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Two-step and explosive synchronization in frequency-weighted Kuramoto model

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ABSTRACT

We explore the dynamics of interacting phase oscillators in the generalized Kuramoto model with frequency-weighted couplings, focusing on the interplay of frequency distribution and network topology on the nature of transition to synchrony. We explore the impact of heterogeneity in the network topology and the frequency distribution. Our analysis includes unimodal (Gaussian, truncated Gaussian, and uniform) and bimodal frequency distributions. For a unimodal Gaussian distribution, we observe that in comparison to fully-connected network, the competition between topological and dynamical hubs hinders the transition to synchrony in the scale-free network, though explosive synchronization eventually happens. However, in the absence of very large frequencies, the transition is gradual. While uniform frequency distributions lead to explosive synchronization. In bimodal distributions, narrow distribution produce a two-step transition. In this case, central frequencies dominate the dynamics, overshadowing the topological features of the network. For wider bimodal distributions, scale-free network exhibits a gradual increase in the order parameter, whereas in fully-connected networks a first-order transition happens. These results specifically elucidate the mechanisms driving two-step and explosive synchronization in frequency-weighted Kuramoto models, offering new insights into managing synchronization phenomena in complex networks like power grids, neural systems, and social systems.

1. Introduction

The Frequency-weighed Kuramoto model incorporates dynamic-dependent interactions, introducing an additional degree of freedom to describe collective phenomena more comprehensively. This model is inspired by the intrinsic nature of phase oscillators, where frequency serves as a defining characteristic. Consequently, it is natural to consider the connections as a function of this unique feature. In this framework, interactions weighted by frequency can induce Explosive Synchronization (ES) under specific network topologies and frequency distributions. ES refers to a first-order phase transition, where the system has an abrupt and discontinuous transition to the synchronized state once the coupling strength exceeds a critical threshold.

Real-world biological and technological networks often exhibit topological structures that discourage explosive synchronization phenomena, typically associated with pathological states. Conversely, social network topologies actively promote the sudden and irreversible emergence of synchronous states [1]. The connectivity pattern of the interaction network significantly shapes collective dynamics, with heterogeneity in degree distribution notably influencing the order of transition to the synchronized state. A positive correlation between the network's structure, i.e., node degree, and its dynamics, i.e., natural frequencies, is known to facilitate ES in heterogeneous topologies, such as scale-free network [2].

Research reveals that local degree–degree correlation primarily contributes to ES [3], particularly when the degrees and natural frequencies of the network's nodes are disassortative [4]. Interestingly, ES can manifest even with partial degree-frequency correlations only for the hubs, the vertices with the highest degrees. This partial correlation not only promotes but also enables explosive synchronization in networks where a full degree-frequency correlation would otherwise prevent it [5]. Additionally, the degree of mixing in networks impacts the nature of the synchronization transition. For instance, studies have shown that when the interaction is dominant from the high-degree to the low-degree (or from the low-degree to the high-degree) nodes, synchronization is enhanced for assortative (or disassortative) degree–degree correlations [6].

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Various factors contribute to the emergence and dynamics of ES. The basin of attraction has been identified as a determinant of hysteresis in ES [7]. Research on the effects of frustration has highlighted its potential as a control parameter for ES [8]. Self-organized correlations also contribute to ES, adding complexity to ES dynamics [9]. Experimental evidence from systems like mercury beating-heart oscillators provides tangible support for the existence of ES [10]. Studies have also explored ES transitions in adaptive and multilayer networks, as well as in complex neural networks, revealing diverse manifestations of this phenomenon [11,12]. Hysteretic transitions happen in the Kuramoto model with inertia. Nodes in a second-order Kuramoto model perform a cascade of transitions toward a synchronous macroscopic state [13–15]. Factors such as disorder and time delay significantly impact the dynamics of interacting oscillators [16]. Introducing quenched disorder to oscillator frequencies induces ES in mildly heterogeneous networks [17]. In addition, considering time-delayed coupling enhances ES dynamics [18]. These studies collectively offer a comprehensive view of the multifaceted nature of ES, encompassing various factors influencing its occurrence and dynamics.

