

# Dispersive analysis of the isospin breaking in the $X(3872) \rightarrow J/\psi \pi^+ \pi^-$ and $X(3872) \rightarrow J/\psi \pi^+ \pi^0 \pi^-$ decays

Jorgivan Morais Dias<sup>1,2,\*</sup> Teng Ji<sup>3,†</sup> Xiang-Kun Dong<sup>3,‡</sup> Feng-Kun Guo<sup>1,4,5,6,§</sup> Christoph Hanhart<sup>7,||</sup>  
Ulf-G. Meißner<sup>3,5,7,8,¶</sup> Yu Zhang<sup>9,1,\*\*</sup> and Zhen-Hua Zhang<sup>10,1,††</sup>

<sup>1</sup>CAS Key Laboratory of Theoretical Physics, *Institute of Theoretical Physics,*  
*Chinese Academy of Sciences, Beijing 100190, China*

<sup>2</sup>Departamento de Física, *Universidade Federal do Piauí, 64049-550 Teresina, Piauí, Brazil*

<sup>3</sup>Helmholtz Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics,  
*Universität Bonn, D-53115 Bonn, Germany*

<sup>4</sup>School of Physical Sciences, *University of Chinese Academy of Sciences, Beijing 100049, China*

<sup>5</sup>Peng Huanwu Collaborative Center for Research and Education, *Beihang University,*  
*Beijing 100191, China*

<sup>6</sup>Southern Center for Nuclear-Science Theory (SCNT), *Institute of Modern Physics,*  
*Chinese Academy of Sciences, Huizhou 516000, China*

<sup>7</sup>Institute for Advanced Simulation (IAS-4), *Forschungszentrum Jülich, D-52425 Jülich, Germany*

<sup>8</sup>Tbilisi State University, 0186 Tbilisi, Georgia

<sup>9</sup>College of Mechanics and Engineering Science, *Hohai University, Nanjing 211100, China*

<sup>10</sup>School of Physics and Center of High Energy Physics, *Peking University, Beijing 100871, China*



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We analyze the latest LHCb data on the  $\pi^+ \pi^-$  spectrum in the isospin-violating  $X(3872) \rightarrow J/\psi \pi^+ \pi^-$  decay, based on dispersion theory to deal with the  $\pi\pi$  final state interactions. Additionally, the isospin breaking effects are properly introduced, allowing for a reliable and accurate extraction of the ratio,  $R_X$ , between the  $X(3872)$  couplings to the  $J/\psi \rho$  and  $J/\psi \omega$  channels from the data. We find very good agreement with the LHCb data for the whole range of the  $\pi^+ \pi^-$  invariant mass, and  $R_X$  is determined to be  $0.26 \pm 0.03$ . Using this value, we make predictions for the  $\pi^+ \pi^0 \pi^-$  mass distribution in the  $X(3872) \rightarrow J/\psi \pi^+ \pi^0 \pi^-$  process, which is currently accessible by the BESIII Collaboration, and update a prediction for the pole positions of the isovector partner states of the  $X(3872)$ ,  $W_{c1}$ , with  $I(J^{PC}) = 1(1^{++})$ .

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## I. INTRODUCTION

The discovery of  $X(3872)$ , also known as  $\chi_{c1}(3872)$ , in 2003 by the Belle Collaboration [1] in the  $J/\psi \pi^+ \pi^-$  invariant mass spectrum from  $B$  meson decays, produced in  $e^+ e^-$  collisions, inaugurated a new era in hadron

spectroscopy physics. Shortly after its discovery, the CDF [2] and D0 [3] Collaborations also confirmed its existence in  $p\bar{p}$  collisions. Since then, many other experiments have investigated its properties in various processes [4–16], making it the best studied hadronic structure among the new hadrons that, like the  $X(3872)$ , behave differently from what would be expected if their quark content were consistent with the conventional constituent quark model (see Refs. [17–26] for recent reviews).

The latest Particle Data Group average values for the mass and width of the  $X(3872)$  are  $(3871.64 \pm 0.06)$  MeV and  $(1.19 \pm 0.21)$  MeV, respectively [27]. However, one should notice that they were obtained from averaging values extracted using the Breit-Wigner (BW) parametrization [10,11,28], which is not appropriate when a resonance is located near the threshold of a channel that it strongly couples to in the  $S$  wave. Using a generalized Flatté parametrization [29], which properly takes the thresholds into account, the LHCb Collaboration obtained the mass and the visible width defined by the full width at

\* Contact author: jorgivan.mdias@itp.ac.cn

† Contact author: teng@hiskp.uni-bonn.de

‡ Contact author: xiangkun@hiskp.uni-bonn.de

§ Contact author: fkguo@itp.ac.cn

|| Contact author: c.hanhart@fz-juelich.de

¶ Contact author: meissner@hiskp.uni-bonn.de

\*\* Contact author: 2110020124@hhu.edu.cn

†† Contact author: zhhzhang@pku.edu.cn

half maximum of the  $X(3872)$  using a fit to the line shape in the  $J/\psi\pi^+\pi^-$  final state from  $b$ -hadron decays as  $3871.69^{+0.00+0.05}_{-0.04-0.13}$  MeV and  $0.22^{+0.07+0.11}_{-0.06-0.13}$  MeV, respectively [10]. The line shape emerged from a pole located on the second sheet displaced only by  $0.06 - i0.13$  MeV from the  $D^0\bar{D}^{*0}$  threshold. Recently, the BESIII Collaboration reported the mass parameter and imaginary part of its pole as  $(3871.63 \pm 0.13^{+0.06}_{-0.05})$  MeV and  $(-0.19 \pm 0.08^{+0.14}_{-0.19})$  MeV, respectively, from the processes  $e^+e^- \rightarrow \gamma X(3872)$ ,  $X(3872) \rightarrow D^0\bar{D}^0\pi^0$ , and  $\pi^+\pi^-J/\psi$  [30]. One sees an intriguing characteristic of the  $X(3872)$ , that is, its mass coincides with the  $D^0\bar{D}^{*0}$  threshold at  $(3871.69 \pm 0.07)$  MeV [27]. In view of the tiny phase spaces, its branching fraction into the  $D^0\bar{D}^{*0}$  channel as well as into  $D^0\bar{D}^0\pi^0$  are remarkably large [30–34], indicating a strong coupling of the  $X(3872)$  to the  $D\bar{D}^*$ .

