An Update on the Isospin-Breaking Effects in the Pion Decay Constant with Staggered Quarks

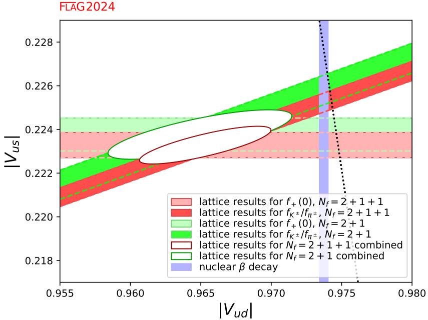
Alessandro Cotellucci

on behalf of the Budapest-Marseille-Wuppertal collaboration

Lattice 2025 Mumbai, November 7th 2025

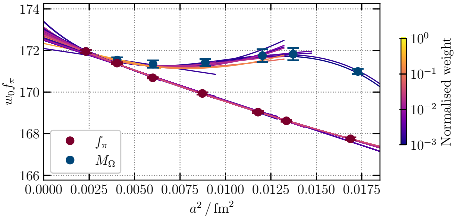
Motivations

FLAG, arXiv preprint (2024)



Tension in the unitarity of the CKM matrix.

F. Stokes, Lattice2025 plenary Nov. 3rd 11:15



Scale setting purpose:
BMW iso QCD $w_0 f_\pi = 0.11438(12)(23)[26] \to 0.2\%$ Precision (Preliminary)

Definitions

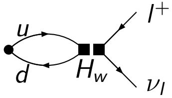
In iso QCD it is computed from the axial current:

$$\langle 0|\, \bar{u}\gamma_4\gamma_5 d\, |\pi(\vec{0})\rangle = M_\pi f_\pi .$$

In QCD+QED it is computed from:

$$F_\pi^2 = \frac{\Gamma_{\pi^+\to l^+\nu_l(\gamma)}}{\frac{G_F^2}{8\pi}|V_{ud}|^2 M_\pi m_l^2 \left(1-\frac{m_l^2}{M_\pi^2}\right)^2} = f_\pi^2\left[1+\delta R_\pi\right]$$

The electroweak Hamiltonian:



$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V_{ud}^* \left[\bar{d}\gamma^\mu (1-\gamma_5)\, u\right]\left[\bar{\nu}_l\gamma_\mu (1-\gamma_5)\, l\right].$$

$\delta R_{K\pi}$ RM123S D. Giusti et al., Phys. Rev. Lett. **120**, 072001 (2018)
$\delta R_\pi, \delta R_K$ RM123S M. Di Carlo et al., Phys. Rev. D **100**, 034514 (2019)
$\delta R_{K\pi}$ RBC/UKQCD P. Boyle et al., JHEP **02**, 242 (2023)
$\delta R_{K\pi}$ RBC/UKQCD X.-Y. Tuo et al., KAON2025 presentation

Current Mixed Determination

Current parametrization:

$$w_0 F_\pi = A + B F_\pi^{-2} M_{\pi^+}^2 + C F_\pi^{-2} \left(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2 \right)/2 \\ + E e_v^2 + F e_v e_s + G e_s^2$$

e_v : valence electric charge, e_s : sea electric charge,

$B,\ C$: mass derivatives

$E,\ F,\ G$: electromagnetic derivatives

Once A, B, C, G, F are computed we can separate:

$$\underbrace{[w_0 F_\pi]_{\mathrm{QCD+QED}}}_{\text{scheme independent}} = \underbrace{[w_0 F_\pi]_{\mathrm{QCD+seaQED}}}_{\text{scheme dependent}} + \underbrace{[w_0 F_\pi]_{\mathrm{valQED}}}_{\text{scheme dependent}}$$

w_0 has not valence quak isopsin-breaking effects.

