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COMMENT • OPEN ACCESS

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Abstract

Recently, Schreiber *et al* (2024 *New J. Phys.* **26** 043012) have presented the theory of an electron lens acting through the Aharonov–Bohm effect in the field-free opening of a static toroidal magnetic field. In this Comment, three key results of this theory are shown to be incorrect due to six basic calculation errors. The errors are corrected, and three correspondingly modified key results are derived. These results are discussed with respect to the hypothetical lens effect and the underlying Aharonov–Bohm effect.

1. Introduction

Recently, Schreiber *et al* [1] have proposed a concept for an electron-optical lens, in which an electron beam propagates through the field-free opening of a static toroidal magnetic field. It is claimed that the hypothetical lens effect is caused by an Aharonov–Bohm effect [2] in the magnetic vector potential of the torus opening. Three theoretical results are presented in [1] as evidence for the hypothetical lens effect: a classical equation of motion, plots of numerically calculated classical trajectories, and plots of numerically calculated phase profiles.

In this Comment it is shown that these key results are incorrect due to six basic calculation errors. The errors, one in the use of differential calculus and five programming errors, are reported in the following three sections. Then, modified key results are reported, which were obtained after correction of all errors. Finally, the modified results are discussed with respect to the hypothetical lens effect [1].

Due to the nature of the errors, no specific physical theory is involved in the error analysis, based entirely on the equations in the supplementary material of [1] and the scripts used for the numerical calculations [3]. A set of essential equations and instructions from one of the scripts that are central to the analysis are reproduced in this Comment, avoiding the reproduction of large parts of the contents of [1] and [3]. It is, however, recommended to place the supplementary material of [1] and the scripts [3] side by side and use the following three sections as a guide to the analysis. In the following, 'figure M1' denotes figure 1 of the main text [1], 'figure S…' and '(S…)' a figure and an equation in the supplementary material of [1], respectively.

2. Classical equation of motion

In [1] a classical equation of motion for an electron in a field-free region is derived. Cylindrical coordinates r, θ, z for radius, azimuth, and axial coordinate of the electron position are employed. The respective components of the magnetic vector potential $\boldsymbol{A}$ are A_r, A_θ, A_z. The result for the radial motion $\theta = \text{const}$ is (S34),

$$\frac{p_0 r''}{(1+r'^2)^{3/2}} = q r' \frac{\partial A_r}{\partial r}, \tag{1}$$

with p_0 the constant kinetic momentum, q the electron charge, and a prime the derivative $\mathrm{d}/\mathrm{d}z$.

The details of the derivation, which are not set out in [1], are shown now in order to check the result (1). For the radial motion $\theta = \text{const}$, with $\theta' = 0$, the reduced refractive index (S32) is

$$n_{\text{red}} = p_0\sqrt{1+r'^2} + q\left(r'A_r + A_z\right). \tag{2}$$

Then, equation (S33)

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial n_{\text{red}}}{\partial r'} = \frac{\partial n_{\text{red}}}{\partial r} \tag{3}$$

yields

$$\frac{p_0 r''}{\left(1+r'^2\right)^{3/2}} = q\left(\frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z}\right) \tag{4}$$

through direct calculation of the partial and total derivatives

$$\frac{\partial n_{\text{red}}}{\partial r} = q\left(r'\frac{\partial A_r}{\partial r} + \frac{\partial A_z}{\partial r}\right), \tag{5}$$

$$\frac{\partial n_{\text{red}}}{\partial r'} = p_0\frac{r'}{\sqrt{1+r'^2}} + qA_r, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{r'}{\sqrt{1+r'^2}} = \frac{r''}{\left(1+r'^2\right)^{3/2}}, \tag{7}$$

$$\frac{\mathrm{d}A_r}{\mathrm{d}z} = \frac{\mathrm{d}r}{\mathrm{d}z}\frac{\partial A_r}{\partial r} + \frac{\partial A_r}{\partial z} = r'\frac{\partial A_r}{\partial r} + \frac{\partial A_r}{\partial z}. \tag{8}$$