Frequency distribution plays a crucial role in defining the dynamics of a network. Symmetric and asymmetric distributions have been explored using the mean-field approach, shedding light on the critical coupling of ES transitions in scale-free networks [19,20]. Moreover, previous studies have investigated the frequency-weighted Kuramoto model with bimodal frequency distributions [21]. The impact of frequency-weighted couplings, which connect coupling strength to the oscillators' natural frequencies, is a key but underexplored factor in understanding various synchronization phenomena. In this paper, our research sheds light on the intricate interplay between network topology and dynamical aspects in influencing synchronization phenomena, focusing on the frequency-weighted Kuramoto model within scale-free and fully-connected networks. Section 2 presents the frequency-weighted Kuramoto model along with the network types (all-to-all and scale-free) and other details used in this study. Section 3 presents the results for different frequency distributions and network topologies, comparing the nature of the transition to synchrony in each case.

2. Method

2.1. Frequency-weighted Kuramoto model

The Kuramoto model with frequency-weighted interactions, describes the dynamics of N coupled limit-cycle oscillators. The phase of the i'th oscillator, denoted by θ_i, evolves according to the following equation:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\lambda|\omega_i|}{k_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i)\ , i = 1, 2, \dots, N \tag{1}$$

where $\{\omega_i\}$ is the natural frequency of i'th oscillator, drawn from a frequency distribution $g(\omega)$. The connectivity pattern is given by the elements of the adjacency matrix A_{ij}s, where $A_{ij} = 1$ for connected nodes, and $A_{ij} = 0$ otherwise. The degree of node i is $k_i = \sum_j A_{ij}$, and λ stands for the coupling strength. A key feature of this model is its frequency-weighted coupling mechanism, where there exists a positive correlation between the coupling strength of oscillators and the absolute value of their natural frequencies. This intricate coupling scheme engenders heterogeneous interactions among the oscillators.

The order parameter r, which quantifies the extent of synchrony within the system, as a function of phase homogeneity is defined as:

$$r = \frac{1}{N}\ |\sum_{j=1}^{N} e^{i\theta_j}\ |\ . \tag{2}$$

The system size is fixed at $N = 1000$, with initial phases randomly assigned between 0 and 2π. We compute the stationary value of the order parameter r for coupling strength λ in an adiabatic manner: progressively increasing the coupling strength until the system reaches its stationary state, then using the outcome as the initial condition for subsequent steps. This procedure is performed for both forward and backward continuations, corresponding to increasing and decreasing coupling strength, respectively.

To investigate the impact of network topology, we study two network configurations: a fully connected network, where $k_i = N - 1$ for all nodes, and a Barabási–Albert (BA) scale-free network. The degree distribution $P(k)$ represents the likelihood that a randomly selected node in the network possesses a degree of k. The degree distribution of scale-free network follows a power low $P(k) \sim k^{-\gamma}$, where γ typically ranges between 2 and 3. This results in the presence of a small number of high-degree vertices and a large number of low-degree ones [6]. The high-degree nodes known as hubs, strongly influence the dynamics. While previous studies have demonstrated the significant influence of heterogeneity in degree distribution on the emergence of ES, it is crucial to acknowledge that it is not the sole factor determining the nature of the transition to synchrony. The distribution of intrinsic frequencies of the oscillators plays a decisive role in determining the onset of synchronization.

To determine which frequency distribution properties dictate the transition's nature, we explore the different levels of heterogeneity. We take frequencies from four distinct distributions: Lorentzian, Gaussian (with and without tails), and uniform distribution. To analyze the influence of different attributes of the intrinsic frequency distribution on the transition to synchrony, we explore the impact of both the *mean* and *width* of the distributions on the transition to the synchronized state. Expanding our investigation beyond zero-centered unimodal distributions, we also explore symmetric bimodal frequency distributions with *means* away from zero.

3. Results

3.1. Unimodal frequency distributions

Using the frequency-weighted Kuramoto model described in Eq. (1), we investigate the impact of symmetric frequency distribution around zero on the transition to synchrony. Fig. 1 presents the evolution of the order parameter, after it reaches stationary state, as a function of the coupling constant for both all-to-all networks (Fig. 1 (a)–(c)) and scale-free networks (Fig. 1 (d)–(f)) under various frequency distributions with zero mean. With a symmetric Gaussian distribution (Fig. 1 (a), (d)), both networks exhibit an abrupt transition to the synchronized state with hysteresis. In the all-to-all network (Fig. 1(a)), the critical coupling in the forward transition to the synchronized state is 2.68, while in the backward continuation, the threshold value is 2, aligning with the exact analytical solution obtained in [22]. Our simulations for various distribution widths (0.01, 0.1, 1, 2) confirm that the distribution width does not affect the critical transition point, or the width of the hysteresis loop. Interestingly, the critical coupling for the transition is smaller in the all-to-all network compared to the scale-free network, suggesting that the heterogeneity of degree distribution in scale-free networks impedes the transition to synchrony. This highlights the influence of structure-dynamics correlation in ES.