Current Mixed Determination

The QCD from RM123S is defined in the GRS scheme:

$$\begin{aligned}[M_\pi]^{\mathrm{GRS}} &= 135.0(2)\ \mathrm{MeV} \\ \left[\frac{1}{2}\left(M_{K^\pm} + M_{K^0}\right)\right]^{\mathrm{GRS}} &= 494.6(1)\ \mathrm{MeV} \\ [f_\pi]^{\mathrm{GRS}} &= 130.65(12)\ \mathrm{MeV} \\ [\delta R_\pi]^{\mathrm{GRS}} &= 0.0153(19).\end{aligned}$$

The quantities in the GRS scheme are computed as:

$$[*]^{\mathrm{GRS}} = [*]^{\mathrm{phys}}_{\mathrm{QCD+QED}} - [*]^{\mathrm{GRS}}_{\mathrm{valQED}} = [*]^{\mathrm{GRS}}_{\mathrm{QCD+seaQED}}\,.$$

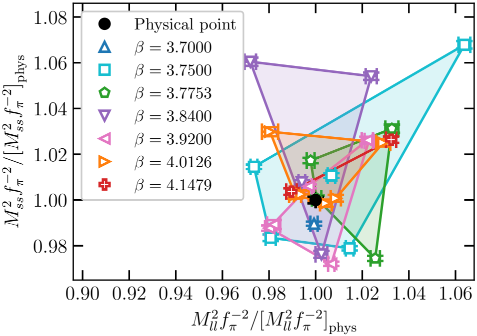
We can combine the BMW and RM123 results using the parametrization:

$$[w_0 F_\pi]_{\mathrm{QCD+QED}} = [w_0 F_\pi]^{\mathrm{GRS}}_{\mathrm{QCD+seaQED}} \sqrt{1 + [\delta R_\pi]^{\mathrm{GRS}}}.$$

Simulation Details

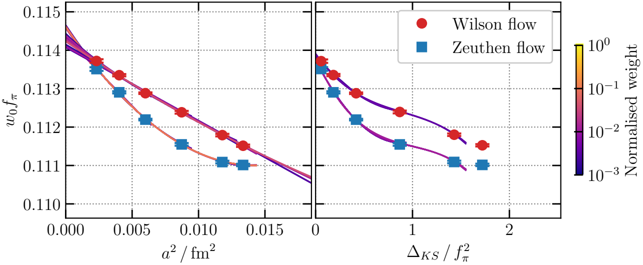
- QCD tree level Symanzik Action;
- $N_f = 2+1+1$ Staggered fermions;
- 4 levels of stout smearing $\rho = 0.125$;
- $L \approx 6$ fm;

F. Stokes, Lattice2025 plenary Nov. 3rd 11:15



Iso QCD determination in FLAG scheme

Preliminary Continuum Extrapolation



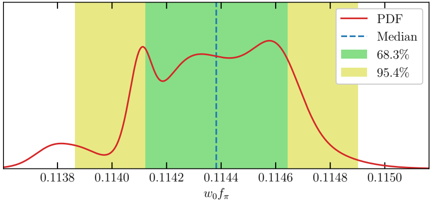


Figure: Continuum limit extrapolation of the iso QCD value of $w_0 f_\pi$.

Iso QCD determination in FLAG scheme

Preliminary Error Budget

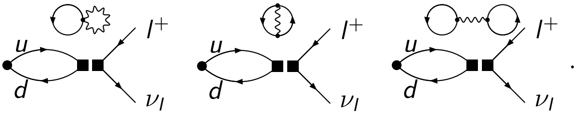
Median	0.11438	
Total error	0.00026	0.22736 %
Statistical error	0.00012	0.10786 %
Systematic error	0.00023	0.20015 %
Type of gradient flow	0.00013	0.10966 %
Pseudoscalar fits	0.00004	0.03144 %
Finite volume XPT order	0.00001	0.01023 %
Lattice spacing cuts	0.00006	0.04956 %
Order of fit polynomials	0.00019	0.16215 %

