Vacuum energy in the effective field theory of general relativity with a scalar field

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A consistency condition of general relativity as an effective field theory in Minkowskian background uniquely fixes the value of the cosmological constant. In two-loop calculations, including the interaction of gravitons with matter fields, it has been shown that this value of the cosmological constant leads to vanishing vacuum energy, under the assumption that the energy-momentum tensor of the gravitational field is given by the pseudotensor of Landau-Lifshitz's classic textbook. Here, we demonstrate that this result also holds when the self-interaction of a scalar field is taken into account. That is, our two-loop-order calculation suggests that in an effective field theory of metric and scalar fields, one arrives at a consistent theory with massless gravitons if the cosmological constant is fixed from the condition of vanishing vacuum energy. Vice versa, imposing the consistency condition in Minkowskian background leads to a vanishing vacuum energy.

It is widely accepted that at low energies, the physics of fundamental interactions is adequately described by effective field theory (EFT) [1–3]. Gravitation can also be included in this formalism by considering the most general effective Lagrangian of metric and matter fields [4–6], which is invariant under all underlying symmetries including the gauge symmetry of massless spin-two particles [7]. It is well-known that for non-vanishing values of the cosmological constant term, Λ, a quantum field theoretical treatment of general relativity with the metric field presented as the Minkowskian background plus the graviton field poses a problem due to the graviton propagator possessing a pole that corresponds to a massive ghost mode [7]. Setting Λ equal to zero does not improve the situation as radiative corrections lead to the re-emergence of the problem with the massive ghost [8]. However, one can represent the cosmological constant as a power series in $\hbar$ with coefficients chosen to yield self-consistent EFT results to all orders in the loop expansion [8]. Thus, the consistency requirement of a perturbative EFT in flat Minkowski background uniquely fixes the cosmological constant as a function of other parameters of the effective Lagrangian. This also necessarily requires considering an EFT in a curved background field if any other value of the cosmological constant is assumed. In this case, the mass term of the graviton can be removed at classical level by imposing the equations of motion with respect to the background graviton field [9]. To the best of our knowledge, a systematic study beyond tree level has not been done due to the lack of an EFT on non-Minkowskian background.

In Refs. [10, 11] it was found by performing two-loop calculations for gravitational interactions only that the vacuum energy vanishes exactly for the value of the cosmological constant corresponding to that of Ref. [8], i.e., for the value that guarantees the vanishing of the graviton mass and of the vacuum expectation value of the graviton field. Provided that this result holds to all orders, also when including interactions among matter fields, the uniquely fixed value of the cosmological constant yielding a self-consistent perturbative EFT on Minkowkian background could also be interpreted as a consequence of imposing the condition of vanishing vacuum energy.

In this work we consider a simple EFT of general relativity with metric and a scalar matter fields. We perform two-loop-order calculations for the value of the cosmological constant leading to a consistent EFT in order to see if the vacuum energy vanishes when self-interactions of the scalar field are taken into account. Notice that while there does not exist a commonly accepted expression of the energy-momentum tensor for the gravitational field (see, e.g., Refs. [12–16]), Refs. [10, 11] and also the current work use the definition of the energy-momentum pseudotensor (EMPT) and of the full four-momentum of the matter and gravitational fields as given in the classic textbook [17].

In the framework of the EFT of general relativity, the action is given via the most general effective Lagrangian of gravitational and matter fields, which is invariant under general coordinate transformations and other underlying symmetries of the considered model,

$$S = \int d^4x \sqrt{-g}\, \{\mathcal{L}_{\mathrm{gr}}(g) + \mathcal{L}_{\mathrm{m}}(g,\psi)\} = \int d^4x \sqrt{-g}\, \left\{ \frac{2}{\kappa^2}(R - 2\Lambda) + \mathcal{L}_{\mathrm{gr,ho}}(g) + \mathcal{L}_{\mathrm{m}}(g,\psi) \right\} = S_{\mathrm{gr}}(g) + S_{\mathrm{m}}(g,\psi). \quad (1)$$

