

## Decoding the structure near the $\pi^+\pi^-$ mass threshold in $\psi(3686) \rightarrow J/\psi\pi^+\pi^-$ decays

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In light of recent high-precision data taken by the BESIII Collaboration, we reconsider the dipion transition  $\psi(3686) \rightarrow J/\psi\pi^+\pi^-$ . The strong pion-pion final-state interactions are taken into account model independently by using dispersion theory. We find that we can reproduce the substructure near the  $\pi^+\pi^-$  threshold observed experimentally without introducing an extra resonance state. While a helicity-flip amplitude plays an important role for the formation of the dip in the invariant-mass distribution, the virtual exchange of the charmoniumlike exotic  $Z_c(3900)$  state improves the fit quality only slightly.

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### I. INTRODUCTION

The dipion transitions between heavy quarkonia are important for understanding both their dynamics and low-energy QCD. Because the heavy quarkonia are expected to be nonrelativistic and compact, the method of the QCD multipole expansion [1–3] is often used to study these transitions, in which the pions are produced due to the hadronization of emitted soft gluons. The QCD multipole expansion maps onto the chiral effective Lagrangian for dipion transitions between heavy quarkonia [4,5], which has been used successfully together with subsequent pion-pion final-state interactions (FSI) to study the decays  $\Upsilon(2S, 3S, 4S, 5S) \rightarrow \Upsilon(1S, 2S, 3S)\pi\pi$  [6–9] and  $\psi(3686) \rightarrow J/\psi\pi\pi$  [10,11].

Very recently, based on  $2712.4(14.4) \times 10^6$   $\psi(3686) \equiv \psi'$  events, BESIII presented a high-precision study of the  $\pi^+\pi^-$  mass spectrum in  $\psi' \rightarrow \pi^+\pi^-J/\psi$  decays. A clear resonancelike structure near the  $\pi^+\pi^-$  mass threshold was

observed for the first time [12]. In the BESIII analysis, the structure was best described by a Breit-Wigner parametrization with a mass of  $(282.6 \pm 0.4_{\text{sta}} \pm 2.5_{\text{sys}})$  MeV and a width of  $(17.3 \pm 0.8_{\text{sta}} \pm 0.4_{\text{sys}})$  MeV [12]. However, a resonance with such a low mass, which is much lower than the mass of the  $f_0(500)$  (for a review, see Ref. [13]), would be at odds with the chiral structure of QCD that suppresses two-pion interactions at low energies. Thus, the new dataset necessitates a reanalysis of the reaction  $\psi' \rightarrow J/\psi\pi\pi$ . Note that the dipion mass spectra in  $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$ , which show a similar two-hump behavior, can be explained naturally by the virtual exchange of the isovector bottomonium exotics  $Z_b(10610)/Z_b(10650)$  [6] (see Refs. [14,15] for earlier studies), and similar double-bump structures were also observed in  $\Upsilon(4S) \rightarrow \Upsilon(1S, 2S)\pi\pi$  decays as discussed in Ref. [16]. The analogous isovector charmoniumlike structure  $Z_c(3900)^\pm$  was discovered in the  $J/\psi\pi$  invariant-mass spectra by the BESIII and Belle Collaborations in 2013 in the process  $e^+e^- \rightarrow J/\psi\pi^+\pi^-$  [17,18]. Therefore, we study the decay  $\psi' \rightarrow J/\psi\pi\pi$ , considering effects of the virtual  $Z_c(3900)$ -exchange mechanism.

It is necessary to account for the two-pion FSI properly. In this article, we use dispersion theory to treat the  $\pi\pi$  FSI [6,19–22]. Instead of the chiral unitary approach [23] that was used in the literature to describe  $\pi\pi$  FSI for the process of interest [15,24], in dispersion theory it is treated in a model-independent way consistent with  $\pi\pi$  scattering data. We will provide a simultaneous description of the

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experimental data for the dipion invariant-mass distribution and the helicity angular distribution of  $\psi' \rightarrow J/\psi\pi\pi$ , which help reveal the nature of the structure near the  $\pi^+\pi^-$  threshold.

## II. THEORETICAL FRAMEWORK

For the contact  $\psi' \rightarrow J/\psi\pi\pi$  interaction, the effective Lagrangian to leading order in the chiral as well as the heavy-quark nonrelativistic expansion reads [5–7,25]

$$\begin{aligned} \mathcal{L}_{\psi'\psi\Phi\Phi} = & g_1 \langle \psi'^\alpha \psi_\alpha^\dagger \rangle \langle u_\mu u^\mu \rangle + h_1 \langle \psi'^\alpha \psi_\alpha^\dagger \rangle \langle u_\mu u_\nu \rangle v^\mu v^\nu \\ & + j_1 \langle \psi'^\mu \psi_\nu^\dagger \rangle \langle u_\mu u^\nu \rangle + c_m \langle \psi'^\mu \psi_\mu^\dagger \rangle \langle \chi_+ \rangle + \text{H.c.}, \end{aligned} \quad (1)$$

where  $v^\mu = (1, \mathbf{0})$  is the velocity of the heavy quark, and  $\psi^\alpha$  and  $\psi'^\alpha$  denote the  $J/\psi$  and  $\psi'$  fields, respectively.  $u_\mu$  collects the Goldstone bosons of the spontaneous breaking of chiral symmetry [26,27],

$$\begin{aligned} u_\mu = & i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger), \quad u^2 = e^{i\Phi/F_\pi}, \\ \Phi = & \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \end{aligned} \quad (2)$$

