Surprises with Karsten-Wilczek and Boriçi-Creutz fermions

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Panopticum of lattice fermions

Classical lattice fermion actions:

- Naive fermions (2^d species in d space-time dimensions)
- Wilson fermions ($N_c 4 N_x N_y N_z N_t \times$ ditto matrix in $d=4$ dimensions)
- Staggered fermions (reduction by $2^{d/2}$, hence size $N_c N_x N_y N_z N_t \times$ ditto)
- Overlap/domain-wall fermions (unique unitary part of $aD_\mathrm{W}-\rho$)

Novel lattice fermion actions:

- Minimally doubled fermions (Karsten-Wilczek, Boriçi-Creutz, ...)
- Ameliorated Wilson fermions (Brillouin, hypercube, ...)
- Staggered fermions with lifting (Adams, Hoelbling, ...)

Issues to be considered:

- Nielsen-Ninomya theorem ("topology")
- suitability for heavy quark physics (dispersion relation, ...)
- Symanzik scaling for $a \to 0$
- suitability for lattice perturbation theory (LPT)
- computational efficiency (MPI/PGAS, OpenMP/OpenACC/cuda, SIMD)

Introduction: Naive and Wilson fermions

- **Naive fermions**

$$D_{\rm nai}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) + m\delta_{x,y}$$

$$D_{\rm nai}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m \quad \text{with} \quad \bar{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$$

- **Wilson fermions**

$$D_{\rm W}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \frac{ra}{2}\sum_\mu \triangle_\mu(x,y) + m\delta_{x,y}$$

$$D_{\rm W}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \frac{r}{a}\sum_\mu \{1-\cos(ap_\mu)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \frac{ra}{2}\sum_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \hat{p}_\mu \equiv \frac{2}{a}\sin(\frac{ap_\mu}{2})$$

Introduction: Karsten-Wilczek and Borici-Creutz fermions

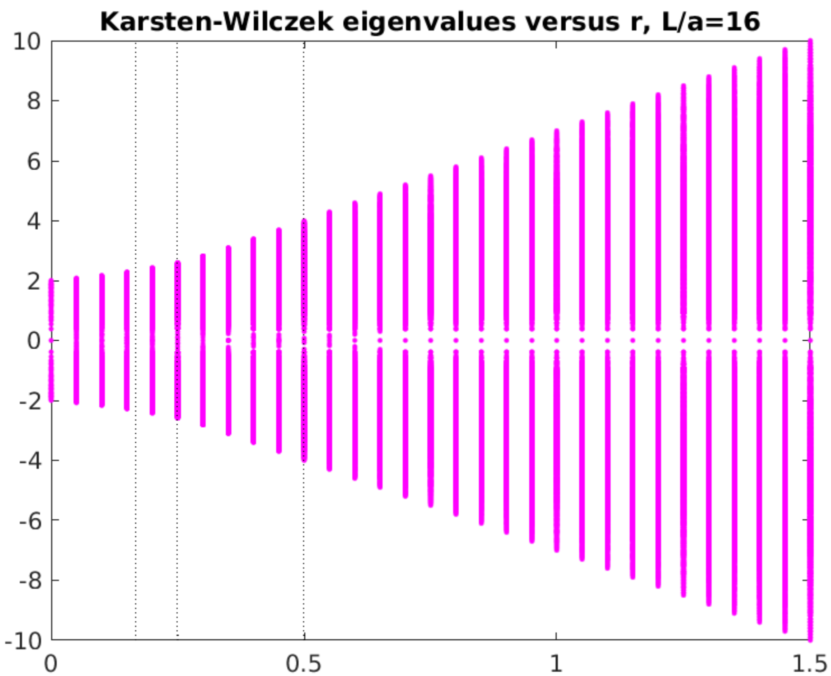
- **Karsten-Wilczek fermions**

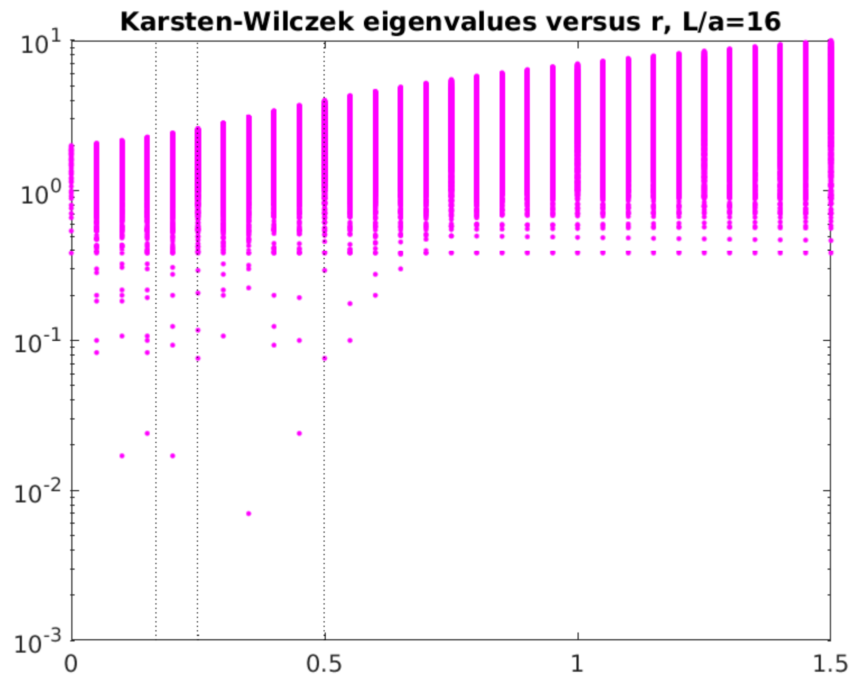
$$
\begin{aligned}
D_{\mathrm{KW}}(x,y) &= \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ra}{2}\gamma_4 \sum_{i=1:3} \triangle_i(x,y) + m\delta_{x,y} \\
D_{\mathrm{KW}}(p) &= \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \mathrm{i}\frac{r}{a}\gamma_4 \sum_{i=1:3} \{1-\cos(ap_i)\} + m \\
&= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ra}{2}\gamma_4 \sum_{i=1:3} \hat{p}_i^2 + m
\end{aligned}
$$

- **Borici-Creutz fermions**

$$
\begin{aligned}
D_{\mathrm{BC}}(x,y) &= \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \mathrm{i}\frac{ra}{2}\sum_\mu \gamma'_\mu \triangle_\mu(x,y) + m\delta_{x,y} \\
D_{\mathrm{BC}}(p) &= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{r}{a}\sum_\mu \gamma'_\mu \{1-\cos(ap_\mu)\} + m \\
&= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \mathrm{i}\frac{ra}{2}\sum_\mu \gamma'_\mu \hat{p}_\mu^2 + m \quad \left[\gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma,\ \Gamma \equiv \frac{1}{\sqrt{d}}\sum_\mu \gamma_\mu\right]
\end{aligned}
$$

• Karsten-Wilczek free-field eigenvalues versus r in 4D





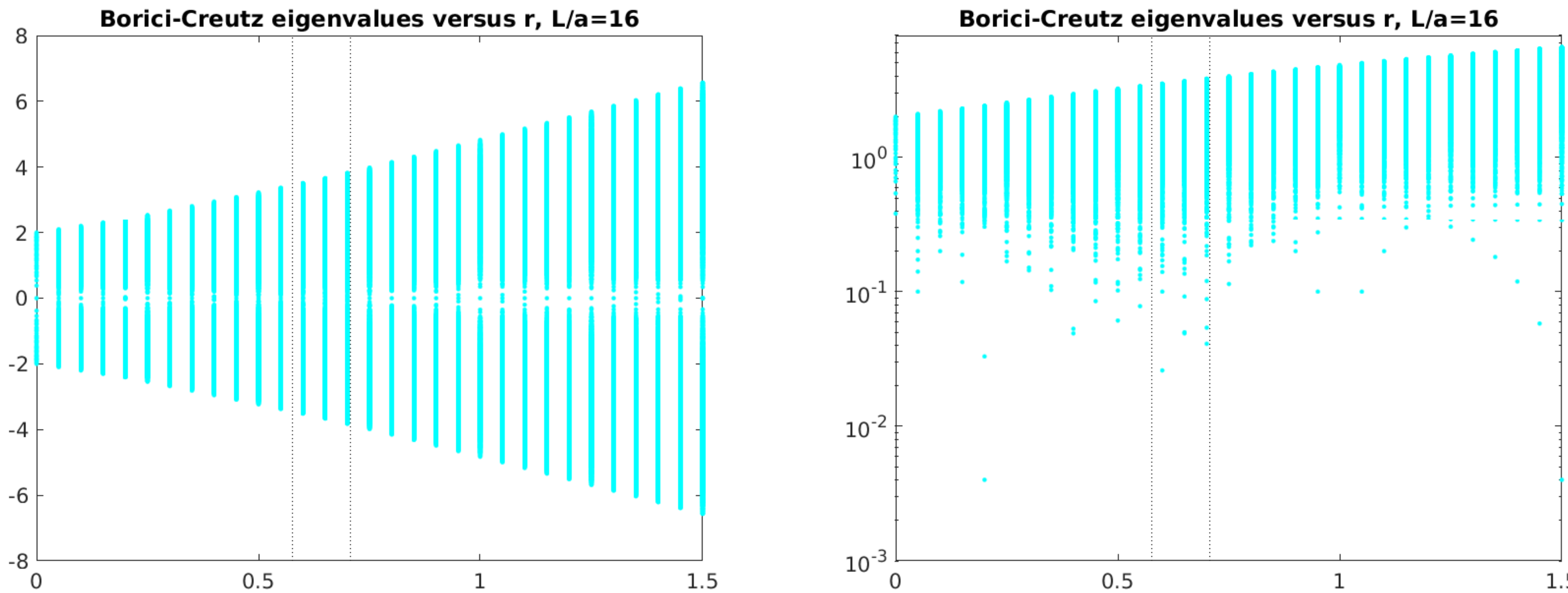
Spectrum at $r = 0$ is naive (i.e. 4-fold staggered) spectrum.

Spectrum at any r is on imaginary axis (chiral symmetry, horizontally displaced).

Spectrum at $r = 1$ covers range $[-7, 7]$ on imaginary axis (worse CN than staggered).

KW species chain is $16 \to 14 \to 8 \to 2$ with transistions at $r = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}$ [2003.10803].

• Borici-Creutz free-field eigenvalues versus r in 4D



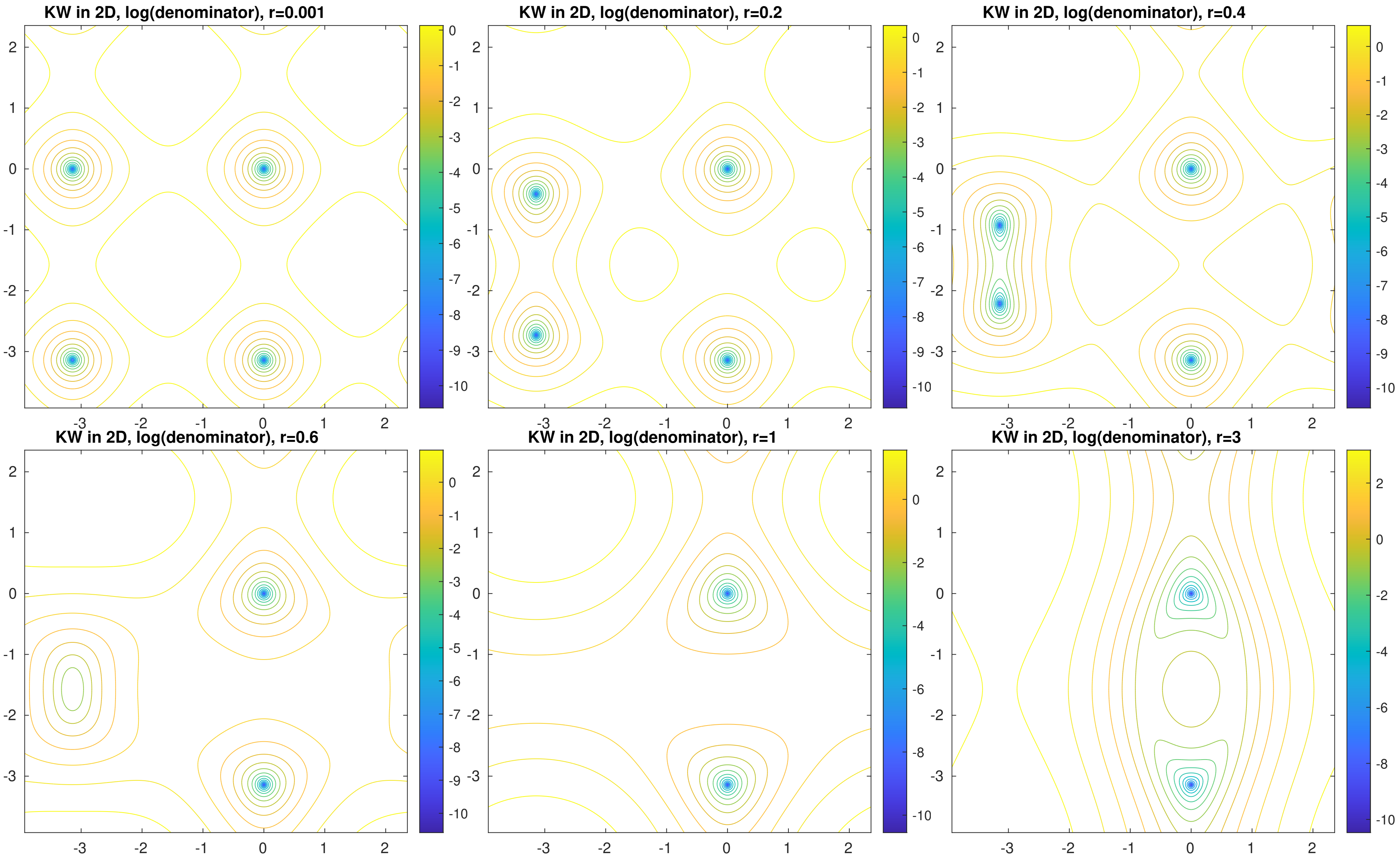
Spectrum at $r = 0$ is naive (i.e. 4-fold staggered) spectrum.

Spectrum at any r is on imaginary axis (chiral symmetry, horizontally displaced).

Spectrum at $r = 1$ covers range $[-4.8284, 2 + 2\sqrt{2}]$ on imag. axis (intermediate CN).

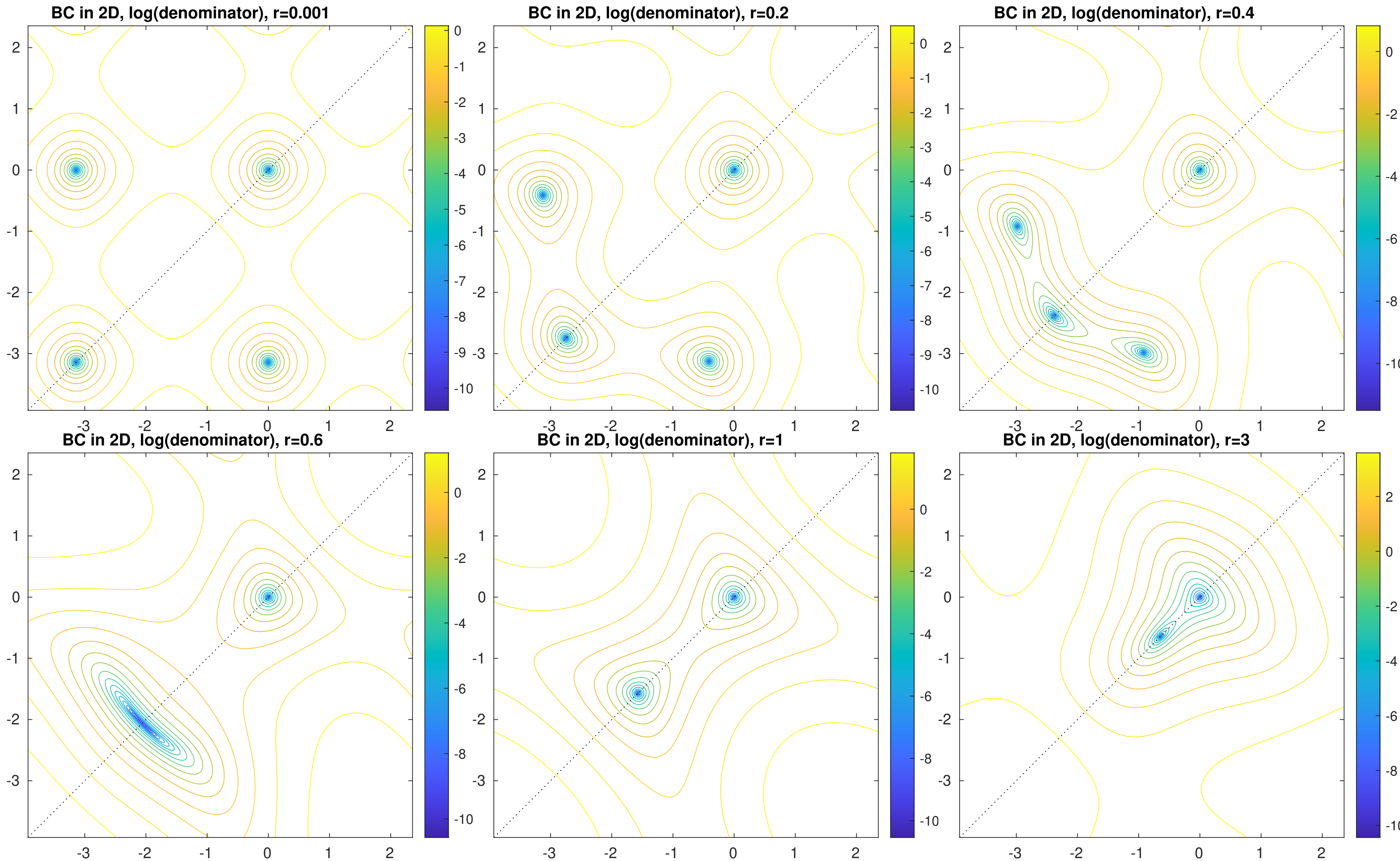
BC species chain is $16 \to 10 \to 2$ with transitions at $r = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}$ [2003.10803].