To explore the interplay between large frequencies and network structure, we cut the tails of the Gaussian distribution. This allows us to examine the effect of large and central frequencies on the transition to synchrony. Note that the network's degree distribution is fixed throughout the study for both fully connected and scale-free networks. Under this circumstance, because of the scale-free topology, the structural heterogeneity is conserved, while cutting the tails decreases the heterogeneity in the frequency distribution. Notably, a truncated Gaussian distribution leads to an explosive transition in the all-to-all network as depicted in Fig. 1(b). This suggests that tail frequencies do not significantly influence the order of transition in the fully-connected

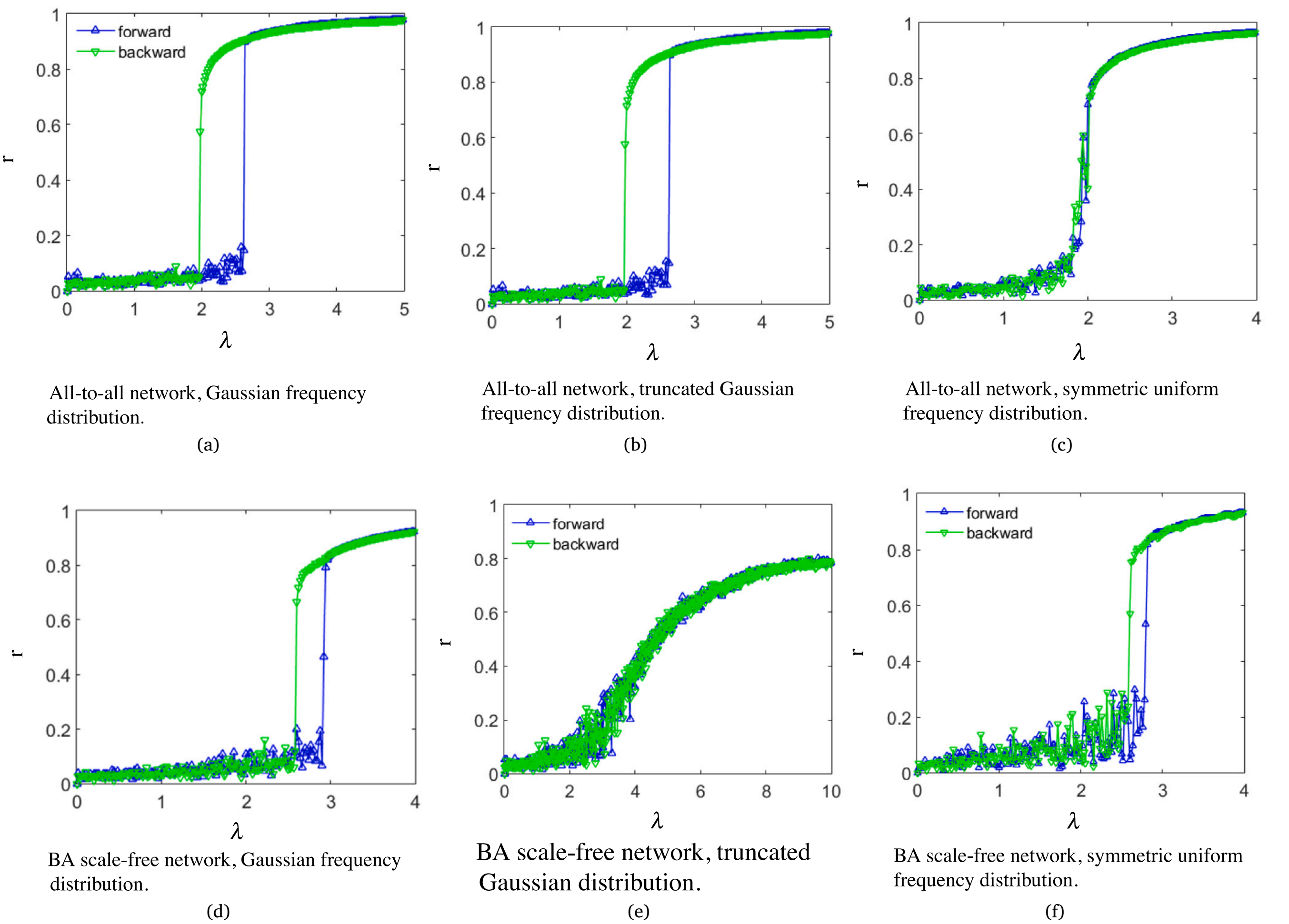

Fig. 1. Stationary value of order parameter, r, versus coupling strength λ for different frequency distributions in two network topologies. Top row fully-connected network. Bottom row BA scale-free network. (a, d): Gaussian distribution, (b), (e): Truncated Gaussian distribution, (c), (f): symmetric uniform frequency distribution.

network. Our simulations across different distribution widths $(0.1, 1)$ confirm that the width does not affect the critical point of transition or the size of the hysteresis loop. However a continuous transition to the synchronized state is observed in scale-free networks Fig. 1(e). This can be attributed to the presence of low-degree nodes belonging to dense sub-graphs interconnected through hubs, resulting in a gradual increase in the order parameter. So, the heterogeneity in the degree distribution, along with the absence of large frequencies, causes a continuous aggregation of oscillators into the cluster of synchrony. Notably, the system exhibits resistance to changing its dynamics, with even very large couplings ($\lambda = 10$), not reaching full synchrony ($r = 1$).

