Three-body universality in the B meson sector

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The charged exotic mesons $Z_b(10610)$ and $Z_b'(10650)$ observed by the Belle collaboration in 2011 are very close to the $B^*\bar{B}$ and $B^*\bar{B}^*$ thresholds, respectively. This suggests their interpretation as shallow hadronic molecules of B and B^* mesons. Using the masses of the $Z_b(10610)$ and $Z_b'(10650)$ as input, we rule out the possibility for universal bound states of three B and B^* mesons arising from the Efimov effect based on their spin-isospin structure. As a consequence, we can predict the phase shifts for the scattering of B and B^* mesons off the exotic mesons $Z_b(10610)$ and $Z_b'(10650)$ to leading order in a non-relativistic effective field theory with contact interactions based on two-body information alone.

I. INTRODUCTION

In 2011, the Belle collaboration reported the discovery of two positively charged mesons in the bottomonium sector, $Z_b(10610)$ and $Z_b'(10650)$ [1]. Their existence was subsequently confirmed by two independent Belle measurements [2, 3]. The masses and widths of these states, as listed in the Review of Particle Physics (RPP), are [4][1]

$$\begin{aligned} M_Z &= (10607.2 \pm 2.0)\ \text{MeV}\ , \quad \Gamma_Z = (18.4 \pm 2.4)\ \text{MeV}\ , \\ M_{Z'} &= (10652.2 \pm 1.5)\ \text{MeV}\ , \quad \Gamma_{Z'} = (11.5 \pm 2.2)\ \text{MeV}\ . \end{aligned} \tag{1}$$

From their production and decay channels [1–3, 5], these mesons must be exotic. Their quark content can not be simply $q\bar{q}$ as for ordinary mesons but must be $b\bar{b}u\bar{d}$. Soon after their discovery it was proposed that their constituents cluster into two bottom mesons which are bound due to hadronic forces [6]. In particular, both Z_b states were interpreted as hadronic molecules with flavor wave functions

$$\begin{aligned} Z_b &= \frac{1}{\sqrt{2}}(B^*\bar{B} + \bar{B}^*B)\ , \\ Z_b' &= B^*\bar{B}^*\ . \end{aligned} \tag{2}$$

For further analyses in this framework and alternative scenarios such as tetraquarks see, e.g., Refs. [7–15] and Refs. [10, 14, 16–20], respectively.

The molecular interpretation of the $Z_b(10610)$ and $Z_b'(10650)$ is supported by the fact that their masses are close to the respective open bottom thresholds defined by the flavor wave functions in Eq. (2). Note that the masses quoted in Eq. (1) are slightly above the corresponding thresholds. A more sophisticated analysis of the invariant mass distributions in an effective field theory with bottom meson loops, however, showed that the Z_b and Z_b' poles are below threshold [11]. This finding is consistent with a recent analysis of the resonance signals based on a formalism consistent with unitarity and analyticity [21–26]. However, the question of whether the Z_b and Z_b' mesons are virtual states, bound states or resonances has not been answered definitely. The interplay of Z_b and Z_b' exchanges with bottom meson loops in Υ decays was further scrutinized in Refs. [27, 28]. For a more detailed discussion of these issues, see the review [29].

Braaten and collaborators argued in Ref. [30] that the closeness to a bottom meson threshold is necessary but not sufficient for the interpretation as a hadronic molecule. They used the Born-Oppenheimer approximation to analyze the substructure of Z_b and Z_b' and concluded that for both states the molecule interpretation is viable [30]. A recent Born-Oppenheimer study based on the Lattice potential suggests the presence of a near-threshold structure in the Z mass range [31]. Similarly, a near-threshold signal in the Z' mass range is pointed out in Ref. [32]. Arguably, one of the most detailed investigations of the Z_b states as hadronic molecules was done in Refs. [11, 12] based on an effective field theory with heavy meson loops that was originally formulated for the charm quark sector [33]. In this framework, a variety of testable predictions to confirm or rule out the molecular nature of these states were given. Some of these predictions will be checked at future high-luminosity experiments.

An analysis of the angular distributions showed that

[1] Note that in the RPP [4], the $Z_b(10610)$ and $Z_b'(10650)$ are denoted as $T_{b\bar{b}1}(10610)$ and $T_{b\bar{b}1}(10650)^+$, respectively.

the quantum numbers $J^P = 1^+$ are favored for the two Z_b states [1]. In addition, their quark content fixes the isospin to be one. Thus, the quantum numbers of both Z_b and Z_b' are $I^G(J^{PC}) = 1^+(1^{+-})$ [4] and the assumption that they are S-wave hadronic molecules of two bottom mesons is tenable. We use an effective field theory with contact interactions to describe the Z_b's. Since their binding momentum $\gamma = \sqrt{2\mu B}$ (with binding energy B and reduced mass of the constituents μ) is much smaller than the pion mass m_π (or at least of that order in case of the Z_b), the constituent bottom mesons which have masses around 5 GeV can be treated as non-relativistic point-like particles which only interact via short-range contact interactions. Thus, one can apply a non-relativistic effective field theory without explicit pions to this system. Similar descriptions of two particle S-wave molecules in the charm sector can be found in Refs. [34, 35] concerning the charm meson molecule $X(3872)$ and in Ref. [36] for the $Z_c(3900)$ whose interpretation as a molecule is still controversial [30].

This so-called pionless EFT contains only contact interactions and was originally developed for nucleons which also display shallow bound states such as the deuteron or the triton [37–40]. The expansion parameter is Q/m_π, where the scale Q is determined by the typical momentum scales of the considered process. Depending on the spin-isospin channel, three-body forces may enter already at leading order in this theory. In the spin-doublet channel of neutron-deuteron scattering, for example, a Wigner-SU(4)-symmetric three-body force is required at leading order for proper renormalization [41–43] and the triton emerges naturally as an Efimov state [44], while three-body forces are strongly suppressed in the spin-quartet channel due to the Pauli principle [45, 46].

The Efimov effect describes the emergence of shallow three-particle bound states (called *trimers*) in a system with resonant interactions characterized by a large scattering length a. It can occur if at least two of the three particle pairs have resonant interactions. In particular, the Efimov effect occurs in systems of a shallow two-particle bound or virtual state of binding momentum $\gamma \sim 1/a$ and a third particle which has resonant interactions with at least one of the constituents of the dimer. For $a \to \infty$, there are infinitely many trimer states with binding energies $B_3^{(n)}$ which are spaced equidistantly [44]: $B_3^{(n+1)}/B_3^{(n)} = \text{const}$. The crucial point is that this constant is universal in the sense that it is independent of the details of the short-range physics in the system. However, its exact value depends on the masses and spin-isospin quantum numbers of the particles as well as the number of resonantly interacting pairs. In a system with finite scattering length, the geometrical spectrum is cut off in the infared and there will only be a finite number of states but the dependence of the states on the scattering length a is also universal.

Whether or not the Efimov effect plays are role in a three-particle system depends on the particular spin-isospin channel. The emergence of the Efimov effect in pionless EFT is closely connected to the requirement of three-body forces for renormalization at leading order.[2] The power counting for three-body forces, in turn, can be obtained from an analysis of the ultraviolet behavior of the corresponding integral equations [42, 43, 48]. The pionless theory contains only contact interactions and is universal. Thus, it can be applied to all processes with purely short-range interactions such as low-energy scattering of D and D^* mesons off the $X(3872)$ [34] or loss processes of ultracold atoms close to a Feshbach resonance [49]. An overview of the Efimov effect in nuclear and particle physics can be found in Ref. [50].

In this work, we assume that the Z_b and Z_b' are bound states and predict $\{Z_b, Z_b'\} - \{B, B^*\}$ scattering in the different spin-isospin channels to leading order in Q in pionless EFT. These predictions could, in principle, be tested in the decays of heavier particles into three B/B^* mesons via final state interactions. If the Z_b and Z_b' are virtual states, this process does not exist and one needs to look at the more complicated three-body scattering of B/B^* mesons. Moreover, we analyze the different channels with regard to the existence of three-body bound states. We note that bound states of three $\bar{B}/\bar{B}^*$ and three B/B^* mesons were previously investigated in Refs. [51–53] using quark models and effective field theory methods. Here, we focus on three-body bound states arising in the $\{Z_b, Z_b'\} - \{B, B^*\}$ scattering channels as a consequence of the Efimov effect [44].

The paper is organized as follows: In Sec. II, we write down an extension of pionless EFT for the $\{Z_b, Z_b'\} - \{B, B^*\}$-system. The intergral equations for the molecule-meson scattering amplitudes are derived in Sec. III and the relation of the amplitudes to observables is discussed in Sec. IV. Our results and concluding remarks are presented in Secs. V and VI, respectively.

II. FORMALISM

To write down an effective Lagrangian density for the $\{Z_b, Z_b'\} - \{B, B^*\}$-system, we start by introducing isospin $I = 1/2$ doublets consisting of the bottom mesons B and B^*:

$$B = \begin{pmatrix} B^+ \\ B^0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{B}^0 \\ B^- \end{pmatrix},$$
$$B^* = \begin{pmatrix} B^{*+} \\ B^{*0} \end{pmatrix}, \quad \bar{B}^* = \begin{pmatrix} \bar{B}^{*0} \\ B^{*-} \end{pmatrix}, \tag{3}$$

where the upper components have $I_3 = +1/2$ and the lower ones have $I_3 = -1/2$. Taking into account that

[2] The case of a covariant formulation was recently investigated in Ref. [47], which arrives at some different conclusions concerning the role of three-body forces.

both Z_b and Z_b' are isospin 1 states, we write down two isospin-triplets:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \quad \text{and} \quad Z' = \begin{pmatrix} Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} . \tag{4}$$

As usual, the physical states, whose electric charges are indicated by the corresponding superscript, are identified as

$$\begin{aligned} -\frac{1}{\sqrt{2}}\left(Z_1 \,+\, iZ_2\right) &\equiv Z^+ \quad \text{with} \quad I_3 = +1\,, \\ Z_3 &\equiv Z^0 \quad \text{with} \quad I_3 = 0\,, \\ \frac{1}{\sqrt{2}}\left(Z_1 \,-\, iZ_2\right) &\equiv Z^- \quad \text{with} \quad I_3 = -1\,, \end{aligned} \tag{5}$$

and analogously for the Z'.