With the azimuthal component of the magnetic field $\boldsymbol{B} = \text{rot}\boldsymbol{A}$,

$$B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \tag{9}$$

and $B_\theta = 0$ due to the magnetic field vanishing outside the torus, equation (4) yields the radial equation of motion

$$r'' = 0. \tag{10}$$

The radial equation of motion (1) presented in [1] differs from equation (10) and therefore is evidently incorrect. Since the details of the derivation are not shown in [1], the exact nature of the error leading to (1) is unclear. The error most probably lies in the incorrect substitution

$$\frac{\mathrm{d}A_r}{\mathrm{d}z} = \frac{\partial A_r}{\partial z} \tag{11}$$

of the total derivative (8), which leads to (1) if all other substitutions are made as above.

3. Numerically calculated classical trajectories

In [1] equation (3) is numerically solved for trajectories initially directed parallel to the z-axis, with the initial values

$$r(z_0) = r_0, \qquad r'(z_0) = 0, \tag{12}$$

at an axial coordinate z_0. Plots of numerically calculated sets of curved trajectories through the torus opening of the lens are displayed in figure S2.

The respective numerical calculations in [1] were performed with the Mathematica script 'ray-trajectories-torus.nb' [3], which contains the following three errors.

Firstly, the function *Blz*, representing the corresponding component in equation (S22) of the model for the magnetic vector potential (S19), exhibits a wrong additional factor $r[z]$ in the denominator.

Secondly, the Mathematica functions *EllipticK* and *EllipticE* are called with the wrong argument k_c (S23), instead of k_c^2 following the different convention for the argument k used in the functions (S24) and (S25). Note: these contain a typographic error with the term $\sin\theta'$ instead of the correct $\sin^2\theta'$.

Thirdly, the function *nred*, representing the reduced refractive index (2), exhibits the wrong term $\sqrt{1+r'}$ instead of the correct $\sqrt{1+r'^2}$.

Through these errors, all sets of curved trajectories through the torus opening of the lens, presented in [1], are incorrect.

Table 1. Variables and corresponding signs in the MATLAB script 'torus_calculations.m' and their equivalents in the loop integral (13). Wrong signs in bold.

Script variable	Sign of sum	Term in (13)	Sign of integral in (13)
rpathbottom	**negative**	1st	positive
rpathtop	**positive**	2nd	negative
opticpath	positive	3rd	positive
zpath	negative	4th	negative

4. Numerically calculated phase profiles

In [1] phase profiles across the toroidal lens are determined using the loop integral (S26)

$$\frac{\hbar}{q_e}\Delta\varphi_r(r_1) = \int_{r_0}^{r_1} A_r(r,z_1)\ \mathrm{d}r - \int_{r_0}^{r_1} A_r(r,z_0)\ \mathrm{d}r + \int_{z_0}^{z_1} A_z(r_0,z)\ \mathrm{d}z - \int_{z_0}^{z_1} A_z(r_1,z)\ \mathrm{d}z, \tag{13}$$

with $\hbar$ the reduced Planck constant and q_e the electron charge. The rectangular loop has radial and axial sides, between r_0 and r_1 and between z_0 and z_1, respectively. Plots of numerically calculated phase profiles $\Delta\varphi_r$ are displayed in figures S6, M1(f) and (g).

The respective numerical calculations in [1] were performed with the MATLAB script 'torus_calculations.m' [3], which contains the following two errors.

Firstly, the radial sums calculated for *rpathbottom* and *rpathtop,* representing approximations of the first and second integral in (13), respectively, receive wrong signs (table 1). The extensive error analysis revealing the sign errors is shown in appendix A.

Secondly, an incorrect term *Bpath,* which contains a double sum approximating the integral

$$\int_{z_0}^{z_1}\int_{r_0}^{r_1} B_\theta(r,z)\ \mathrm{d}r\mathrm{d}z, \tag{14}$$

is added to the value of the loop integral (13). The term shall replace that part of the integral in which the loop runs through the toroidal magnetic field B_θ (S18), where the magnetic vector potential is not given by equation (S19). Yet, through Stokes' theorem (S6), the term (14) is equal to the value of the entire loop integral (13). As a result, the large steps of the phase profiles at the radial positions of the torus are incorrectly doubled in magnitude. This affects the display of all phase profiles in [1], which are normalised to the step magnitude, so that in particular the magnified profiles, figures S6(e), (f) and M1(g), appear at one half of their true magnitude.