As no charged partner of the  $X(3872)$  has been reported so far [28,35], the  $X(3872)$  is expected to be an isoscalar in the isospin symmetric limit. However, in the isospin breaking world, the mass eigenstate is a mixture of different isospin eigenstates. Measurements on isospin breaking processes are crucial to determining how large the admixture is. For the  $X(3872)$ , in this sense relevant measurements are provided by its branching fractions decaying into the modes  $J/\psi\pi^+\pi^-\pi^0$  and  $J/\psi\pi^+\pi^-$ ,  $\mathcal{B}[X(3872) \rightarrow J/\psi 3\pi]/\mathcal{B}[X(3872) \rightarrow J/\psi\pi^+\pi^-]$ ,

$$\begin{cases} 1.0 \pm 0.4 \pm 0.3 & \text{Belle,} \\ 0.7 \pm 0.3(1.7 \pm 1.3) & \text{BABAR } B^+(B^0) \text{ events,} \\ 1.6^{+0.4}_{-0.3} \pm 0.2 & \text{BESIII,} \end{cases} \quad (1)$$

reported by the Belle [15], BABAR [5], and BESIII [16] Collaborations.<sup>1</sup> Given the positive  $C$  parity of the  $X(3872)$  [14],  $C$ -parity conservation and Bose-Einstein statistics imply that the  $\pi^+\pi^-$  pair in the  $J/\psi\pi^+\pi^-$  final state must be an isovector, coming mainly from the  $\rho^0$  meson. Accordingly, the  $3\pi$  channel is expected to be saturated by the  $\omega$  meson. It is worthwhile to notice that a large part of the isospin breaking comes from the huge phase space difference between the  $X(3872) \rightarrow J/\psi\omega$  and  $X(3872) \rightarrow J/\psi\rho^0$  [36]. Thus, the true measure of the isospin breaking effects at the dynamical level should be, instead of the ratio of branching fractions in Eq. (1), the ratio between the  $X(3872)$  couplings to the  $J/\psi\rho$  and  $J/\psi\omega$  channels, that is [36]

$$R_X \equiv \frac{g_{X\psi\rho}}{g_{X\psi\omega}}. \quad (2)$$

For studies of the isospin breaking in the multiquark (either molecular or nonmolecular) configurations of the  $X(3872)$ , see Refs. [37–52].

Obtaining the value of  $R_X$  reliably and accurately is of utmost importance for understanding the mechanism behind the observable in Eq. (1) and the very nature of the  $X(3872)$ . In particular,  $R_X$  has been utilized as a crucial input to determine the isoscalar and isovector low-energy constants (LECs) of the  $D\bar{D}^*$  interactions [42,44,47,53,54], which can be used to predict the pole positions of the heavy quark spin partners [44,47,53,55] and the isovector partner  $W_{c1}$  [54] of the  $X(3872)$  in the hadronic molecular picture.

The ratio  $R_X$  was first estimated to be  $0.29 \pm 0.02$  [36] and  $0.30 \pm 0.07$  [56] in 2005 using the experimental value of  $\mathcal{B}[X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-]/\mathcal{B}[X(3872) \rightarrow J/\psi\pi^+\pi^-]$  from Belle [15], where the two processes are mediated by the  $\rho$  and  $\omega$  resonances using the BW parametrization. Such parametrization for the  $\rho$  meson is precarious as the broad bump in the line shapes from the  $\rho$  resonance cannot be well described by the BW function [57]. In addition,  $\rho$ - $\omega$  mixing was shown to have a significant impact on the two-pion channel [58]. This led to an improved value of the pertinent ratio,  $R_X = 0.26^{+0.08}_{-0.05}$ , by fitting the data from Belle [28] and BABAR [5] on the invariant mass distributions of  $\pi^+\pi^-$  and  $\pi^+\pi^0\pi^-$  in the  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  and  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  decays, respectively. The recent LHCb experiment [59] updated the  $\pi^+\pi^-$  invariant mass distribution in  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  and estimated  $R_X$  to be  $0.29 \pm 0.04$  utilizing a similar strategy as in Ref. [58], but it set the  $X(3872)$  mass to be 4 GeV, much larger than the Flatté result [10], to extend the phase space. In Ref. [60], the updated LHCb data [59] for  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  as well as previous BABAR data [5] for  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  were analyzed simultaneously, where the  $\omega$  meson contribution via  $\omega \rightarrow \pi^+\pi^-$  was taken into account through a complex-valued effective coupling instead of  $\rho$ - $\omega$  mixing. The BW parametrization, supplemented with a dipole form factor, for the  $\rho$  and  $\omega$  mesons was applied again in this analysis, and  $R_X$  was extracted to be  $0.25 \pm 0.01$  for a running  $\rho$  width and  $0.30 \pm 0.01$  for a constant  $\rho$  width in the  $\rho$  propagator. So far, there is no analysis on the  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  decay properly treating the broad  $\rho$  resonance and the  $\rho$ - $\omega$  mixing at the same time.