To separate the effect of heterogeneity in structure and frequency distribution, we consider a uniform frequency distribution around zero. In the fully connected network, the transition is almost continuous, Fig. 1(c). Comparing this to Gaussian and truncated Gaussian distributions (Fig. 1(a), (b)), we observe that frequency heterogeneity is the primary factor driving hysteresis in a fully connected network. However, degree heterogeneity in the scale-free network facilitates the hysteresis behavior even when there is no heterogeneity in the frequency distribution, Fig. 1(f).

3.2. Bimodal frequency distribution

To explore the impact of clustering in intrinsic frequencies, we take frequencies from bimodal distributions, including bimodal Gausssian, Lorentzian, and uniform distributions. In Ref. [23], the effect of the location of the center of the bimodal Lorentzian distribution was studied while holding the width constant. In our study, we do the opposite: by fixing the location of the center of distribution, we investigate the role of distribution width.

The probability density function of the bimodal Lorentz distribution is defined as:

$$P = \frac{1}{\pi}\left(\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2} + \frac{\gamma}{(\omega+\omega_0)^2+\gamma^2}\right) \tag{3}$$

Here, γ represents the width parameter of each Lorentzian peak and $\pm\omega_0$ denote their center frequencies. It is important to note that $g(\omega)$ is bimodal if and only if the peaks are sufficiently far apart compared to their widths. Specifically, one needs $\omega_0 > \gamma/\sqrt{3}$; Otherwise, the distribution is unimodal.

In Fig. 2 we present the long-time averaged value of the order parameter, defined in Eq. (2), for both all-to-all and scale-free networks. Frequencies are taken from bimodal Gaussian and bimodal Lorentzian distributions, and we examine the effect of the distribution width γ. Small values of γ correspond to a narrow distribution, while large γ values indicate a wider distribution. In both all-to-all and scale-free networks, a narrow bimodal Lorentzian distribution results in a smooth and two-step transition to the synchronized state. When γ is small, the system has two transitions: first, from unsynchronized to two-cluster synchrony, where oscillators within each cluster are synchronized but

