S FACTORING ALGORITHM

Hardware challenges

- Algorithm needs to be compiled into realizable 1- and 2-qubit gates



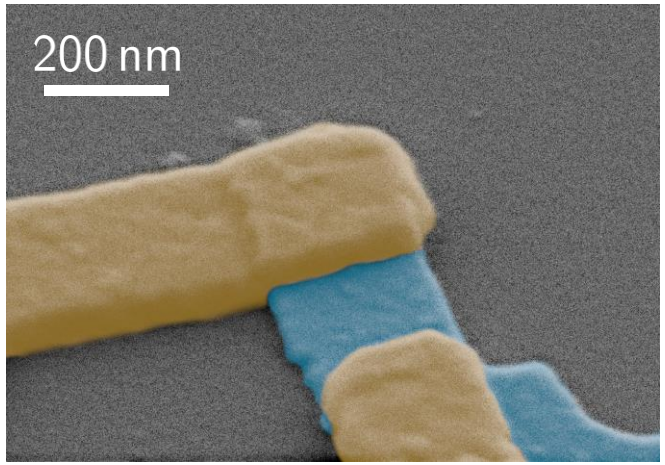
- Takes 2L+3 or 2L+1 qubits
  Beauregard, Quantum Inf. Comput. 3, 175 (2003)
  Gidney, arXiv:1706.07884 (2018)
  (or ~ 1.5L qubits with a trick)
  Zalka, arXiv:quant-ph/0601097 (2006)
  instead of L+1 qubits
- **Recent new development:** Maybe only L/2 qubits?
  Chevignard, Fouque and Schrottenloher, cryptoeprint:2024/022 (2024)

But still: Massive increase in number of gates!

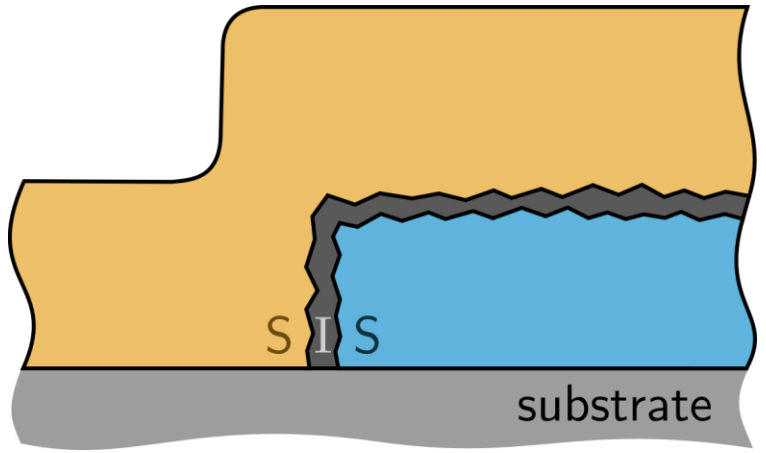
Bits in N	N	# Gates
4	15	9 308
10	1007	181 652
25	33554089	4 657 463

HARDWARE CHALLENGES: JOSEPHSON HARMONICS

Josephson (1962)



?



Experiments with QCs

Fluxonium QND Readout
Phys. Rev. Applied 15, 064030 (2021)

Josephson Harmonics
Nat. Phys. 20, 815 (2024)

$$H = 4E_C n^2 - \cancel{E_J \cos\varphi} - \sum_m E_{Jm} \cos\, m\varphi$$

Paper: Willsch & Rieger et al.,
Nat. Phys. 20, 815 (2024)

Talk: https://youtu.be/dWu3lCwyXBM

Conclusion: Standard Model does not work!

- Fundamental challenge!
- But also an opportunity?

HARDWARE CHALLENGES: ERRORS

Idea: Erroneous NISQ devices
→ Try variational algorithm?