Sea-sea Contribution

The renormalized sea-quark electromagnetic derivative is computed from:

$$G = \left[w_0 F_\pi - B F_\pi^{-2} M_{\pi^+}^2 - C F_\pi^{-2} \left(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2\right)/2\right]_{\text{sea-sea}} .$$

The sea-sea diagrams are:



They have been estimated using the following ensembles and QED_L.

action	β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
4stout	3.7000	0.1315	24×48	volume/24	0.057291	27.899	904
			48×64	volume/48	0.057291	27.899	904
	3.7753	0.1116	56×84	dir00	0.047615	27.843	510
				dir01	0.048567	28.400	726
				dir02	0.046186	26.469	300
				dir03	0.049520	27.852	887
	3.8400	0.0952	64×96	dir00	0.043194	28.500	1110
				dir02b	0.043194	30.205	1072
				dir04	0.040750	28.007	1036
				dir05	0.039130	26.893	1035
4hex	0.7300	0.1120	56×84	phys2/56	0.06061	33.728	1305

Sea-sea Contribution

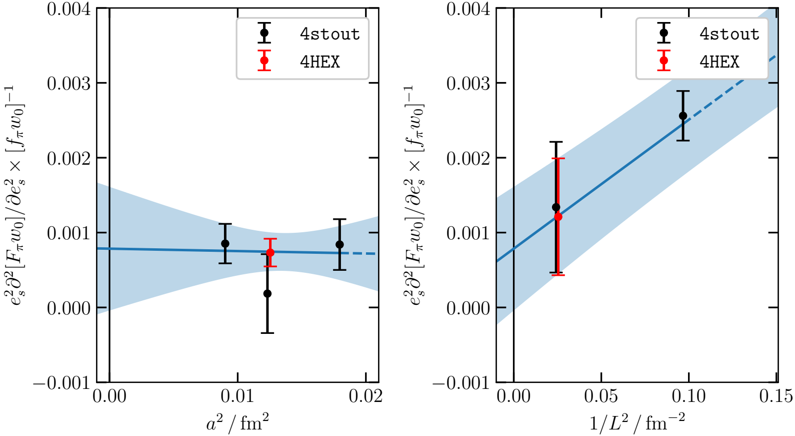


Figure: Renormalized sea quark electromagnetic derivative of $w_0 F_\pi$ divided by its QCD value.

Error Budget of the current determination

The QCD+QED value of $w_0 F_\pi$:

$$[w_0 F_\pi]_{\text{QCD+QED}} = 0.11527(15)(26)[31] \text{ (Preliminary)}.$$

The systematic error is:

$$(26) = (23)_{\text{QCD}}(11)_{\text{QED-Valence}}(5)_{\text{QED-Sea}} \text{ (Preliminary)}.$$

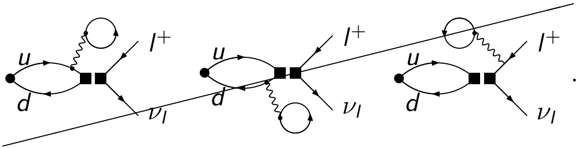
Combined with the physical value of F_π we get:

$$[w_0]_{\text{QCD+QED}} = 0.17270(22)(40)[46] \text{ fm (Preliminary)}$$

The systematic error is:

$$(40) = (34)_{\text{QCD}}(17)_{\text{QED-Valence}}(8)_{\text{QED-Sea}}(7)_{\text{exp}} \text{ (Preliminary)}$$

Sea-valence derivative has been estimated to be 20% of the sea-sea derivative.