In view of the above discussion, here we perform an analysis of the LHCb data for the decay  $X(3872) \rightarrow J/\psi\pi^+\pi^-$ , where a dispersive approach is applied to describe the universal nature of the  $\pi\pi$  final state interaction (FSI), through which the  $\rho^0$  resonance enters. This approach allows us to analyze the LHCb data accurately and extract the value of  $R_X$  in a reliable manner. The value

<sup>1</sup>The cuts on the  $3\pi$  invariant mass are  $m_{3\pi} > 0.75$  GeV for Belle [15],  $m_{3\pi} \in [0.74, 0.7965]$  GeV for the  $B^+$  events and  $\in [0.74, 0.8055]$  GeV for the  $B^0$  events for BABAR [5], and  $m_{3\pi} \in (0.71, 0.81)$  GeV for BESIII [16], respectively. One also notices that the  $3\pi$  distribution in the selected region of the BABAR measurement peaks at around 0.76 GeV and is significantly broader than the  $\omega$  width.

found for this important quantity in this way is significantly smaller than those quoted above.

This paper is structured as follows. In Sec. II, we discuss the  $\pi\pi$  FSI and how it is included in the  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  amplitude, along with the proper isospin-breaking effects. Our results of the fit to the LHCb data are discussed in Sec. III. Section IV presents our prediction for the isospin-conserving  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  decay and the updates on the predictions of the  $W_{c1}$  states, the isovector partner of the  $X(3872)$ . Finally, Sec. V presents a brief summary.

## II. THE $X(3872) \rightarrow J/\psi\pi^+\pi^-$ AMPLITUDE

In this section, we discuss the construction of the decay amplitude used in the evaluation of the  $\pi^+\pi^-$  invariant mass distribution in the  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  decay. We begin with the implementation of the  $\pi\pi$  FSI. Next, we discuss the inclusion of the factor that encodes isospin breaking and its correspondence with the ratio  $R_X$ , which is the quantity in the focus of this investigation.

### A. Universal $\pi\pi$ FSI

The  $\pi^+\pi^-$  FSI plays an important role in describing the process  $X(3872) \rightarrow J/\psi\pi^+\pi^-$ . In this particular case, the pions interact in the  $P$  wave ( $\ell = 1$ ). For a given partial wave, the phase of the  $\pi\pi$  FSI amplitude (or pion form factor) in the elastic regime equals to the  $\pi\pi$  scattering phase shift  $\delta_\ell(s)$  modulo  $n\pi$  with  $n$  an integer (Watson's theorem [61]), with  $\sqrt{s} \equiv m_{\pi^+\pi^-}$  the invariant mass of the  $\pi^+\pi^-$  pair in their center-of-mass (c.m.) frame. Consequently, the  $\pi\pi$  FSI is described by a universal function called the Omnès function  $\Omega(s)$  [62], which, in our case, is given in terms of the  $P$ -wave elastic phase shift  $\delta_1^1(s)$  as

$$\Omega(s) = \exp \left[ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s' - s - i\varepsilon)} \right]. \quad (3)$$

Since we are interested in analyzing LHCb data for  $X(3872) \rightarrow J/\psi\pi^+\pi^-$ , where the  $\pi\pi$  invariant mass is limited by the phase space to be  $\sqrt{s} \lesssim 0.775$  GeV, inelastic effects can be safely neglected, and we can use the  $P$ -wave elastic scattering phase shift from Ref. [63]. For a treatment of the pion vector form factor including the high-energy region where inelasticities become important (particularly above 1 GeV), we refer to Ref. [64].

In terms of the Omnès function,  $\Omega(s)$ , the amplitude of  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  can be constructed as

$$\mathcal{M}_{X \rightarrow J/\psi\pi\pi} = \mathcal{N} \varepsilon_{ijk} \varepsilon_\psi^i \varepsilon_X^j q_\pi^k P(s) \Omega(s), \quad (4)$$

where  $\varepsilon_\psi$  and  $\varepsilon_X$  are the polarization vectors for the  $J/\psi$  and  $X(3872)$ , respectively,  $q_\pi$  is the c.m. momentum of the  $\pi^+$ , and  $\mathcal{N}$  represents the overall strength, which will serve

as the normalization constant in the fitting later. The function  $P(s)$  appears, since the linear unitarity relation for the form factor fixes it only up to a function that does not have a right-hand cut, most easily parametrized by a polynomial. In Refs. [65,66], the  $\pi\pi$  FSI was taken into account in the reactions  $e^+e^- \rightarrow \pi^+\pi^-$  and  $\eta \rightarrow \pi^+\pi^-\gamma$  together with a linear polynomial (see Ref. [67] for a related discussion). In Ref. [68], it was demonstrated that a prominent left-hand cut can call for a second order polynomial. However, since there is no obvious meson exchange providing a left-hand cut contribution here, we employ

$$P(s) = 1 + \alpha s. \quad (5)$$

The slope  $\alpha$  will be left as a free parameter to be constrained by the fit to the LHCb data.

### B. Including the isospin breaking effects

The Omnès representation discussed above captures only the  $\rho$  resonance associated with  $\pi\pi$  isovector interactions in the elastic region and does not encode any isospin breaking contribution from the  $\omega$  meson via  $\omega \rightarrow \pi^+\pi^-$ . The effects of this isospin breaking, typically of  $\mathcal{O}(10^{-3})$ , are overcome near the  $\omega$  pole by a factor  $M_\omega/\Gamma_\omega \sim 90$  induced by the  $\omega$  propagator (see Ref. [57] for a detailed discussion). Therefore, it can vary the  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  amplitude significantly, as the LHCb data [59] indeed suggest.