Using a similar analysis for spin, we can write down a non-relativistic effective Lagrangian $\mathcal{L}$ up to leading order (LO). It contains all B and B^* mesons as degrees of freedom. Additionally, there are two auxiliary dimer fields Z and Z' representing the Z_b and Z_b', respectively. Since we are interested in ZB scattering,[3] in general, we have to include three-body forces as well. As will be discussed below, however, their explicit form is not required to leading order. Taking into account the spin and isospin structure and the the particle content of the Z_b and Z_b' (cf. Eq. (2)) one finds:

$$\begin{aligned} \mathcal{L} = {} & B_\alpha^\dagger\left(i\partial_t + \frac{\nabla^2}{2M_B}\right)B_\alpha \,+\, \bar{B}_\alpha^\dagger\left(i\partial_t + \frac{\nabla^2}{2M_B}\right)\bar{B}_\alpha \\ & + B_{i\alpha}^{*\dagger}\left(i\partial_t + \frac{\nabla^2}{2M_{B^*}}\right)B_{i\alpha}^* \,+\, \bar{B}_{i\alpha}^{*\dagger}\left(i\partial_t + \frac{\nabla^2}{2M_{B^*}}\right)\bar{B}_{i\alpha}^* \\ & + Z_{iA}^\dagger \Delta Z_{iA} \,+\, Z_{iA}'^\dagger \Delta' Z_{iA}' \\ & - g\Big[Z_{iA}^\dagger\Big(\bar{B}_{j\alpha}^*\,\delta_{ij}(\tau_2\tau_A)_{\alpha\beta}\,B_\beta \\ & \qquad\qquad + \bar{B}_\alpha\,\delta_{ij}(\tau_2\tau_A)_{\alpha\beta}\,B_{j\beta}^*\Big) \,+\, h.c.\Big] \\ & - g'\Big[Z_{iA}'^\dagger\,\bar{B}_{j\alpha}^*\,(U_i)_{jk}(\tau_2\tau_A)_{\alpha\beta}\,B_{k\beta}^* \,+\, h.c.\Big] + \ldots\,, \end{aligned} \tag{6}$$

where the ellipsis denotes higher-oder terms, lowercase Latin letters $(i, j, k... \in \{1, 2, 3\})$ are spin-1 indices, Greek lowercase letters $(\alpha, \beta, \gamma... \in \{1, 2\})$ are isospin-1/2 indices, and uppercase Latin letters $(A, B, C... \in \{1, 2, 3\})$ denote isospin 1 for the dimer fields. The matrices τ_A are Pauli matrices acting in isospin space and the matrices U_i are the generators of the rotation group acting on the spin-1 representation. Furthermore, we introduce two coupling constants g and g' for the interaction between the dimer fields and their constituents. The coefficients of the kinetic terms of Z and Z', Δ and Δ', are also constants. At leading order in Q, $\Delta^{(\prime)}$ and $g^{(\prime)}$ are not independent and only kept for convenience. Furthermore, both auxiliary fields are not dynamic. However, their bare propagators are dressed by bottom meson loops, so that the full propagators

$$iS_{Z^{(\prime)}}(p_0, \vec{p}) = \frac{i}{\left(S^0_{Z^{(\prime)}}\right)^{-1} \,+\, \Sigma^{(\prime)}}\,, \tag{7}$$

can be expressed in terms of the bare ones $S^0_{Z,Z'}$ and the self-energies Σ and Σ' which are functions of the four momentum $p = (p_0, \vec{p})$. They are ultraviolet divergent and need to be regulated using a momentum cutoff Λ. Using the reduced masses of Z_b and Z_b',

$$\mu = \frac{M_B + M_{B^*}}{M_B M_{B^*}} \quad \text{and} \quad \mu' = \frac{M_{B^*}}{2}\,, \tag{8}$$

and their kinetic molecule masses $M_Z = M_B + M_{B^*}$ and $M_{Z'} = 2M_{B^*}$, one can calculate their self-energies. The self-energy Σ of the Z_b is given by

$$\Sigma(p_0, \vec{p}) = \frac{2g^2\mu}{\pi}\left[-\sqrt{-2\mu\left(p_0 - \frac{\vec{p}^2}{2M_Z}\right) - i\varepsilon} \,+\, \frac{2}{\pi}\Lambda\right]\,, \tag{9}$$

where Λ is a cutoff used to regulate the loop integral for the self-energy and $1/\Lambda$ suppressed terms have been neglected. The self-energy Σ' of the Z_b' is obtained from Eq. (9) if all parameters are replaced by their "primed" counterparts. Inserting the self-energies into Eq. (7) one can match the scattering amplitudes

$$-iT^{(\prime)} = (-ig^{(\prime)})^2 iS_{Z^{(\prime)}}\left(\frac{k^2}{2\mu^{(\prime)}}, 0\right)\,, \tag{10}$$

with their first order effective range expansions (ERE)

$$\left(T^{(\prime)}\right)^{(1)}_{ERE} = -\frac{\pi}{2\mu^{(\prime)}}\frac{1}{\frac{1}{a^{(\prime)}} + ik}\,, \tag{11}$$

to obtain the B meson scattering lengths a and a' in the flavor channels of the Z_b and Z_b' (cf. Eq. (2)), respectively. We find

$$a^{(\prime)} = \frac{\pi\Delta^{(\prime)}}{2\left(g^{(\prime)}\right)^2\mu^{(\prime)}} \,+\, \frac{2}{\pi}\Lambda\,, \tag{12}$$

where the binding momenta are defined as

$$\gamma^{(\prime)} \equiv \frac{1}{a^{(\prime)}} = \mathrm{sgn}(B^{(\prime)})\sqrt{2\mu^{(\prime)}|B^{(\prime)}|}\,. \tag{13}$$

Here, the quantity $B = m_1 + m_2 - M_{12}$ represents the binding energy, which is positive for a bound state and negative for a virtual state. Note, that these definitions are chosen in a way that one takes care of both, bound and virtual states (i.e. a virtual state corresponds to a

[3] Note that ZB is used as a placeholder for all $\{Z_b, Z_b'\}$ – $\{B, B^*\}$ scattering processes.

negative scattering length). Now one can write the full propagators of both molecules in terms of their binding momentum:

$$iS_Z(p_0,\vec{p}) = -i\frac{\pi}{2g^2\mu}\frac{1}{-\gamma+\sqrt{-2\mu\left(p_0-\frac{\vec{p}^2}{2M_Z}\right)-i\varepsilon}}\,,$$
$$iS_{Z'}(p_0,\vec{p}) = -i\frac{\pi}{2g'^2\mu'}\frac{1}{-\gamma'+\sqrt{-2\mu'\left(p_0-\frac{\vec{p}^2}{2M_{Z'}}\right)-\,i\varepsilon}}\,. \tag{14}$$

The wave function renormalization constants, W and W', for both molecules are given by the residue of the bound state pole of the respective propagators in Eq. (14):

$$W^{(\prime)} = \frac{\pi\gamma^{(\prime)}}{2\left(g^{(\prime)}\right)^2\left(\mu^{(\prime)}\right)^2}\,. \tag{15}$$

Higher-order corrections can, in principle, be taken into account by including additional operators in the Lagrangian (6) [37–40]. The first correction comes from the effective range term which is not known for the Z_b and Z_b'.

III. MOLECULE-MESON SCATTERING AMPLITUDES

We are interested in the universal properties of the systems of three B/B^* mesons. This includes scattering processes, such as the scattering of B and B^* particles off the Z_b and Z_b', as well as bound states of three B/B^* mesons. The corresponding information can be extracted from the integral equations for ZB scattering where possible bound states appears as simple poles in the scattering amplitude below threshold. If such bound states exist, they must be bound due to the Efimov effect [44].

At LO, it is sufficient work with integral equations for the ZB scattering amplitudes that contain only two-body interactions. In channels without shallow trimer states, three-body interactions are strongly suppressed [45, 46]. Observables become independent of the cutoff Λ used to regulate the loop integrals for large momenta. If three-body bound states are present and the Efimov effect occurs, however, the integral equations with two-body interaction only will display a strong cutoff dependence and a three-body interaction is required for renormalization already at leading order [41, 42]. The running of the three-body interaction is governed by a limit cycle and thus vanishes at special, log-periodically spaced values of the cutoff. In particular, at leading order it is always possible to tune these three-body terms to zero by working at an appropriate value of the cutoff Λ [54]. The value of Λ can then be directly related to the three-body parameter Λ_* which specifies the three-body force [41, 42]. This particular behaviour can also be found using machine learning, see [55].

The presence of bound states can therefore be investigated by investigating the cutoff dependence of the scattering amplitudes in different channels of ZB scattering. If no cutoff dependence is found, shallow three-body bound states due to the Efimov effect are not present. The ZB scattering amplitudes can then be predicted to leading order from two-body information alone.