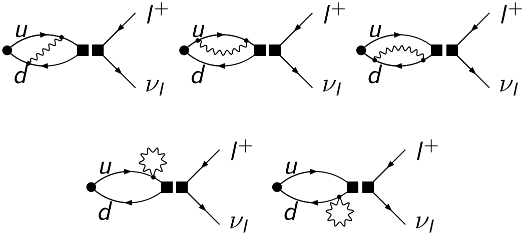


- room for improvement of the QED valence determination;
- independent determination of δR_π and check FVE M. Di Carlo et al., Phys. Rev. D **105**, 074509 (2022);

Valence-valence Contributions

Infrared safe contributions

The valence correction to the axial-pseudoscalar two-point function is:



Extraceted from the effective correction to $w_0 F_\pi$:

$$\frac{G_{ud}^{PA_4}(x_4)^2}{G_{ud}^{PP}(x_4)} = M_\pi Z_A^{-2} F_\pi^2 e^{-M_\pi T/2} \frac{\sinh^2\left[M_\pi\left(T/2 - t - 1/2\right)\right]}{\cosh\left[M_\pi\left(T/2 - t\right)\right]}.$$

action	β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
4stout	3.7000	0.1315	24×48	volume/24	0.057291	27.899	48
			32×64	volume/32	0.057291	27.899	48
			48×64	volume/48	0.057291	27.899	48
			64×64	volume/64	0.057291	27.899	48
	3.8400	0.0952	64×96	dir00	0.043194	28.500	48

Valence-valence Contribution

Infrared safe contribution

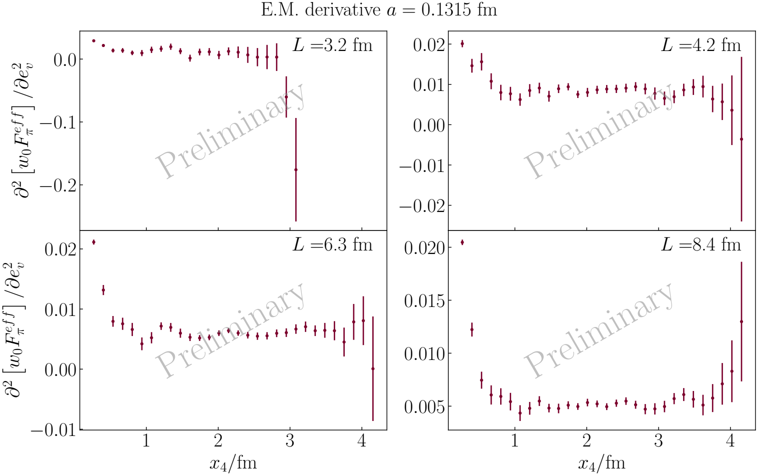


Figure: Infrared safe contribution to the valence electromagnetic derivative of $w_0 F_\pi$ for $a = 0.1315$ fm.

Valence-valence Contribution

Infrared safe contribution

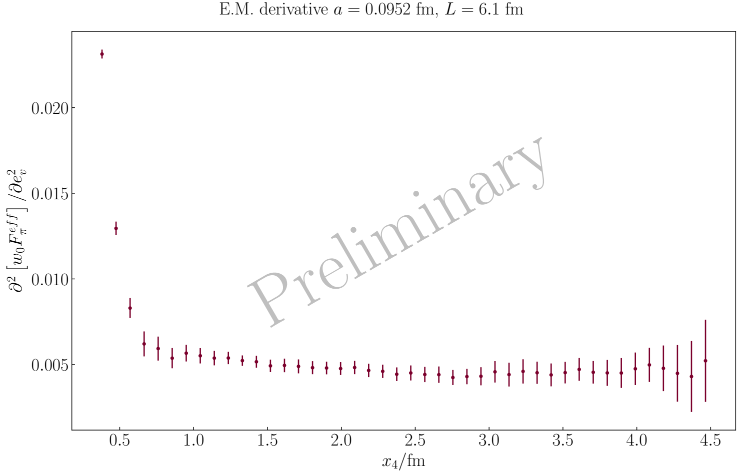


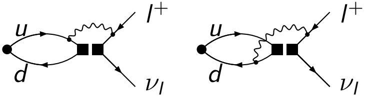
Figure: Infrared safe contribution to the valence electromagnetic derivative of $w_0 F_\pi$ for $a = 0.0952$ fm.