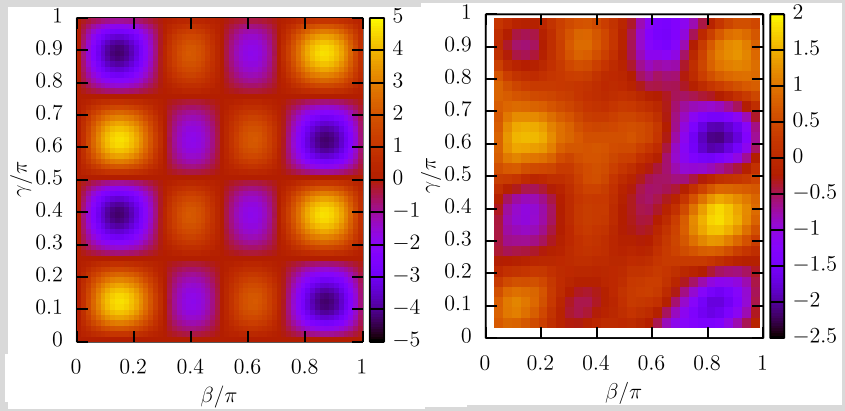
Variational Quantum Factoring
N = 1099551473989 ~ 2^{41}
Anschuetz et al., arXiv:1808.08927 (2018)
Karamlou et al., npj Quantum Inf. 7, 156 (2021)

Careful: only 3 qubits!

Benchmarking QCs

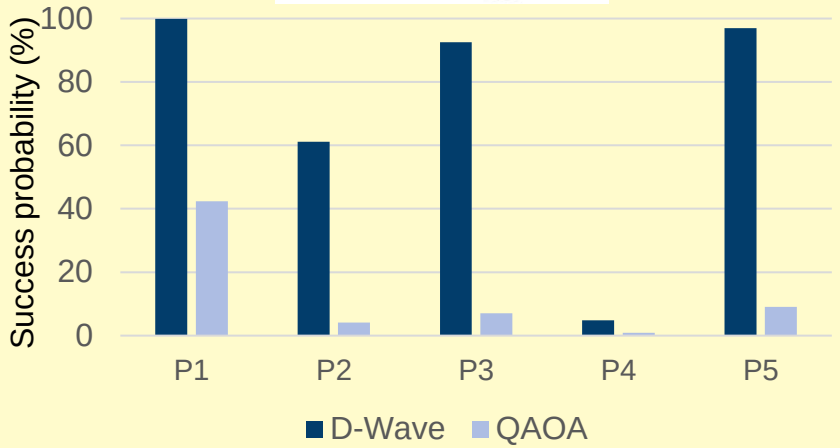
Benchmarking digital QCs
Comput. Phys. Commun. 220, 44 (2017)
Benchmarking the QAOA
Quantum Inf. Process. 19, 197 (2020)
Benchmarking analog QCs
Quantum Inf. Process. 21, 141 (2022)

Gate-based quantum computer



Simulation IBM-Q device

Quantum annealer



Conclusion:

- If the problem is already in QUBO/Ising form
  - Do not use digital QCs
  - Try analog QCs

The D-Wave quantum annealer (= experiment) outperforms QAOA executed on a simulator (= under optimal conditions)

