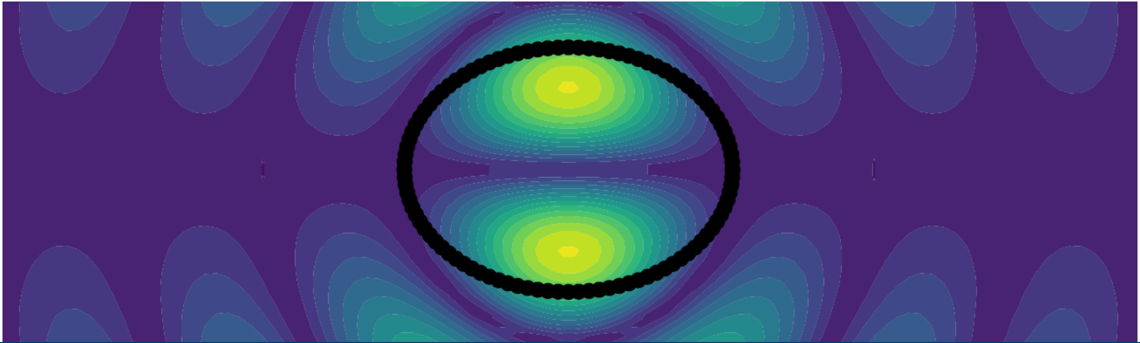

NON-SCATTERING WAVE NUMBERS VERSUS TRANSMISSION EIGENVALUES

(joint work with Lukas Pieronek)

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INTRODUCTION

Problem setup

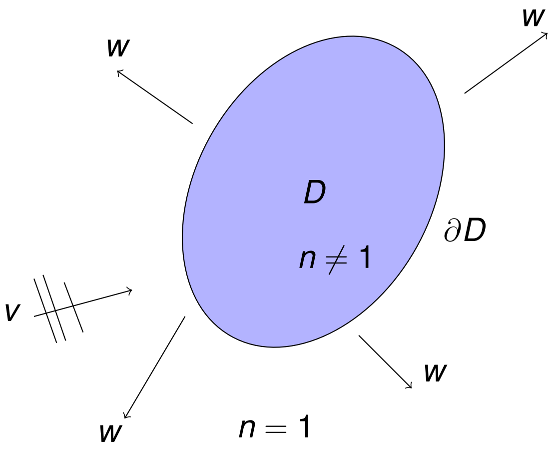
- Incident field v satisfies

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^2 \,.$$

- Total field $u = w + v$ satisfies

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^2 \,.$$

- Scattered field w satisfies the Sommerfeld radiation condition.
- $k > 0$ is the wave number.
- n (constant) is the index of refraction.
- D is simply-connected and star-shaped.



INTRODUCTION

Problem setup

- Scattered field satisfies

$$w(x) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}} \frac{\mathrm{e}^{\mathrm{i}k\|x\|}}{\sqrt{\|x\|}} w^{\infty}(\widehat{x}) + \mathcal{O}\left(\|x\|^{-3/2}\right), \quad \|x\| \to \infty$$

  uniformly with respect to $\hat{x} = x/\|x\| \in \mathbb{S}^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$.
- Far-field $w^{\infty}(\widehat{x})$ is defined on the unit circle $\mathbb{S}^1$.

INTRODUCTION

Problem setup (first introduced by Kirsch (1986) and Colton & Monk (1988))

- Take $v = u^i(x;\widehat{d},k) = \mathrm{e}^{\mathrm{i}kx\cdot\widehat{d}}$ as incident plane wave with direction $\widehat{d}\in\mathbb{S}^1$.
- Then, we have $w^\infty(\widehat{x};\widehat{d},k)$.
- Inverse problem: Given the wave number $k>0$ and the far-field patterns $w^\infty(\widehat{x};\widehat{d},k)$ for all $\widehat{x},\widehat{d}\in\mathbb{S}^1$ determine the shape of the scattering obstacle D.
- Far-field operator

$$F_k[g](\widehat{x}) = \int_{\mathbb{S}^1} w^\infty(\widehat{x};\widehat{d},k)g(\widehat{d})\,\mathrm{d}s(\widehat{d})\,,\quad \widehat{x}\in\mathbb{S}^1$$

- Kirsch's factorization method: Is the far-field operator $F_k : L^2(\mathbb{S}^1)\to L^2(\mathbb{S}^1)$ injective?