Pole position drift for KW fermions in 2D (annihilate at r=0.5)



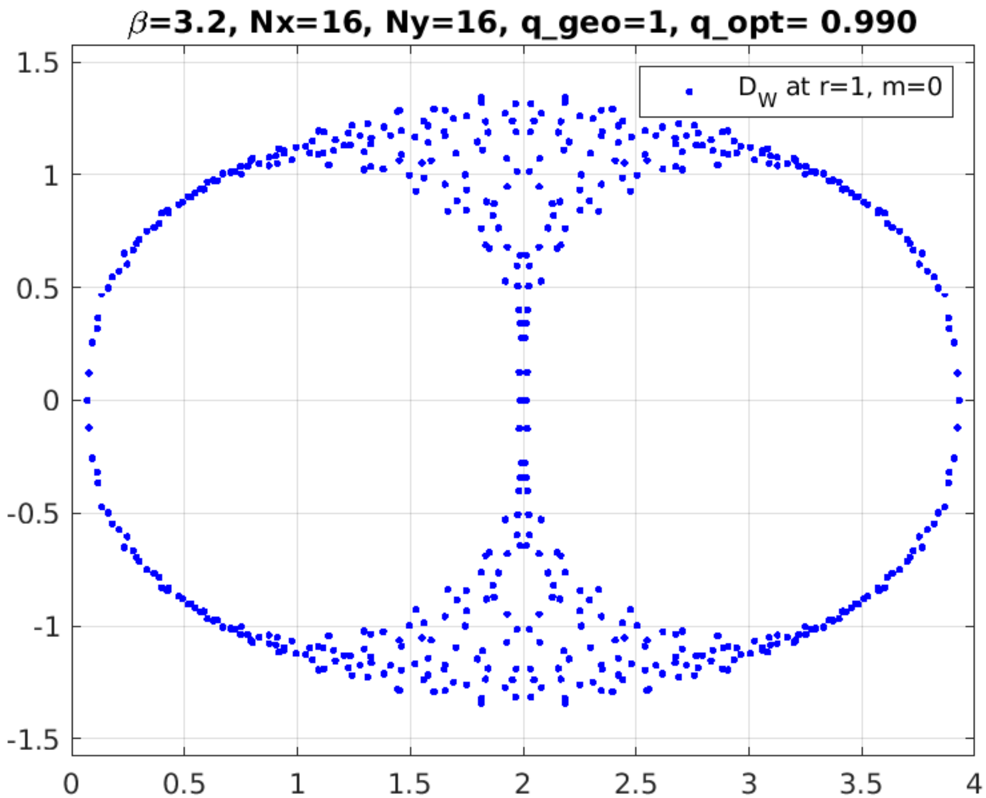
Contour plots of denominator $\sum_{i=1}^{d-1} \bar{p}_i^2 + (\bar{p}_d + \frac{ar}{2} \sum_{i=1}^{d-1} \hat{p}_i^2)^2$ [arXiv:2003.10803].

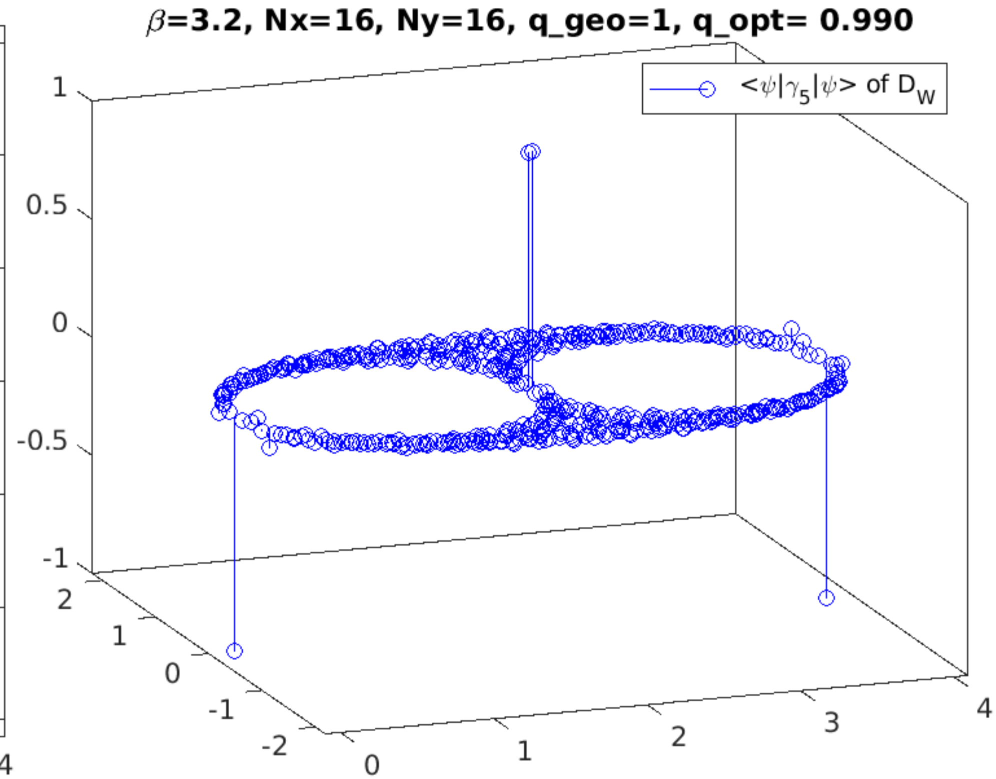
Pole position drift for BC fermions in 2D (merge at r=0.57735)



Contour plots of $\sum_\lambda \bar{p}_\lambda^2 - ar \sum_\lambda \bar{p}_\lambda \hat{p}_\lambda^2 + \frac{a^2 r^2}{4} \sum_\lambda \hat{p}_\lambda^4 + \frac{2ar}{d} \sum_{\rho,\sigma} \bar{p}_\rho \hat{p}_\sigma^2$ [2003.10803].

Eigenvalues and topology with Wilson fermions

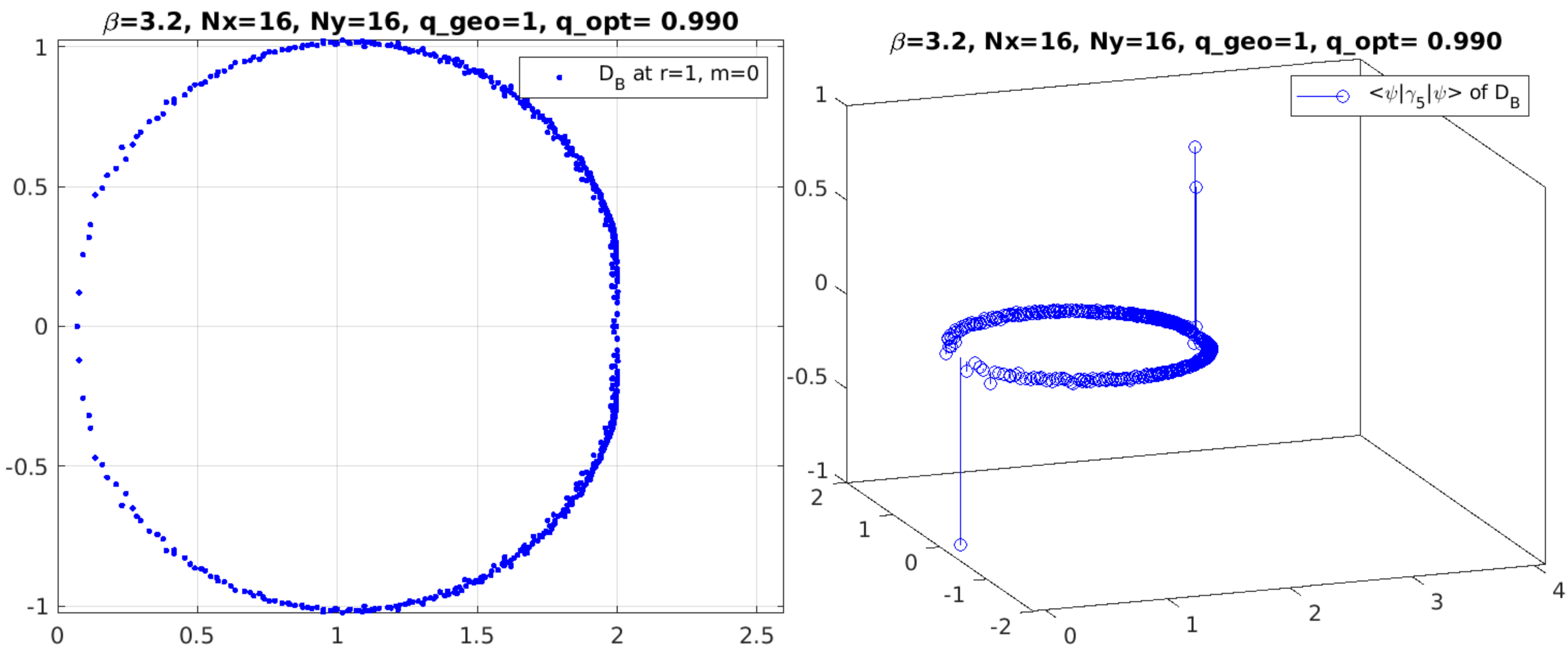




- $|q|$ would-be zero modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle\psi|.|\psi\rangle \equiv \langle L|.|R\rangle$ for non-chiral D [Hip et al 2001]
- plot (and subsequent ones) taken from [arXiv:2203.15699] with J. Weber

Add-on: central-branch yields 2 (4D: 6) species (no chiral symmetry despite $m_{\text{crit}} = 0$)

Eigenvalues and topology with Brillouin fermions

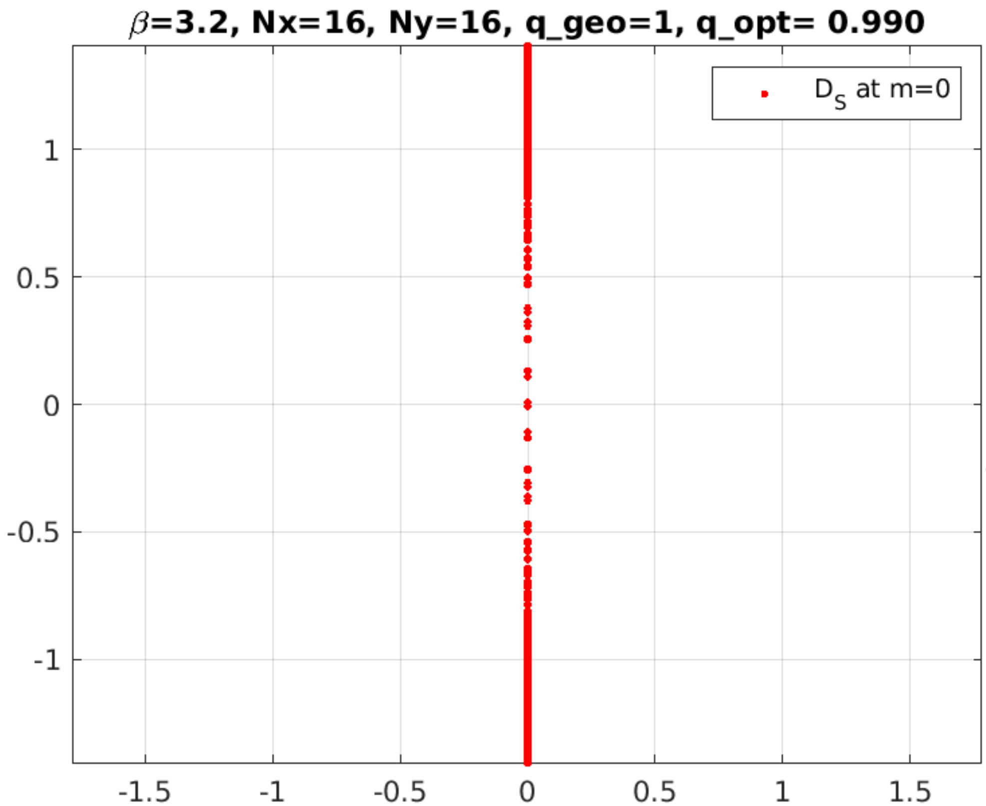


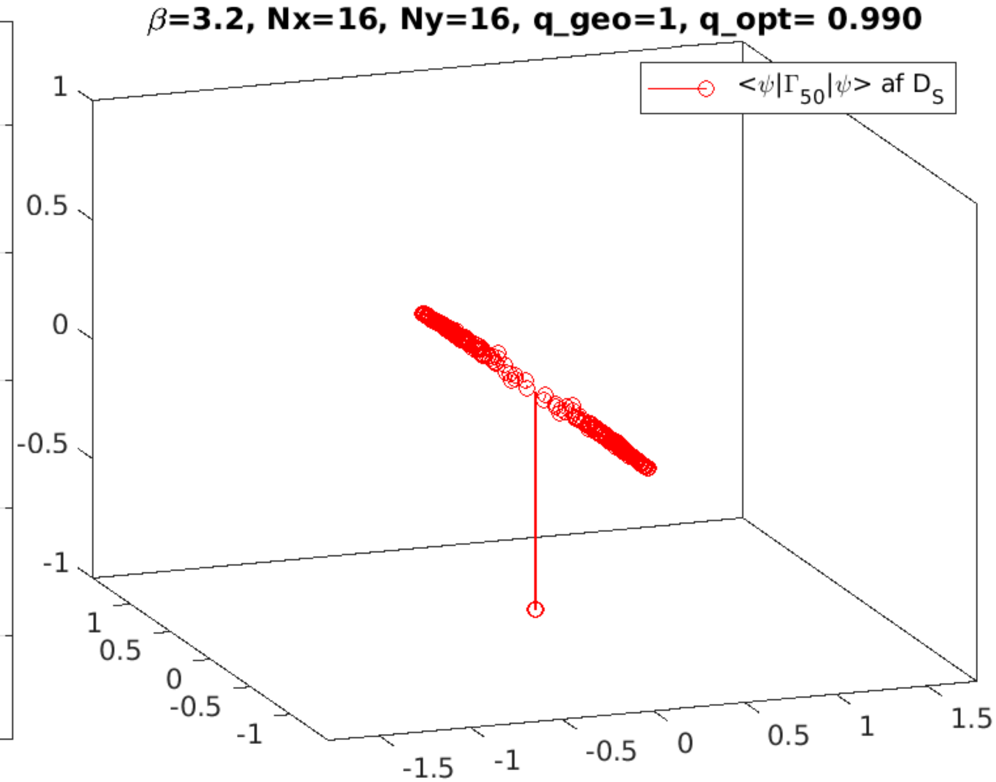
$D_{\rm B} = \sum_\mu \gamma_\mu \nabla_\mu - \frac{r}{2}\triangle$ like Wilson but ∇_μ and $\triangle$ with hypercubic stencil (3^d-points)

- $|q|$ would-be zero modes in physical branch (unchanged in 4D), additive mass shift
- L/R-eigenmode sandwich $\langle\psi|.|\psi\rangle \equiv \langle L|.|R\rangle$ for non-chiral D (compare 1302.0773)

Add-on: use as overlap-kernel, already close to shifted-unitary [arXiv:1701.00726].

Eigenvalues and topology with staggered fermions





$2|q|$ would-be zero modes (changes to $4|q|$ in 4D), remnant chiral symmetry $U(1)_\epsilon$

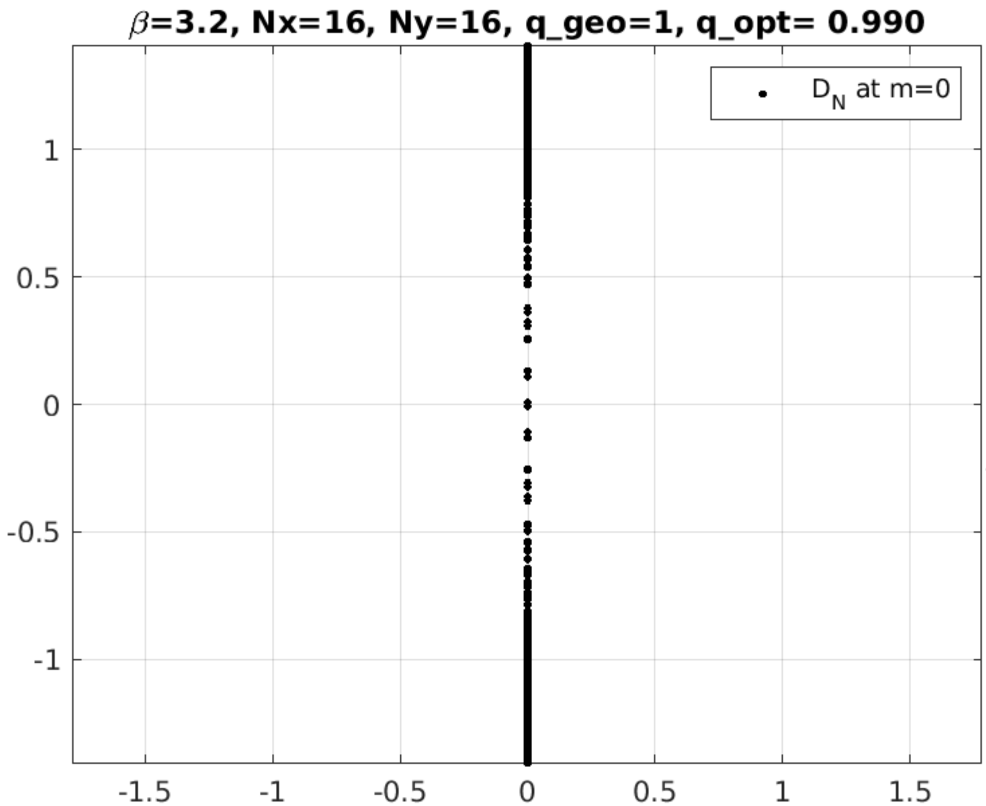
$\epsilon \equiv \gamma_5 \otimes \xi_5$ not sensitive to topology (see "backup pages" for meaning of $\gamma_\mu \otimes \xi_\nu$)