Here, $\kappa^2 = 32\pi G$, with Newton's constant $G = 6.70881 \times 10^{-39}$ GeV^{-2}, Λ is the cosmological constant, ψ and $g^{\mu\nu}$ denote the matter and metric fields, respectively, $g = \det g^{\mu\nu}$, and R is the scalar curvature. An infinite number of gravitational self-interactions involving higher orders in derivatives are contained in $\mathcal{L}_{\mathrm{gr,ho}}(g)$ and $\mathcal{L}_{\mathrm{m}}(g,\psi)$ is the effective Lagrangian of the matter fields interacting with the metric and the vielbein tetrad fields. Experimental

evidence suggests that the contributions of $\mathcal{L}_{\mathrm{gr,ho}}(g)$ and of non-renormalizable interactions of $\mathcal{L}_{\mathrm{m}}(g, \psi)$ to physical quantities are heavily suppressed.

The action of the matter part of the model considered here includes an infinite number of terms, of which we show below only those contributing in our specific calculations:

$$S_{\mathrm{m}} = \int d^4x \sqrt{-g} \left\{ \frac{g^{\mu\nu}}{2} \partial_\mu H \partial_\nu H - \frac{m^2}{2} H^2 + \frac{g}{3!} H^3 + fH \right\}, \tag{2}$$

where H is the scalar field. The low-energy EFT is obtained by representing the metric field as the sum of the Minkowskian background and the quantum fields [18]

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\lambda h^{\lambda\nu} - \kappa^3 h^\mu_\lambda h^\lambda_\sigma h^{\sigma\nu} + \cdots , \end{aligned} \tag{3}$$

and we calculate physical quantities perturbatively by expanding in κ and other coupling constants treating them independently.

The energy-momentum tensor of the matter fields coupled to the gravitational field is given by

$$T_{\mathrm{m}}^{\mu\nu}(g, \psi) = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}}. \tag{4}$$

The energy-momentum tensor corresponding to Eq. (2) has the form

$$T_m^{\mu\nu} = \partial_\mu H \partial_\nu H - g^{\mu\nu} \left\{ \frac{g^{\alpha\beta}}{2} \partial_\alpha H \partial_\beta H - \frac{m^2}{2} H^2 + \frac{g}{3!} H^3 + fH \right\}. \tag{5}$$

The pseudotensor of the gravitational field (alone) is given as

$$T_{\mathrm{gr}}^{\mu\nu}(g) = \frac{4}{\kappa^2} \Lambda g^{\mu\nu} + T_{LL}^{\mu\nu}(g), \tag{6}$$

where $T_{LL}^{\mu\nu}(g)$ is defined via [17]