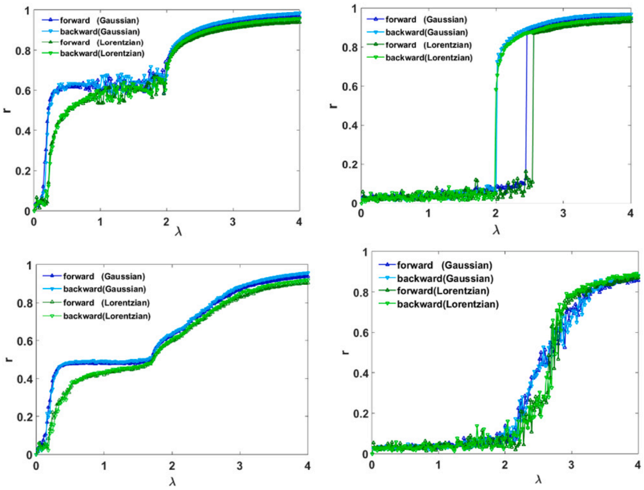

Fig. 2. Stationary value of order parameter r versus coupling strength λ in (top) all-to-all and (bottom) scale-free network, with bimodal Gaussian (blue) and bimodal Lorentz (green) distribution. Picks are locatied at 2 and −2, for different scale parameter (γ). In the left column $\gamma = 0.1$, and in the right column $\gamma = 2$.

clusters themselves are not. The second transition occurs with further increase in the coupling, which is at $\lambda = 2$, where the two clusters get synchronized. In the intermediate state, further increase in the coupling does not significantly alter the overall order parameter. This is because even though the degree of synchrony between the nodes in each cluster increases, the opposite sign neutralizes the final effect on the overall order parameter. The formation of synchronous clusters is also discussed in reference [22]. As shown in the left column of Fig. 2, for a narrow distribution in both networks, the Gaussian distribution causes a steeper transition compared to the Lorantzian distribution. Compared to the fully-connected network, the scale-free network has a lower order parameter in the intermediate step and shows less fluctuations. In scale-free networks, there is competition between the topological hubs with a large number of connections, and dynamical hubs with large frequencies. Scale-free networks exhibit behavior similar to fully connected networks for a narrow distribution as shown by the upper panel in the right column of Fig. 2. However, the presence of high structural clustering in scale-free networks, coupled with a wide bimodal distribution, leads to a continuous transition to synchrony (lower panel in the right column of Fig. 2). This contrasts with fully connected networks, where wider distributions induce ES with hysteresis. The heterogeneous degree distribution in scale-free networks inhibits ES, preventing a two-step transition observed in fully-connected networks. This observation emphasizes that the spread of the two modes of the frequency plays an important role in the characteristics of the transition.

In social systems, for instance, strongly dominant parties attract individuals; while in their absence, individuals form smaller groups, much like ES, where increased coupling leads to the formation of small groups of oscillators until they unify into a single cluster upon reaching a threshold.

Fig. 3 represents the simulation result for a bimodal uniform distribution of frequencies in a fully connected network. Here, we investigate the effect of uniform bipartite intrinsic frequencies in a network with homogeneous degrees. We plot the overall order parameter r and two sub-order-parameters r_1 and r_2 corresponding to the two clusters. After the first transition, the oscillators inside each cluster are synchronized with order parameters ($r_1 \sim 1$) and ($r_2 \sim 1$). However, the two clusters are not in sync with each other, leading to a two-step transition in the overall order parameter. By increasing the coupling constant, after the intermediate step, the two clusters get synchronized.

To delve into the system's underlying dynamics in addition to the order parameter, which is a global measure of synchrony, we measure the correlation matrix which provides local phase configuration information. It measures the coherency between oscillator i and j and determines the local phase configuration. $D_{ij} = 1$ indicates full synchrony, and $D_{ij} = -1$ indicates oscillators in an anti-phase state, that is, $\theta_i = \theta_j \pm \pi$:

$$D_{ij} = \lim_{\Delta t \longrightarrow \infty} \frac{1}{\Delta t} \int_{t_s}^{\Delta t + t_s} \cos\left(\theta_i(t) - \theta_j(t)\right) \quad (4)$$

where t_s is the time after which the dynamics of the system reach a stationary state. In our simulations we consider $t_s = 5 \times 10^5$ and the averaging time window, $\Delta t = 3 \times 10^4$. Fig. 4 presents the correlation matrix defined by Eq. (4), revealing three distinct dynamical phases of the system. At low coupling values ($\lambda = 0.1, 0.8$), the system exhibits a random phase state with no correlation between the phases of the oscillators. At intermediate coupling values ($\lambda = 1.2, 1.4$), a two-cluster state emerges, where oscillators within each cluster synchronize with each other but not with those in the other cluster. Finally, at high coupling values ($\lambda = 3.6$), the system achieves a fully synchronized state, where both clusters synchronize.