According to Refs. [57,64,69–71], the  $\rho$ - $\omega$  mixing can be introduced as

$$\mathcal{M}_{X \rightarrow J/\psi\pi\pi} = \mathcal{N} \varepsilon_{ijk} \varepsilon_\psi^i \varepsilon_X^j q_\pi^k P(s) \Omega(s) [1 + \kappa_X G_\omega(s)], \quad (6)$$

where  $G_\omega(s)$  is the propagator of  $\omega$ ,

$$G_\omega(s) = \frac{1}{s - M_\omega^2 + iM_\omega\Gamma_\omega}. \quad (7)$$

The parameter  $\kappa_X$  encodes the isospin-breaking effects in the present case;  $M_\omega$  and  $\Gamma_\omega$  stand for the  $\omega$  meson mass and its decay width, respectively. Here we use a constant width for the  $\omega$  [we checked that the energy dependence of  $\Gamma_\omega(\sqrt{s})$ , whose explicit expression is shown in Appendix B, has negligible effects on the final results].

Crucial for this analysis is the connection between the parameters  $\kappa_X$  and  $R_X$ . This is done by performing a matching between the amplitude in Eq. (6), Laurent expanded around the  $\rho$  pole, and the amplitude corresponding to the decay of  $X$  via  $\rho$  including the  $\rho$ - $\omega$  mixing, given by [58]

$$\begin{aligned} \mathcal{M}_{X \rightarrow J/\psi\pi^+\pi^-}^{\text{BW}} = & -g_{XJ/\psi\rho} g_{\rho\pi^+\pi^-} \varepsilon_{ijk} \varepsilon_X^i \varepsilon_\psi^{*j} q_\pi^k P(s) \\ & \times G_\rho(s) \left( 1 - \frac{\varepsilon_{\rho\omega}}{R_X} G_\omega(s) \right), \end{aligned} \quad (8)$$



with  $g_{XJ/\psi\rho}$  the  $X(3872)$  coupling to the  $J/\psi\rho$  mode. The parameter  $\epsilon_{\rho\omega}$  measures the  $\rho$ - $\omega$  mixing. Using the results in Ref. [72], its value is determined to be  $3.35(8) \times 10^{-3} \text{ GeV}^2$ ; see Appendix A for details. It turns out the uncertainty of  $\epsilon_{\rho\omega}$  has negligible effect compared to the statistical error of  $R_X$  from the fitting. In Eq. (8),  $G_\rho$  is the propagator of  $\rho$  in the BW form,

$$G_\rho(s) = \frac{1}{s - M_\rho^2 + iM_\rho\Gamma_\rho(\sqrt{s})}, \quad (9)$$

with  $\Gamma_\rho(\sqrt{s})$  the energy-dependent width of  $\rho$ , as detailed in Appendix B.

On the other hand, around the  $\rho$  pole, the amplitude  $\mathcal{M}_{X \rightarrow J/\psi\pi^+\pi^-}$  in Eq. (6) can be expanded as

$$\begin{aligned} \mathcal{M}_{X \rightarrow J/\psi\pi^+\pi^-} &= \mathcal{N} \epsilon_{ijk} \epsilon_X^i \epsilon_\psi^{*j} q_\pi^k \mathcal{R}P(s) G_\rho(s) \\ &\times [1 + \kappa_X G_\omega(s)] + \text{regular terms}, \end{aligned} \quad (10)$$

where  $\mathcal{R}$  is the residue of the Omnès function at the  $\rho$  pole. Thus, by performing the matching, we obtain  $\mathcal{N}\mathcal{R} = -g_{XJ/\psi\rho}g_{\rho\pi^+\pi^-}$  and especially

$$\kappa_X = -\frac{\epsilon_{\rho\omega}}{R_X}. \quad (11)$$

Thus, once  $\kappa_X$  is obtained from the fit, Eq. (11) directly provides the value for the ratio  $R_X$ .

### III. FITS TO THE LHCb DATA

Once the decay amplitude associated with the process  $X(3872) \rightarrow J/\psi\pi^+\pi^-$  is defined, we can write the invariant mass distribution of the  $\pi^+\pi^-$  pair as

$$\frac{d\Gamma_{X \rightarrow J/\psi\pi^+\pi^-}}{dm_{\pi^+\pi^-}} = \frac{p_{J/\psi} q_\pi}{32\pi^3 M_X^2} \frac{1}{3} \sum_{\text{spin}} |\mathcal{M}_{X \rightarrow J/\psi\pi^+\pi^-}|^2, \quad (12)$$

where  $M_X = 3871.69 \text{ MeV}$  is the mass of the  $X(3872)$ ,  $p_{J/\psi}$  is the momentum of  $J/\psi$  in the  $X(3872)$  rest frame,  $\sum_{\text{spin}}$  corresponds to the sum over the polarizations of the  $X(3872)$  and  $J/\psi$ , and the amplitude  $\mathcal{M}_{X \rightarrow J/\psi\pi^+\pi^-}$  is given by Eq. (6).