We go on to derive the intergral equations for ZB scattering. Besides their quark content the isospin doublets B and $\bar{B}$ have the same spin and isospin degrees of freedom. The small difference in the masses of their constituents is neglected. If electromagnetic effects are not taken into account, they behave identically when they are scattered off a Z state. The same argument holds for the doublets B^* and $\bar{B}^*$. Hence, it is sufficient to analyze the four remaining scattering processes Z_bB, Z_bB^*, $Z_b'B$ and $Z_b'B^*$. Since both Z_b and Z_b' belong to an isospin-triplet and all relevant bottom mesons have $I = 1/2$, the isospin structure of all four scattering amplitudes is exactly the same and each corresponding process has an isospin-3/2 and an isospin-1/2 channel. In contrast, the spin structure is different because B and $\bar{B}$ contain pseudoscalar particles while the components of B^* and $\bar{B}^*$ have spin 1. Hence, the S-wave scattering of a B off a Z_b or a Z_b' only occurs in a spin-triplet channel whereas the scattering of a B^* has a spin-singlet, spin-triplet and spin-quintet channel.

The scattering amplitudes T can be decomposed in a series of partial waves as

$$T(E,\vec{k},\vec{p}) = \sum_{L=0}^{\infty}(2L+1)\,T_{(L)}(E,k,p)\,P_L(\cos\theta)\,, \tag{16}$$

where P_L is a Legendre polynomial, θ is the angle between $\vec{k}$ and $\vec{p}$, $k = |\vec{k}|$, and $p = |\vec{p}|$. As we focus on trimer states generated by the Efimov effect which have $L = 0$ and meson-molecule scattering processes at low energies, we project onto S-waves and ignore all contributions from higher angular momenta.

It is most convenient to work in the center-of-mass system of the molecule and the meson. The coupled integral equations describing molecule-meson scattering contain only diagrams of the type shown in Fig. 1. Thus, we can use the assignment of momenta given in that figure for all ZB channels. The total energy E is given by

$$E = \frac{k^2}{2(m_1+m_2)} + \frac{k^2}{2m_3} - \frac{\gamma_{12}^2}{2\mu_{12}} \tag{17}$$

where γ_{ij} is the binding momentum of the bound state of two mesons i and j, which has a reduced mass of $\mu_{ij} = (m_im_j)/(m_i+m_j)$.

A. $\mathbf{Z_bB}$ scattering

We start with the simplest scattering process where two of the three bottom mesons are pseudoscalars. The

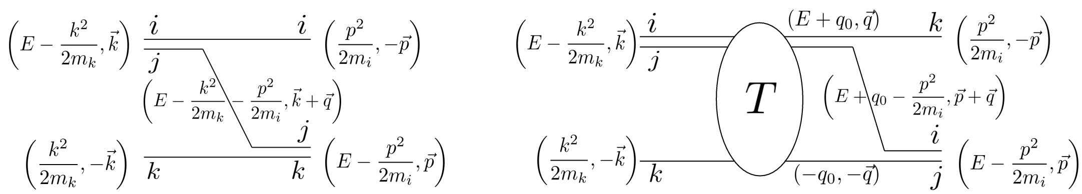

FIG. 1. Topologies which appear in the coupled integral equations describing molecule-meson scattering. Energies and momenta assigned to the lines are given as (energy, momentum). E is the center of mass energy, γ_{ij} denotes the binding momentum of the bound state of two mesons i and j, and μ_{ij} is their reduced mass. The associated momenta can be used in all ZB scattering processes.

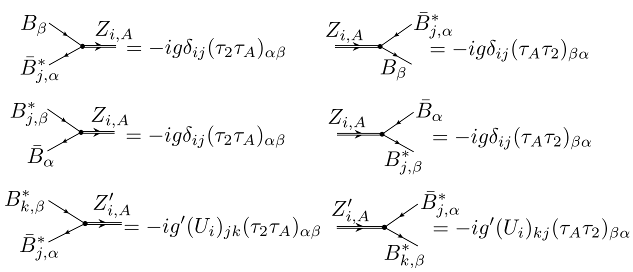

FIG. 2. Feynman rules that follow the Lagrangian density in Eq. (6).

integral equation for the corresponding scattering amplitude T is shown in Fig. 3. Before proceeding further, let us present the independent tree-level amplitudes for the $Z^{(\prime)}B^{(*)}$ systems. Using the vertex factors, see Fig. 2, which follow from the Lagrangian density Eq. (6), we have

$$
\begin{aligned}
i\mathcal{M}^{iA\alpha\to jB\beta}_{ZB\to ZB}(E,\vec{k},\vec{p}) &= \frac{-ig^2\delta_{ij}(\tau_A\tau_B)_{\beta\alpha}}{E-\frac{k^2}{2M_B}-\frac{p^2}{2M_B}-\frac{(\vec{k}+\vec{p})^2}{2M_{B^*}}+i\varepsilon}, \\
i\mathcal{M}^{imA\alpha\to jnB\beta}_{ZB^*\to ZB^*}(E,\vec{k},\vec{p}) &= \frac{-ig^2\delta_{jm}\delta_{in}(\tau_A\tau_B)_{\beta\alpha}}{E-\frac{k^2}{2M_{B^*}}-\frac{p^2}{2M_{B^*}}-\frac{(\vec{k}+\vec{p})^2}{2M_B}+i\varepsilon}, \\
i\mathcal{M}^{imA\alpha\to jB\beta}_{ZB^*\to Z'B}(E,\vec{k},\vec{p}) &= \frac{-igg'(U_j)_{im}(\tau_A\tau_B)_{\beta\alpha}}{E-\frac{k^2}{2M_{B^*}}-\frac{p^2}{2M_B}-\frac{(\vec{k}+\vec{p})^2}{2M_{B^*}}+i\varepsilon}, \\
i\mathcal{M}^{iA\alpha\to jmB\beta}_{Z'B\to ZB^*}(E,\vec{k},\vec{p}) &= \frac{-igg'(U_i)_{mj}(\tau_A\tau_B)_{\beta\alpha}}{E-\frac{k^2}{2M_B}-\frac{p^2}{2M_{B^*}}-\frac{(\vec{k}+\vec{p})^2}{2M_{B^*}}+i\varepsilon}, \\
i\mathcal{M}^{imA\alpha\to jnB\beta}_{Z'B^*\to Z'B^*}(E,\vec{k},\vec{p}) &= \frac{-ig'^2(U_iU_j)_{nm}(\tau_A\tau_B)_{\beta\alpha}}{E-\frac{k^2}{2M_{B^*}}-\frac{p^2}{2M_{B^*}}-\frac{(\vec{k}+\vec{p})^2}{2M_{B^*}}+i\varepsilon}.
\end{aligned}
\tag{18}
$$

The integral equation for the ZB scattering is then given by

$$
\begin{aligned}
t^{jB\beta}_{iA\alpha}(E,\vec{k},\vec{p}) &= \mathcal{M}^{iA\alpha\to jB\beta}_{ZB\to ZB}(E,\vec{k},\vec{p}) \\
&+ \int\frac{d^4q}{(2\pi)^4}\,\frac{i\mathcal{M}^{\ell C\rho\to jB\beta}_{ZB\to ZB}(E,\vec{q},\vec{p})}{-q_0-\frac{q^2}{2M_B}+i\varepsilon} \\
&\times\frac{\pi}{2g^2\mu}\frac{t^{\ell C\rho}_{iA\alpha}(E,\vec{k},\vec{q})}{-\gamma+\sqrt{-2\mu\left(E+q_0-\frac{q^2}{2M_Z}\right)-i\varepsilon}},
\end{aligned}
\tag{19}
$$

where $t^{jB\beta}_{iA\alpha}$ is the scattering amplitude including the full spin-isospin structure. Integrating over the q_0 component and multiplying with the wave function renormalization, we obtain

$$
\begin{aligned}
T^{jB\beta}_{iA\alpha}(E,\vec{k},\vec{p}) = &-\frac{\pi\gamma}{2\mu^2}\frac{(\tau_A\tau_B)_{\beta\alpha}\,\delta_{ij}}{E-\frac{k^2}{2M_B}-\frac{p^2}{2M_B}-\frac{(\vec{k}+\vec{p})^2}{2M_{B^*}}+i\varepsilon} \\
&-\frac{\pi}{2\mu}\int\frac{d^3q}{(2\pi)^3}\,\frac{T^{\ell C\rho}_{iA\alpha}(E,\vec{k},\vec{q})}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
&\times\frac{(\tau_C\tau_B)_{\beta\rho}\,\delta_{\ell j}}{E-\frac{q^2}{2M_B}-\frac{p^2}{2M_B}-\frac{(\vec{q}+\vec{p})^2}{2M_{B^*}}+i\varepsilon},
\end{aligned}
\tag{20}
$$

with $T^{jB\beta}_{iA\alpha}\equiv W\,t^{jB\beta}_{iA\alpha}$. Evaluating the projection of $T^{jB\beta}_{iA\alpha}$ onto a general partial wave,

$$
\frac{1}{2}\int_{-1}^{1}d\cos\theta\,P_L(\cos\theta)\,T(E,\vec{k},\vec{p})=T_{(L)}(E,k,p)\,, \tag{21}
$$

for $L=0$, we obtain the integral equation for the S-wave Z_bB scattering amplitude

$$
\begin{aligned}
T_{(0)}{}^{jB\beta}_{iA\alpha}(E,k,p) = &-\frac{\pi\gamma}{2\mu^2}\boxed{(\tau_A\tau_B)_{\beta\alpha}\,\delta_{ij}} \\
&\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right] \\
&-\frac{1}{4\pi\mu}\int_0^\Lambda dq\frac{q^2\boxed{(\tau_C\tau_B)_{\beta\rho}\,\delta_{\ell j}}\,T_{(0)}{}^{\ell C\rho}_{iA\alpha}(E,k,q)}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
&\times\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right] \\
\equiv&\boxed{C_0{}^{jB\beta}_{iA\alpha}}\mathcal{M}_0+\int_0^\Lambda dq\,\mathcal{M}_1\boxed{C_0{}^{jB\beta}_{\ell C\rho}}T_{(0)}{}^{\ell C\rho}_{iA\alpha}(E,k,q)\,,
\end{aligned}
\tag{22}
$$