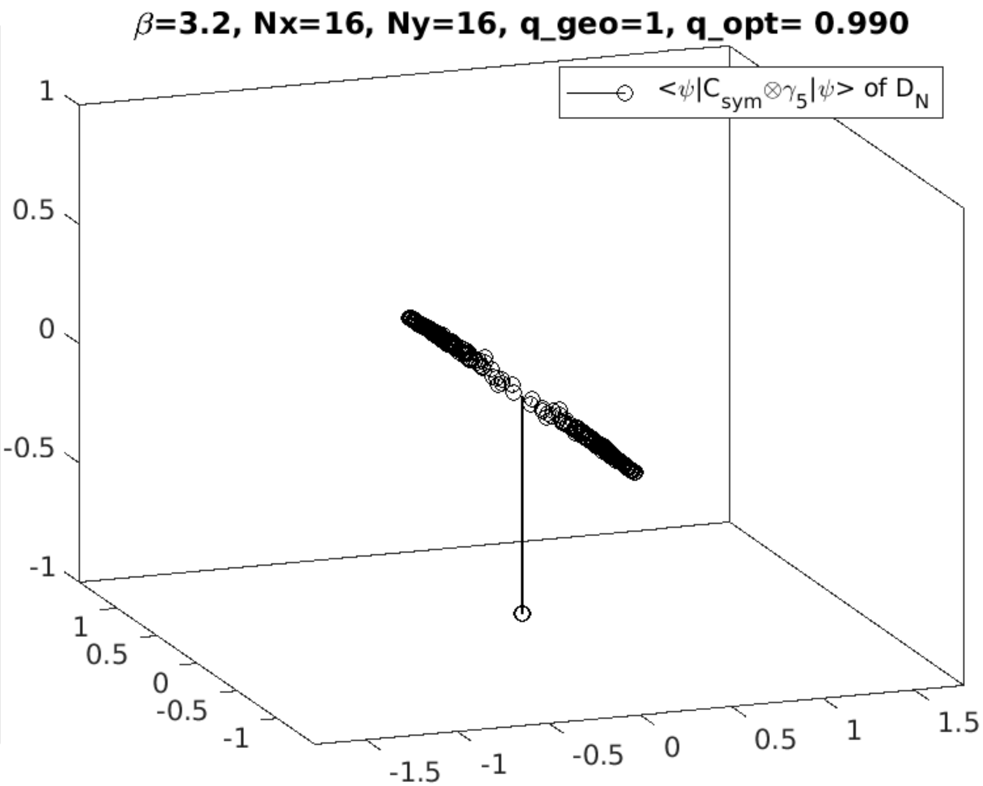
$\Gamma_5 \simeq \gamma_5 \otimes 1$ crafted to "turn around" chirality of second mode (both point down)

$\Xi_5 \simeq 1 \otimes \xi_5$ not sensitive to topology (Γ_5 and Ξ_5 depend on gauge-field U)

$1 \equiv 1 \otimes 1$ not sensitive to topology (like ϵ not shown)

Eigenvalues and topology with naive fermions

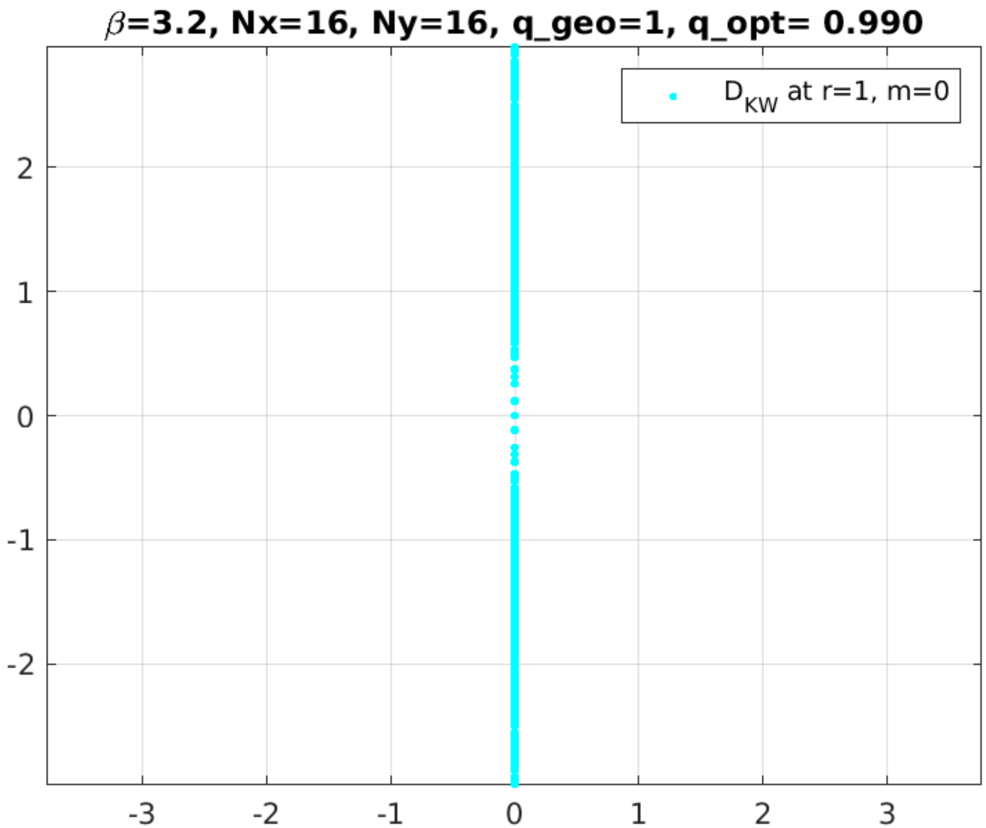


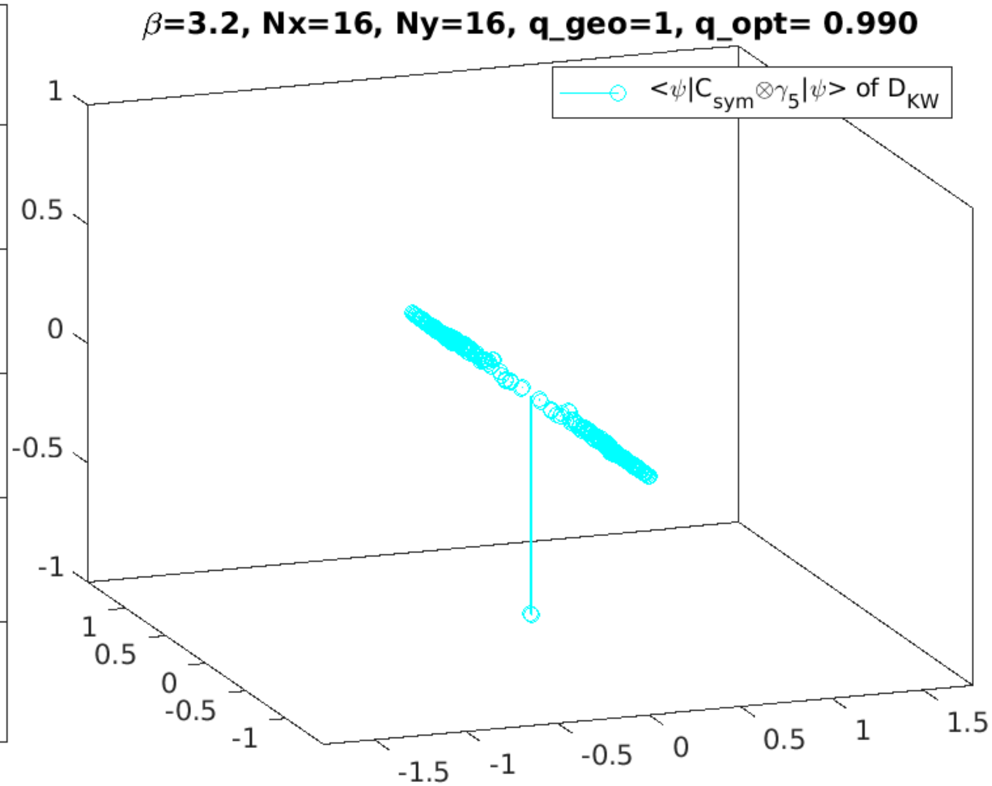


$$D_{\text{naive}} = \sum_\mu \eta_\mu \nabla_\mu$$

- eigenvalue spectrum like staggered, but 2-fold extra degeneracy (4-fold in 4D)
- γ_5-chiralities exactly zero (like ϵ-chiralities for staggered)
- suitable chirality operator is $X = C_{\text{sym}} \otimes \gamma_5$ (4 needles down, 16 in 4D)

Eigenvalues and topology with Karsten-Wilczek fermions



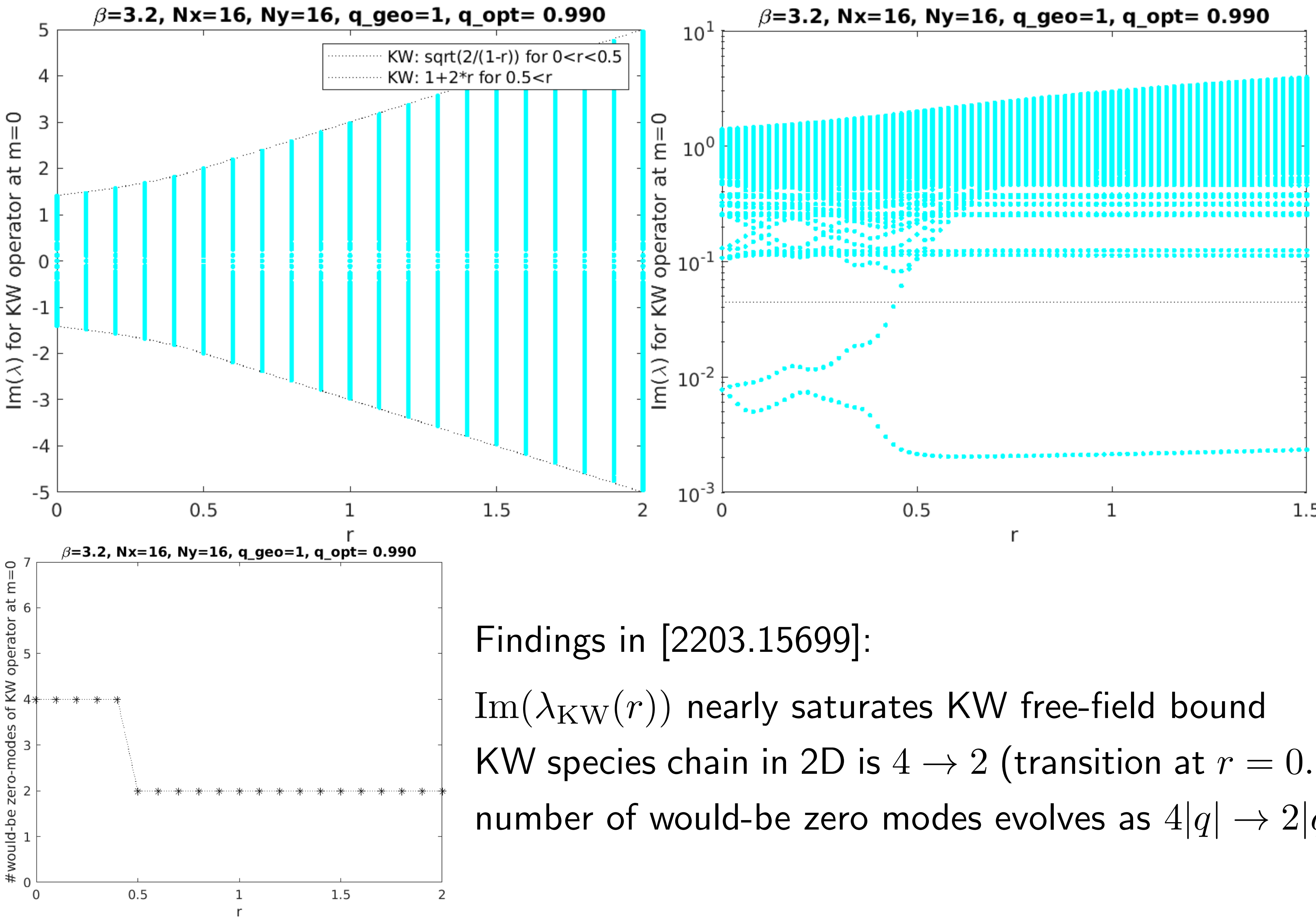


- $2|q|$ would-be zero modes at $r = 1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of $D_{\rm KW}$ not sensitive to γ_5

Operator X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of $D_{\rm KW}$

Options are $X = \frac{1}{2}(C_1 + C_2)^2 \otimes \gamma_5$ and $X = C_{\rm sym} \otimes \gamma_5$ with $C_\mu \equiv \frac{1}{2}\triangle_\mu + 1$

• Transition $D_{\text{naive}} \to D_{\text{KW}}$ as a function of r on $|q| = 1$ configuration



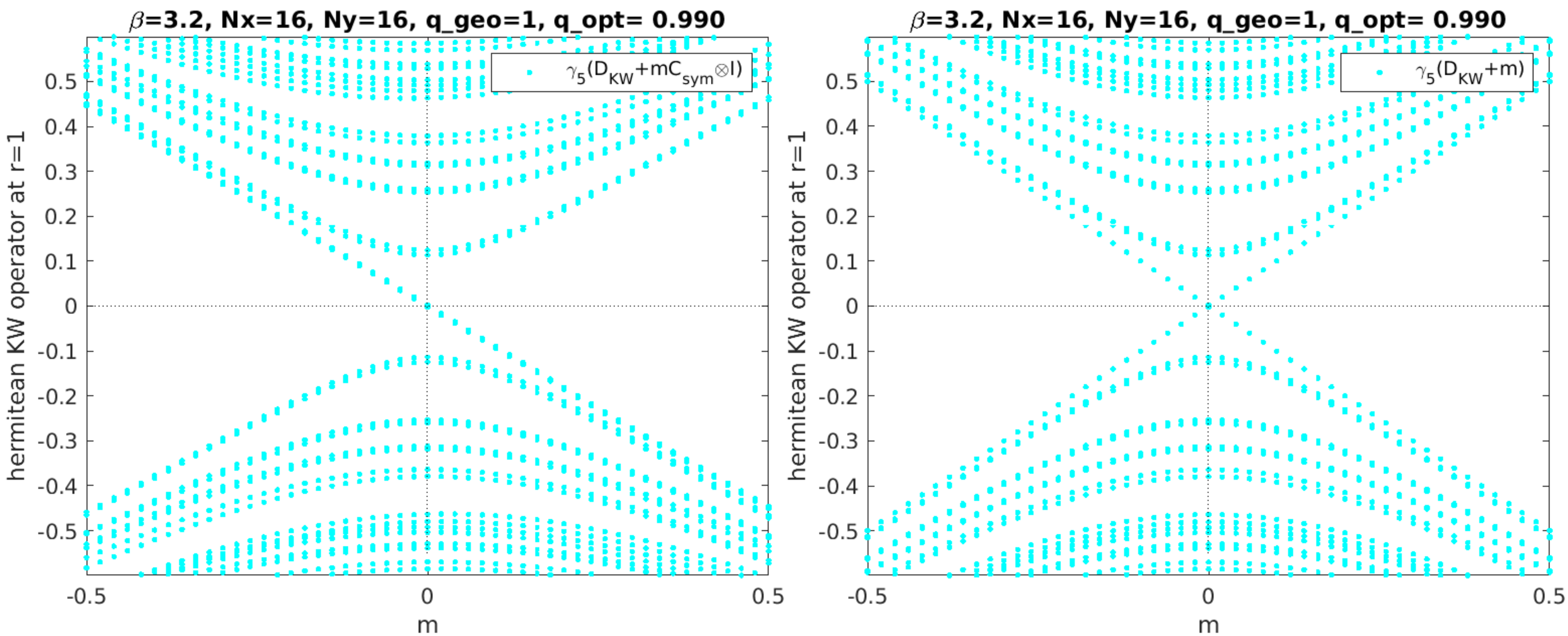
Findings in [2203.15699]:

$\text{Im}(\lambda_{\text{KW}}(r))$ nearly saturates KW free-field bound

KW species chain in 2D is $4 \to 2$ (transition at $r = 0.5$)

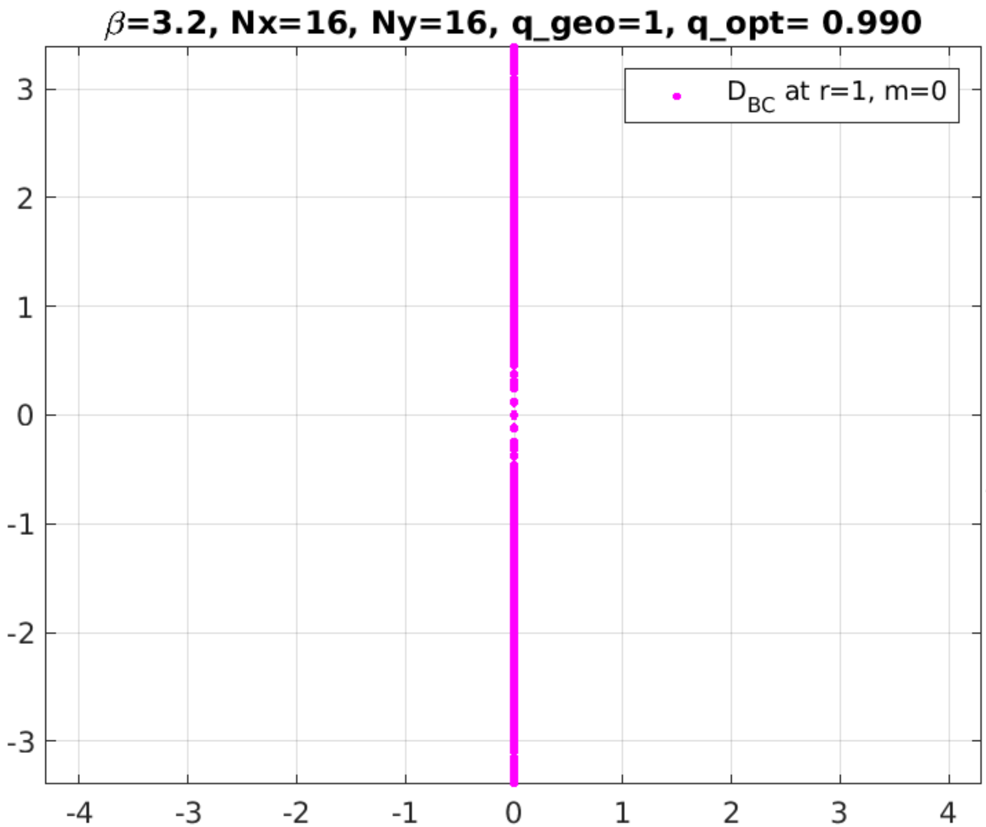
number of would-be zero modes evolves as $4|q| \to 2|q|$