Using Eq. (12), averaged over each bin of  $m_{\pi^+\pi^-}$ , we performed a fit to the corresponding  $\pi^+\pi^-$  distribution data reported by the LHCb Collaboration [59] to determine the parameters:  $\mathcal{N}$ , which sets a global normalization constant,  $\alpha$ , corresponding to the slope of the linear polynomial  $P(s)$  in front of the Omnès function  $\Omega(s)$ , and  $R_X$ , which defines the ratio between the  $X(3872)$  couplings to the  $J/\psi\rho$  and  $J/\psi\omega$  channels. Moreover, in order to perform the fit, we have considered the experimental energy resolution as well as the efficiency reported in Ref. [59]. For comparison, we also perform a fit using the BW parametrization in Eq. (8).

TABLE I. Results from the best fit to the LHCb data [59] using the Omnès or BW parametrization for the  $\rho$  meson. The uncertainties are propagated from the  $1\sigma$  statistical errors of the data.

Parametrization	$\alpha \text{ (GeV}^{-2}\text{)}$	$R_X$	$\chi^2/\text{d.o.f.}$
Omnès	$0.70 \pm 0.32$	$0.26 \pm 0.03$	1.29
BW	$1.30 \pm 0.47$	$0.30 \pm 0.03$	1.32

The best fits lead to  $\chi^2/\text{d.o.f.} = 1.29$  for the Omnès parametrization and 1.32 for the BW parametrization, where d.o.f. denotes the number of degrees of freedom. The parameter values obtained from the fit, together with the corresponding  $1\sigma$  uncertainties propagated from the statistical errors of the data, are listed in Table I. We have checked that the parameters, within the uncertainties, are insensitive to the energy dependence of the  $\omega$  decay width, as concluded in Ref. [60], although in that analysis the energy-dependent case provided a slightly larger value than the constant one. The central value of  $R_X$  obtained from our fit using the Omnès parametrization is smaller than the one extracted in Ref. [59],  $0.29 \pm 0.04$ , which is close to our results using the BW parametrization. As the Omnès parametrization, which contains not only the  $\rho$  pole but also regular terms, is more proper than the BW one, the value of  $R_X$  extracted with the Omnès parametrization is regarded as our final result. The visible difference of the central values shows the importance of using a more proper parametrization.

In Fig. 1, we show the comparison between the line shape of the  $\pi^+\pi^-$  distribution in Eq. (12) (red solid line) and the corresponding spectrum measured by LHCb (black dots with error bars) [59]. The almost invisible error band is the  $1\sigma$  error region corresponding to the uncertainties of the fitted parameters. An excellent agreement with the data is obtained across the entire mass range of the spectrum, including the high-energy region around the peak at 770 MeV, which is dominated by the  $\rho$  meson. This behavior becomes more evident when analyzing the line shape of the  $\pi^+\pi^-$  distribution considering only the  $\rho$  contribution (blue dashed line), highlighting a peak precisely in the region where the  $\rho$  should dominate the spectrum. It is important to emphasize that the  $\rho$  contribution arises naturally in our amplitude, as it is fully encoded in the pion-pion rescattering effects captured by the Omnès function  $\Omega(s)$ . Furthermore, the green dot-dashed line in Fig. 1 corresponds to the line shape solely due to the  $\omega$  resonance, which, although small compared to the  $\rho$  meson one, is still sizeable to the spectrum under study.

### IV. PREDICTIONS

#### A. The $X \rightarrow J/\psi\pi^+\pi^0\pi^-$ spectrum

Once  $R_X$  is extracted from the data, it can be used to predict the line shape of the  $\pi^+\pi^0\pi^-$  mass distribution in the decay  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$ . In particular, we will

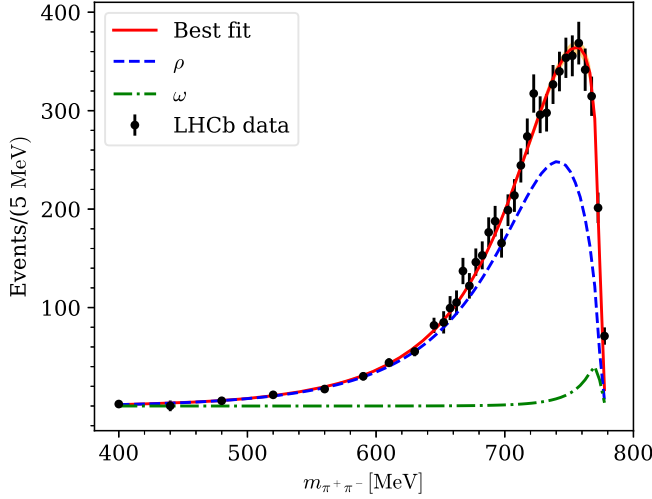


FIG. 1. Comparison of the best fit result of the  $\pi^+\pi^-$  invariant mass distribution (red solid line), given in Eq. (12), with the corresponding data from the LHCb Collaboration [59]. The almost invisible band corresponds to the  $1\sigma$  error region. The blue dashed line corresponds solely to the  $\rho$  meson contribution, while the green dot-dashed one features the  $\omega$  contribution to the spectrum, obtained by dropping the unity inside the square brackets in Eq. (6). Note that due to interference, the red distribution is not equal to the sum of the blue and green ones.