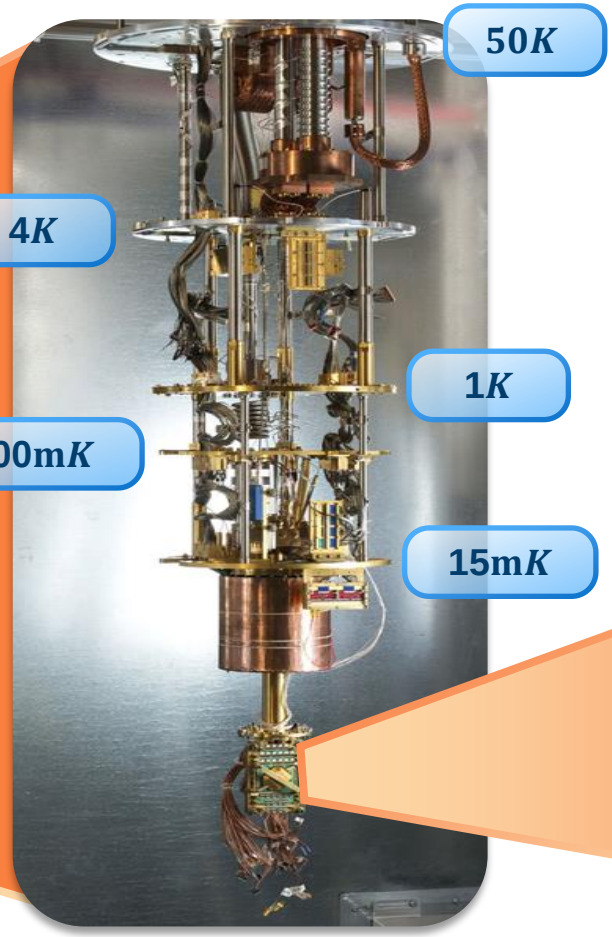
ANALOG QCS

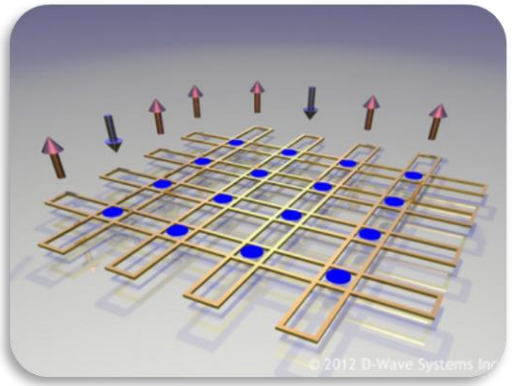
Device: D-Wave “Advantage” Quantum Annealer

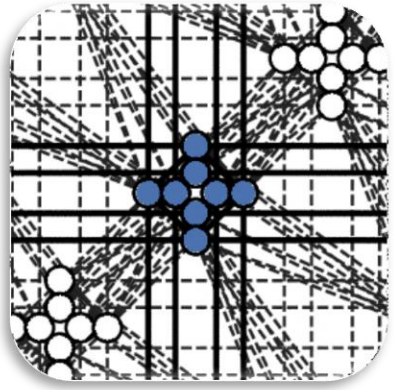
D-Wave The Quantum Computing Company™ **Advantage**

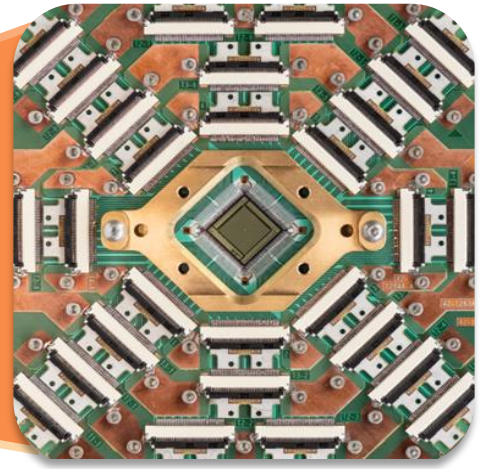
Quantum Processing Unit

5640 superconducting qubits
Pegasus topology with 40.484 couplers









FACTORING ON ANALOG QCS

Mehtods

- Problem needs to be expressed as QUBO / Ising
- We investigated **3 approaches**:

QUBO: $\min_{x_i=0,1} \left(\sum_{i \leq j}^{N-1} x_i Q_{ij} x_j + c \right)$

Ising: $\min_{s_i=\pm 1} \left(\sum_{i=0}^{N-1} h_i s_i + \sum_{i<j}^{N-1} J_{ij} s_i s_j + C \right)$

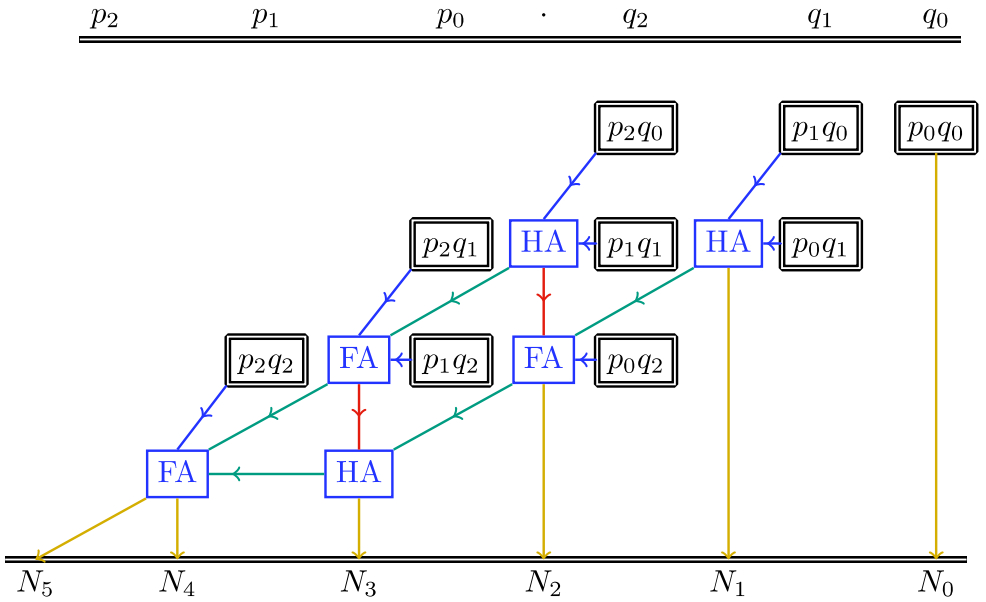
Direct Method

Peng et al., Phys. Rev. Lett. 101, 220405 (2008)