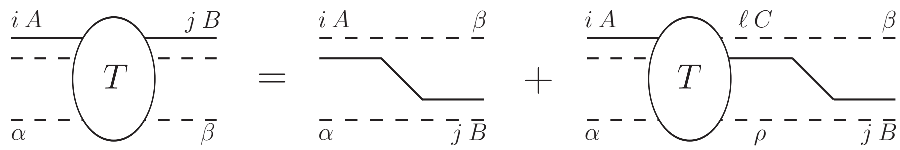

FIG. 3. Integral equation of the amplitude T of Z_bB scattering with incoming spin index i and isospin indices A, α. The respective indices of the outgoing particles are j, B, and β. Pseudoscalar mesons are depicted as dashed lines and spin-1 bottom mesons as solid lines, respectively.

where Λ is the ultraviolet cutoff discussed above. The last equality defines the amplitudes $\mathcal{M}_0$, $\mathcal{M}_1$ and the coefficients $C_0{}^{jB\beta}_{iA\alpha}$ and $C_0{}^{jB\beta}_{\ell C\rho}$; for the latter we use boxed notations to make the corresponding definitions more transparent (similar notations will be used later). The logarithmic function Q_0 originates from the one-meson exchange contributions, whose S-wave projection leads to integrals of the type

$$Q_0(\beta) \equiv \frac{1}{2}\int_{-1}^{+1} dx \frac{P_0(x)}{x+\beta} = \frac{1}{2}\ln\left(\frac{\beta+1}{\beta-1}\right) . \qquad (23)$$

1. $I = 3/2$, $S = 1$ scattering channel

We can now choose a specific Z_bB scattering channel. While there is just one spin channel ($S = 1$), the isospin I can either be equal to 3/2 or equal to 1/2 since we are coupling isospin-1 to isospin-1/2. We start with the former. Following Ref. [56], we project out the desired channel by evaluating:

$$T^{I,S}_{(0)} \equiv \frac{1}{(2S+1)(2I+1)} \sum_{\tilde m\tilde\eta,\tilde n\tilde\lambda} \mathcal{O}^{\dagger}_{\tilde n\tilde\lambda,\tilde j\tilde\beta}\, T_{(0)}{}^{\tilde j\tilde\beta}_{\tilde i\tilde\alpha}\, \mathcal{O}_{\tilde m\tilde\eta,\tilde i\tilde\alpha}\ , \qquad (24)$$

where $\tilde i$, $\tilde j$, $\tilde m$, and $\tilde n$ represent general spin indices in the given operators, while $\tilde\alpha$, $\tilde\beta$, $\tilde\eta$, and $\tilde\lambda$ denote general isospin indices. Note that for elastic scattering, the initial and final states must be identical, requiring $\tilde\eta = \tilde\lambda$ and $\tilde m = \tilde n$. The projectors for the Z_bB scattering are given by

$$\begin{aligned} &\mathcal{O}^{S=1}_{j,i}(1\otimes 0 \to 1) = \delta_{ij}\ ,\\ &\mathcal{O}^{I=1/2}_{\beta,A\alpha}\left(1\otimes\frac{1}{2}\to\frac{1}{2}\right) = \frac{-1}{\sqrt{3}}(\tau_A)_{\alpha\beta}\ ,\\ &\mathcal{O}^{I=3/2}_{j\beta,A\alpha}\left(1\otimes\frac{1}{2}\to\frac{3}{2}\right) = \frac{1}{3}[(\tau_j\tau_A)_{\alpha\beta}+\delta_{Aj}\delta_{\alpha\beta}]\ . \end{aligned} \qquad (25)$$

Applying the above projections to the S-wave Z_bB integral equation of Eq. (22), one gets

$$\begin{aligned} T^{I=\frac{3}{2},S=1}_{(0)} &\equiv \frac{1}{12}\sum_{\substack{mD\eta,\\ nE\lambda}} \left(\mathcal{O}^{I=3/2}_{E\lambda,B\beta}\mathcal{O}^{S=1}_{n,j}\right)^{\dagger} T_{(0)}{}^{jB\beta}_{iA\alpha} \times\left(\mathcal{O}^{I=3/2}_{D\eta,A\alpha}\mathcal{O}^{S=1}_{m,i}\right)\\ &= \frac{1}{12}\frac{1}{9}\left[\sum_{m,n}\delta_{jn}\delta_{ij}\delta_{im}\right]\left[\sum_{\substack{D\eta,\\ E\lambda}}\left((\tau_B\tau_E)_{\lambda\beta}+\delta_{BE}\delta_{\lambda\beta}\right)\right.\\ &\quad\left.\times(\tau_A\tau_B)_{\beta\alpha}\left((\tau_D\tau_A)_{\alpha\eta}+\delta_{DA}\delta_{\alpha\eta}\right)\right]\mathcal{M}_0\\ &\quad+\frac{1}{12}\int_0^{\Lambda} dq\,\mathcal{M}_1\sum_{\substack{mD\eta,\\ nE\lambda}}\left(\mathcal{O}^{I=3/2}_{E\lambda,B\beta}\mathcal{O}^{S=1}_{n,j}\right)^{\dagger} C_0{}^{jB\beta}_{\ell C\rho} T_{(0)}{}^{\ell C\rho}_{iA\alpha}\\ &\quad\times\left(\mathcal{O}^{I=3/2}_{D\eta,A\alpha}\mathcal{O}^{S=1}_{m,i}\right)\\ &= \frac{1}{12}\frac{1}{9}\times 3\times 72\,\mathcal{M}_0 + 2\int_0^{\Lambda} dq\,\mathcal{M}_1\frac{1}{12}\\ &\quad\times\sum_{\substack{mD\eta,\\ nE\lambda}}\left(\mathcal{O}^{I=3/2}_{E\lambda,C\rho}\mathcal{O}^{S=1}_{n,j}\right)^{\dagger} T_{(0)}{}^{jC\rho}_{iA\alpha}\left(\mathcal{O}^{I=3/2}_{D\eta,A\alpha}\mathcal{O}^{S=1}_{m,i}\right)\\ &= 2\,\mathcal{M}_0 + 2\int_0^{\Lambda} dq\,\mathcal{M}_1 T^{I=\frac{3}{2},S=1}_{(0)}, \end{aligned} \qquad (26)$$

where summation over repeated indices is implied and the identity $\left[(\tau_B\tau_E)_{\lambda\beta}+\delta_{BE}\delta_{\lambda\beta}\right](\tau_C\tau_B)_{\beta\rho} = 2\left[(\tau_C\tau_E)_{\lambda\rho}+\delta_{CE}\delta_{\lambda\rho}\right]$ is used in the second-to-last step. Consequently, the S-wave Z_bB scattering amplitude for $I = 3/2$ and $S = 1$ satisfies the integral equation

$$\begin{aligned} T^{I=\frac{3}{2},S=1}_{(0)}(E,k,p) =& -2\times\frac{\pi\gamma}{2\mu^2}\\ &\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]\\ &-\frac{2}{4\pi\mu}\int_0^{\Lambda} dq\frac{q^2\, T^{I=\frac{3}{2},S=1}_{(0)}(E,k,q)}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_Z}\right)-i\varepsilon}}\\ &\times\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]. \end{aligned} \qquad (27)$$

2. $I = 1/2$, $S = 1$ scattering channel

Similarly, one can project onto the second isospin channel $I = 1/2$ (cf. Ref. [56]):

$$
\begin{aligned}
T_{(0)}^{I=\frac{1}{2},S=1} =& \frac{1}{6}\frac{1}{3}\left[\sum_{m,n}\delta_{jn}\delta_{ij}\delta_{im}\right]\sum_{\eta,\lambda}(\tau_B\tau_A\tau_B\tau_A)_{\lambda\eta}\mathcal{M}_0 \\
&+\frac{1}{6}\int_0^\Lambda dq\,\mathcal{M}_1\sum_{\eta,\lambda}\left(\frac{-1}{\sqrt{3}}(\tau_B\tau_C\tau_B)_{\lambda\rho}\right)\delta_{jn}\delta_{j\ell}T_{(0)\,iA\alpha}^{\;\ell C\rho} \\
&\times\left(\mathcal{O}_{\eta,A\alpha}^{I=1/2}\mathcal{O}_{m,i}^{S=1}\right) \\
=& \frac{1}{6}\frac{1}{3}\times 3\times(-6)\,\mathcal{M}_0+(-1)\int_0^\Lambda dq\,\mathcal{M}_1\frac{1}{6} \\
&\times\sum_{m\eta,n\lambda}\left(\mathcal{O}_{\lambda,C\rho}^{I=1/2}\mathcal{O}_{n,j}^{S=1}\right)^\dagger T_{(0)\,iA\alpha}^{\;jC\rho}\left(\mathcal{O}_{\eta,A\alpha}^{I=1/2}\mathcal{O}_{m,i}^{S=1}\right) \\
=& -\mathcal{M}_0-\int_0^\Lambda dq\,\mathcal{M}_1 T_{(0)}^{I=\frac{1}{2},S=1}, \qquad (28)
\end{aligned}
$$

with the identity $\tau_B\tau_C\tau_B = -\tau_C$. Consequently, the S-wave Z_bB scattering amplitude in the $I = 1/2$ and $S = 1$ channel is given by

$$
\begin{aligned}
T_{(0)}^{I=\frac{1}{2},S=1}(E,k,p) =& \frac{\pi\gamma}{2\mu^2} \\
&\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right] \\
&+\frac{1}{4\pi\mu}\int_0^\Lambda dq\frac{q^2\,T_{(0)}^{I=\frac{1}{2},S=1}(E,k,q)}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
&\times\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]. \qquad (29)
\end{aligned}
$$