$$\begin{aligned} (-g)T_{LL}^{\mu\nu}(g) &= \frac{2}{\kappa^2} \left(\frac{1}{8} g^{\lambda\sigma} g^{\mu\nu} g_{\alpha\gamma} g_{\beta\delta} \, \mathfrak{g}^{\alpha\gamma}{}_{,\sigma} \, \mathfrak{g}^{\beta\delta}{}_{,\lambda} - \frac{1}{4} g^{\mu\lambda} g^{\nu\sigma} g_{\alpha,\gamma} g_{\beta\delta} \, \mathfrak{g}^{\alpha\gamma}{}_{,\sigma} \, \mathfrak{g}^{\beta\delta}{}_{,\lambda} - \frac{1}{4} g^{\lambda\sigma} g^{\mu\nu} g_{\beta\alpha} g_{\gamma\delta} \, \mathfrak{g}^{\alpha\gamma}{}_{,\sigma} \, \mathfrak{g}^{\beta\delta}{}_{,\lambda} \right. \\ &+ \frac{1}{2} g^{\mu\lambda} g^{\nu\sigma} g_{\beta\alpha} g_{\gamma\delta} \, \mathfrak{g}^{\alpha\gamma}{}_{,\sigma} \, \mathfrak{g}^{\beta\delta}{}_{,\lambda} + g^{\beta\alpha} g_{\lambda\sigma} \, \mathfrak{g}^{\nu\sigma}{}_{,\alpha} \, \mathfrak{g}^{\mu\lambda}{}_{,\beta} + \frac{1}{2} g^{\mu\nu} g_{\lambda\sigma} \, \mathfrak{g}^{\lambda\beta}{}_{,\alpha} \, \mathfrak{g}^{\alpha\sigma}{}_{,\beta} \\ &\left. - g^{\mu\lambda} g_{\sigma\beta} \, \mathfrak{g}^{\nu\beta}{}_{,\alpha} \, \mathfrak{g}^{\sigma\alpha}{}_{,\lambda} - g^{\nu\lambda} g_{\sigma\beta} \, \mathfrak{g}^{\mu\beta}{}_{,\alpha} \, \mathfrak{g}^{\sigma\alpha}{}_{,\lambda} + \mathfrak{g}^{\lambda\sigma}{}_{,\sigma} \, \mathfrak{g}^{\mu\nu}{}_{,\lambda} - \mathfrak{g}^{\mu\lambda}{}_{,\lambda} \, \mathfrak{g}^{\nu\sigma}{}_{,\sigma} \right), \end{aligned} \tag{7}$$

with $\mathfrak{g}^{\mu\nu} = \sqrt{-g} \, g^{\mu\nu}$ and $\mathfrak{g}^{\mu\nu}{}_{,\lambda} = \partial \mathfrak{g}^{\mu\nu}/\partial x^\lambda$.

The conserved full four-momentum of the matter and the gravitational fields is defined via the full EMPT $T^{\mu\nu} = T_{\mathrm{m}}^{\mu\nu}(g, \psi) + T_{\mathrm{gr}}^{\mu\nu}(g)$ as [17]

$$P^\mu = \int (-g) \, T^{\mu\nu} dS_\nu, \tag{8}$$

where the integration covers any hypersurface containing the whole three-dimensional space. Thus, for vanishing vacuum expectation value of $(-g) \, T^{\mu\nu}$, the energy of the vacuum will be zero. This expectation value can be represented via the following path integral:

$$\langle 0 | (-g) T^{\mu\nu} | 0 \rangle = \int \mathcal{D}g \, \mathcal{D}\psi \, (-g) \left[T_{\mathrm{gr}}^{\mu\nu}(g) + T_{\mathrm{m}}^{\mu\nu}(g, \psi) \right] \exp \left\{ i \int d^4x \sqrt{-g} \left[\mathcal{L}(g, \psi) + \mathcal{L}_{\mathrm{GF}} \right] \right\}, \tag{9}$$

where

$$\mathcal{L}_{\mathrm{GF}} = \xi \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h^\nu_\nu \right) \left(\partial^\beta h_{\mu\beta} - \frac{1}{2} \partial_\mu h^\alpha_\alpha \right) \tag{10}$$