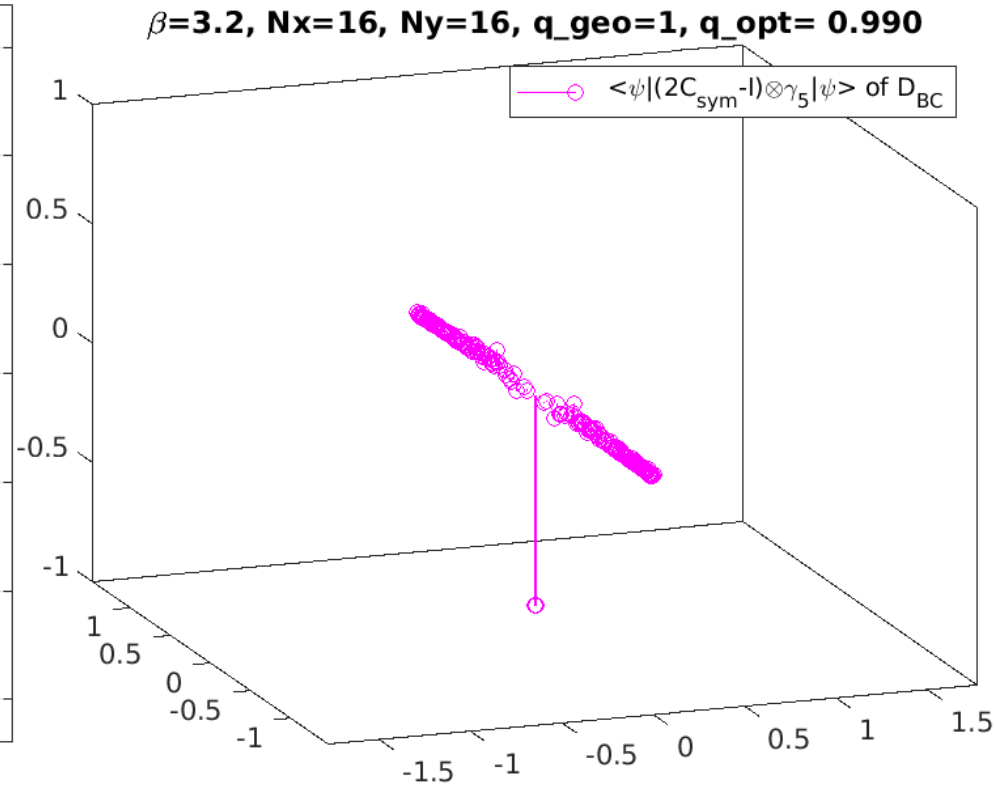
- **Spectral flow with Karsten-Wilczek fermions**



- eigenvalues of $H_{\mathrm{KW}} \equiv \gamma_5(D_{\mathrm{KW}} + mC_{\mathrm{sym}} \otimes 1)$ versus am
- choice matches $X = C_{\mathrm{sym}} \otimes \gamma_5$ being a good chirality operator for D_{KW}
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- near-degeneracy much better than for BC fermions (cf. below)
- dull choice $\gamma_5(D_{\mathrm{KW}} + m)$ amounts to wrong chirality operator

Eigenvalues and topology with Borici-Creutz fermions



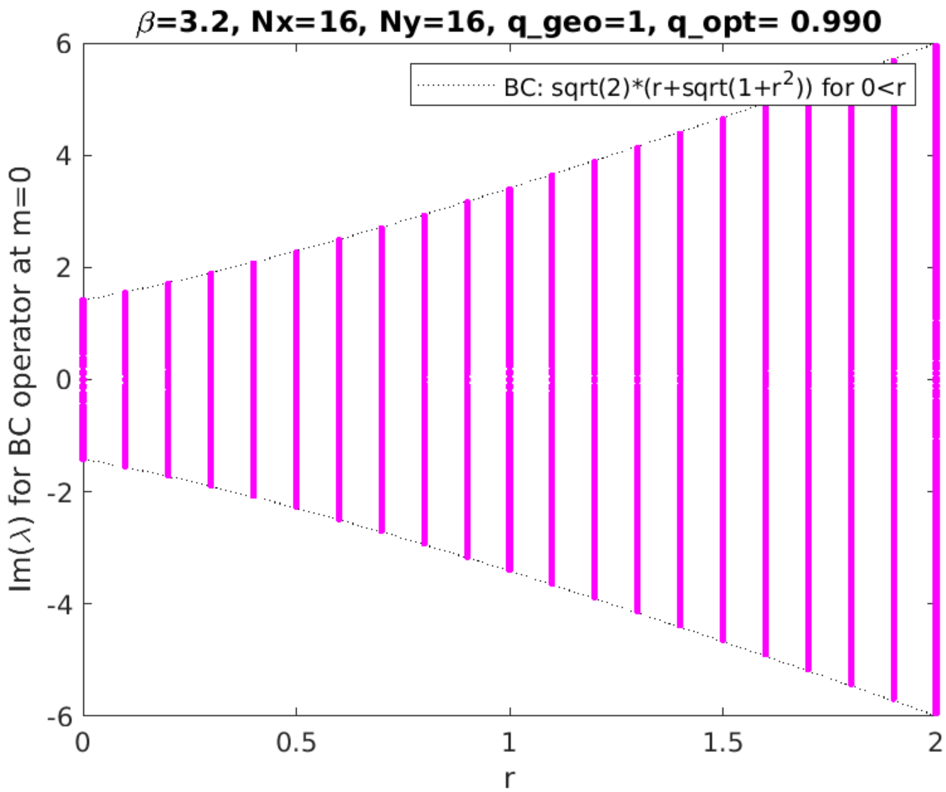


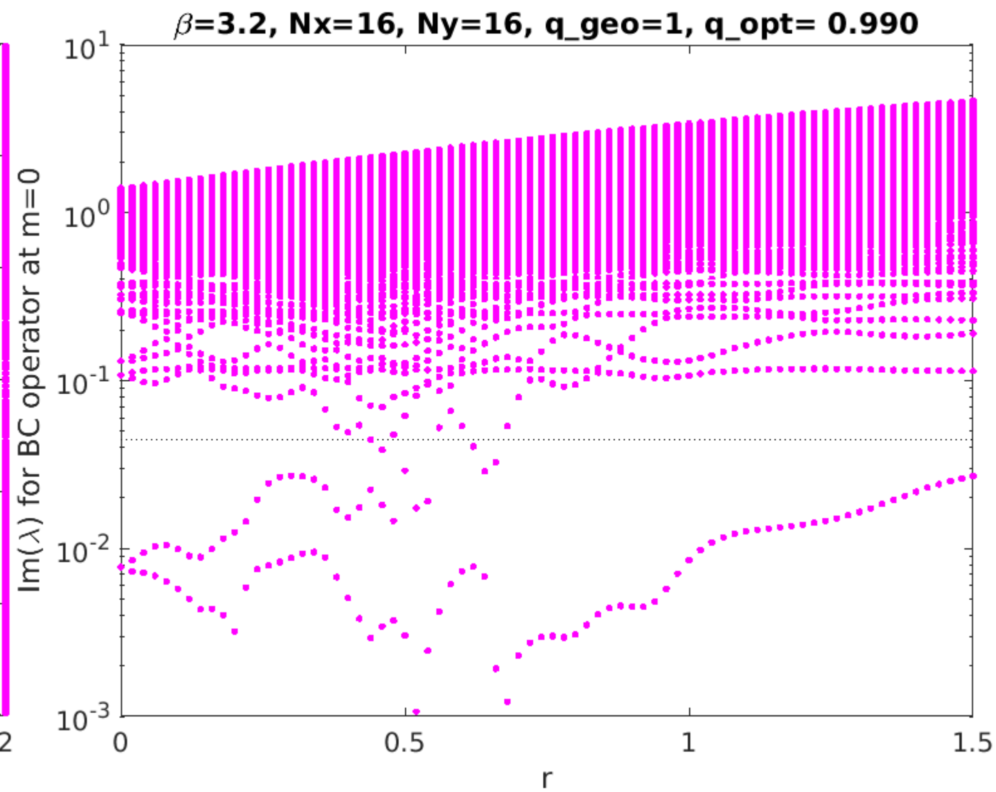
- $2|q|$ would-be zero modes at $r = 1$ (unchanged in 4D), remnant chiral symmetry
- pertinent L/R-eigenmodes of $D_{\rm BC}$ not sensitive to γ_5 (not shown)

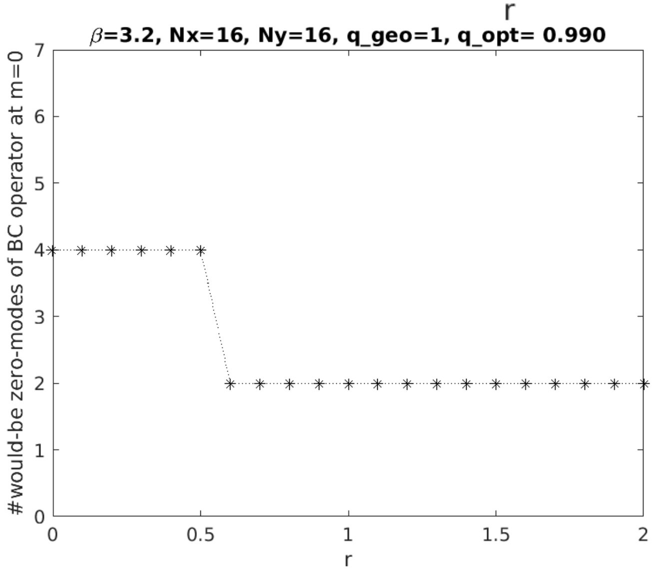
Operator X can be crafted to have $\langle L|X|R\rangle \neq 0$ with L/R-eigenmodes of $D_{\rm BC}$

Options are $X = \frac{1}{2}(C_1 + C_2)^2 \otimes \gamma_5$ and $X = (2C_{\rm sym} - 1) \otimes \gamma_5$ with $C_\mu \equiv \frac{1}{2}\triangle_\mu + 1$

- **Transition $D_{\text{naive}} \to D_{\text{BC}}$ as a function of r on $|q|=1$ configuration**







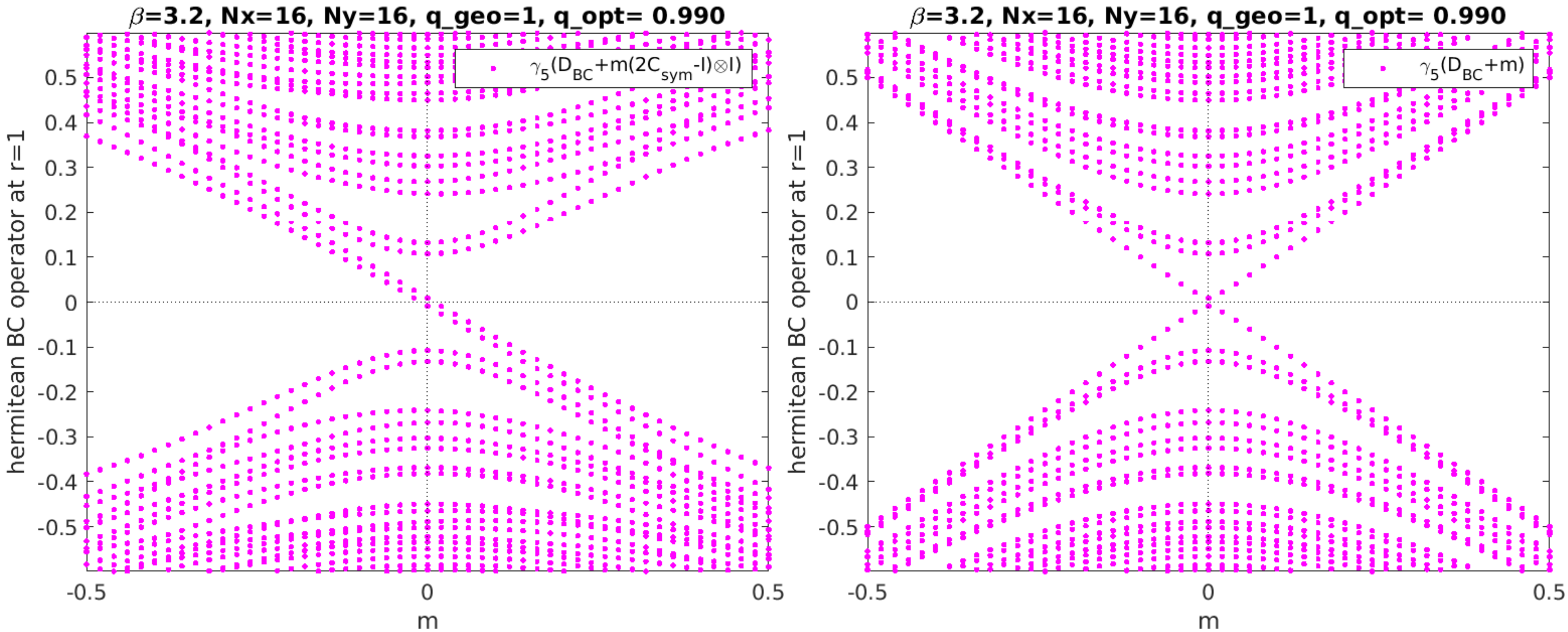
Findings in [2203.15699]:

$\text{Im}(\lambda_{\text{KW}}(r))$ nearly saturatess BC free-field bound

KW species chain in 2D is $4 \to 2$ (transition at $r=\frac{1}{\sqrt{3}}$)

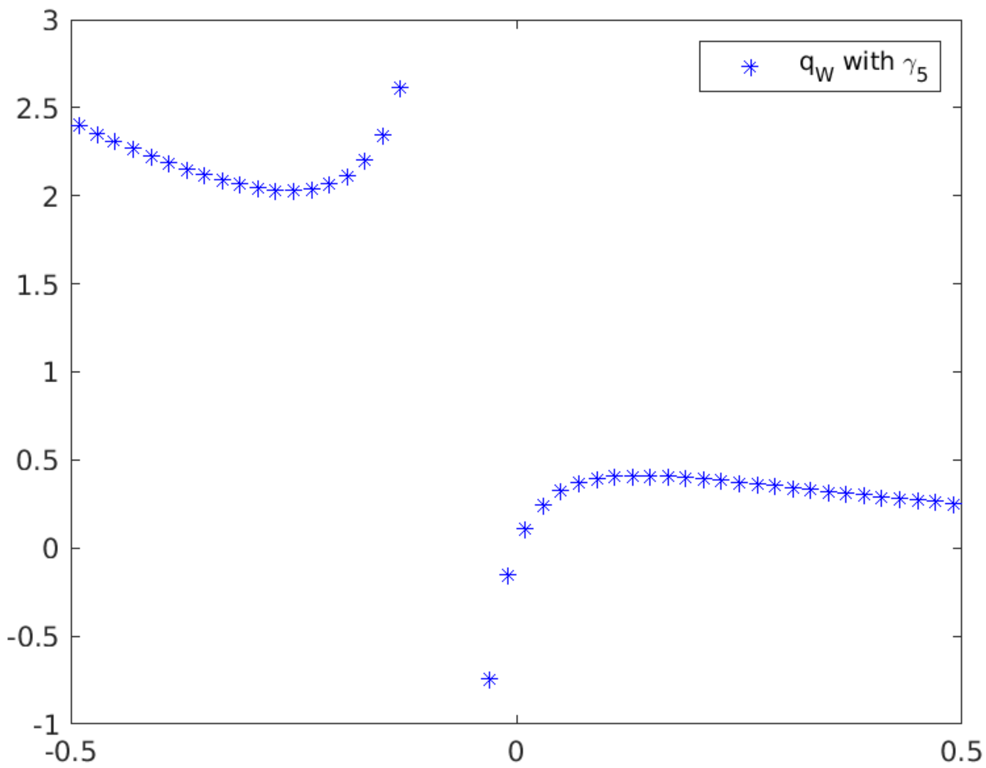
number of would-be zero modes evolves as $4|q| \to 2|q|$

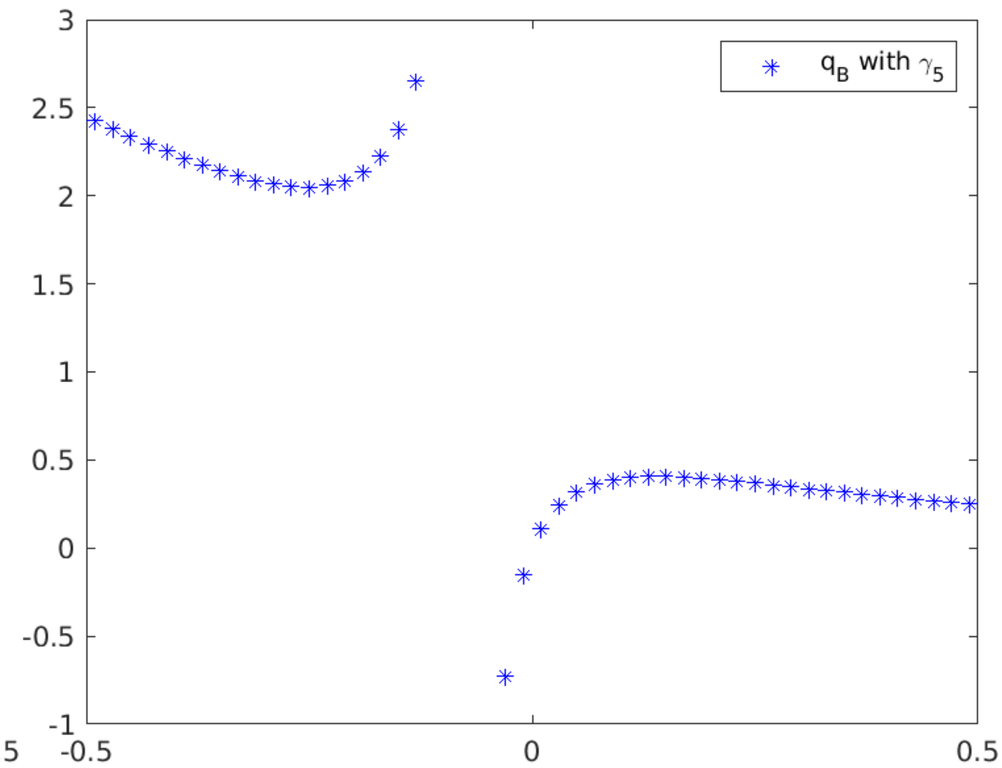
- ## Spectral flow with Borici-Creutz fermions