$$\min_{p,q} (N - pq)^2$$

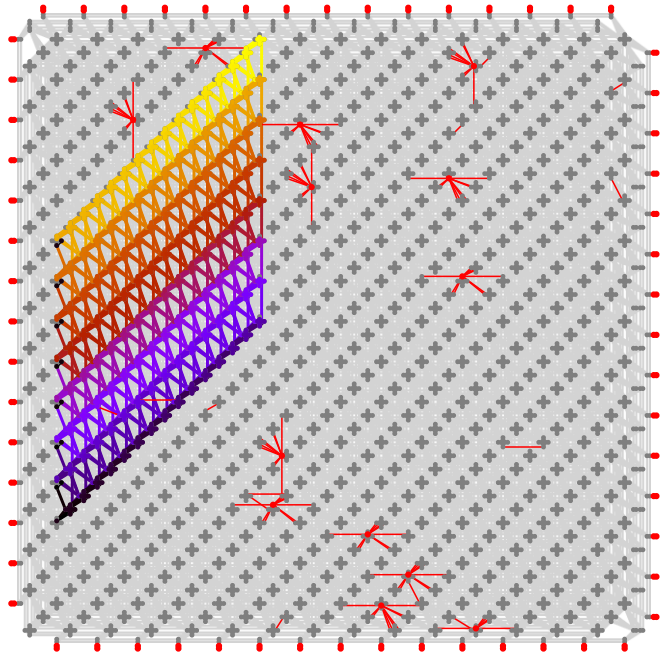
Multiplication Circuit Method

Andriyash et al, D-Wave TR 14-1002A-B (2016)