B. $\mathbf{Z_bB^*}$ scattering

Next, we consider Z_bB^* scattering. The pseudoscalar doublet B is then replaced by the vector doublet B^*. Hence, there are three spin channels: $S = 0, 1$, and 2. The isospin structure is the same as in Z_bB scattering discussed in the previous section. From Fig. 4, we find the coupled integral equations for the Z_bB^* scattering amplitude T_1:

$$
\begin{aligned}
(T_1)_{(0)\,ikA\alpha}^{\;j\ell B\beta}(E,k,p) =& -\frac{\pi}{2}\frac{\gamma}{\mu^2}\boxed{(\tau_A\tau_B)_{\beta\alpha}\,\delta_{jk}\delta_{i\ell}} \\
&\times\left[-\frac{M_B}{kp}Q_0\left(-\frac{M_B}{kp}(E-\frac{k^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right] \\
&-\frac{1}{4\pi}\int_0^\Lambda dq\frac{q^2\left[-\frac{M_B}{qp}Q_0\left(-\frac{M_B}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_{B^*}}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
&\times\frac{1}{\mu}\boxed{(\tau_C\tau_B)_{\beta\rho}\,\delta_{jr}\delta_{\ell m}}\,(T_1)_{(0)\,ikA\alpha}^{\;mrC\rho}(E,k,q) \\
&-\frac{1}{4\pi}\int_0^\Lambda dq\frac{q^2\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu'})-i\varepsilon\right)\right]}{-\gamma'+\sqrt{-2\mu'\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_{Z'}}\right)-i\varepsilon}} \\
&\times\frac{\sqrt{\gamma}}{\mu\sqrt{\gamma'}}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,(U_m)_{\ell j}}\,(T_2)_{(0)\,ikA\alpha}^{\;mC\gamma}(E,k,q) \\
\equiv& \boxed{C_{0\,ikA\alpha}^{\;j\ell B\beta}}\mathcal{M}_{10}+\int_0^\Lambda dq\,\mathcal{M}_{11}\boxed{C_{0\,mrC\rho}^{\;j\ell B\beta}}(T_1)_{(0)\,ikA\alpha}^{\;mrC\rho}(E,k,q) \\
&+\int_0^\Lambda dq\,\mathcal{M}_{12}\boxed{C_{1\,mC\gamma}^{\;j\ell B\beta}}(T_2)_{(0)\,ikA\alpha}^{\;mC\gamma}(E,k,q)\,, \qquad (30)
\end{aligned}
$$

and

$$
\begin{aligned}
(T_2)_{(0)\,ikA\alpha}^{\;jB\beta}(E,k,p) =& -\frac{\pi}{2}\frac{\sqrt{\gamma\gamma'}}{\mu\mu'}\boxed{(\tau_A\tau_B)_{\beta\alpha}\,(U_j)_{ik}} \\
&\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu'}-\frac{p^2}{2\mu})-i\varepsilon\right)\right] \\
&-\frac{1}{4\pi}\int_0^\Lambda dq\frac{q^2\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu'}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_{B^*}}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
&\times\frac{\sqrt{\gamma'}}{\mu'\sqrt{\gamma}}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,(U_j)_{mr}}\,(T_1)_{(0)\,ikA\alpha}^{\;mrC\gamma}(E,k,q) \\
\equiv& \boxed{C_{2\,ikA\alpha}^{\;jB\beta}}\mathcal{M}_{20} \\
&+\int_0^\Lambda dq\,\mathcal{M}_{21}\boxed{C_{2\,mrC\gamma}^{\;jB\beta}}(T_1)_{(0)\,ikA\alpha}^{\;mrC\gamma}(E,k,q)\,, \qquad (31)
\end{aligned}
$$

where the S-wave projection and wave function renormalization factors have already been applied. The projections onto isospin 3/2 and isospin 1/2 are the same as in the previous section. Then the isospin part of the projection operators given by Eq. (25) does still work for Z_bB^* scattering. Since there are three different spin channels which can be combined with both isospin states, one finds six scattering channels in total. The projectors for the scalar, vector and tensor amplitudes are given by [56]

$$
\begin{aligned}
\mathcal{O}_{ji}^{S=0}\,(1\otimes 1\to 0) &= \frac{-1}{\sqrt{3}}\delta_{ij}\,, \\
\mathcal{O}_{\ell,mn}^{S=1}\,(1\otimes 1\to 1) &= \frac{-1}{\sqrt{2}}(U_\ell)_{mn}\,, \\
\mathcal{O}_{\ell k,mn}^{S=2}\,(1\otimes 1\to 2) &= \frac{1}{2}[\delta_{\ell m}\delta_{kn}+\delta_{\ell n}\delta_{km}-\frac{2}{3}\delta_{\ell k}\delta_{mn}]\,, \qquad (32)
\end{aligned}
$$

where $[(U_i)_{jk}]^\dagger = (U_i^\dagger)_{kj} = (U_i)_{kj}$ and $(U_i)_{jk} = -i\epsilon_{ijk}$. Using the same strategy as presented in the Z_bB case, one can obtain the projection coefficients for the S-wave Z_bB^* coupled integral equations by applying the above operators to Eq. (30) and (31). The results are collected in Tab. 1. Note that for the inelastic transition T_2, the

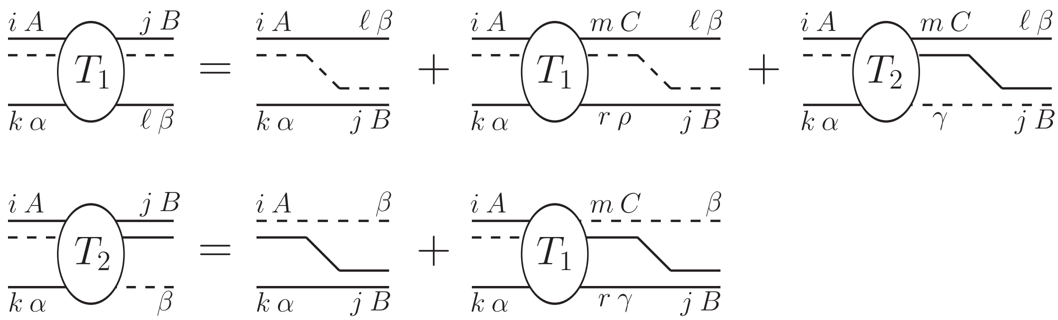

FIG. 4. Coupled integral equations for the amplitude T_1 of $Z_b B^*$ scattering with incoming spin indices i, k and isospin indices A, α, and the corresponding outgoing indices j, ℓ and B, β.

projection is defined as

$$(T_2)_{(0)}^{I=1/2,S=1} = \sum_{\substack{g\eta,\\ h\lambda}} \left(\mathcal{O}^{I=1/2}_{\lambda,B\beta}\mathcal{O}^{S=1}_{h,j}\right)^\dagger (T_2)_{(0)\,ikA\alpha}^{\;\;jB\beta} \times \left(\mathcal{O}^{I=1/2}_{\eta,A\alpha}\mathcal{O}^{S=1}_{g,ik}\right) . \quad (33)$$

As expected, there is no T_2 contribution to the $S = 0$

TABLE 1. Coefficients of the partial-wave projected integral equation for S-wave $Z_b B^*$ scattering. One finds that $C_0^I = C_1^I = C_2^I$ and $C_1^S = C_2^S$, as expected.

Channel	$C_0^I C_0^S$	$C_1^I C_1^S$	$C_2^I C_2^S$
$I = 1/2,\ S = 0$	$(-1)\times 1$	$(-1)\times 0$	$(-1)\times 0$
$I = 3/2,\ S = 0$	2×1	2×0	2×0
$I = 1/2,\ S = 1$	$(-1)\times(-1)$	$(-1)\times\sqrt{2}$	$(-1)\times\sqrt{2}$
$I = 3/2,\ S = 1$	$2\times(-1)$	$2\times\sqrt{2}$	$2\times\sqrt{2}$
$I = 1/2,\ S = 2$	$(-1)\times 1$	$(-1)\times 0$	$(-1)\times 0$
$I = 3/2,\ S = 2$	2×1	2×0	2×0

and $S = 2$ channel because the scattered particles in this channel can only couple to a total spin of $S = 1$, i.e., $C_1^{S=0,2} = C_2^{S=0,2} = 0$. The final integral equations for all six scattering channels are then expressed in the following general form with the spin-isospin factors listed in Tab. 1,

$$\begin{aligned}(T_1)_{(0)}^{I,S}(E,k,p) &= C_0^I C_0^S\, \mathcal{M}_{10}(k,p) \\ &+ \int_0^\Lambda dq\, C_0^I C_0^S \mathcal{M}_{11}(p,q)(T_1)_{(0)}^{I,S}(E,k,q) \\ &+ \int_0^\Lambda dq\, C_1^I C_1^S \mathcal{M}_{12}(p,q)(T_2)_{(0)}^{I,S}(E,k,q)\,, \\ (T_2)_{(0)}^{I,S}(E,k,p) &= C_2^I C_2^S\, \mathcal{M}_{20}(k,p) \\ &+ \int_0^\Lambda dq\, C_2^I C_2^S \mathcal{M}_{21}(p,q)(T_1)_{(0)}^{I,S}(E,k,q)\,. \end{aligned} \quad (34)$$

The five scalar amplitudes $\mathcal{M}_{10}$, $\mathcal{M}_{11}$, $\mathcal{M}_{12}$, $\mathcal{M}_{20}$ and $\mathcal{M}_{21}$ are defined by Eq. (30) and (31).