- eigenvalues of $H_{\rm BC} \equiv \gamma_5(D_{\rm BC} + m[2C_{\rm sym} - 1] \otimes 1)$ versus am
- choice matches $X = [2C_{\rm sym} - 1] \otimes \gamma_5$ being a good chirality operator for $D_{\rm BC}$
- sign of slope for $|\lambda| \ll 1$ reflects chirality (cf. needle down)
- near-degeneracy much worse than for KW fermions (cf. above)
- dull choice $\gamma_5(D_{\rm BC} + m)$ amounts to wrong chirality operator

Topological charge via Wilson/Brillouin fermion



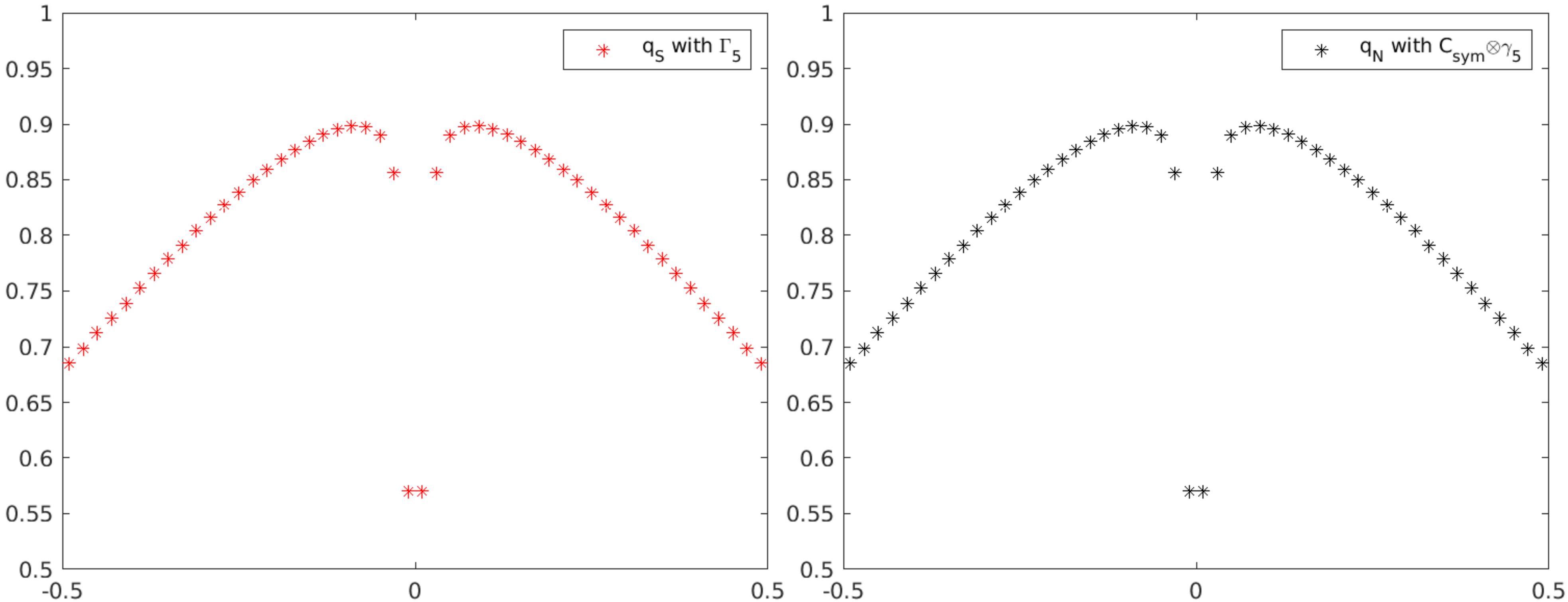


$$q_{\mathrm{W}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{W}}+m)^{-1} I \otimes \gamma_5] \quad , \qquad q_{\mathrm{B}}[U] = -m\,\mathrm{tr}[(D_{\mathrm{B}}+m)^{-1} I \otimes \gamma_5]$$

- apparent pole structure plausible (see App. C of arXiv:2203.15699) from

$$q_{\mathrm{W}} \simeq m \frac{2(2r+m_{\mathrm{crit}})^2}{(m-m_{\mathrm{crit}})(2r+m_{\mathrm{crit}})(4r+2m_{\mathrm{crit}})} = \frac{m}{m-m_{\mathrm{crit}}}$$

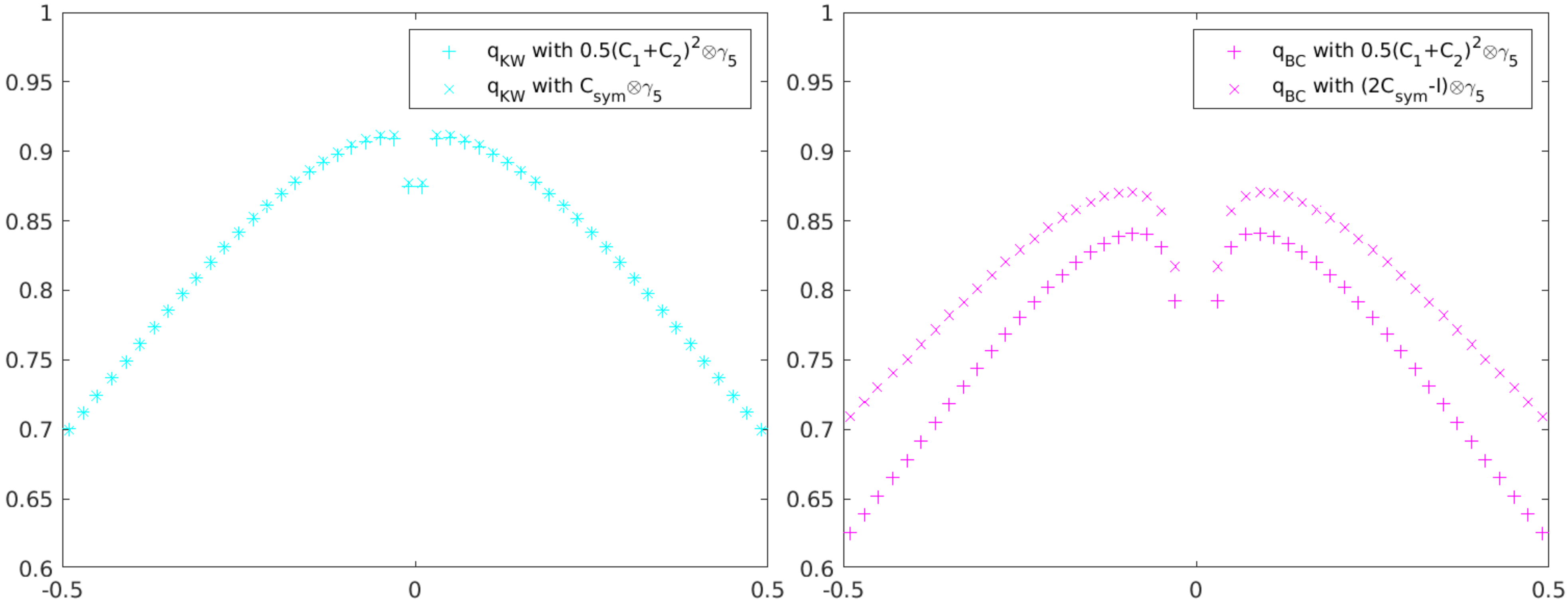
Topological charge via staggered/naive fermion



$$q_\mathrm{S}[U] = -\frac{m}{2}\,\mathrm{tr}[(D_\mathrm{S}+m)^{-1}\Gamma_{50}] \quad , \qquad q_\mathrm{N}[U] = -\frac{m}{4}\,\mathrm{tr}[(D_\mathrm{N}+m)^{-1}C_\mathrm{sym}\otimes\gamma_5]$$

- two-species formulation requires factor $\frac{1}{2}$ [staggered]
- four-species formulation requires factor $\frac{1}{4}$ [naive]
- no additive mass shift with $q_\mathrm{S}[U]$ or $q_\mathrm{N}[U]$ (chiral symmetry)

Topological charge via KW/BC fermion



$$q_{\rm KW}[U] = -\frac{m}{2}\,{\rm tr}[(D_{\rm KW}+m)^{-1}X_{\rm KW}]\,, \quad q_{\rm BC}[U] = -\frac{m}{2}\,{\rm tr}[(D_{\rm BC}+m)^{-1}X_{\rm BC}]$$

$$X_{\rm KW} = \begin{cases} \frac{1}{2}(C_1+C_2)^2 \otimes \gamma_5 \\ C_{\rm sym} \otimes \gamma_5 \end{cases}, \quad X_{\rm BC} = \begin{cases} \frac{1}{2}(C_1+C_2)^2 \otimes \gamma_5 \\ (2C_{\rm sym}-1) \otimes \gamma_5 \end{cases}$$

- two-species formulation requires factor $\frac{1}{2}$ [KW and BC]
- no additive mass shift for both KW an BC (chiral symmetry)

Summary ("part 1")

KW and BC fermions have matrix size like Wilson fermions ($N_c 4 N_x N_y N_z N_t$ in 4D).
KW and BC fermions have exact chiral symmetry (eigenvalues on imaginary axis).
KW and BC fermions have condition number less favorable than staggered fermions.

They have $2|q|$ would-be zero modes with *opposite chiralities* (like staggered) in 2D.
This figure remains $2|q|$ in 4D (while staggered fermions have $4|q|$ in 4D).

With an appropriate chirality operator
all lattice fermions perceive correct $q_{\mathrm{top}}[U]$.

Dispersion relations of 4D fermion actions

- **Dispersion relation of naive fermion**

$$D_{\text{nai}} \;=\; \sum_\mu \gamma_\mu \nabla_\mu + m \;=\; \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m$$

$$G_{\text{nai}} \;=\; \frac{-\mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m}{(\mathrm{i}\sum_\rho \gamma_\rho \bar{p}_\rho + m)(-\mathrm{i}\sum_\sigma \gamma_\sigma \bar{p}_\sigma + m)} = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m}{\bar{p}^2 + m^2}$$

$$aE = \sqrt{\mathrm{asinh}\Big(\sum_i \sin^2(ap_i) + (am)^2\Big)}$$

- **Dispersion relation of Wilson fermion**

At $r = 1$ the DR for Wilson fermion simplifies to

$$2\cosh(aE)\Big[d + am - \sum_i \cos(ap_i)\Big] = 1 + \sum_i \sin^2(ap_i) + \Big[d + am - \sum_i \cos(ap_i)\Big]^2$$

which one solves for $aE > 0$ by means of $\mathrm{acosh}(x) = \ln(x + \sqrt{x^2 - 1})$ for $x > 1$.

- **Dispersion relation of KW fermion**

$$G_{\mathrm{KW}} = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu - \mathrm{i}\frac{ar}{2}\gamma_d \sum_{i=1}^{d-1} \hat{p}_i^2 + m}{\sum_{i=1}^{d-1} \bar{p}_i^2 + (\bar{p}_d + \frac{ar}{2}\sum_{i=1}^{d-1} \hat{p}_i^2)^2 + m^2}$$

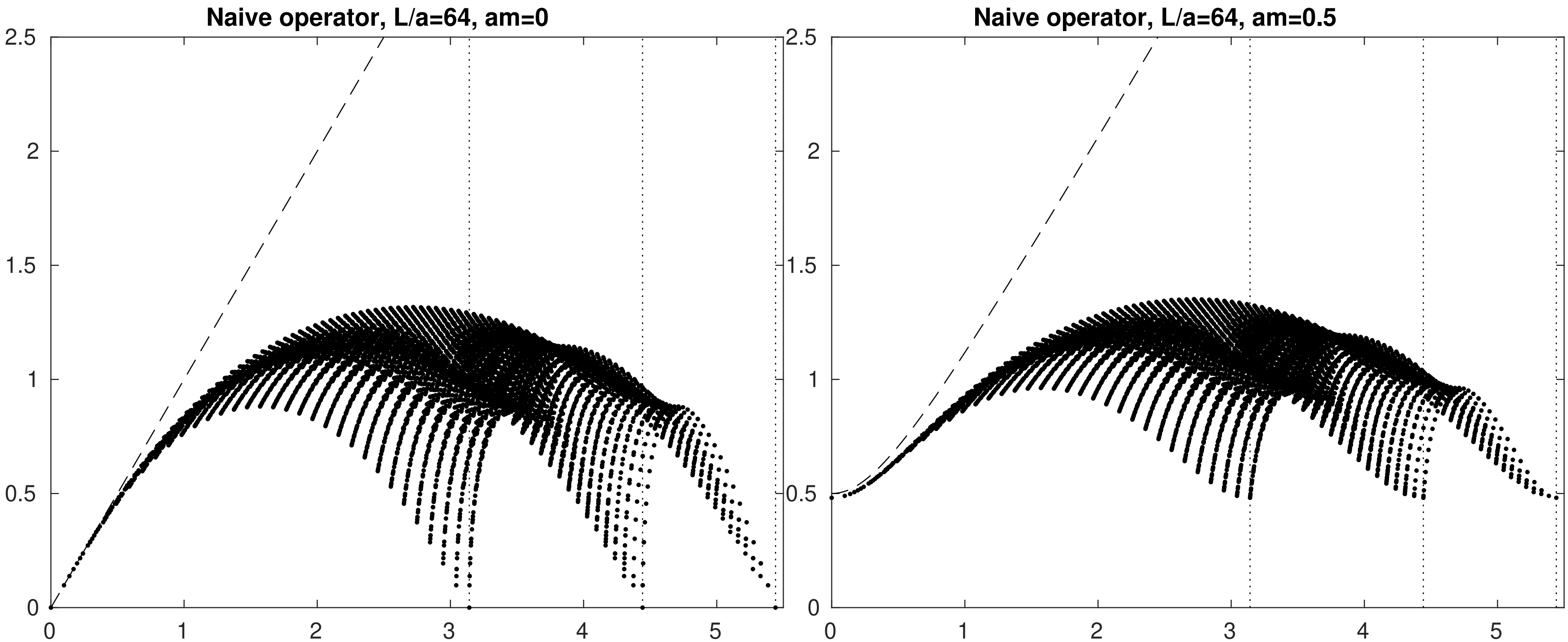
$$\sinh(aE) = \mathrm{i}r \sum_{i=1}^{d-1} \{1 - \cos(ap_i)\} \pm \sqrt{\sum_{i=1}^{d-1} \sin^2(ap_i) + (am)^2}$$

- **Dispersion relation of BC fermion**

$$G_{\mathrm{BC}} = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu - \mathrm{i}\frac{ar}{2}\sum_\mu \gamma'_\mu \hat{p}_\mu^2 + m}{\sum_\lambda \bar{p}_\lambda^2 - ar\sum_\lambda \bar{p}_\lambda \hat{p}_\lambda^2 + \frac{a^2r^2}{4}\sum_\lambda \hat{p}_\lambda^4 + \frac{2ar}{d}\sum_{\rho,\sigma} \bar{p}_\rho \hat{p}_\sigma^2 + m^2}$$

$$\begin{aligned} 0 \;&=\; \sum_i \Big[\sin(ap_i) - r\{1 - \cos(ap_i)\}\Big]^2 + \Big[\mathrm{i}\sinh(aE) - r\{1 - \cosh(aE)\}\Big]^2 \\ &+\; \frac{4r}{d}\sum_{i,j} \sin(ap_i)\{1 - \cos(ap_j)\} + \frac{4\mathrm{i}r}{d}\sinh(aE)\sum_j \{1 - \cos(ap_j)\} \\ &+\; \frac{4r}{d}\sum_i \sin(ap_i)\{1 - \cosh(aE)\} + \frac{4\mathrm{i}r}{d}\sinh(aE)\{1 - \cosh(aE)\} + (am)^2 \end{aligned}$$

• Dispersion relation of naive fermion

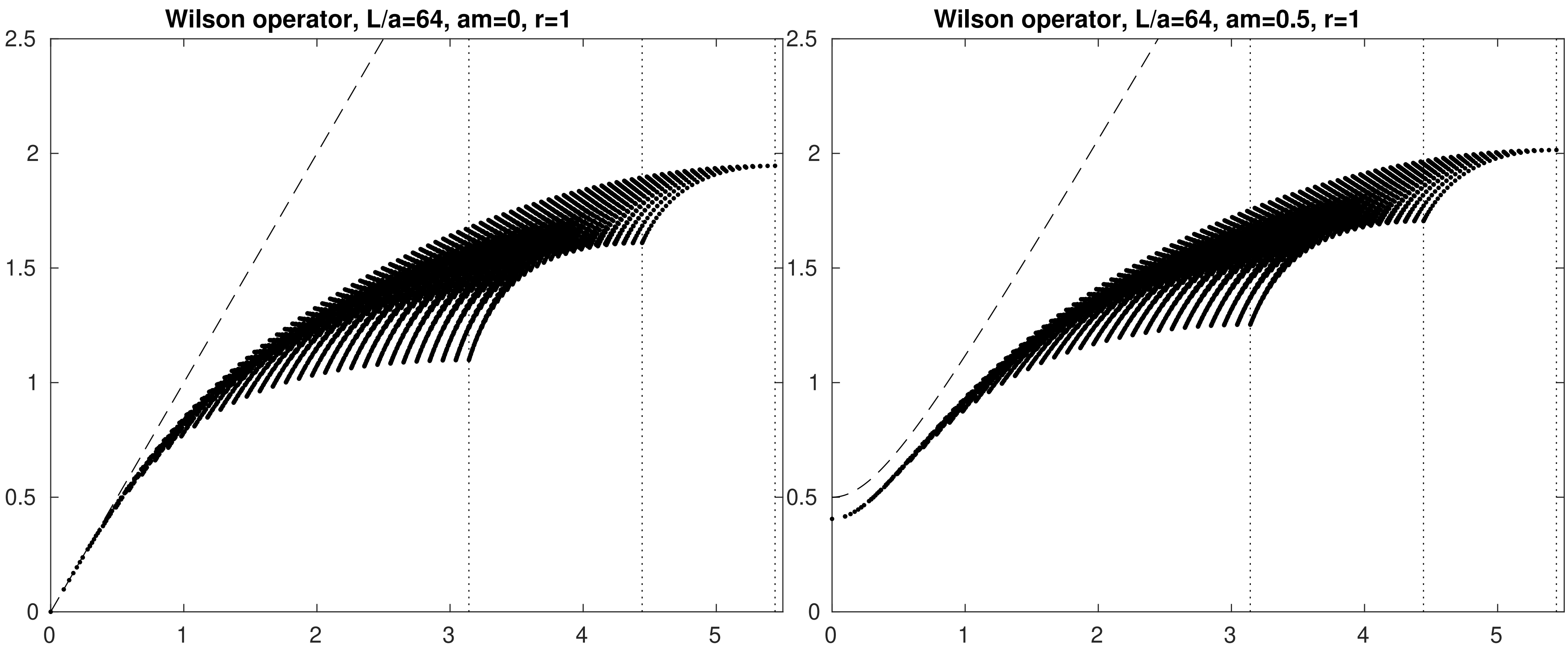


Momentum configurations with $|\vec{p}| = 0, \pi, \sqrt{2}\pi, \sqrt{3}\pi, 2\pi$ realize 1,4,6,4,1 species.