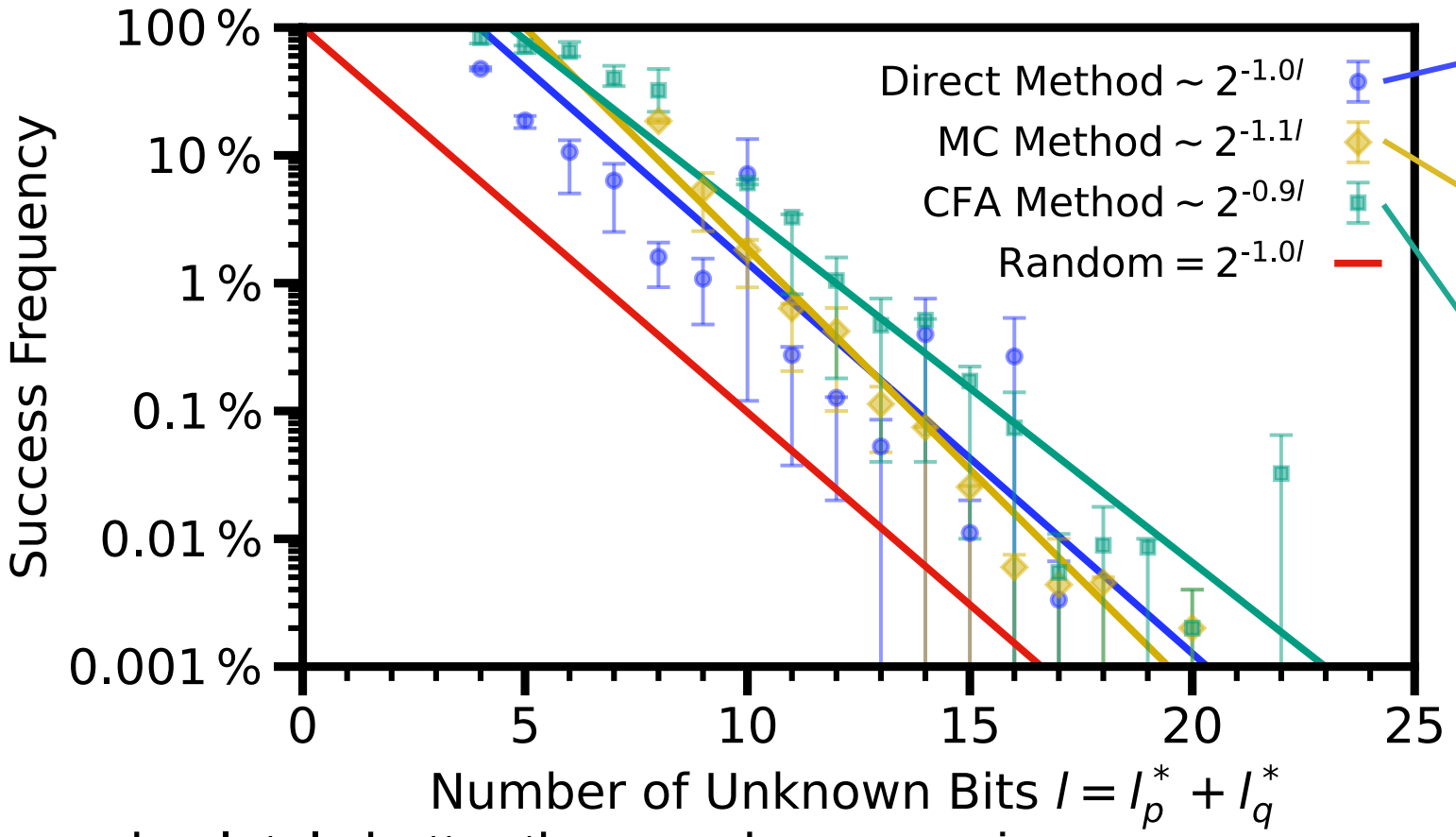
Controlled Full-Adder Method

Ding et al., Sci. Rep. 14, 1 (2024)