Through these errors, all radial phase profiles across the toroidal lens, presented in [1], are incorrect.

Finally, it should be noted that the approximation of the integrals (13) by sums causes a tiny numerical error dependent on the radial and axial step sizes *deltar2D* and *deltaz2D,* respectively. For the phase profiles across the toroidal lens (figures S6, M1(f) and (g)), this error, determined through variation of the step sizes, increases with the radial coordinate from zero at $r/r_l = 0$ to 4.1×10^{-5} at $r/r_l = 0.5$, with r_l the major radius of the torus.

5. Modified results after correction of all errors

5.1. Correct classical equation of motion

The correct equation of motion is (10), which does not depend on the magnetic vector potential.

5.2. Correct classical trajectories

The correct trajectories fulfilling the correct radial equation of motion (10) and the initial conditions (12) are evidently

$$r(z) = r_0. \tag{15}$$

They are invariably straight, run parallel to the z-axis at a constant distance r_0, and are independent of the kinetic momentum p_0 (figure 1).

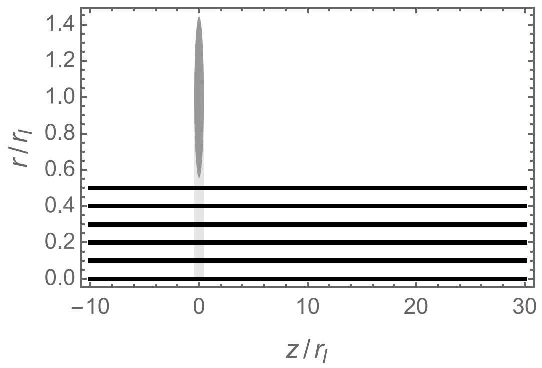

Figure 1. Straight trajectories through the torus opening, directed parallel to the z-axis, with z the axial coordinate, r the radial coordinate, and r_l the major radius of the torus. The torus at $z = 0$, with minor radius $r_s = 0.444\ r_l$, appears distorted due to the different axial and radial scales.

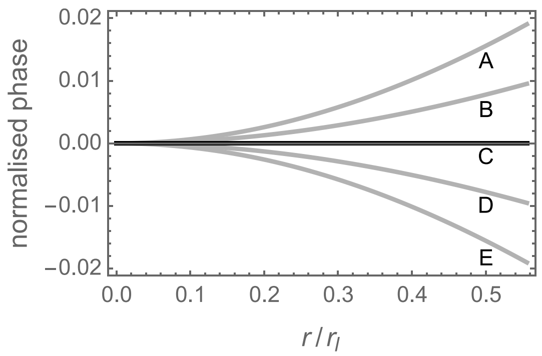

Figure 2. Phase versus radial coordinate r across the torus opening, normalised to the phase step at the major radius r_l of the torus. (A) Wrong signs, correct magnitude. (B) Wrong signs, wrong magnitude. (C) Correct profile according to (13). (D) Wrong signs, wrong magnitude, potential reversed. (E) Wrong signs, correct magnitude, potential reversed.

5.3. Correct phase profile

The correct phase profile according to (13) is flat across the torus opening, and the phase is vanishing, independent of the orientation of the magnetic vector potential (figure 2, profile C). The profile was calculated with the same parameters as in the MATLAB script 'torus_calculations.m' [3] used in [1], but with the correct signs (table 1) and with the correct magnitude.

The impact of the sign errors (table 1) and the magnitude error related to the term (14) is illustrated through the other phase profiles displayed in figure 2. The incorrect profiles (A) and (E), calculated with the wrong signs, have positive and negative curvature, depending on the two opposite orientations of the magnetic vector potential. Their magnitude is twice as large as that of the incorrect, curved profiles (B) and (D), which correspond with the profiles displayed in figures S6(e), (f) and M1(g).