Useful feature for heavy-quark physics: cut-off effects at $|a\vec{p}| = 0$ are quadratic:

$$aE = am\left\{1 - \frac{1}{6}(am)^2 + \frac{3}{40}(am)^4 + O((am)^6)\right\}$$

• Dispersion relation of Wilson fermion

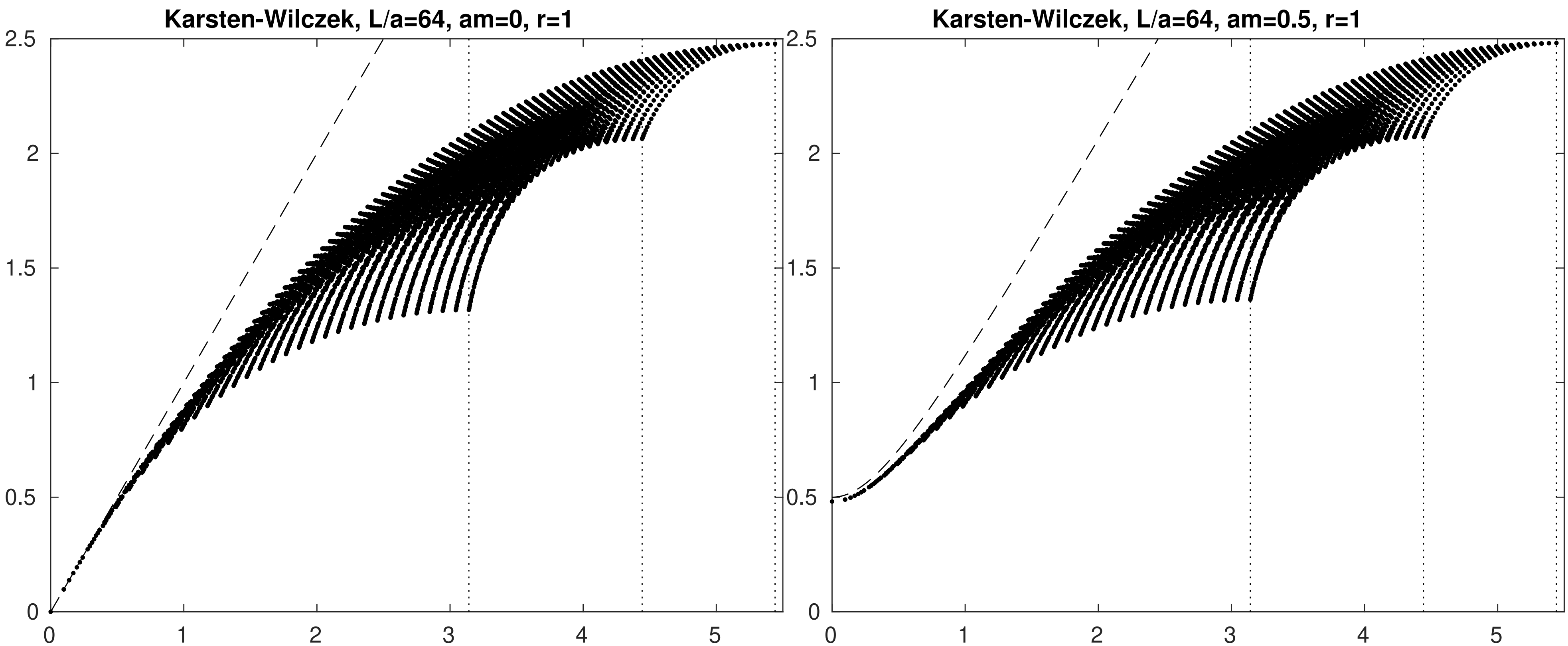


Inconvenient feature for heavy-quark physics: cut-off effects at $|a\vec{p}| = 0$ are linear:

$$aE = am\Big\{1 - \frac{r}{2}am + \frac{3r^2-1}{6}(am)^2 - \frac{[5r^2-3]r}{8}(am)^3 + O((am)^4)\Big\}$$

Non-zero momenta up to $|a\vec{p}| = O(1)$ seem affected by common mismatch im am.

- ## Dispersion relation of KW fermion

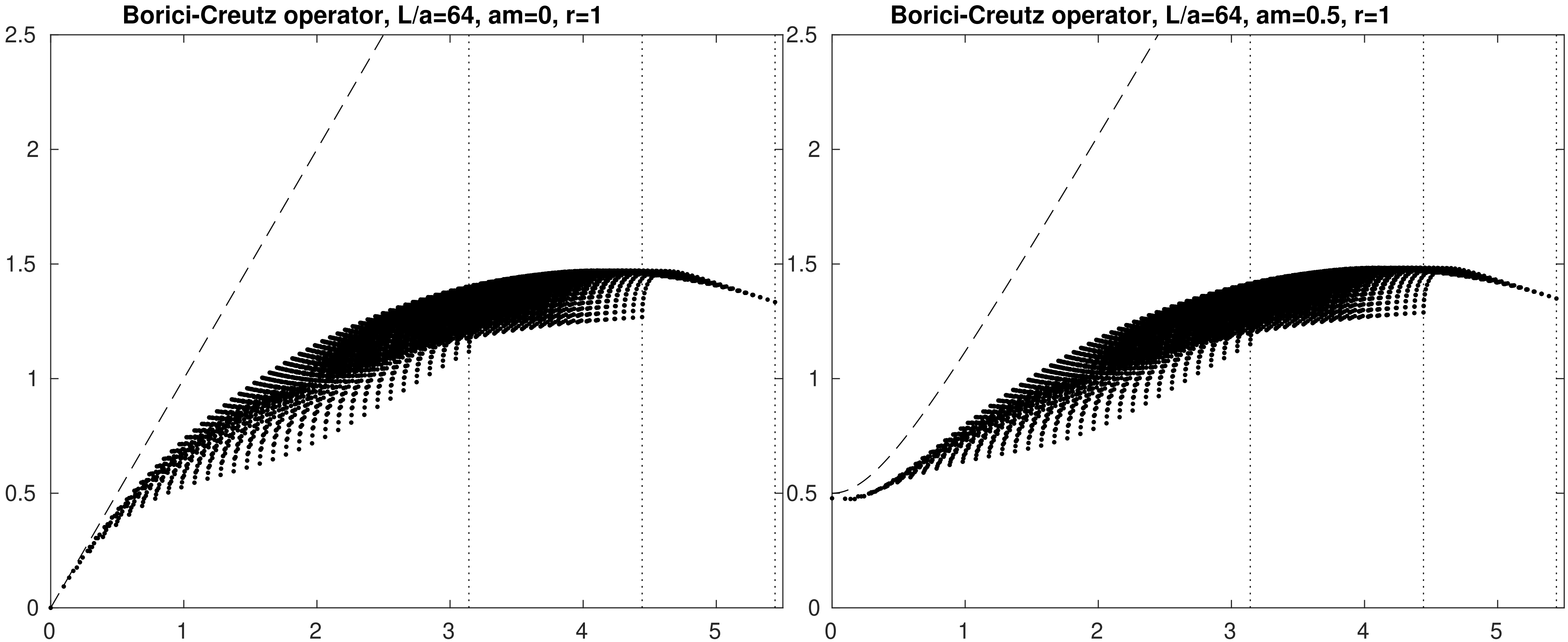


Feature for heavy-quark physics: cut-off effects at $|a\vec{p}| = 0$ are quadratic:

$$aE = am\left\{1 - \frac{1}{6}(am)^2 + \frac{3}{40}(am)^4 + O((am)^6)\right\}$$

Non-zero momenta up to $|a\vec{p}| = O(1)$ seem well represented [arXiv:2003.10803].

• Dispersion relation of BC fermion



Feature for heavy-quark physics: cut-off effects in real/imag part are linear/quadratic

$$aE = am\left\{1+\frac{\mathrm{i}r}{4}am-\frac{3r^2+16}{96}(am)^2+\frac{\mathrm{i}[r^3-3r]}{16}(am)^3-\frac{805r^4-960r^2-768}{10240}(am)^4\right\}$$

Unclear/questionable features for tiny momenta at $am \simeq 0.5$ [arXiv:2003.10803].

Summary ("part 2")

Naive fermions have (for any am) very nice dispersion relation (despite doublers).

Wilson fermions at $am = 0$ have very nice dispersion relation (an no doublers).

Wilson fermions at $am > 0$ have large $O(a)$ cutoff effects (visible at $a\vec{p} = \vec{0}$).

KW fermions have (for any am) very nice dispersion relation.

KW fermions have same perturbative expansion in am as Wilson fermions at $a\vec{p} = \vec{0}$.

BC fermions have not-so-nice features at any am and $a\vec{p}$.

BC fermions have factors of i in perturbative expansion in am at $a\vec{p} = \vec{0}$.

KW fermions are interesting for heavy-quark physics.

BC fermions likely to cause troubles in physics applications.

Schwinger Model: QED in 2D with any N_f

SM at $N_f = 0$ simulated with Metropolis/overrelax/instanton-hit/parity-hit.
Topological charge autocorrelation time is $O(1)$ at any β [arXiv:1203.2560].

Wilson gauge action per site:

$s_{\mathrm{wil}}(x) = 1 - \mathrm{Re}(U(x)) = 1 - \cos(\theta(x))$

Plaquette at position $x = (x_1, x_2)$:

$U(x) = U_1(x) U_2(x+e_1) U_1^\dagger(x+e_2) U_2^\dagger(x)$

$U(x) = \exp(\mathrm{i}\theta(x))$

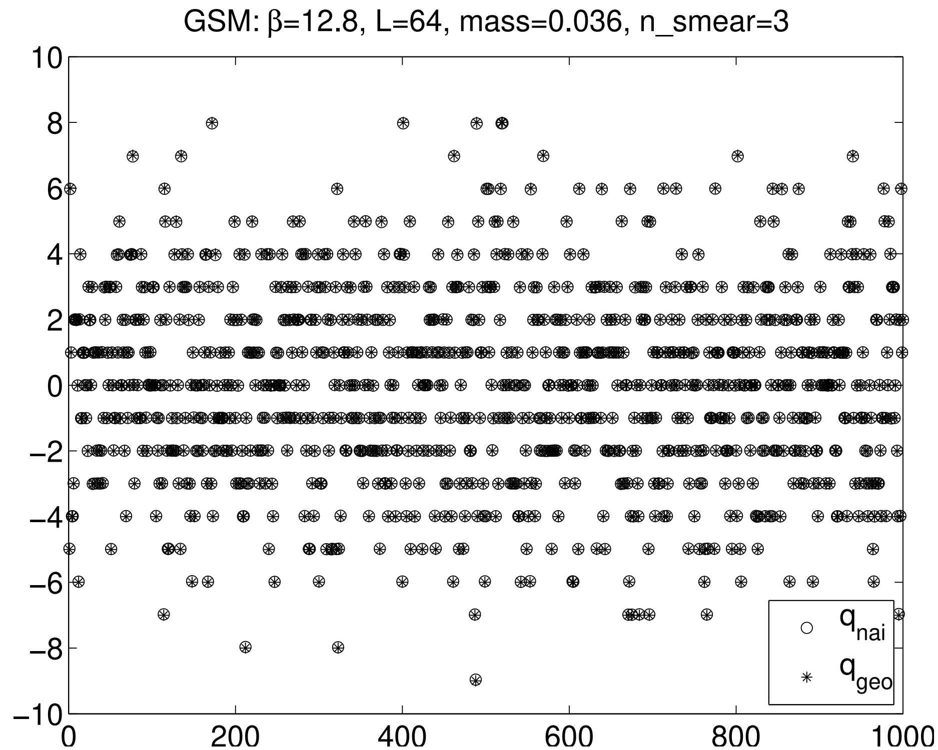
Two gluonic topological charges:

$q_{\mathrm{raw}}^{(n)} = \sum \sin(\theta^{(n)}(x))/(2\pi) \in \mathbf{R}$ ("fth/Z")

$q_{\mathrm{geo}}^{(n)} = \sum \theta^{(n)}(x)/(2\pi) \in \mathbf{Z}$ ("geometric")

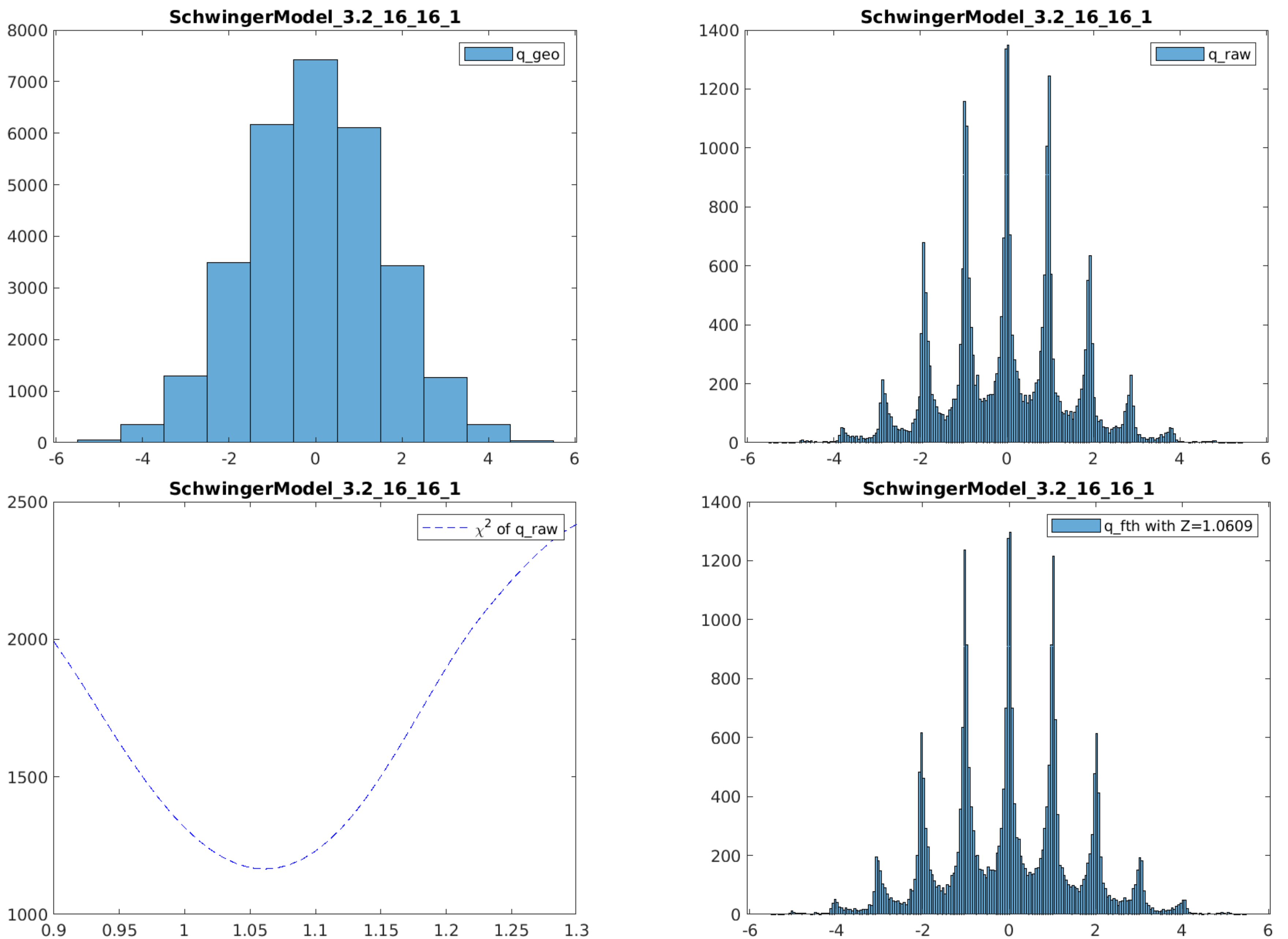
$\theta^{(n)}$ plaquette angle after n smearings

$q_{\mathrm{opt}}(x)$ is clover-leaf version of $q_{\mathrm{raw}}(x)$



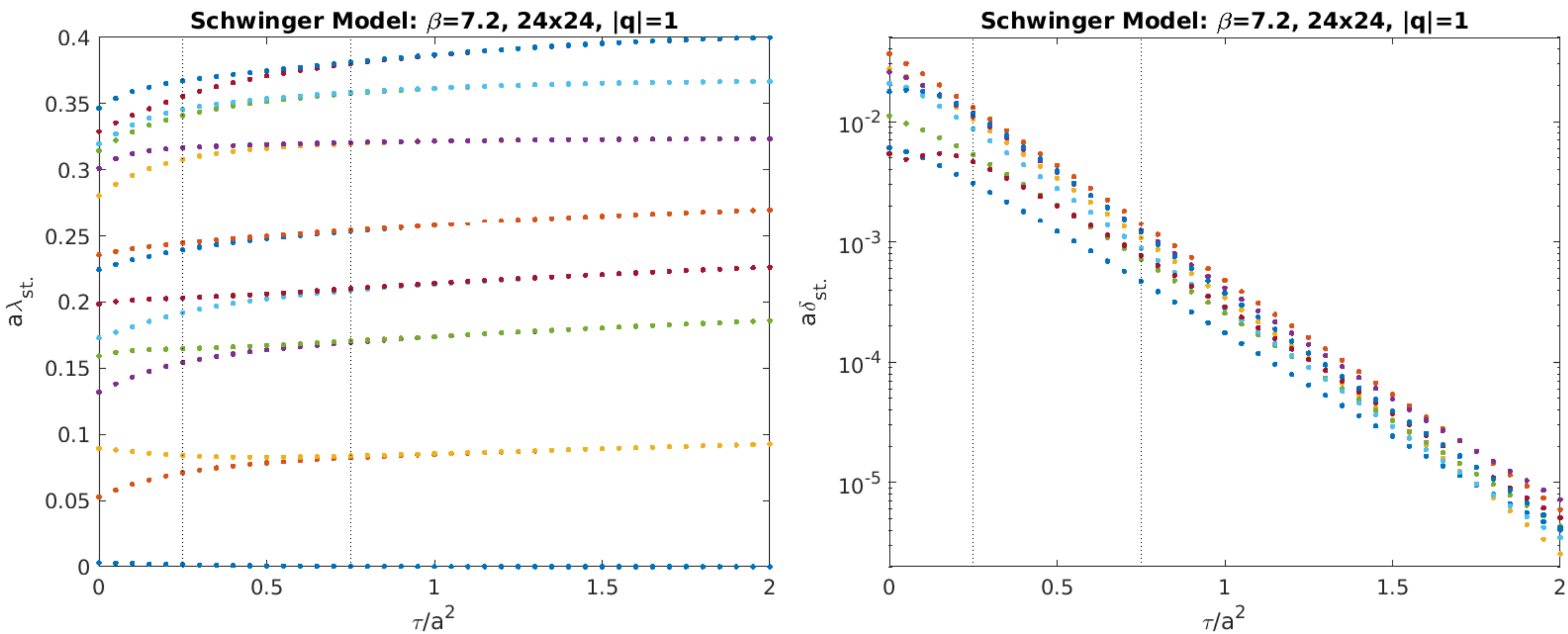
Operators use $n = 0, 1, 3$ steps of $\rho = 0.25$ stout-smearing [Morningstar Peardon 2003].