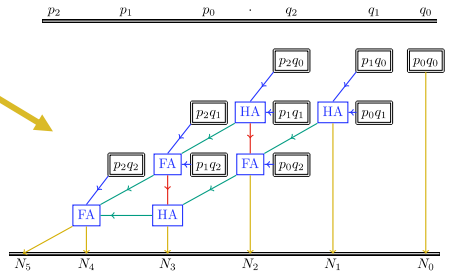


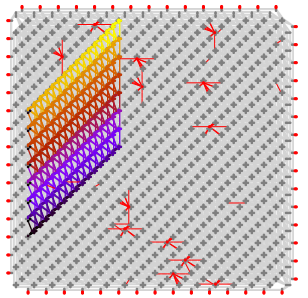
ANALOG QCS

Results



$$\min_{p,q} (N - pq)^2$$

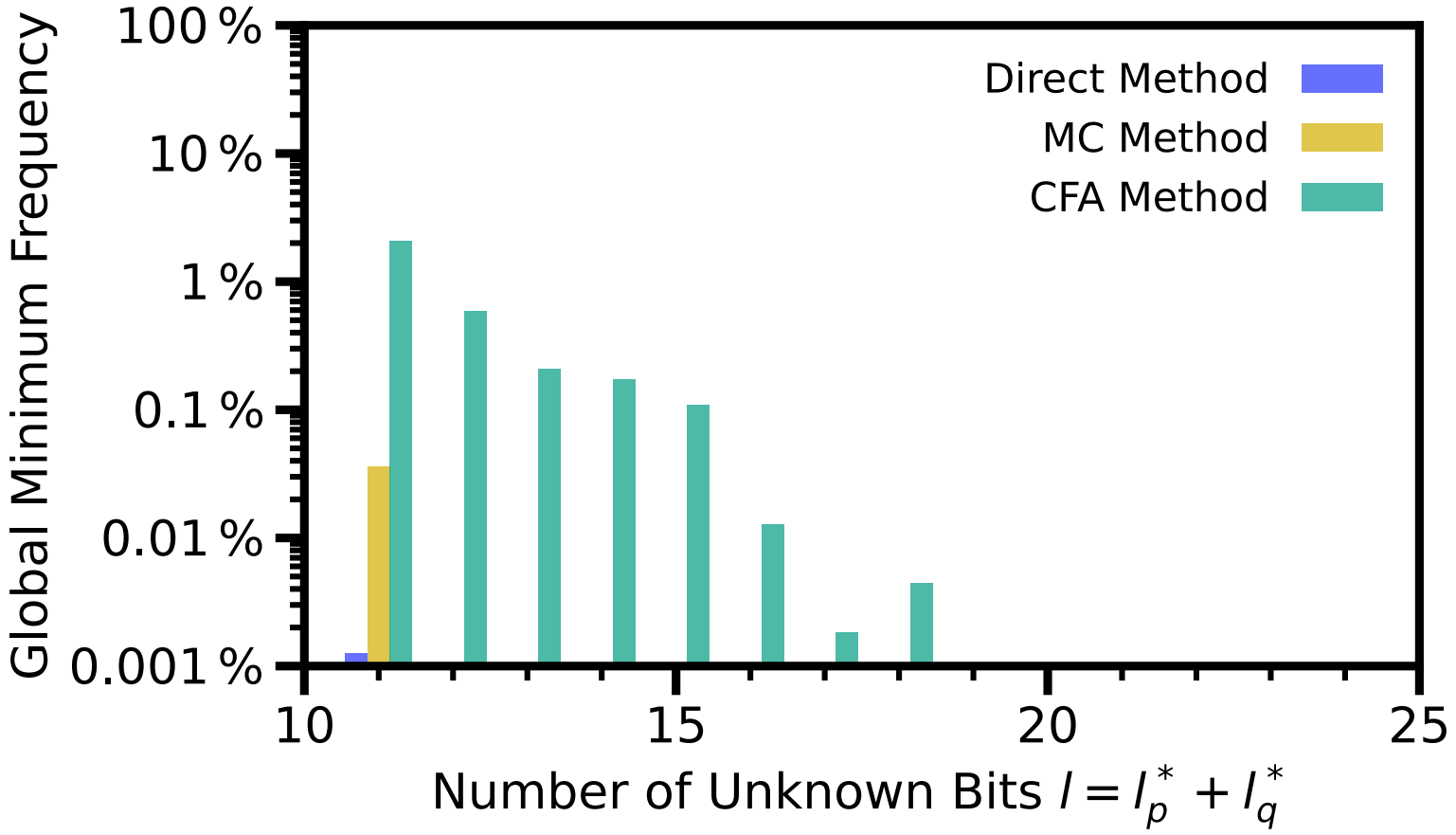


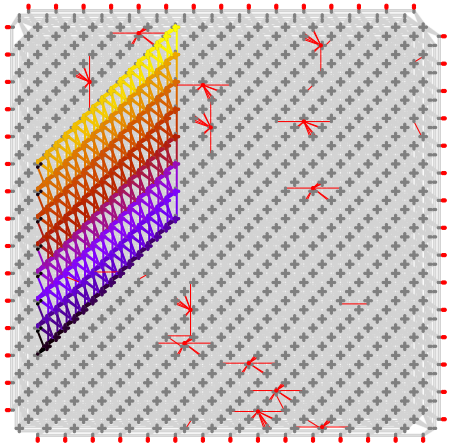


- All methods are absolutely better than random guessing
- CFA method also scales better
- **BUT:** All methods scale exponentially!

ANALOG QCS

Results





- CFA method finds true ground state much more often!
- **Reason:** Custom embedding on the hardware

SUMMARY

Factoring on Quantum Computers

- **Many different approaches**

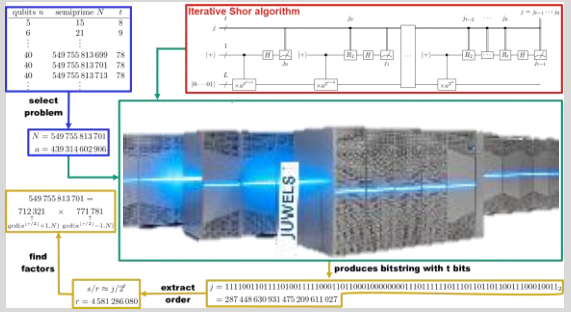
Future perspectives:

- Large RSA integers
  - Likely require quantum error correction
  - Likely require millions of qubits

Gidney and Ekerå, Quantum 5, 433 (2021)

  - Likely not factorizable anytime soon

- **But:** Larger semiprimes than we can factor with Shor on HPC maybe soon ☺



- <u>Present evidence:</u> Only Shor and derivatives seem to scale

Shor, Proc. Found. Comp. Sci. 35, 124 (1994)
Ekerå and Håstad, PQCrypto 2017, 347 (2017)
CFS, cryptoeprint:2024/022 (2024)

- **Many challenges**
  - Software and hardware errors on digital QCs
  - New method on analog QCs

THANK YOU FOR YOUR ATTENTION

Further References:

Willsch et al., Mathematics 11, 4222 (2023)
Hanussek, DOI:10.34734/FZJ-2024-05254 (2024)
Hanussek, jugit.fz-juelich.de/qip/jupsifactoring (2024)
Willsch et al., arXiv:2410.14397 (2024)