4. Concluding remarks

We investigated the effect of frequency distribution and network topology on the nature of the transition to synchronization under the frequency-weighted model, which considers the positive correlation between coupling strength and intrinsic dynamics of the nodes. By examining intrinsic frequencies from Gaussian, truncated Gaussian, uniform, and bimodal distributions across all-to-all and scale-free networks, we found that both network topology and frequency distribution critically influence synchronization dynamics. All-to-all and scale-free topology both show ES with hysteresis under Gaussian frequency distribution. However, scale-free networks exhibited continuous transitions in the absence of large frequencies, while fully connected networks showed hysteretic behavior. In the uniform frequency distribution, fully connected network shows a continuous transition to synchrony while the scale-free network exhibits a hysteretic behavior.

To clarify the mechanism, we conducted simulations for bimodal uniform distribution. While narrow frequency distributions lead to two-step transition to synchrony, wider distributions can result in either explosive or continuous transitions depending on the network topology. In a fully-connected network, the transition is explosive, whereas a sclefree topology leads to a continuous transition. The results are similar for both Lorentzian and Gaussian distributions.

These findings provide crucial insights into the mechanisms driving synchronization in complex systems, highlighting the significant role of network structure and frequency distribution. Understanding these dynamics is essential for predicting and controlling sudden transitions in real-world systems, such as neural network synchronization, power grids stability, and coordinated social behaviors. This knowledge can inform the design of more resilient infrastructures and strategies for mitigating risks associated with abrupt changes in system dynamics. For example, in power grid design, insights from this study could help engineers develop strategies to prevent cascading failures by predicting and mitigating sudden synchronization events. In neuroscience, this knowledge can improve our understanding of neural synchronization associated with certain brain disorders, potentially leading to better diagnostic tools or treatments. For instance, explosive synchronization has been proposed as a potential mechanism underlying the hypersensitive brain in fibromyalgia, where a small perturbation to a network can lead to an abrupt state transition [24]. Furthermore, in social network analysis, recognizing the conditions that lead to rapid shifts in collective behavior can aid in managing information spread or controlling panic during crises.

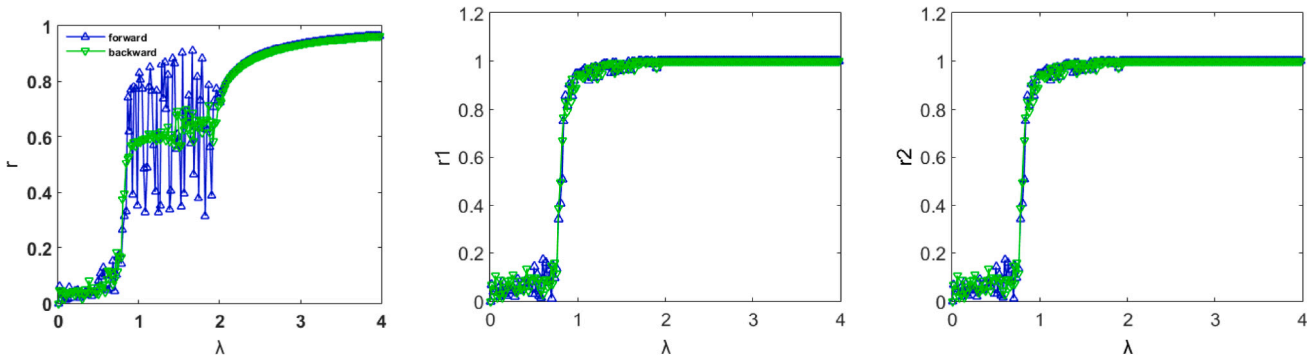

Fig. 3. Synchronization in all-to-all network with bimodal uniform frequency distribution: $1 < \omega_i < 2$ and $-2 < \omega_i < -1$. Each sub-cluster gets synchronized independently, leading to an intermediate overall order parameter.