INTRODUCTION

Problem setup (first introduced by Kirsch (1986) and Colton & Monk (1988))

- The far-field operator $F_k : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is injective and has dense range if and only if $k > 0$ is not a non-scattering wave number (NSWN).
- Interior transmission problem (ITP):

$$\begin{aligned} \Delta v + k^2 v &= 0 \quad \text{in } D\,, \\ \Delta w + n k^2 w &= 0 \quad \text{in } D\,, \\ v &= w \quad \text{on } \partial D\,, \\ \partial_\nu v &= \partial_\nu w \quad \text{on } \partial D\,. \end{aligned}$$

- Given $n \in L^\infty(D)$, $k > 0$ is called (real) transmission eigenvalue (TE) if the ITP is solved for non-trivial $v, w \in L^2(D)$ such that $(v - w) \in H_0^2(D)$.

MOTIVATION
Problem setup

- Let $\mathcal{N}$ be the set of NSWNs and $\mathcal{T}$ be the set of TEs.
- Obviously, we have $\mathcal{N} \subseteq \mathcal{T}$.
- Many methods are available to numerically compute TEs and corresponding v in D such as
  - finite element method,
  - boundary integral equations,
  - inside-outside-duality method, and
  - modified method of fundamental solutions.
- If we can extend v from D to $\mathbb{R}^2 \backslash \overline{D}$, then a TE is also a NSWN.

MOTIVATION

What is known (the extreme cases)?

- Unit disk, $n = 4$:

$$\mathcal{N} = \mathcal{T} = \{2.9026, 3.3842, 3.4121, 3.9765, \dots\}\,.$$

With $\widehat{k} \approx 3.3842$, the function

$$v(\tilde{r}, \theta) = J_0(\widehat{k} \cdot \tilde{r})$$

used as an incident field does not produce a scattering field. w inside D is given by

$$w(\tilde{r}, \theta) = \frac{J_0(\widehat{k})}{J_0(2\widehat{k})} J_0(2\widehat{k} \cdot \tilde{r})\,.$$

- Unit square, $n = 4$: $\mathcal{N} = \emptyset$ and

$$\mathcal{T} = \{5.4761, 6.1003, 6.1844, 6.6510, \dots\}\,.$$

MOTIVATION

What do we investigate?

- Can we construct a method to determine whether v can be extended to some $B_r(0) \supset \overline{D}$?
- If v cannot be extended, then TE k will not be a NSWN.
- Otherwise, k is a possible NSWN candidate.

THE PROCEDURE

The constrained minimization problem

$$\sigma_{N,\lambda,r} := \min_{v,w\in V_N(k)\times W_N(\sqrt{n}k)} \left\{ \|v-w\|^2_{H^{\frac{3}{2}}(\partial D)} + \|\partial_\nu(v-w)\|^2_{H^{\frac{1}{2}}(\partial D)} + \lambda\|v\|^2_{L^2(B_r(0)\setminus\overline{D})} \right\}$$

s.t.

$$\|v-w\|^2_{H^{\frac{3}{2}}(\partial D)} + \|\partial_\nu(v-w)\|^2_{H^{\frac{1}{2}}(\partial D)} + \|v\|^2_{L^2(D)} + \|w\|^2_{L^2(D)} + \lambda\|v\|^2_{L^2(B_r(0)\setminus\overline{D})} = 1\,, \tag{1}$$

- $\lambda > 0$ is a Tikhonov parameter to penalize blow-up behavior of v in the exterior.
- k is a given TE of D.
- $V_N(k)$ and $W_N(\sqrt{n}k)$ are N-dependent-dimensional subspaces of global Helmholtz solutions with wave numbers k and $\sqrt{n}k$, respectively.