follow the discussion in Ref. [58] in defining the amplitude  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$ . In this case, the amplitude can be divided into two contributions: one due to the  $\omega$  resonance and the other due to the  $\rho$  resonance via isospin breaking, where the quantity  $R_X$  enters. Thus, around the peak of the distribution, which is also close to the  $\omega$  pole, we have

$$\mathcal{M}_{X \rightarrow J/\psi\omega} = g_{XJ/\psi\omega} \epsilon_{ijk} \epsilon_X^i \epsilon_\psi^{*j} \epsilon_\omega^{*k} (1 - \epsilon_{\rho\omega} R_X G_\rho), \quad (13)$$

where  $g_{XJ/\psi\omega}$  represents the coupling of  $X(3872)$  to  $J/\psi\omega$ . The differential decay width corresponding to the  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  decay via the  $\omega$  intermediate state reads

$$\frac{d\Gamma_{X \rightarrow J/\psi 3\pi}}{dm_{3\pi}} = \frac{1}{4\pi^2 M_X^2} \frac{1}{3} \sum_{\text{spin}} |\mathcal{M}_{X \rightarrow J/\psi\omega}(m_{3\pi}^2)|^2 p_{J/\psi} \times |G_\omega(m_{3\pi}^2)|^2 m_{3\pi}^2 \Gamma_{\omega \rightarrow 3\pi}(m_{3\pi}), \quad (14)$$

with  $\mathcal{M}_{X \rightarrow J/\psi\omega}$  the amplitude given by Eq. (13) and  $\Gamma_{\omega \rightarrow 3\pi}$  defined in Eq. (B3).

Figure 2 shows our prediction for the  $\pi^+\pi^0\pi^-$  spectrum from the four-body  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  decay. As can be seen, it exhibits a sharp peak in the high-energy part of the distribution, which then abruptly drops off due to the phase-space boundary. In this region, the distribution is supported only by a small portion of the  $\omega$  pole (the vertical gray dashed line shows the nominal  $\omega$  mass), specifically from its tail [36], since the  $\omega$  nominal mass lies outside the

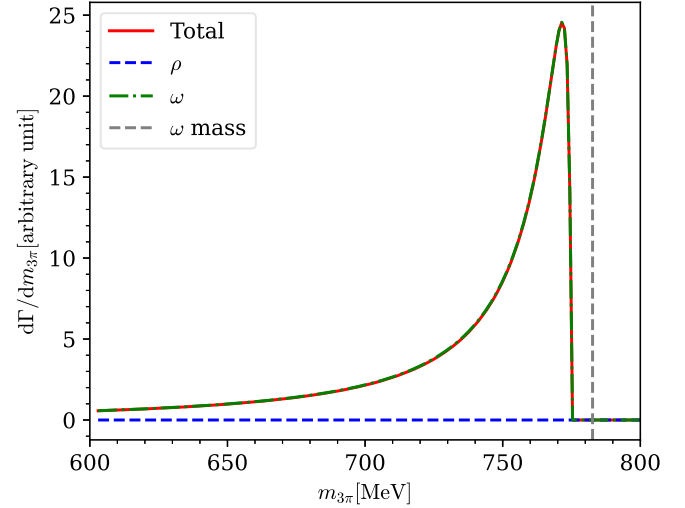


FIG. 2. Prediction for the  $\pi^+\pi^0\pi^-$  invariant mass distribution of the  $X(3872)$  decay, as given by Eq. (14). The  $1\sigma$  error band from the errors of the parameters is too narrow to be seen. The gray dashed one locates the nominal  $\omega$  mass.

physical boundary allowed by the phase space. In addition, unlike the previous case, the contribution from the  $\rho$  meson is very small and does not affect the line shape of the three-pion spectrum, which is not surprising since the  $\rho$  contribution to the  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  is doubly suppressed by the isospin-violating coupling of  $X(3872)$  to  $J/\psi\rho$  and the small  $\rho$ - $\omega$  mixing.

It should be noted that in principle, the isovector state  $W_{c1}$ , proposed to exist in Ref. [54], should also contribute to this spectrum, as well as the  $\pi\pi J/\psi$  spectrum discussed before, as will be explained in the next paragraph. Since this state decays predominantly into  $\rho J/\psi$ , it could lead to a modification of the  $3\pi J/\psi$  spectrum via a mixing from the  $\rho$  to the  $\omega$ , driven by the same amplitude already discussed above. Unfortunately we are not able to generally quantify this impact here, since the production strength of the  $W_{c1}^0$  relative to that of the  $X(3872)$  is reaction dependent.

It is clear that the peak in Fig. 2 is much narrower than the one in the *BABAR* data [5], as observed previously in Ref. [58]. Recently, the BESIII Collaboration reported the  $\pi^+\pi^0\pi^-$  distribution from the  $e^+e^- \rightarrow \gamma\pi^+\pi^0\pi^- J/\psi$  reaction [16]. A narrow peak is clearly visible around 0.78 GeV of the  $\pi^+\pi^0\pi^-$  spectrum, which is mainly due to the  $\omega$  meson. However, the peak contains not only the events from  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$  but also other contributions, such as the  $X(3915) \rightarrow J/\psi\omega$ , and thus a direct comparison of our prediction with the BESIII data is currently not possible.

## B. Updating predictions on the isovector $W_{c1}$

It was predicted in Ref. [54] that there should be isovector  $D\bar{D}^*$  hadronic molecules  $W_{c1}^0$  and  $W_{c1}^\pm$ . The quantum numbers of the neutral member is  $J^{PC} = 1^{++}$ .

The prediction has been backed by recent lattice calculations in Ref. [73].