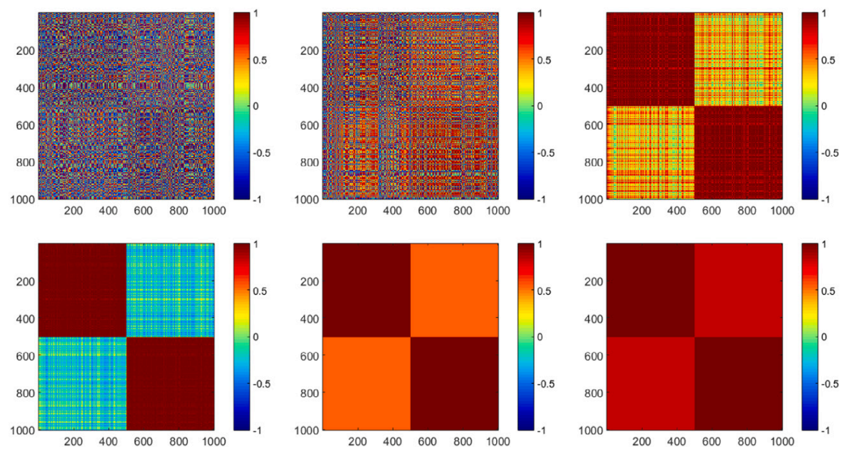

Fig. 4. Correlation matrix for different coupling constant in all-to-all network with symmetric bimodal uniform distribution. $-2 < \omega_i < -1$ and $1 < \omega_i < 2$ and different coupling strength λ. The couplings from left to right in the first row are $\lambda = 0.1, 0.8, 1.2$ and in the second row: $\lambda = 1.4, 2.4, 3.6$..

Future research could explore the effects of different network topologies, such as small-world or random networks, to see how these structures influence two-step and explosive synchronization. Noise has a significant influence on the dynamics of synchronization in Kuramoto mode, as noted in previous studies [25]. Further research could incorporate the effect of noise in frequency-weighted model to investigate the effect on the stability of intermediate states and to gain insights into the robustness of synchronization against external perturbations. Another valuable direction would be to apply these findings to real-world data, such as analyzing power grid failures, brain wave patterns in neural disorders, or behavioral synchronization in social networks, to validate the theoretical models and enhance practical understanding.

CRediT authorship contribution statement

Sara Ameli: Writing – review & editing, Writing – original draft, Visualization, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Sara Ameli reports was provided by Research Centre Jülich. Sara Ameli reports a relationship with Research Centre Jülich that includes: employment. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix. Analytical solution

The analytical solution for frequency weighted Kuramoto model has been done in [22] using a mean-field approach, a critical equation for bimodal Lorentz distribution is obtained as follows:

$$\frac{2\pi}{k} = \ln \frac{\sqrt{1+\delta^2}}{\lambda}\left[\frac{(1-\lambda)\lambda}{(1-\lambda)^2+\delta^2} + \frac{(1+\lambda)\lambda}{(1+\lambda)^2+\delta^2}\right]. \tag{A.1}$$

By solving this equation, the critical coupling is found to be:

$$\frac{4}{\sqrt{1+\delta^2}}. \tag{A.2}$$

In which δ is $\frac{\omega_0}{\gamma}$. By doing so, for $\gamma = 0.1$ the critical coupling will be 0.87, for $\gamma = 0.5$, 1.79 and for $\gamma = 2$, 2.83.

For the backward phase transition, using the self-consistency method, the mean-field equation

$$\frac{d\theta_i}{dt} = \omega_i + k|\omega_i|\sin(\psi - \theta_i) \tag{A.3}$$

In the stationary state, the system splits into two synchronous clusters having the same phase $(\arcsin \frac{1}{kr})$ with opposite sign, provided that $g(\omega)$ is symmetric and centered around zero. The order parameter then will be:

$$r = (e^{i\theta_p} + e^{i\theta_n})/2 = cos(\theta_p) = \sqrt{1 - (\frac{1}{kr})^2} \tag{A.4}$$

Then two branches of r are:

$$r_1(k) = \frac{\sqrt{2}}{2}\sqrt{1 + \sqrt{1 - 4/k^2}} \tag{A.5}$$

$$r_2(k) = \frac{\sqrt{2}}{2}\sqrt{1 - \sqrt{1 - 4/k^2}} \tag{A.6}$$

Being r_1 stable and r_2 unstable solutions. In the case of the scale-free network, for both narrow and wide frequency distributions, the system shows a continuous transition to the synchronized state but persisting to stay in the disordered state in the case of wide peaks.

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