THE PROCEDURE

The constrained minimization problem

We use Fourier-Bessel functions (FBF) (in polar coordinates $(\tilde{r}, \phi)$)

$$
\begin{aligned}
V_N(k) &= \left\{ \alpha_0 J_0(k\tilde{r}) + \sum_{p=1}^{N} J_p(k\tilde{r}) \big(\alpha_p \exp(\mathrm{i}p\phi) + \beta_p \exp(-\mathrm{i}p\phi) \big) \right\} , \\
W_N(\sqrt{n}k) &= \left\{ \gamma_0 J_0(\sqrt{n}k\tilde{r}) + \sum_{p=1}^{N} J_p(\sqrt{n}k\tilde{r}) \big(\gamma_p \exp(\mathrm{i}p\phi) + \delta_p \exp(-\mathrm{i}p\phi) \big) \right\} ,
\end{aligned} \tag{2}
$$

They are entire solutions to the Helmholtz equations and additionally L^2-biorthogonal on disks of arbitrary radii.

THE PROCEDURE

The constrained minimization problem

Theorem

Let D be a smooth domain that is star-shaped with respect to a ball, let $k > 0$ be some TE of D, $B_r(0) \supset \overline{D}$ for $r > 0$ and $(v_{N,\lambda,r}, w_{N,\lambda,r}) \in V_N(k) \times W_N(\sqrt{n}k)$ be the corresponding minimizer of (1) on $B_r(0)$ for $\lambda > 0$ and $N \in \mathbb{N}$ with minimum $\sigma_{N,\lambda,r}$. Then k is a NSWN if and only if

$$\lim_{\tilde{\lambda}\to 0} \lim_{N\to\infty} \inf_{\lambda<\tilde{\lambda}} \frac{\sigma_{N,\lambda,r}}{\lambda} < \infty$$

for all $r > 0$ such that $\overline{D} \subset B_r(0)$.

If the numerical computation of $\frac{\sigma_{N,\lambda,r}}{\lambda}$ for large N, small λ, and large r, stays small, indicates a possible NSWN. Otherwise, we do not have a NSWN.

NUMERICAL IMPLEMENTATION

The constrained minimization problem

- Let $r > 0$ such that $\overline{D} \subset B_r(0)$. We use the computational points:
  - boundary points $x_i \in \partial D$ for $1 \leq i \leq N_B$,
  - interior points $y_i \in D$ for $1 \leq i \leq N_I$,
  - exterior points $z_i \in B_r(0) \setminus \overline{D}$ for $1 \leq i \leq N_E$.
- Further, let $N \in \mathbb{N}$ (can be extended to $N \in \mathbb{N}/2$) such that $N \ll N_B, N_I, N_E$ and let

$$\{\phi_{j,k}\}_{1 \leq j \leq 2N+1} \subset V_N(k) \; ,$$
$$\{\psi_{j,\sqrt{n}k}\}_{1 \leq j \leq 2N+1} \subset W_N(\sqrt{n}k)$$

be the canonical basis of Fourier Bessel functions from (2) for some (numerically) exact TE k.