The inputs of the calculations in Ref. [54] are the  $X(3872)$  mass and the value of  $R_X$  reported by LHCb [59]. With the new  $R_X$  value in Table I, we update the predictions here (for details of the calculations, we refer to Ref. [54]). All the poles are located on the unphysical Riemann sheets (RSs) of the corresponding scattering  $T$  matrix. We use the signs of the imaginary part of the c.m. three-momenta to denote the RSs. The  $W_{c1}^0$  pole is located on  $RS_{+-}$  (i.e., the fourth RS) of the  $C = +D^0\bar{D}^{*0}-D^+D^{*-}$  coupled-channel  $T$  matrix, and the  $W_{c1}^-$  pole is located on  $RS_-$  (i.e., the second RS) of the  $G = +D^0D^{*-}$  single-channel  $T$  matrix. The pole positions are

$$\begin{aligned} W_{c1}^0 &: 3881.7_{-0.7}^{+1.0} + i(1.2_{-0.7}^{+0.8}) \text{ MeV}, \\ W_{c1}^\pm &: 3862.5_{-10.3}^{+6.4} - i(0.07 \pm 0.00) \text{ MeV}, \end{aligned} \quad (15)$$

where we have only shown the  $W_{c1}^0$  pole on the upper half energy plane, which is closer to the physical region than the one in the lower half plane [74].

The  $W_{c1}^0$  pole is  $(10.0_{-0.7}^{+1.0})$  MeV above the  $D^0\bar{D}^{*0}$  threshold and  $(1.8_{-0.7}^{+1.0})$  MeV above the  $D^+D^{*-}$  threshold. The  $W_{c1}^-$  pole is  $13.3_{-6.4}^{+10.3}$  MeV below the  $D^0D^{*-}$  threshold. It is compatible with the lattice QCD result  $6.7_{-6.7}^{+19.5}$  MeV obtained with a pion mass about 280 MeV in Ref. [73]. There is also a shadow pole [74,75] of the  $X(3872)$  at  $3861.2_{-10.1}^{+6.2} - i(0.17_{-0.03}^{+0.02})$  MeV on  $RS_{--}$  (i.e., the third RS) in the  $D^0\bar{D}^{*0}-D^+D^{*-}$  coupled-channel  $T$  matrix.

## V. SUMMARY

Using dispersion theory to implement the  $\pi^+\pi^-$  FSI in the decay  $X(3872) \rightarrow J/\psi\pi^+\pi^-$ , we performed an analysis of recent data from the LHCb Collaboration to reinvestigate the isospin breaking effects in this reaction and extracted the ratio between the couplings of  $X(3872)$  to the  $J/\psi\rho$  and  $J/\psi\omega$  channels, encoded in the parameter  $R_X$ . The parameter provides a measure of isospin violation at the  $X(3872) \rightarrow J/\psi V$  vertex ( $V = \rho, \omega$ ). Our result for  $R_X = 0.26 \pm 0.03$  is valuable to determine the LECs of the  $D\bar{D}^*$  interaction. With the extracted  $R_X$  value, we updated the predictions on the isovector  $J^{PC} = 1^{++}$   $W_{c1}$  poles.

Additionally, we made predictions for the  $3\pi$  invariant mass distribution in the four-body decay  $X(3872) \rightarrow J/\psi\pi^+\pi^0\pi^-$ . Measurements of this observable are accessible in experiments such as BESIII. Note that there should also be a contribution from the decay of the predicted  $W_{c1}^0$  to the spectra discussed in this work. However, a quantitative prediction for this effect needs additional knowledge about the relative production strength of  $X(3872)$  and  $W_{c1}^0$  in a given process, which could in principle be deduced

from an analysis of improved data hopefully available in the near future.

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## DATA AVAILABILITY

No data were created or analyzed in this study.

## APPENDIX A: $\rho$ - $\omega$ MIXING ANGLE

Let  $\tilde{\epsilon}_{\rho\omega}$  represent the mixing in Eq. (3.3) of Ref. [76], where the one-photon pole contribution is excluded,

$$F_{\pi}^{V,e^+e^-}(s) = \left(1 + \tilde{\epsilon}_{\rho\omega} \frac{s}{M_\omega^2 - s - iM_\omega\Gamma_\omega}\right) F_{\pi}^V(s), \quad (A1)$$

whose value was determined to be

$$\tilde{\epsilon}_{\rho\omega} = \begin{cases} 2.00(7) \times 10^{-3}, \\ 1.99(3) \times 10^{-3}, \end{cases} \quad (A2)$$

in Refs. [76] and [72], respectively. In the following, we use the most updated value, i.e., the one in the second line. Adding back the one-photon contribution, the complete  $\rho$ - $\omega$  mixing angle,  $\theta_{\rho\omega}$ , reads

$$\theta_{\rho\omega} = \tilde{\epsilon}_{\rho\omega} - e^2 g_{\gamma\omega}^2 = [2.00(7) - 0.34(0)] \times 10^{-3} \quad (A3)$$

$$= 1.66(7) \times 10^{-3}, \quad (A4)$$

where we have used the values of the partial decay width of  $\rho/\omega \rightarrow e^+e^-$  in Ref. [27] to calculate the couplings of photon and vector mesons,

$$g_{\gamma\rho} = \sqrt{\frac{3\Gamma_{\rho \rightarrow e^+e^-}}{4\pi\alpha^2 m_\rho}} = 0.201(1), \quad (A5)$$

$$g_{\gamma\omega} = \sqrt{\frac{3\Gamma_{\omega \rightarrow e^+e^-}}{4\pi\alpha^2 m_\omega}} = 0.0606(9). \quad (A6)$$

The above  $\theta_{\rho\omega}$  in Eq. (A4) is related to the  $\epsilon_{\rho\omega}$  parameter in Eq. (8) as

$$\epsilon_{\rho\omega} = \theta_{\rho\omega} \frac{g_{\gamma\omega}}{g_{\gamma\rho}} m_\omega^2 = 3.35(8) \times 10^{-3} \text{ GeV}^2, \quad (\text{A7})$$

with a relative error of about 2%.