Schwinger Model: topological charge distributions



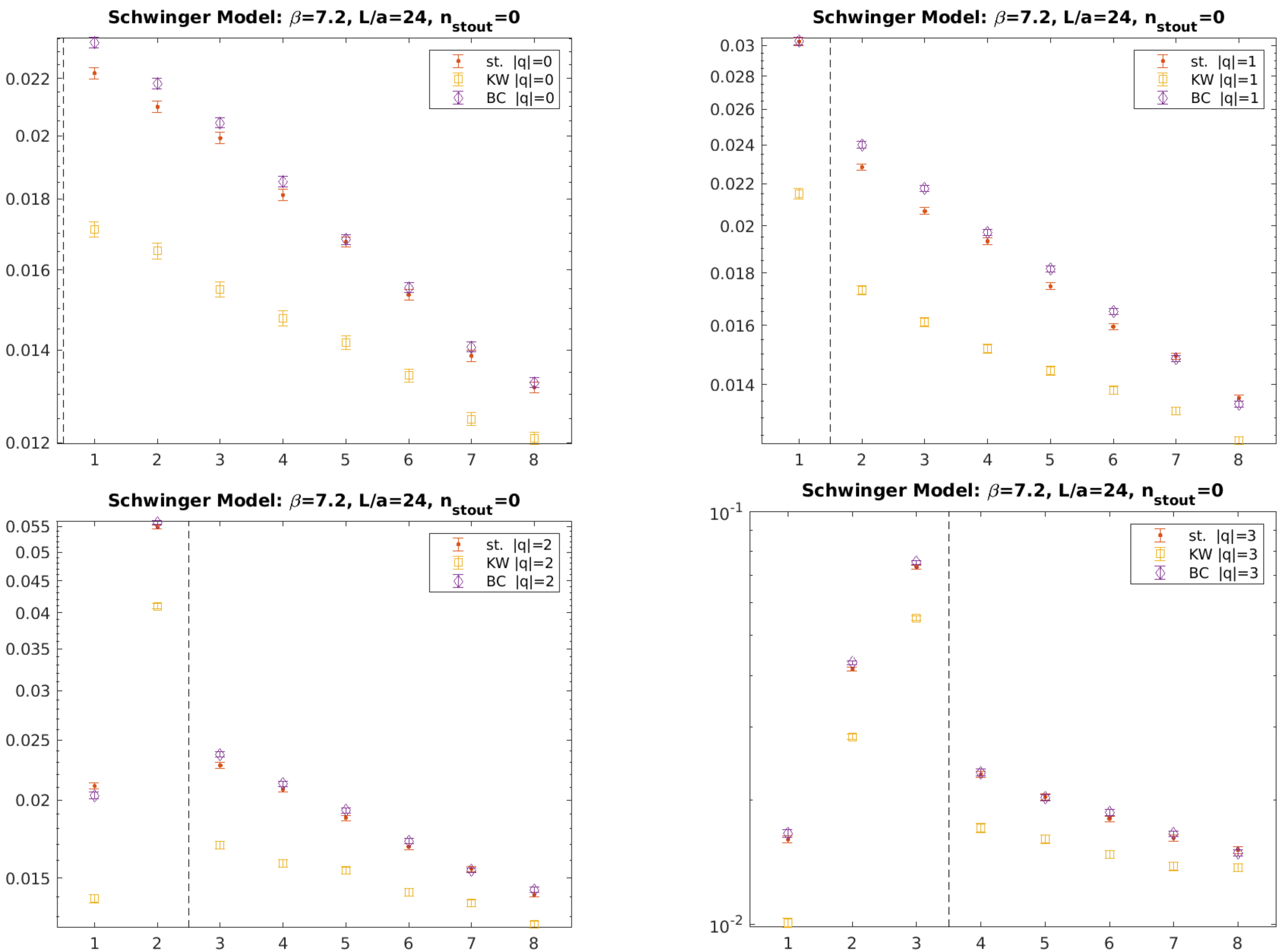
Taste splittings: $a\delta_{\rm stag}$ under gradient flow

Eigenvalues $a\lambda_1, ..., a\lambda_{15}$ of $D_{\rm stag}/\mathrm{i}$ on a $|q|=1$ configuration at $(\beta, L/a) = (7.2, 24)$ versus gradient flow time τ/a^2. Note that λ_1 pairs with $-\lambda_1$, while $\lambda_2 \simeq \lambda_3$ pair, and so on. Splittings defined with proper pairing: $\delta_1 = 2\lambda_1$, $\delta_2 = \lambda_3 - \lambda_2$, ... for $|q| = 1$.

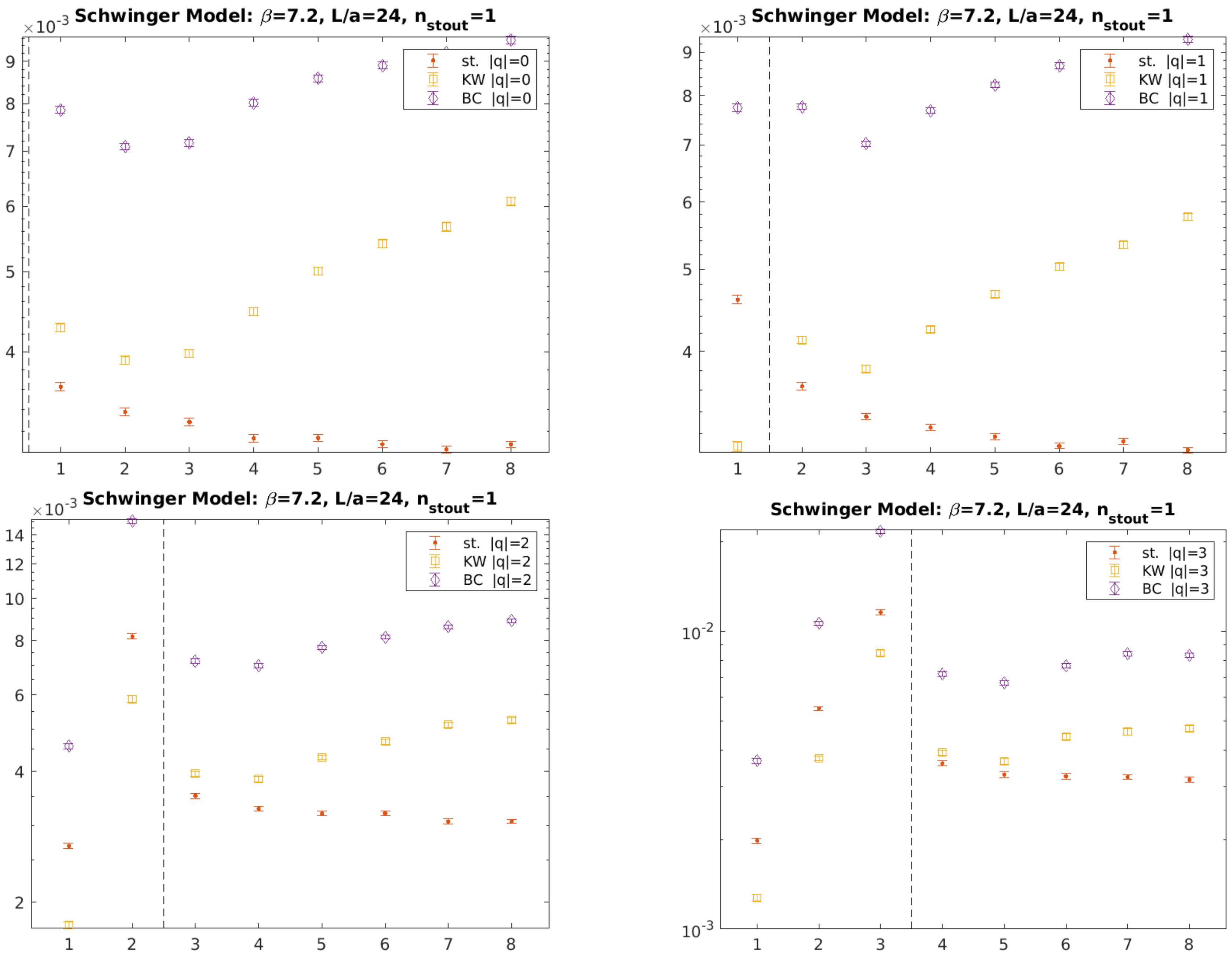


Main investigation carried out with $n = 0, 1, 3$ steps of $\rho = 0.25$ stout smearing, paramount to $\tau/a^2 = 0, 0.25, 0.75$ (up to small discretization effects).

Taste splittings: $a\delta$ on "central ensemble" with $n_{\rm stout} = 0$



Taste splittings: $a\delta$ on "central ensemble" with $n_{\text{stout}} = 1$



Schwinger Model: ensemble details

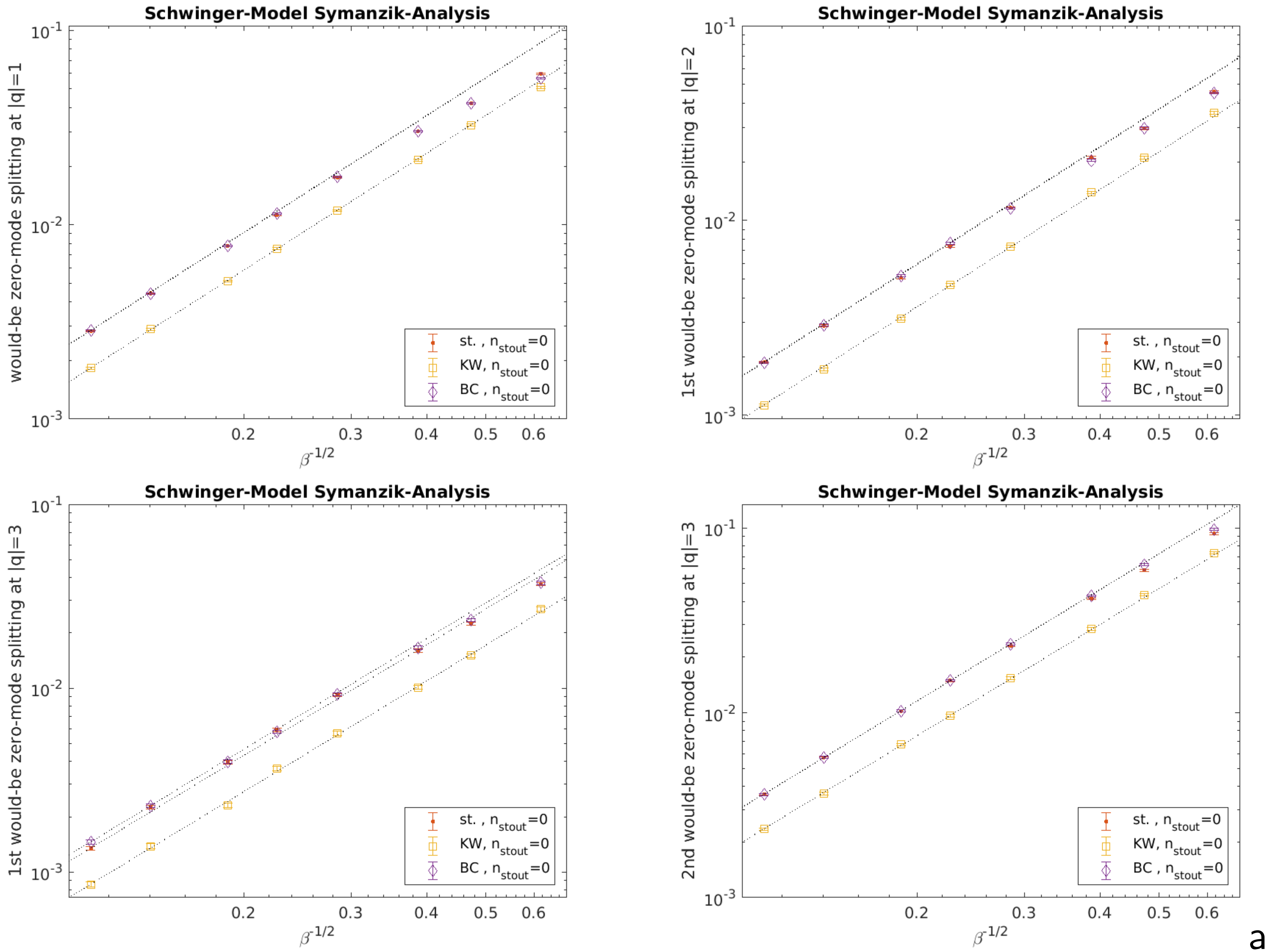
β	3.2	5.0	7.2	12.8	20.0	28.8	51.2	80.0
L/a	16	20	24	32	40	48	64	80
n_{stout}	0,1,3	0,1,3	0,1,3	0,1,3	0,1,3	0,1,3	0,1,3	0,1,3
$s^{(0)}_{\text{wils}}$	0.17625	0.10662	0.07230	0.03989	0.02533	0.01752	0.00981	0.00627
$p_{\text{inst.hit}}$	0.750(2)	0.737(2)	0.729(2)	0.725(2)	0.726(2)	0.722(2)	0.721(2)	0.721(1)

Table 1: Ensembles used in the "cut-off effect" study; they implement constant physical volume through $(L/a)^2/\beta = 80$. For every choice of $(\beta, L/a)$ three ensembles of 10 000 configurations are generated, to be used with 0, 1 or 3 steps of $\rho = 0.25$ stout smearing, respectively. The analytic result $s^{(0)}_{\text{wils}}$ is taken from [Elser:2001pe].

β	7.2	7.2	7.2	7.2	7.2
L/a	16	20	24	32	40
n_{stout}	1	1	1	1	1
$p_{\text{inst.hit}}$	0.597(2)	0.677(2)	0.729(2)	0.799(2)	0.838(2)

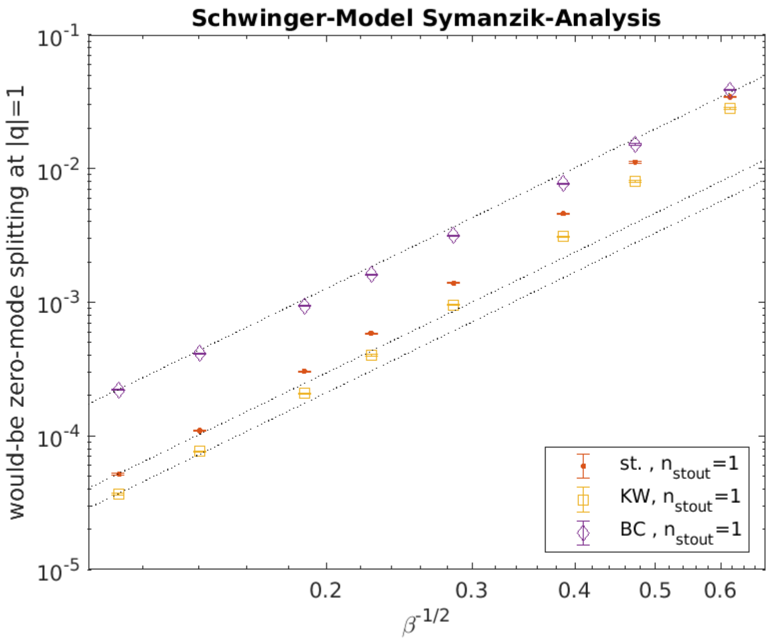
Table 2: Ensembles used in the "finite volume" study; each one contains 10 000 configurations and is used after a single step of $\rho = 0.25$ stout smearing. Also the acceptance ratio of the instanton hit update at the respective $(\beta, L/a)$ is given.