NUMERICAL IMPLEMENTATION

The constrained minimization problem

We set for $1 \leq j \leq 2N+1$

$$
\begin{aligned}
(V_N(\partial D))_{i,j} &= \phi_{j,k}(x_i)\ ,\ 1 \leq i \leq N_B\ ,\\
(W_N(\partial D))_{i,j} &= \psi_{j,\sqrt{n}k}(x_i)\ ,\ 1 \leq i \leq N_B\ ,\\
(\partial_\nu V_N(\partial D))_{i,j} &= \partial_\nu \phi_{j,k}(x_i)\ ,\ 1 \leq i \leq N_B\ ,\\
(\partial_\nu W_N(\partial D))_{i,j} &= \partial_\nu \psi_{j,\sqrt{n}k}(x_i)\ ,\ 1 \leq i \leq N_B\ ,\\
(V_N(D))_{i,j} &= \partial_\nu \phi_{j,k,N}(y_i)\ ,\ 1 \leq i \leq N_I\ ,\\
(W_N(D))_{i,j} &= \partial_\nu \psi_{j,\sqrt{n}k}(y_i)\ ,\ 1 \leq i \leq N_I\ ,\\
\left(V_N(B_r(0)\setminus\overline{D})\right)_{i,j} &= \partial_\nu \phi_{j,k}(z_i)\ ,\ 1 \leq i \leq N_E
\end{aligned}
$$

NUMERICAL IMPLEMENTATION

The constrained minimization problem

- Construct block matrix

$$M_{FBF}(N,\lambda,r) := \begin{pmatrix} V_N(\partial D) & W_N(\partial D) \\ \partial_\nu V_N(\partial D) & \partial_\nu W_N(\partial D) \\ \lambda \cdot V_N(B_r(0)\setminus\overline{D}) & 0 \\ V_N(D) & 0 \\ 0 & W_N(D) \end{pmatrix} \in \mathbb{C}^{N_{tot}\times(4N+2)}\,. \qquad (3)$$

  with $N_{tot} := 2N_B + N_E + 2N_I$ and $\lambda > 0$.

- QR decomposition

$$M_{FBF}(N,\lambda,r) = Q(N,\lambda,r)R(N,\lambda,r) = \begin{pmatrix} Q_B(N,\lambda,r) \\ Q_E(N,\lambda,r) \\ Q_I(N,\lambda,r) \end{pmatrix} R(N,\lambda,r)\,.$$

NUMERICAL IMPLEMENTATION

The constrained minimization problem

- Set

$$Q_{B,E}(N,\lambda,r) = \begin{pmatrix} Q_B(N,\lambda,r) \\ Q_E(N,\lambda,r) \end{pmatrix} ,$$

- Numerical solution to (1) becomes

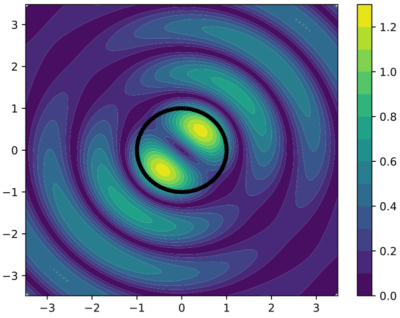
$$\min_{\substack{q\in\mathbb{C}^{4N+2}, \\ \|q\|=1}} \|Q_{B,E}(N,\lambda,r)q\| = s_{\min}(N,\lambda,r) , \qquad (4)$$

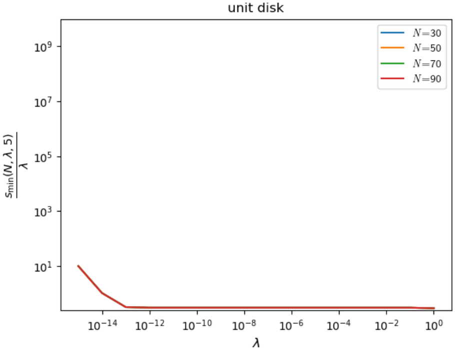
  where $s_{\min}(N,\lambda,r)$ denotes the minimal singular value of $Q_{B,E}(N,\lambda,r)$

- It represents the discrete equivalent of $\sigma_{N,\lambda,r}$.