6. Discussion

The modified results, calculated after correction of all errors in [1], indicate that the electron beam propagating through the field-free torus opening is not deflected by the magnetic vector potential. Since the classical trajectories are straight (figure 1) and the phase profile is flat (figure 2, profile C), there is no theoretical evidence for a lens effect.

The original results in [1], containing the calculation errors shown above, comprise curved classical trajectories (figure S2) and curved phase profiles (figure S6, M1(f) and (g)). Therefore, they give the false impression of electrons being deflected by the magnetic vector potential and of a respective lens effect.

Further, the phase (13) is vanishing across the opening (figure 2, profile C), which indicates that the magnetic vector potential is conservative in the cylindrical region $r < r_l - r_s$ of the torus opening. The respective assumption in [1] of a non-conservative magnetic vector potential in the torus opening, which leads to an Aharonov–Bohm effect [2], is obviously incorrect.

Finally, no other theoretical evidence is presented in [1] for the hypothetical lens effect.

It remains to mention a contradiction in the application of Stokes' theorem (S6) in [1] and [3]. On the one hand it is argued in [1] that 'Stokes' theorem cannot actually be applied here due to the magnetic vector potential being non-continuously differentiable when the whole system is considered.' On the other hand, Stokes' theorem is explicitly used in the MATLAB script 'torus_calculations.m' [3] in the attempt to replace that part of the integral (13) in which the loop runs through the toroidal magnetic field by the term (14), see section 4. The contradiction is resolved through the usual convention of electromagnetic theory [4], see appendix B, in which Stokes' theorem is valid throughout, even across the step discontinuity of the magnetic field.

7. Conclusions

The results of the theory in [1], namely the equation of motion (1), the numerically calculated trajectories (figure S2), and the numerically calculated phase profiles (figures S6, M1(f) and (g)), are incorrect due to calculation errors.

There is no theoretical evidence for the hypothetical lens effect [1], a non-conservative magnetic vector potential, or an Aharonov–Bohm effect [2] in the field-free opening of a static toroidal magnetic field.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

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Appendix A. Analysis of sign errors in the script 'torus_calculations.m'

The analysis of the sign errors in the MATLAB script 'torus_calculations.m' [3] (section 4) is based on the compilation of important script variables (table 2) and instructions (table 3), which are labelled below by their line numbers. Central to the data structure of the script is a system of radial and axial indices, which convert to radial and axial coordinates through the arrays *r2D* and *z2D,* respectively.

The correspondence of the sums in the script with the integrals in equation (13), compiled in table 1, is determined in the following way. The sum in line #118 corresponds with the third integral, because the constant radial index $Nr2D/2 + 1$ converts to $r2D(Nr2D/2 + 1)$ and corresponds with the constant radial coordinate $r_0 = 0$ (table 2). Therefore, the lower and upper axial summation indices $lenstopind < lensbottomind$, converting to $z2D(lenstopind) < z2D(lensbottomind)$, correspond with the lower and upper axial integration limits $z_0 < z_1$, respectively (table 2). Then, the sum in line #132 corresponds with the second integral, because the constant axial index *lenstopind* corresponds with the constant axial coordinate z_0. Further, the sum in line #133 corresponds with the first integral, because the constant axial index *lensbottomind* corresponds with the constant axial coordinate z_1. Finally, the remaining sum in line #119 corresponds with the fourth integral.

The signs of the sums (table 1) are determined without the explicit minus sign for the electron charge $-qe$ on the right-hand side of the instructions, in accordance with equation (13) where the electron charge q_e appears on the left-hand side. Then, the comparison of the second and fourth column of table 1 reveals the sign errors of the terms *rpathbottom* and *rpathtop.*

Note: In lines #132 and #133 an incorrect step size *deltaz2D,* instead of *deltar2D,* is used for the radial sums. The error has no effect in the numerical calculations in [1] only because the two values are chosen to be equal.

Appendix B. Stokes' theorem

In [1] it is argued that Stokes' theorem (S6) is invalid in the present case, because the step discontinuity of the magnetic field at the torus surface would cause discontinuous derivatives of the magnetic vector potential there.