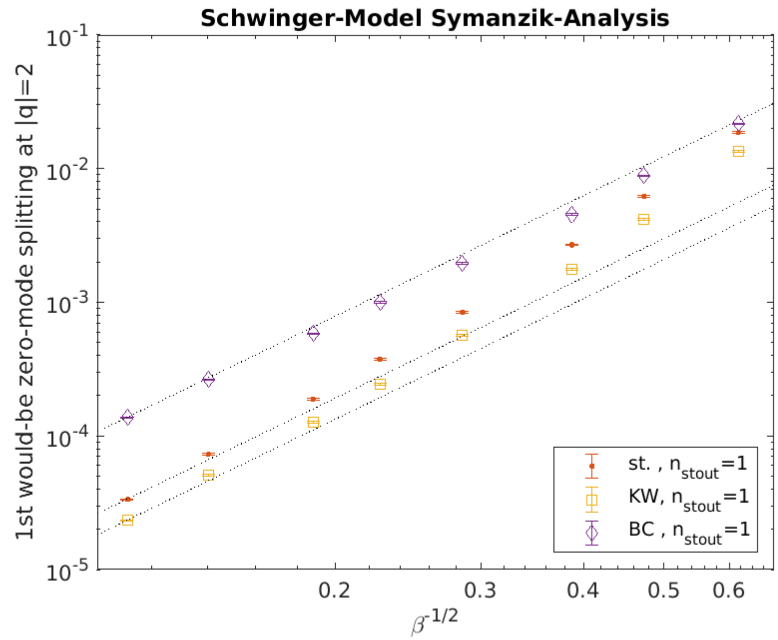
Taste splittings: would-be zero mode scaling for $n_{\text{stout}} = 0$

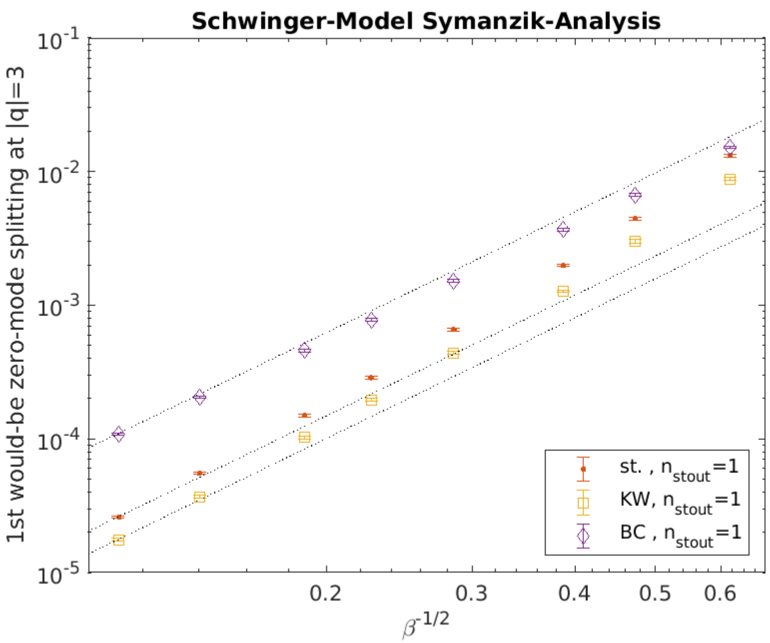


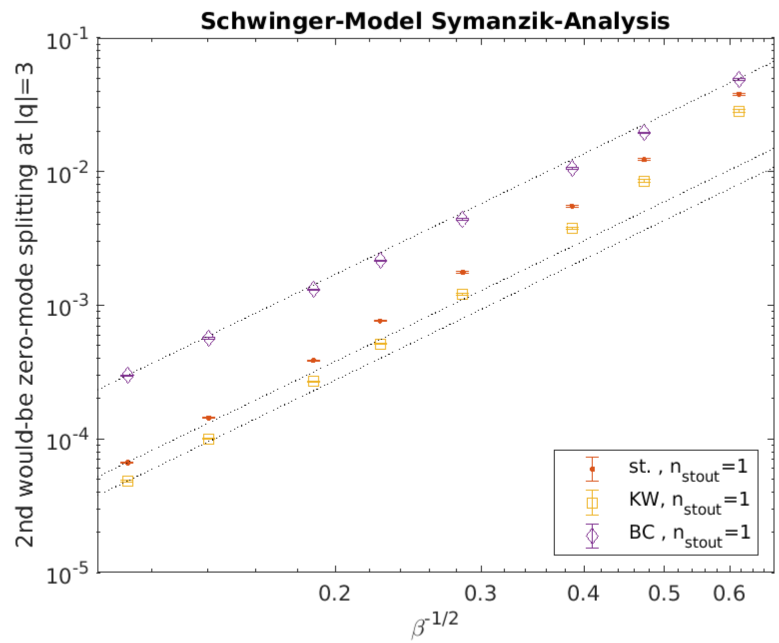
all a^2

Taste splittings: would-be zero mode scaling for $n_{\text{stout}} = 1$



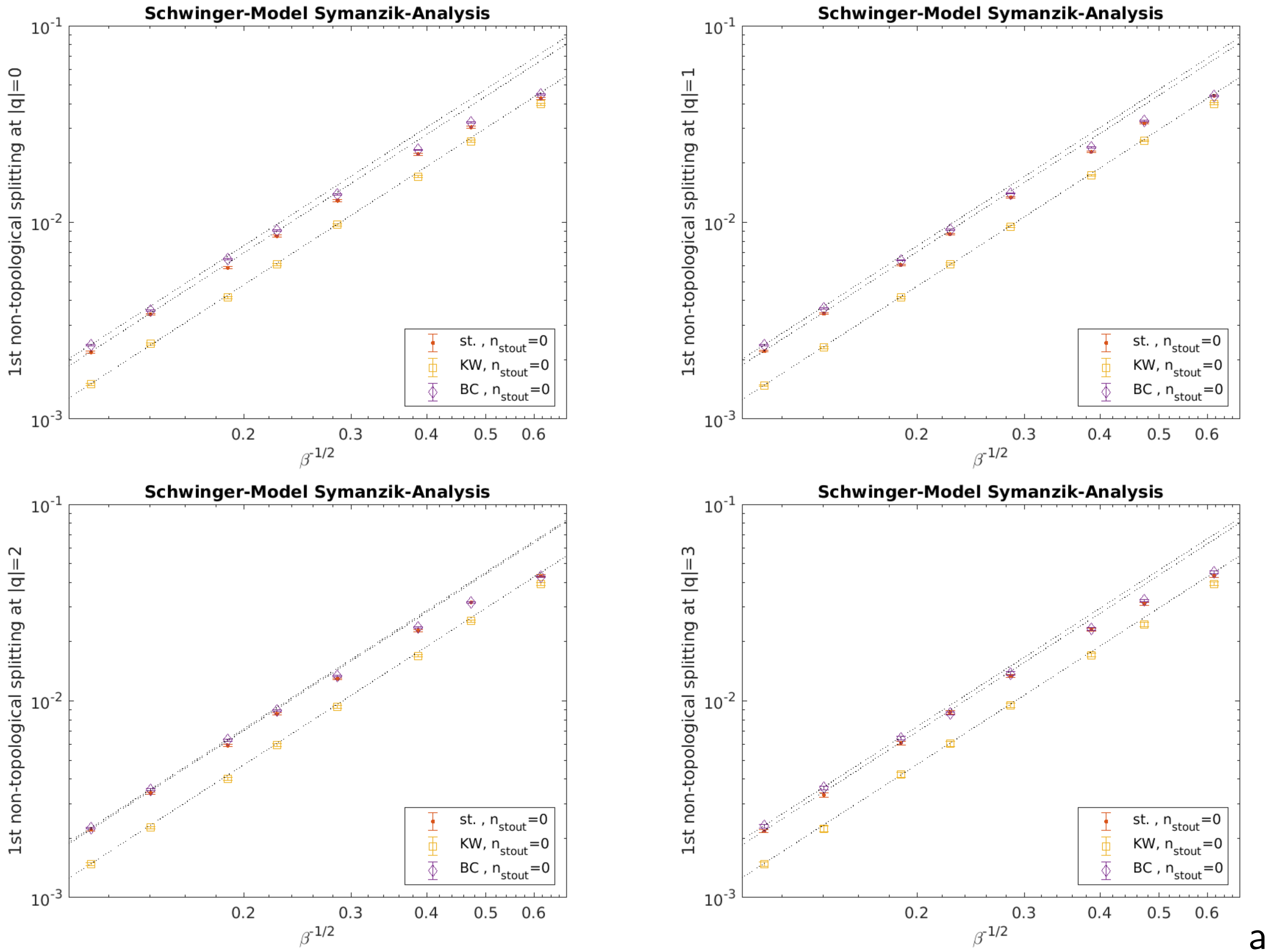






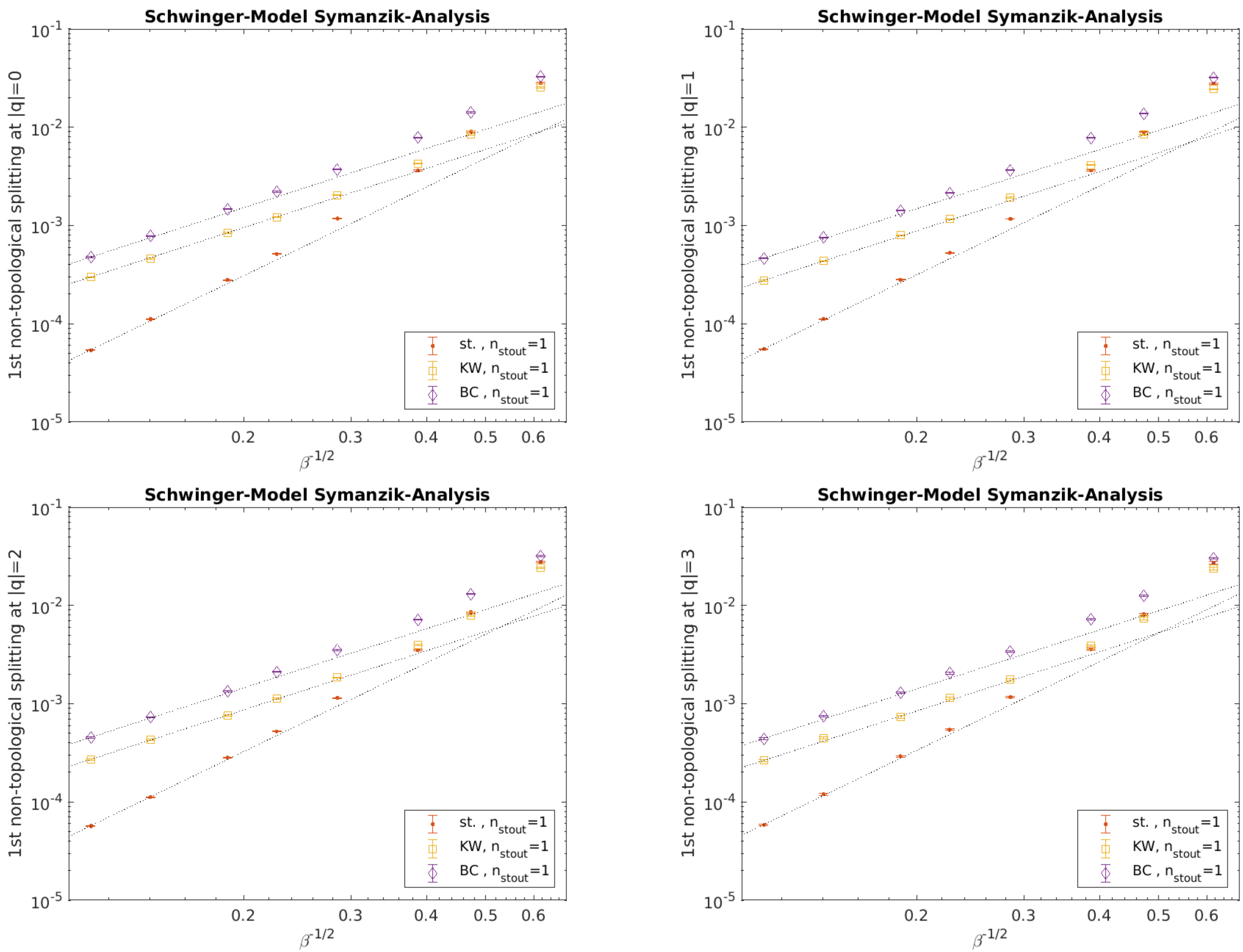
all a^3

Taste splittings: non-topological mode scaling for $n_{\mathrm{stout}} = 0$



all a^2

Taste splittings: non-topological mode scaling for $n_{\text{stout}} = 1$



$a^{2,3}$

Summary ("part 3")

In 2D all of $D_{\rm stag}, D_{\rm KW}, D_{\rm BC}$ are minimally doubled, since on any $U_{|q|=1}$ one finds

- 2 would-be zero modes of $D_{\rm stag}$ with *opposite chiralities* (invisible to ϵ)
- 2 would-be zero modes of $D_{\rm KW}$ with *opposite chiralities* (invisible to γ_5)
- 2 would-be zero modes of $D_{\rm BC}$ with *opposite chiralities* (invisible to γ_5)

but in all cases appropriate chirality operators can be defined [arXiv:2203.15699].

In quenched SM intra-taste splittings $\delta_{\rm stag}, \delta_{\rm KW}, \delta_{\rm BC}$ are measured over a wide β-range, considering would-be zero modes and non-topological modes separately.

	$n_{\rm stout}=0$		$n_{\rm stout}=1,3$	
	wbz	ntm	wbz	ntm
$\delta_{\rm stag}\propto a^{\#}$	1	1	2	2
$\delta_{\rm KW}\propto a^{\#}$	1	1	2	1
$\delta_{\rm BC}\propto a^{\#}$	1	1	2	1

Power $\#$ in Symanzik scaling law $\delta\propto a^{\#}$ depends on would-be zero mode (wbz) versus non-topological mode (ntm) and/or smearing level (which is disturbing).

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Flavored mass/lifting terms

$$C_\mu(x,y) = \frac{1}{2}[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}] = \frac{1}{2}a^2\triangle_\mu(x,y) + \delta_{x,y}$$

$M_S = 1$	Scalar	(0-link)	1
$M_V = \sum_{\text{sym}} C_\mu$	Vector	(1-link)	$\frac{1}{4}[C_1+C_2+C_3+C_4]$
$M_T = \sum_{\text{sym}}\sum_{\text{per}} C_\mu C_\nu$	Tensor	(2-link)	see detail
$M_A = \sum_{\text{sym}}\sum_{\text{per}} C_\mu C_\nu C_\rho$	Axial	(3-link)	see detail
$M_P = \sum_{\text{per}} C_\mu C_\nu C_\rho C_\sigma$	Pseudo	(4-link)	$\frac{1}{24}[C_1C_2C_3C_4 + \text{perms}] = C_{\text{sym}}$

detail T: $\frac{1}{12}[C_1C_2 + \text{perm}] + \ldots + \frac{1}{12}[C_3C_4 + \text{perm}]$
6 square brackets [...] each of which contains 2 terms

deatil A: $\frac{1}{24}[C_2C_3C_4 + \text{perms}] + \ldots$
4 square brackets [...] each of which contains 6 terms

Brillouin fermion: dim=5 term (Laplacian) is $M_V+M_T+M_A+M_P$
Brillouin fermion: dim=4 term is $\sum_\mu \gamma_\mu \nabla_\mu^{\text{iso}}$ instead of $\sum_\mu \gamma_\mu \nabla_\mu^{\text{std}}$

Creutz, Kimura, Misumi (10, 11)

Review of staggered mass/lifting terms

The $(\gamma_\mu \otimes 1)$ and $(\gamma_5 \otimes 1)$ "taste singlet" operators are defined by

$$\begin{aligned}
\Gamma_\mu(x,y) \;\equiv\; \Gamma_{\mu 0}(x,y) &= \frac{1}{2}\eta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat{\mu},y} + U_\mu^\dagger(x-\hat{\mu})\delta_{x-\hat{\mu},y}\Big] \\
\Gamma_5(x,y) \;\equiv\; \Gamma_{50}(x,y) &= \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}}\Gamma_1\Gamma_2\Gamma_3\Gamma_4
\end{aligned}$$

and the $(1 \otimes \xi_\mu)$ and $(1 \otimes \xi_5)$ "spinor singlet" operators are defined by

$$\begin{aligned}
\Xi_\mu(x,y) \;\equiv\; \Gamma_{0\mu}(x,y) &= \frac{1}{2}\zeta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat{\mu},y} + U_\mu^\dagger(x-\hat{\mu})\delta_{x-\hat{\mu},y}\Big] \\
\Xi_5(x,y) \;\equiv\; \Gamma_{05}(x,y) &= \frac{1}{4!}\sum_{\mathrm{perm}} \epsilon_{\mathrm{perm}}\Xi_1\Xi_2\Xi_3\Xi_4
\end{aligned}$$

with the consequence that both Γ_{50} and Γ_{05} are 4-hop operators. Furthermore, the latter two operators relate to each other by a simple Γ_{55} operation (from left or right).

Acceptable mass terms are proportional to $(1 \otimes 1)$ or $(1 \otimes \xi_5)$ or possibly $(1 \otimes \xi_\mu \xi_\nu)$.

Adams species lifting

In practice it is advantageous to introduce the commutators in spinor and taste space

$$\Gamma_{\mu\nu}(x,y) \equiv \frac{\mathrm{i}}{2}\Big(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\Big) \qquad \longleftrightarrow \qquad \gamma_{\mu\nu}\otimes 1$$

$$\Xi_{\mu\nu}(x,y) \equiv \frac{\mathrm{i}}{2}\Big(\Xi_\mu\Xi_\nu - \Xi_\nu\Xi_\mu\Big) \qquad \longleftrightarrow \qquad 1\otimes\xi_{\mu\nu}$$

respectively, with $\gamma_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$ a.k.a. $\sigma_{\mu\nu}$ and $\xi_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\xi_\mu, \xi_\nu]$, which yields

$$\Gamma_{50}(x,y) \simeq -\frac{1}{6}\Big(\Gamma_{12}\Gamma_{34} - \Gamma_{13}\Gamma_{24} + \Gamma_{14}\Gamma_{23} + \Gamma_{23}\Gamma_{14} - \Gamma_{24}\Gamma_{13} + \Gamma_{34}\Gamma_{12}\Big)$$

$$\Gamma_{05}(x,y) \simeq -\frac{1}{6}\Big(\Xi_{12}\Xi_{34} - \Xi_{13}\Xi_{24} + \Xi_{14}\Xi_{23} + \Xi_{23}\Xi_{14} - \Xi_{24}\Xi_{13} + \Xi_{34}\Xi_{12}\Big)$$

Adams: Promote 2 of the 4 tastes of D_{stag} to doublers by $\Gamma_{05} = \Xi_5 \simeq (1\otimes\xi_5)$.
Key observation is that the remaining 2 species share *one chirality*.

Corollary: It makes sense to apply overlap construction to shifted kernel $D_{\mathrm{A}} - \rho$.
The resulting operator will be doubled, and the two species will be chiral.