Conclusions and Outlook

- The pion decay constant is still a reliable scale setting quantity for the precise determination of w_0

$$[w_0]_{\mathrm{QCD+QED}} = 0.17270(22)(40)[46]\ \mathrm{fm}\ \text{(Preliminary)};$$

- Current precision of the sea-quark effects promising;
- We are working on the infrared divergent diagrams



;

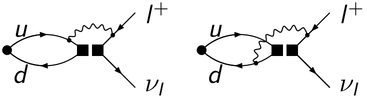
- We are working on the renormalization strategy for the electro-weak hamiltonian;
- We plan a detailed study of the finite volume effects with different volumes (up to 10.8 fm);
- The sea-valence diagrams will be included in the final work;

Conclusions and Outlook

- The pion decay constant is still a reliable scale setting quantity for the precise determination of w_0

$$[w_0]_{\text{QCD+QED}} = 0.17270(22)(40)[46] \text{ fm (Preliminary)};$$

- Current precision of the sea-quark effects promising;
- We are working on the infrared divergent diagrams



;

- We are working on the renormalization strategy for the electro-weak hamiltonian;
- We plan a detailed study of the finite volume effects with different volumes (up to 10.8 fm);
- The sea-valence diagrams will be included in the final work;

Thank you for your attention!

Backup: Set of Ensembles for iso QCD f_π

β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
3.7000	0.1315	48×64	dir00	0.057291	27.899	904
3.7500	0.1191	56×96	dir00	0.049593	28.038	315
			dir01	0.049593	26.939	516
			dir02	0.051617	29.183	504
			dir03	0.051617	28.038	522
			dir05	0.055666	28.083	215
3.7753	0.1116	56×84	dir00	0.047615	27.843	510
			dir01	0.048567	28.400	505
			dir02	0.046186	26.469	507
			dir03	0.049520	27.852	385
3.8400	0.0952	64×96	dir00	0.043194	28.500	510
			dir02b	0.043194	30.205	436
			dir04	0.040750	28.007	1503
			dir05	0.039130	26.893	500

Table: List of the ensembles used in this work, with gauge coupling, lattice spacing, lattice size, ensemble tag, strange-quark mass, mass ratio of strange and light quarks and number of configurations.

Backup: Set of Ensembles for iso QCD f_π

β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
3.9200	0.0787	80×128	dir02	0.032440	27.679	506
			dir04	0.034240	27.502	512
			dir01b	0.032000	26.512	1001
			dir02b	0.032440	27.679	327
			dir03b	0.033286	27.738	1450
			dir04b	0.034240	27.502	500
4.0126	0.0640	96×144	phys1	0.026500	27.634	446
			phys2	0.026500	27.124	551
			phys1b	0.026500	27.634	2248
			phys2b	0.026500	27.124	1000
			phys3	0.027318	27.263	985
			phys4	0.027318	28.695	1750
4.1479	0.0483	128×192	phys1	0.019370	27.630	2792
			phys2	0.019951	27.104	2225

Table: List of the ensembles used in this work, with gauge coupling, lattice spacing, lattice size, ensemble tag, strange-quark mass, mass ratio of strange and light quarks and number of configurations.

Backup: Fit Functions

The fit functions used for the continuum extrapolation respect are polynomials in a^2 and Δ_{KS} following these rules:

- $A\left(a^2, \Delta_{\mathrm{KS}}\right)$ at most cubic in a^2 or Δ_{KS};
- $A\left(a^2, \Delta_{\mathrm{KS}}\right)$ $A_4 a^4 + A' \Delta_{KS}$ not allowed;
- $A\left(a^2, \Delta_{\mathrm{KS}}\right)$ $\left(a^2\right)^n \Delta_{\mathrm{KS}}^m$ as long as no order is skipped and no more than 3 non-constant terms;
- $B\left(a^2, \Delta_{\mathrm{KS}}\right)$ and $C\left(a^2, \Delta_{\mathrm{KS}}\right)$ constant or linear in a^2 or Δ_{KS}.