C. $\mathbf{Z_b'B}$ scattering

The scattering process $Z_b'B \to Z_b'B$ has the same isospin structure as described in the previous sections. This yields the isospin factors C_0^I in the two amplitudes shown in Fig. 5. Although the only spin channel is $S = 1$ and thus the projection operator is the same as in Sec. III A, the vertices appearing in the amplitudes are different. This leads to different spin factors in these amplitudes. Firstly, the S-wave $Z_b'B$ scattering amplitudes satisfy the integral equation,

$$\begin{aligned}(T_1)_{(0)\,iA\alpha}^{\;\;jB\beta}(E,k,p) =& -\frac{1}{4\pi}\int_0^\Lambda dq \\ &\times \frac{q^2\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu'}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_{B^*}}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\ &\times \frac{\sqrt{\gamma'}}{\mu'\sqrt{\gamma}}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,(U_j)_{\ell m}}\,(T_2)_{(0)\,iA\alpha}^{\;\;\ell mC\gamma}(E,k,q) \\ \equiv& +\int_0^\Lambda dq\,\mathcal{M}_{12}\boxed{C_{3\;\ell mC\gamma}^{\;jB\beta}}(T_2)_{(0)\,iA\alpha}^{\;\;\ell mC\gamma}(E,k,q)\,, \end{aligned} \quad (35)$$

and

$$\begin{aligned}(T_2)_{(0)\,iA\alpha}^{\;\;jkB\beta}(E,k,p) =& -\frac{\pi}{2}\frac{\sqrt{\gamma\gamma'}}{\mu\mu'}\boxed{(\tau_A\tau_B)_{\beta\alpha}\,(U_i)_{kj}} \\ &\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu}-\frac{p^2}{2\mu'})-i\varepsilon\right)\right] \\ &-\frac{1}{4\pi}\int_0^\Lambda dq\frac{q^2\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu'})-i\varepsilon\right)\right]}{-\gamma'+\sqrt{-2\mu'\left(E-\frac{q^2}{2M_B}-\frac{q^2}{2M_{Z'}}\right)-i\varepsilon}} \\ &\times\frac{\sqrt{\gamma}}{\mu\sqrt{\gamma'}}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,(U_\ell)_{kj}}\,(T_1)_{(0)\,iA\alpha}^{\;\;\ell C\gamma}(E,k,q)\end{aligned}$$

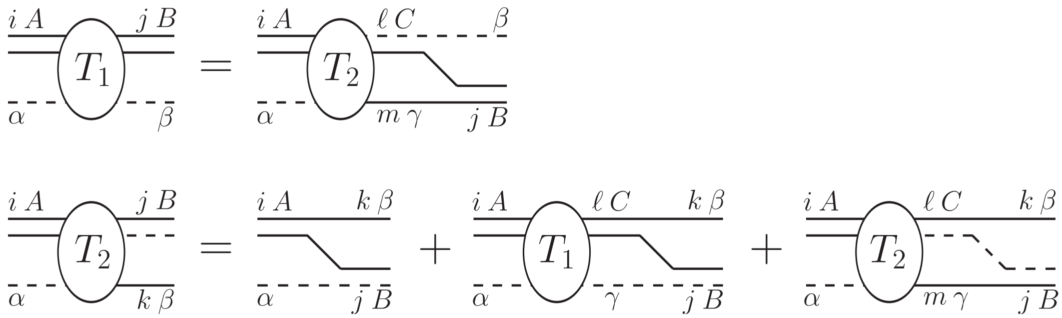

FIG. 5. Coupled integral equations for the amplitude T_1 of $Z_b'B$ scattering with incoming spin index i and isospin indices A, α, and the corresponding outgoing indices and j and B, β.

$$
\begin{aligned}
& -\frac{1}{4\pi}\int_0^\Lambda dq \frac{q^2\left[-\frac{M_B}{qp}Q_0\left(-\frac{M_B}{qp}(E-\frac{q^2}{2\mu}-\frac{p^2}{2\mu})-i\varepsilon\right)\right]}{-\gamma+\sqrt{-2\mu\left(E-\frac{q^2}{2M_{B^*}}-\frac{q^2}{2M_Z}\right)-i\varepsilon}} \\
& \quad\times\frac{1}{\mu}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,\delta_{jm}\delta_{\ell k}}\,(T_2)_{(0)\,iA\alpha}^{\;\ell mC\gamma}(E,k,q) \\
& \equiv \boxed{C_{4\,iA\alpha}^{\;jkB\beta}}\mathcal{M}_{20} \\
& \quad+\int_0^\Lambda dq\,\mathcal{M}_{21}\boxed{C_{4\,\ell C\gamma}^{\;jkB\beta}}(T_1)_{(0)\,iA\alpha}^{\;\ell C\gamma}(E,k,q) \\
& \quad+\int_0^\Lambda dq\,\mathcal{M}_{22}\boxed{C_{5\,\ell mC\gamma}^{\;jkB\beta}}(T_2)_{(0)\,iA\alpha}^{\;\ell mC\gamma}(E,k,q)\,, \qquad (36)
\end{aligned}
$$

Similarly, by applying the spin-isospin projection onto these equations, we obtain

$$
\begin{aligned}
(T_1)_{(0)}^{I,S}(E,k,p) &= +\int_0^\Lambda dq\,C_3^IC_3^S\mathcal{M}_{12}(p,q)(T_2)_{(0)}^{I,S}(E,k,q), \\
(T_2)_{(0)}^{I,S}(E,k,p) &= C_4^IC_4^S\,\mathcal{M}_{20}(k,p) \\
&\quad+\int_0^\Lambda dq\,C_4^IC_4^S\mathcal{M}_{21}(p,q)(T_1)_{(0)}^{I,S}(E,k,q) \\
&\quad+\int_0^\Lambda dq\,C_5^IC_5^S\mathcal{M}_{22}(p,q)(T_2)_{(0)}^{I,S}(E,k,q)\,, \qquad (37)
\end{aligned}
$$

with four scalar amplitudes $\mathcal{M}_{12}$, $\mathcal{M}_{20}$, $\mathcal{M}_{21}$ and $\mathcal{M}_{22}$ whose expressions are given by Eq. (35) and (36). Note that $C_3^I = C_4^I = C_5^I = C_0^I$. The spin coefficients are given by Tab. 2.

TABLE 2. Coefficients of the partial-wave projected integral equation for S-wave $Z_b'B$ scattering.

Channel	$C_3^IC_3^S$	$C_4^IC_4^S$	$C_5^IC_5^S$
$I=1/2,\ S=1$	$(-1)\times\sqrt{2}$	$(-1)\times\sqrt{2}$	$(-1)\times(-1)$
$I=3/2,\ S=1$	$2\times\sqrt{2}$	$2\times\sqrt{2}$	$2\times(-1)$

D. $\mathbf{Z_b'B^*}$ scattering

The last scattering process includes two vector particles. Thus, there are six channels in total (three spin states $S=0,1,2$ combined with the isospin 1/2 and 3/2 states). The integral equation for this process is shown in Fig. 6. After wave function renormalization and S-wave projection we find

$$
\begin{aligned}
& T_{(0)\,ikA\alpha}^{\;j\ell B\beta}(E,k,p) = -\frac{\pi}{2}\frac{\gamma'}{\mu'^{\,2}}\boxed{(\tau_A\tau_B)_{\beta\alpha}\,(U_iU_j)_{\ell k}} \\
& \quad\times\left[-\frac{M_{B^*}}{kp}Q_0\left(-\frac{M_{B^*}}{kp}(E-\frac{k^2}{2\mu'}-\frac{p^2}{2\mu'})-i\varepsilon\right)\right] \\
& -\frac{1}{4\pi}\int_0^\Lambda dq\frac{q^2\left[-\frac{M_{B^*}}{qp}Q_0\left(-\frac{M_{B^*}}{qp}(E-\frac{q^2}{2\mu'}-\frac{p^2}{2\mu'})-i\varepsilon\right)\right]}{-\gamma'+\sqrt{-2\mu'\left(E-\frac{q^2}{2M_{B^*}}-\frac{q^2}{2M_{Z'}}\right)-i\varepsilon}} \\
& \quad\times\frac{1}{\mu'}\boxed{(\tau_C\tau_B)_{\beta\gamma}\,(U_mU_j)_{\ell n}}\,T_{(0)\,ikA\alpha}^{\;mnC\gamma}(E,k,q) \\
& \equiv\boxed{C_{6\,ikA\alpha}^{\;j\ell B\beta}}\mathcal{M}_0+\int_0^\Lambda dq\,\mathcal{M}_1\boxed{C_{6\,mnC\gamma}^{\;j\ell B\beta}}T_{(0)\,ikA\alpha}^{\;mnC\gamma}(E,k,q)\,, \\
& \qquad (38)
\end{aligned}
$$

The spin projection operators are known from Sec. III B and given in Eq. (32). When applied to the amplitude of the above equation one ends up with the following spin-isospin factors, see Tab. 3. Then the projected integral equation for the S-wave $Z_b'B^*$ scattering reads

TABLE 3. Coefficients of the partial-wave projected integral equation for S-wave $Z_b'B^*$ scattering.