In the usual convention of electromagnetic theory [4], however, any step discontinuity can be understood as a limit of a sequence of functions, each of which can be chosen to have continuous first-order derivatives in approaching a step function. In the same way, throughout the limit process, all components of the magnetic vector potential can be chosen to have continuous first-order partial derivatives at the torus

Table 2. Terms and variables in the MATLAB script 'torus_calculations.m' and their equivalents in the loop integral (13).

Script term/variable	Value	Symbol	Comment
qe	$1.60217662 \times 10^{-19}$	e	Elementary charge
		$q_e = -e$	Electron charge in (13)
hbar	$1.0545718 \times 10^{-34}$	$\hbar$	Reduced Planck constant
pathint		$\Delta\varphi_r$	Phase value of the loop integral (13), *rpathbottom + rpathtop + opticpath + zpath + Bpath*
Atmapr		A_r	Radial component of magnetic vector potential
Atmapz		A_z	Axial component of magnetic vector potential
Btmaptheta		B_θ	Azimuthal component of magnetic field, see (14)
Bpath		see (14)	Term causing the magnitude error, see section 4
rl	900×10^{-9}	r_l	Major radius of the torus
rs	400×10^{-9}	r_s	Minor radius of the torus
Nr2D/2 + 1	1025		Constant radial index in sum for *opticpath*, lower index in sums for *rpathtop* and *rpathbottom*
rr			Upper index in sums for *rpathtop* and *rpathbottom*
r2D			Array of radial coordinates
r2D(Nr2D/2 + 1)	0	r_0	Constant radial coordinate of 3rd term in (13), lower integration limit in 1st and 2nd term in (13)
r2D(rr)		r_1	Constant radial coordinate of 4th term in (13), upper integration limit in 1st and 2nd term in (13)
deltar2D	3.515625×10^{-9}	dr	Step size of radial coordinate
lenstopind	1		Constant axial index in sum for *rpathtop*, lower index in sums for *opticpath* and *zpath*
lensbottomind	1400		Constant axial index in sum for *rpathbottom*, upper index in sums for *opticpath* and *zpath*
z2D			Array of axial coordinates
z2D(lenstopind)	-4 *rl*	z_0	Constant axial coordinate of 2nd term in (13), lower integration limit in 3rd and 4th term in (13)
z2D(lensbottomind)	1.46484375 *rl*	z_1	Constant axial coordinate of 1st term in (13), upper integration limit in 3rd and 4th term in (13)
deltaz2D	3.515625×10^{-9}	dz	Step size of axial coordinate

Table 3. Excerpts from the MATLAB script 'torus_calculations.m' for the case $rr > Nr2D/2 + 1$ corresponding with $r_1 > r_0$.

line	instruction
118	*opticpath = −qe/hbar*sum(Atmapz(lenstopind:lensbottomind,Nr2D/2 + 1))*deltaz2D;*
119	*zpath = −1*−qe/hbar*sum(Atmapz(lenstopind:lensbottomind,:),1)*deltaz2D;*
124	*zpathtorusedge = zeros(1,length(r2D));*
132	*rpathtop(rr) = −qe/hbar*sum(Atmapr(lenstopind,Nr2D/2 + 1:rr))*deltaz2D;*
133	*rpathbottom(rr) = −1*−qe/hbar*sum(Atmapr(lensbottomind,Nr2D/2 + 1:rr))*deltaz2D;*
134	*Bpath(rr) = −qe/hbar*sum(sum(Btmaptheta(lenstopind:lensbottomind,Nr2D/2 + 1:rr)))*deltaz2D*deltar2D;*
135	*pathint = opticpath + zpath + rpathtop + rpathbottom + Bpath + zpathtorusedge;*

surface. Further, it should be noted that the idealisation of a step discontinuity does not play a role in real experiments due to the finite extension of real charge and current distributions as causes for electromagnetic potentials and fields.

With the convention [4], Stokes' theorem (S6) is valid without restriction for the case of the toroidal solenoid, even if the surface of integration is extended such that it cuts the torus surface. Therefore, the criticism in [1] regarding Stokes' theorem is unfounded in the present case.

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