NUMERICAL RESULTS

The unit disk, $n = 4$, $k \approx 2.9026$, $r = 5$

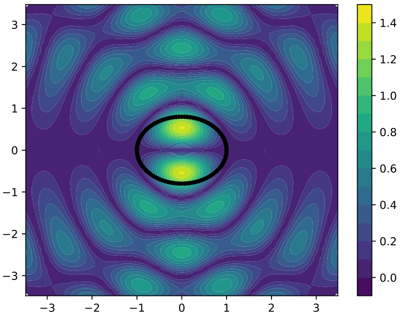


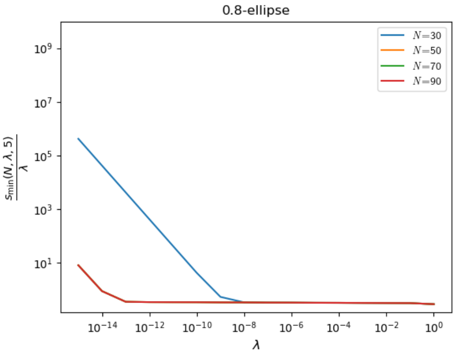


We see in the $\lambda \mapsto (s_{\min}(N, \lambda, 5)/\lambda)$-plots that for each fixed $\lambda > 0$ the corresponding point on the graph approaches a reasonably small value as N grows. We know that all TEs are NSWN, hence $k \approx 2.9026$ is a NSWN. Our numerical method confirms this.

NUMERICAL RESULTS

The ellipse with half-axis 1 and 0.8, $n = 4$, $k \approx 3.4852$, $r = 5$

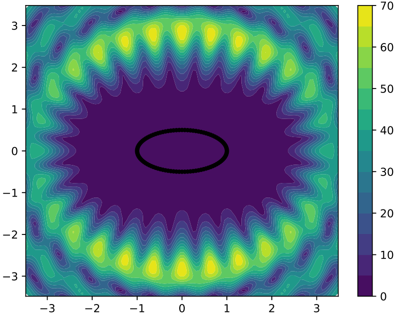


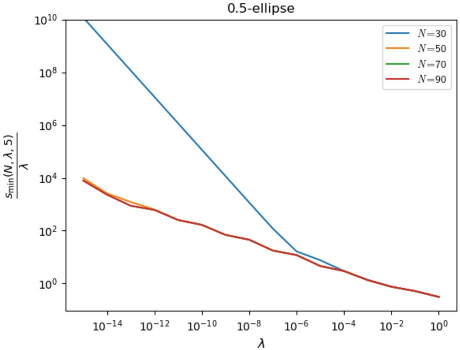


We see in the $\lambda \mapsto (s_{\min}(N, \lambda, 5)/\lambda)$-plots that for each fixed $\lambda > 0$ the corresponding point on the graph approaches a reasonably small value as N grows for decreasing λ. According to Theorem 1, we expect $k \approx 3.4852$ to be a NSWN.

NUMERICAL RESULTS

The ellipse with half-axis 1 and 0.5, $n = 4$, $k \approx 5.4092$, $r = 5$

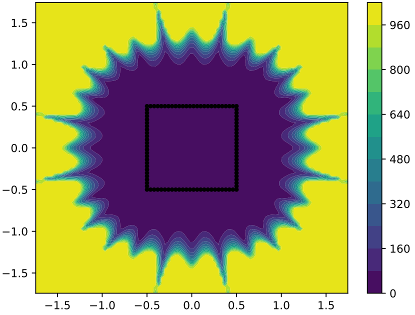


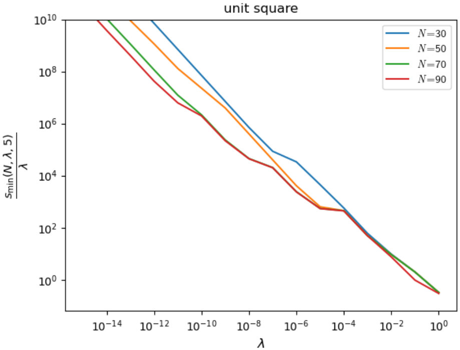


We see in the $\lambda \mapsto (s_{\min}(N, \lambda, 5)/\lambda)$-plots that for any $\lambda < 10^{-4}$ the corresponding point on the graph increases independent of the growing N. According to Theorem 1, we expect $k \approx 5.4092$ to be a pure TE.