(I,S)	$(\frac{1}{2},0)$	$(\frac{1}{2},1)$	$(\frac{1}{2},2)$
$C_6^IC_6^S$	$(-1)\times(-2)$	$(-1)\times 1$	$(-1)\times 1$
(I,S)	$(\frac{3}{2},0)$	$(\frac{3}{2},1)$	$(\frac{3}{2},2)$
$C_6^IC_6^S$	$2\times(-2)$	2×1	2×1

$$
T_{(0)}^{I,S}(E,k,p) = C_6^IC_6^S\,\mathcal{M}_0(k,p)
$$

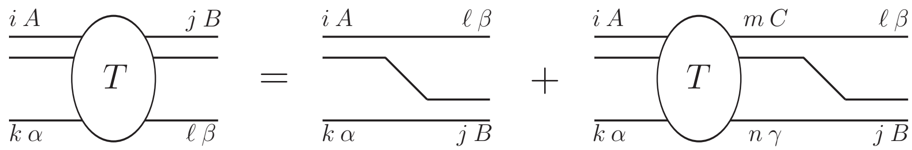

FIG. 6. Integral equation for the amplitude T of $Z_b'B^*$ scattering with incoming spin indices i, k and isospin indices A, α, and the corresponding outgoing indices j, ℓ and B, β.

$$+\int_0^\Lambda dq\, C_6^I C_6^S \mathcal{M}_1(p,q) T_{(0)}^{I,S}(E,k,q)\,, \tag{39}$$

with two scalar amplitudes $\mathcal{M}_0$ and $\mathcal{M}_1$ defined by Eq. (38).

IV. RELATION TO OBSERVABLES

After discretization, the integral equations from Sec. III reduce to inhomogeneous matrix equations of the form $\mathbf{T} = \mathbf{R} + \mathcal{M}\mathbf{T}$, where all quantities implicitly depend on the energy E. For an integral equation describing ZB scattering in a single channel there are three relevant regions of the center of mass energy E. Denoting the binding energy of the relevant molecule Z_b or Z_b' as $B(Z)$, we have:

(i) $-B(Z) \leq E \leq 0$: here, the elastic scattering of a bottom meson off a molecule is the only process that is allowed. In terms of the center-of-mass momentum k this energy region translates to $0 \leq k \leq k_{break}$, where k_{break} is the breakup momentum of the molecule.

(ii) $-\infty < E < -B(Z)$: for energies below the molecule-meson scattering threshold, trimer states can appear.

(iii) $0 < E < \infty$: for positive energies the molecule can break apart and three-particle singularities have to be taken into account. This regime is beyond the scope of our work.

In a system of two coupled integral equations where both Z_b and Z_b' are involved, the scenario above must be generalized. One has to replace in the second case $-\infty < E < -B(Z)$ by $-\infty < E < -\max(B, B')$ because a stable trimer state must lie below both dimer thresholds. In the first case one has to take care of the relation between the two binding energies B and B'. Purely elastic two-body scattering Z_bB^*, for example, only takes place for $B \geq B'$. Namely, in the energy region $-B \leq E \leq -B'$. In the other case, i.e. for $B < B'$ both molecule states can be formed out of the three bottom mesons and inelastic reactions become possible. Such reactions will not be considered here. In $Z_b'B$ scattering, the situation is reversed.

In the following, we consider a system of two coupled integral equations of the type derived in Sec. III. The simpler case with just one such equation can straightforwardly be deduced from this.

A. Elastic ZB scattering

In the energy region (i), where elastic ZB scattering is dominant, the corresponding amplitudes $T_1(p)$ and $T_2(p)$ can be found by solving the inhomogeneous matrix equation $\mathbf{T} = \mathbf{R} + \mathcal{M}\mathbf{T}$ for a given momentum k, i.e. at a given center of mass energy $E \sim k^2$. Besides the amplitudes themselves there are two additional observables of interest in the ZB scattering process: the meson-molecule scattering length a_3 and the phase shift $\delta_L(k)$ where L is the relative angular momentum between Z and B. Since we focus on S-wave scattering, we define the S-wave phase shift as $\delta(k) \equiv \delta_0(k)$.

For the determination of these two quantities, we use the relation

$$T_1(k, p = k) = \frac{2\pi}{\mu_3} \frac{1}{k\cot\delta - ik}\,, \tag{40}$$

with the effective range expansion

$$k\cot\delta = -\frac{1}{a_3} + \mathcal{O}(k^2)\,. \tag{41}$$

Thus, the scattering length a_3 is given by

$$a_3 = -\frac{\mu_3}{2\pi}\, T_1(0,0)\,, \tag{42}$$

and the scattering phase shift can be determined by inverting Eq. (40).

B. Trimer states

For negative energies below the two particle threshold, there are no poles in the kernels of the integral equations. A three-particle bound state with binding energy B_3 shows up as a simple pole in the two amplitudes T_1 and T_2 which are combined in $\mathbf{T}$. One can parametrize the amplitudes in the vicinity of the pole as

$$T_1(k,p) = \frac{B(k)\, B_1(p)}{E + B_3} + \text{regular terms},$$

$$T_2(k,p) = \frac{B(k)\,B_2(p)}{E+B_3} + \text{regular terms}, \quad \text{for } E \to -B_3\,. \tag{43}$$

Inserting this into the coupled integral equations for $T_1(k,p)$ and $T_2(k,p)$ and matching the coefficients of the pole in $(E+B_3)$, we obtain a homogeneous integral equation for $B(p)$ which has nontrivial solutions only for a discrete (and possibly empty) set of negative bound state energies. After discretization, this turns into a homogeneous matrix equation of the form $\mathbf{B} = \mathcal{M}(E)\mathbf{B}$.

V. RESULTS

The question of whether the Z_b and Z_b' mesons are virtual states, bound states or resonances has not been answered definitely (see, e.g., Ref. [24] for an analysis of recent experimental data on the production and decay channels of the Z and Z' in an effective field theory framework that incorporates constraints from unitarity and analyticity). Here we assume that the Z and Z' are bound states and solve the (coupled) integral equations derived in the previous section. Since their binding energies, required as input for these calculations, are uncertain, we follow the strategy of Ref. [57] and assume the ranges:

$$\begin{aligned} B &= 5.0 \pm 2.5\ \text{MeV}\,, \\ B' &= 1.0 \pm 0.5\ \text{MeV}\,. \end{aligned} \tag{44}$$

Using Eq. (13), this leads to the binding momenta

$$\begin{aligned} \gamma &= 162.8^{+36.6}_{-47.7}\ \text{MeV}\,, \\ \gamma' &= 73.0^{+16.4}_{-21.4}\ \text{MeV}\,. \end{aligned} \tag{45}$$

for the Z_b and Z_b' molecules, respectively. One observes that the central value for the $Z_b(10610)$ is larger than the pion mass, so the applicability of an EFT without explicit pions is not obviuous. However, due to the large uncertainties of γ, a binding momentum of the Z_b below M_π is not excluded. As a consequence, one can use pionless EFT as a model to obtain first insights on the properties of the ZB systems. The sensitivity to the input values is illustrated by showing results for the central values and the upper and lower bounds in Eq. (44). As discussed in detail in Ref. [29], there is some uncertainty about the precise location of these poles, so our analysis should be updated when precise data becomes available.

A. Bound states of three $\mathbf{B/B^*}$ mesons

We searched for solutions of the homogeneous integral equations (cf. Subsec. IV B) corresponding to bound states of three B/B^* mesons in all spin and isospin channels of the Z_bB and $Z_b'B$ ($I = 1/2, 3/2$, $S = 1$), as well as Z_bB^* and $Z_b'B^*$ ($I = 1/2, 3/2$, $S = 0,1,2$) systems discussed in Sec. III, repectively. No such solutions were found. As a consequence, there is no Efimov effect with three B/B^* mesons. Heuristically, this can be understood from the effective number of interacting pairs, which is smaller than two in all channels. Moreover, the amplitudes are independent of the cutoff Λ for sufficiently large Λ and three-body forces do not enter at leading order.

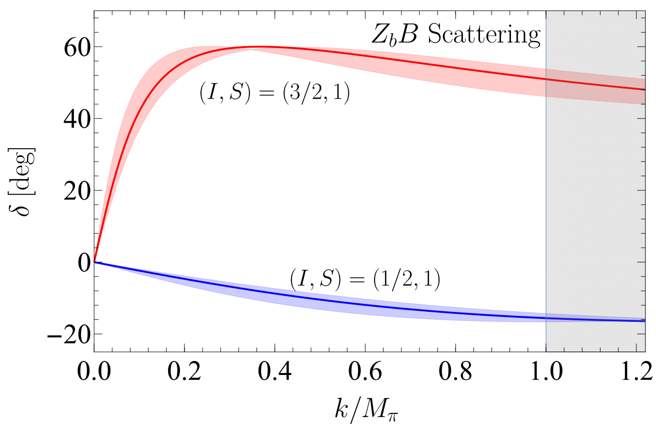

FIG. 7. S-wave phase shift δ as function of the momentum k for the $I = 3/2$ (solid line) and $I = 1/2$ channels (dashed line) of elastic Z_bB scattering. The bands are obtained by varying the input binding energies in the ranges given in Eq. (44). Note that the pionless EFT expansion breaks down for momenta of order M_π as indicated by the shaded area.