For comparison, using the formulae in Ref. [58] and  $\text{Br}(\omega \rightarrow 2\pi) = 1.52(8)\%$  extracted in Ref. [77], we get

$$\epsilon_{\rho\omega} \approx \sqrt{m_\omega m_\rho \Gamma_\rho \Gamma_{\omega \rightarrow 2\pi}} = 3.43(10) \times 10^{-3} \text{ GeV}^2, \quad (\text{A8})$$

with a relative error of about 3%. The two values agree with each other within  $1\sigma$ , and the difference in the central values is about  $(3.43 - 3.35)/3.40 \approx 2\%$ .

The uncertainty of  $\epsilon_{\rho\omega}$  has little influence compared to the statistical uncertainties of  $R_X$  from the fitting, which is about 10%, and thus we can safely ignore it.

## APPENDIX B: ENERGY DEPENDENCE OF $\Gamma_\omega$

For the  $\omega$  decay width, we consider two modes,  $\pi^+\pi^0\pi^-$  and  $\pi\gamma$ , with the branching fractions  $\mathcal{B}[\omega \rightarrow 3\pi] = 89.2\%$  and  $\mathcal{B}[\omega \rightarrow \pi\gamma] = 8.35\%$  [27],

$$\Gamma_\omega(m) = \Gamma_{\omega \rightarrow 3\pi}(m) + \Gamma_{\omega \rightarrow \pi^0\gamma}(m). \quad (\text{B1})$$

For the  $\pi\gamma$  mode, we have [58]

$$\Gamma_{\omega \rightarrow \pi\gamma}(m) = \Gamma_{\omega \rightarrow \pi\gamma}^{(0)} \left[ \frac{M_\omega(m^2 - M_\pi^2)}{m(M_\omega^2 - M_\pi^2)} \right]^3, \quad (\text{B2})$$

with  $\Gamma_{\omega \rightarrow \pi\gamma}^{(0)} = 0.725 \text{ MeV}$ . For the  $\pi^+\pi^0\pi^-$  mode, we follow Refs. [78,79] and have

$$\Gamma_{\omega \rightarrow 3\pi}(m) = \frac{m}{192\pi^3} \int_{E_+^{\min}(m)}^{E_+^{\max}(m)} dE^+ \int_{E_-^{\min}(m, E_+)}^{E_-^{\max}(m, E_+)} dE^- \mathcal{E}(m, E_+, E_-) |F(m, E_+, E_-)|^2, \quad (\text{B3})$$

where  $E_+$  and  $E_-$  correspond to the c.m. energies of the outgoing  $\pi^+$  and  $\pi^-$ , respectively, and

$$\mathcal{E}(m, E_+, E_-) = (E_+^2 - M_{\pi^+}^2)(E_-^2 - M_{\pi^+}^2) - \frac{1}{4} [m^2 - 2m(E_+ + E_-) + 2E_+E_- + 2M_{\pi^+}^2 - M_{\pi^0}^2], \quad (\text{B4})$$

with

$$\begin{aligned} E_+^{\min} &= M_{\pi^+}, E_+^{\max}(m) = \frac{m^2 - M_{\pi^0}(2M_{\pi^+} + M_{\pi^0})}{2M}, \\ E_-^{\max, \min}(m, E_+) &= \frac{1}{2(m^2 + M_{\pi^+}^2 - 2ME_+)} ((m - E_+)(m^2 + 2M_{\pi^+}^2 - M_{\pi^0}^2 - 2mE_+) \\ &\quad \pm \{(E_+^2 - M_{\pi^+}^2)[m^2 + M_{\pi^0}(2M_{\pi^+} - M_{\pi^0}) - 2mE_+][m^2 - M_{\pi^0}(2M_{\pi^+} + M_{\pi^0}) - 2mE_+]\}^{1/2}). \end{aligned} \quad (\text{B5})$$

The expression for the amplitude  $F(m, E_+, E_-)$  reads

$$F(m, E_+, E_-) = -\frac{3}{4\pi^2} \frac{g_{\rho\pi^+\pi^-}^3}{F_\pi} \sum_{a=\pm,0} G_\rho(Q_a^2), \quad (\text{B6})$$

with  $G_\rho$  the propagator of  $\rho$  in the BW form given in Eq. (9),<sup>2</sup> and

$$Q_\pm^2 = m^2 + M_{\pi^+}^2 - 2mE_\pm, \quad (\text{B7})$$

$$Q_0^2 = M_{\pi^+}^2 - m^2 + 2m(E_+ + E_-), \quad (\text{B8})$$

where  $F_\pi = 92.1 \text{ MeV}$  is the pion decay constant, and the  $\rho\pi\pi$  coupling constant  $g_{\rho\pi^+\pi^-}$  can be fixed by the experimental  $\rho \rightarrow \pi^+\pi^-$  width as  $g_{\rho\pi^+\pi^-}^2/4\pi \simeq 0.50$ . The running decay width of  $\rho$  reads [79] (see also Ref. [80])

$$\Gamma_\rho(m) \simeq \Gamma_{\rho \rightarrow 2\pi}(m) = \Gamma_\rho(M_\rho) \frac{M_\rho^2}{m^2} \left( \frac{m^2 - 4M_\pi^2}{M_\rho^2 - 4M_\pi^2} \right)^{3/2}, \quad (\text{B9})$$

<sup>2</sup>Since the energy dependence of the width from the  $\omega$  meson is tiny, it is safe here to use the BW form for the  $\rho$  propagator.

since the  $\rho$  decays primarily into  $\pi^+\pi^-$  with  $\mathcal{B}[\rho \rightarrow \pi^+\pi^-] \simeq 100\%$ .



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