NUMERICAL RESULTS

The unit square, $n = 4$, $k \approx 6.1003$, $r = 5$





We see in the $\lambda \mapsto (s_{\min}(N, \lambda, 5)/\lambda)$-plots that for any $\lambda < 10^{-4}$ the corresponding point on the graph increases independent of the growing N. According to Theorem 1, we expect $k \approx 6.1003$ to be a pure TE. We know that there is no NSWN.

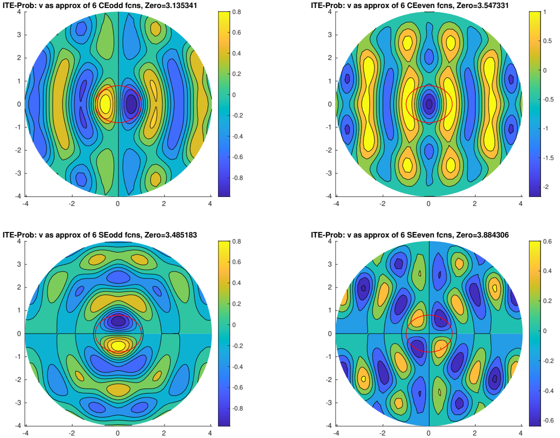
NUMERICAL RESULTS

The ellipse using Mathieu functions

- Consider separation of variables using elliptic coordinates.
- Leads to radial and angular Mathieu functions (there are four different ones).
- ITP leads to the computation of a determinant of a matrix of infinite size containing expressions involving such Mathieu functions.

NUMERICAL RESULTS

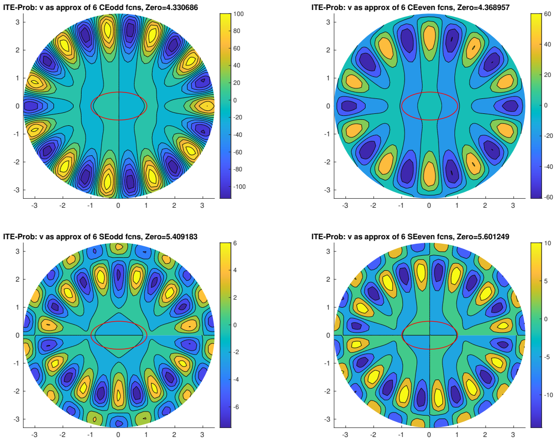
The 0.8-ellipse using Mathieu functions



Claim: All TEs are NSWNs. The eigenfunctions can be analytically extended far away from the boundary.

NUMERICAL RESULTS

The 0.5-ellipse using Mathieu functions



Claim: All TEs are not NSWNs. The eigenfunctions can be analytically extended close to the boundary but not far away.

SUMMARY

- Presented a numerical method to analytically extend the eigenfunction v corresponding to a TE locally.
- Numerical and theoretical results are in agreement with the disk and the unit square.
- Numerical results for the ellipse are the same when using Mathieu-functions directly.
- Theorem 1 gives us numerical tendency to say whether a TE is a NSWN or not.
- Might give insight whether we really have $\emptyset \neq \mathcal{N} \subset \mathcal{T}$ for other domains.

OUTLOOK

- However, the stability of our method is only valid for disc-like domains as for other domains the FBF-biorthogonality is lost.
- Hence, due to an increase of the condition number of $M_{FBF}(N, \lambda, r)$ the quantity $s_{\min}(N, \lambda, r)/\lambda$ is prone to errors and due to large N.
- Try to use a domain-dependent ansatz instead (ML) to improve condition number.

- Further investigate whether there is critical half-axis for an ellipse such that we have the extreme cases (unit disk vs. unit square). We claim there is!

- **Looking forward to your ideas, comments, and inspiring conversations.**

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