Next we focus on ZB scattering in the different channels. Due to the suppression of three-body forces, this is completely predicted by the Z_b and Z_b' binding energies to leading order and the cutoff Λ in the integral equations in Sec. III can be removed. Note that we will not show numerical results for $Z_b'B$ scattering, since a purely elastic scattering process without coupling to the Z_bB^* system is not possible (cf. the discussion in Sec. IV).

B. Discussion of $\mathbf{Z_bB}$ scattering

The elastic scattering Z_bB is completely described by the formulae in Sec. IV A characterized by two observables: the Z_bB scattering length a_3 and S-wave phase shift $\delta(k)$. The scattering length in the $I = 3/2$, $S = 1$ channel is given by

$$a_3^{I=\frac{3}{2},S=1} = -14.0^{+2.6}_{-6.0}\ \text{fm}\,, \tag{46}$$

and the corresponding phase shift in this channel is shown as a function of k in Fig. 7. The bands are obtained by varying the input binding energies in the ranges given in Eq. (44). Note that the pionless EFT expansion breaks down for momenta of order M_π and our results in the shaded region of Fig. 7 should only be taken as an indication of the general trend. The positive phase shift corresponds indicates an attractive interaction between the

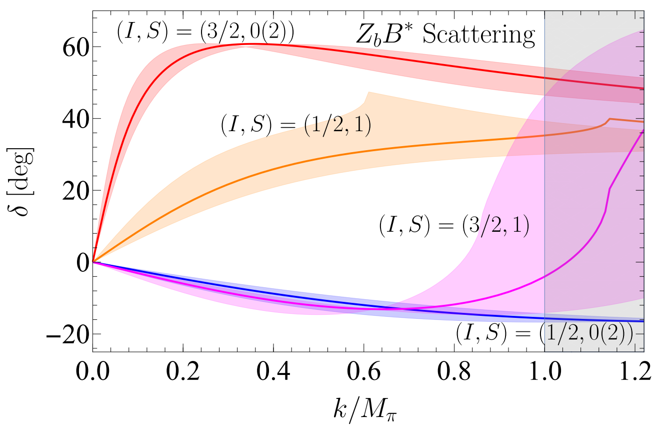

FIG. 8. S-wave phase shift δ as function of the momentum k for all six channels in elastic Z_bB^* scattering. The bands are obtained by varying the input binding energies in the ranges given in Eq. (44). Note, that the $S = 0$ and $S = 2$ spin channels yield the same result. Moreover, the pionless EFT expansion breaks down for momenta of order M_π as indicated by the shaded area.

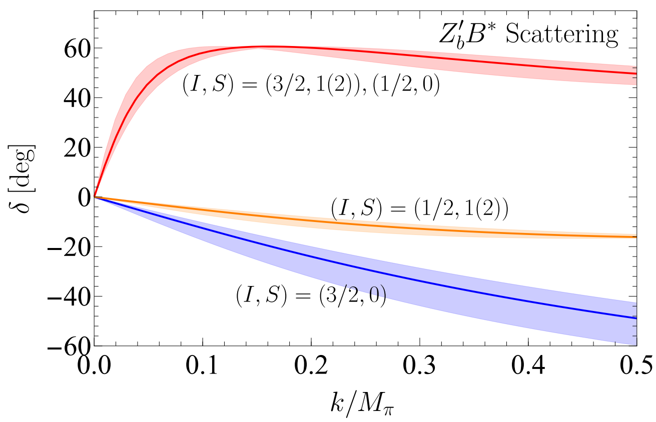

FIG. 9. S-wave phase shift δ as function of the momentum k for all six channels in elastic $Z_b'B^*$ scattering. The bands are obtained by varying the input binding energies in the ranges given in Eq. (44). Note, that for each isospin state the $S = 1$ and $S = 2$ spin channels yield the same result and furthermore that the $I = 1/2$, $S = 0$ result is equivalent to that of $I = 3/2$, $S = 1, 2$.

two scattered particles. However, as discussed above the interaction is not strong enough to induce a three-body bound state. For a increasing attraction of the meson-molecule interaction, the scattering length a_3 tends to minus infinity and jumps to plus infinity when a bound state appears (see, e.g., Ref. [49]). The scattering length $a_3^{I=\frac{3}{2},S=1}$ is large but negative such that only a little more attraction would be needed to form a universal trimer state.

From the negative phase shift in Fig. 7 for the $I = 1/2$, $S = 1$ channel, we conclude that the ZB interaction in this channel is weakly repulsive. The corresponding scattering length is

$$a_3^{I=\frac{1}{2},S=1} = 0.6^{+0.3}_{-0.1} \text{ fm} . \tag{47}$$

C. Discussion of $\mathbf{Z_bB^*}$ scattering

In the same way as for Z_bB scattering, one can analyze the scattering observables in the Z_bB^* system. We calculate the molecule-meson scattering length and the phase shift in all six isospin-spin channels. Due the purely S-wave interaction at leading order, the projection onto some of the isospin and spin states leads to identical prefactors. Therefore only four independent amplitudes remain. The corresponding phase shifts are shown in Fig. 8. Note that the pionless EFT expansion breaks down for momenta of order M_π indicated by the shaded region. The $I = 3/2, S = 0, 2$ and $I = 1/2, S = 1$ phase shifts indicate an attractive interaction between the Z_b and B^*. However, the attraction again is not strong enough to produce trimer states. The corresponding scattering lengths are:

$$a_3^{I=\frac{3}{2},S=0,2} = -14.8^{+2.8}_{-6.3} \text{ fm} , \tag{48}$$

$$a_3^{I=\frac{1}{2},S=0,2} = 0.6^{+0.3}_{-0.1} \text{ fm} , \tag{49}$$

$$a_3^{I=\frac{3}{2},S=1} = 0.8^{+0.4}_{-0.1} \text{ fm} , \tag{50}$$

$$a_3^{I=\frac{1}{2},S=1} = -2.2^{+0.6}_{-1.5} \text{ fm} , \tag{51}$$

in agreement with the absence of trimer states.

D. Discussion of $\mathbf{Z_b'B^*}$ scattering

Finally, we turn to $Z_b'B^*$ scattering. In Fig. 9, we show the $Z_b'B^*$ scattering phase shifts up to the Z_b' breakup momentum, $k \approx 0.5M_\pi$ where the scattering is purely elastic. The meson-molecule scattering lengths in the different spin-isospin channels are given by

$$a_3^{I=\frac{3}{2},S=1,2} = a_3^{I=\frac{1}{2},S=0} = -32.5^{+6.1}_{-13.6} \text{ fm} , \tag{52}$$

$$a_3^{I=\frac{1}{2},S=1,2} = 1.3^{+0.6}_{-0.3} \text{ fm} , \tag{53}$$

$$a_3^{I=\frac{3}{2},S=0} = 3.2^{+1.4}_{-0.6} \text{ fm} , \tag{54}$$

respectively. One observes that the absolute value of the scattering length in the $I = 3/2$, $S = 1, 2$ and $I = 1/2$, $S = 0$ channels is an order of magnitude larger than in all other processes and channels. The large negative value of the scattering length reflects the steep rise of the phase shift below $k \approx 0.1M_\pi$. It indicates that the $I = 3/2$, $S = 1, 2$ and $I = 1/2$, $S = 0$ channels in the $Z_b'B^*$ system are very close to the emergence of trimer states due to the Efimov effect but that the attraction is not quite enough.

VI. CONCLUSIONS

In this work, we have investigated the bound states and scattering processes of B and B^* mesons off the $Z_b(10610)$ and the $Z_b'(10650)$. Using an pionless EFT with short-range contact interactions, we have derived the integral equations for the corresponding scattering amplitudes to leading order in the EFT expansion. Furthermore, we investigated the ultraviolet behavior of the scattering amplitudes and ruled out the possibility of bound states of three bottom mesons due to the Efimov effect in all considered channels. As a consequence, there are no three-body forces at leading order, and we were able to predict the phase shifts and scattering lengths for the elastic scattering of Z_bB, Z_bB^*, and $Z_b'B^*$. Our analysis showed the the $Z_b'B^*$ channel, in particular, is close to supporting an Efimov state and has a very large scattering length. Our predictions could, in principle, be tested via the final state interactions in the decays of heavier particles into three B/B^* mesons (cf. the discussion in Ref. [34]) or in lattice simulations. Because of the universality of large scattering length physics, they apply to any system with short-range interactions and the same spin-isospin structure.

In the future, it would be interesting to calculate the effective range corrections to our results. While this is straightforward in principle, at present there is no experimental information on the effective ranges available such that only order of magnitude estimates are feasible. Since the $Z_b(10610)$ is at the border of applicability of pionless EFT, an extension to include explicit pions analog to XEFT for the $X(3872)$ in the charm sector [58, 59] should be considered. With respect to future lattice calculations, it would then also be interesting to investigate the light quark mass dependence and finite volume effects in a framework with explicit pions [60–62]. A related process is the short-distance production of three B/B^* mesons. This process is also observable when the Z_b and/or Z_b' mesons are virtual states and will be considered in a forthcoming publication. For an investigation in the D meson sector, see Ref. [63].

Universal three-body states bound by the Efimov effect have been found in various areas of physics, ranging from nuclear physics to ultracold atoms [50, 64–66]. While the search for hadronic molecules bound by the Efimov effect has not been successful so far, it remains an intriguing possibility to form shallow three-body hadronic molecules with universal properties.

ACKNOWLEDGEMENTS

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