is the gauge-fixing term with parameter ξ, and the integration measure includes the Faddeev-Popov determinant.
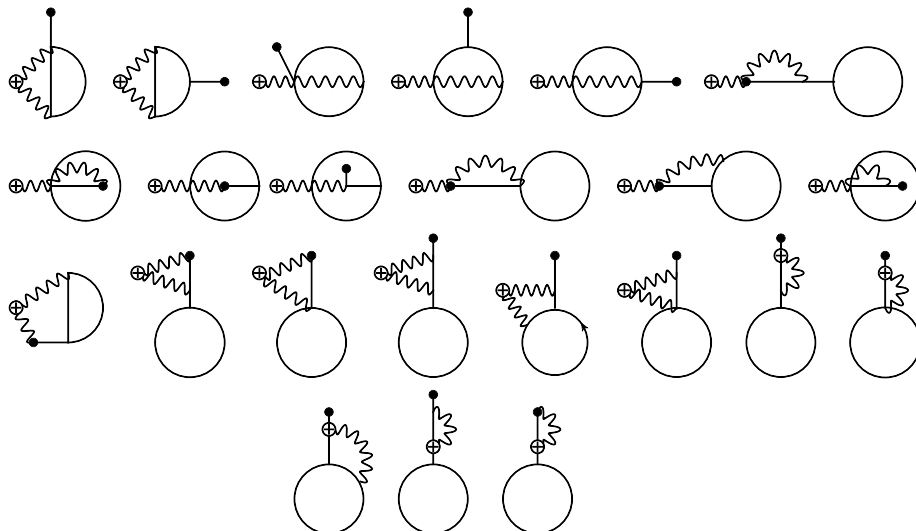
FIG. 1: Topologies of two-loop diagrams whose sums (but not the separate diagrams) lead to the same contribution to the cosmological constant for the graviton tadpole and the matrix element of $(-g) T^{\mu\nu}$. The wavy and solid lines correspond to gravitons and scalars, respectively. The filled dots stand for vertices generated by the fH term of the Lagrangian. The cross corresponds to either external graviton line or a $(-g) T^{\mu\nu}$ insertion.

According to Ref. [8] the cosmological constant Λ is uniquely fixed by the condition that the vacuum expectation value of the quantum field $h_{\mu\nu}$ vanishes, and this also removes the graviton mass from the dressed propagator as a result of a Ward identity [8]. To study the implications for the vacuum energy at two loop order we represent Λ as

$$\Lambda = \sum_{i=0}^{\infty} \hbar^i \Lambda_i \,. \tag{11}$$

The vacuum expectation value of $(-g)T^{\mu\nu}$ at tree order vanishes for $\Lambda_0 = 0$, and this choice also removes the graviton mass from the propagator at tree level. It is a straightforward consequence of Eq. (4) that the same value of Λ_1 cancels the one-loop contribution to the vacuum expectation value of the graviton field $h_{\mu\nu}$ and the vacuum energy. The first non-trivial result is obtained at two-loop order. Calculations taking into account only gravitational interaction have been done in Refs. [10, 11]. In this work we consider the order-fg two-loop contributions to the vacuum expectation values of $(-g) T^{\mu\nu}$ and of the graviton field $h_{\mu\nu}$, and we find that the same value of Λ_2 leads to an exact cancelation of both of them.[1] Again, it is a trivial consequence of Eq. (4) that separate diagrams with the same structure, where the vertices with external graviton and with $(-g) T^{\mu\nu}$ insertion couple only to scalar lines, give equal Λ_2-contributions. On the other hand, only the sums of diagrams of the topologies shown in Fig. 1 lead to identical expressions when imposing the two conditions. That is, we first fix Λ by imposing on the corresponding diagrams the condition that the expectation value of $h_{\mu\nu}$ vanishes, and next by imposing on corresponding diagrams the condition that the vacuum expectation value of $(-g) T^{\mu\nu}$ vanishes. We find that individual diagrams contributing to a given topology in Fig. 1 yield different contributions to the cosmological constant Λ_2 in the two cases, but considering the sums of the two sets of diagrams results in the same value of the cosmological constant.

A self-consistent EFT should lead to finite physical quantities after renormalizing (an infinite number of) parameters of the effective Lagrangian. Therefore, it is mandatory that the unique value of the cosmological constant that defines the perturbative EFT of the Standard Model, coupled to gravitons on a Minkowskian flat background, leads to a finite expression of the energy (density) of the vacuum to all orders of perturbation theory. Based on the two-loop order results including the one of the current work, we expect that this finite value should be at least of three-loop-order, and thus very small. Moreover, it seems natural that an adequate theory formulated in the language of mathematics should assign zero energy to the vacuum state [21]. Therefore turning the argument around we expect that by demanding

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[1] In calculations we used the program FeynCalc [19, 20].

that the vacuum energy should be vanishing to all orders we obtain a self-consistent perturbative low-energy EFT of matter and gravitational fields on a flat Minkowskian background.

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