where  $F_\pi = 92.2$  MeV is the pion decay constant,  $\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u$ , with  $\chi = 2B \text{diag}(m_u, m_d)$ , and  $B$  is a constant related to the quark condensate in the chiral limit. The leading-order  $Z_c \psi' \pi$  and  $Z_c J/\psi \pi$  interaction Lagrangians are proportional to the pion energy [28],

$$\begin{aligned} \mathcal{L}_{Z_c \psi' \pi} = & C_{Z_c \psi' \pi} \psi'^i \langle Z_c^{\dagger i} u_\mu v^\mu \rangle + \text{H.c.}, \\ \mathcal{L}_{Z_c \psi \pi} = & C_{Z_c \psi \pi} \psi^i \langle Z_c^{\dagger i} u_\mu v^\mu \rangle + \text{H.c.} \end{aligned} \quad (3)$$

We define  $C_{\psi'\psi}^{Z_c} \equiv C_{Z_c \psi' \pi} C_{Z_c \psi \pi}$  as the product of the coupling constants for the exchange of the  $Z_c$ .

The doubly differential decay width for the process  $\psi'(p_a) \rightarrow J/\psi(p_b)\pi(p_c)\pi(p_d)$  can be written as

$$\frac{d^2\Gamma}{d\sqrt{s}d\cos\theta} = \frac{\sqrt{s}\kappa(s)}{32(2\pi)^3 M_{\psi'}^3} \frac{1}{3} \sum_{\lambda, \lambda'} |\mathcal{M}^{\lambda, \lambda'}(s, t, u)|^2, \quad (4)$$

where the Mandelstam variables are  $s = (p_c + p_d)^2$ ,  $t = (p_a - p_c)^2$ , and  $u = (p_a - p_d)^2$ , and  $\theta$  is defined as the angle between  $\pi^+$  and  $\psi'$  in the  $\pi\pi$  c.m. frame,  $\kappa(s) = \sqrt{1 - 4m_\pi^2/s} \lambda^{1/2}(M_{\psi'}^2, s, M_\psi^2)$ , and  $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$  is the Källén triangle function.  $\mathcal{M}^{\lambda, \lambda'}(s, t, u) \equiv \mathcal{M}_{\mu\nu}(s, t, u) \epsilon_{\lambda'}^\mu \epsilon_\lambda^{*\nu}$  is the helicity amplitude, and  $\lambda'$  and  $\lambda$  denote the helicities of  $\psi'$  and  $J/\psi$ , respectively. To proceed, we perform the  $s$ -channel partial-wave decomposition of the amplitude

$$\begin{aligned} \mathcal{M}^{\lambda, \lambda'}(s, t, u) = & \sum_{l=0}^{\infty} d_{\lambda-\lambda', 0}^l(\theta) S_l^{1/2}(s) \\ & \times [H^{\lambda, \lambda', l}(s) + \hat{H}^{\lambda, \lambda', l}(s)], \end{aligned} \quad (5)$$

where  $l$  denotes the relative orbital angular momentum of the pions, and  $d_{\lambda-\lambda', 0}^l(\theta)$  is the Wigner- $d$  function. In order to take the near-threshold Coulomb enhancement of the  $\pi^+\pi^-$  pair into account, we employ the Sommerfeld factors [29–31]

$$\begin{aligned} S_0(s) = & \frac{2\pi x}{1 - \exp(-2\pi x)}, \\ S_{l>0}(s) = & S_0(s) \times \prod_{b=1}^l \left(1 + \frac{x^2}{b^2}\right), \end{aligned} \quad (6)$$

with  $x = \alpha m_\pi / \sqrt{s - 4m_\pi^2}$ , where  $\alpha = e^2/(4\pi) \approx 1/137$  is the fine-structure constant.  $H^{\lambda, \lambda', l}(s)$  is the right-hand-cut part and represents the  $s$ -channel  $\pi\pi$  rescattering. The ‘‘hat function’’  $\hat{H}^{\lambda, \lambda', l}(s)$  encodes the left-hand-cut contribution derived from the partial-wave projection of the  $Z_c$ -exchange amplitude obtained from Eq. (3). Charge-parity conservation requires that  $l$  must be even, and we only take the  $S$ - and  $D$ -wave components into account in this study.

In the regime of elastic  $\pi\pi$  rescattering, the partial-wave unitarity relations read

$$\text{Im} H^{\lambda, \lambda', l}(s) = [H^{\lambda, \lambda', l}(s) + \hat{H}^{\lambda, \lambda', l}(s)] \sin \delta_l^0(s) e^{-i\delta_l^0(s)}, \quad (7)$$

where  $\delta_l^0(s)$  is the  $\pi\pi$  isoscalar phase shift of angular momentum  $l$ .

The dispersive solution to Eq. (7) is [32]

$$\begin{aligned} H^{\lambda, \lambda', l}(s) = & \Omega_l^0(s) \left\{ M^{\lambda, \lambda', l}(s) \right. \\ & \left. + \frac{s^n}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dx \hat{H}^{\lambda, \lambda', l}(x) \sin \delta_l^0(x)}{x^n |\Omega_l^0(x)|(x-s)} \right\}, \end{aligned} \quad (8)$$

where the subtraction terms  $M^{\lambda, \lambda', l}(s)$  are matched to the partial-wave projection of the low-energy chiral amplitudes obtained from Eq. (1). The single-channel Omnès function reads [33]

$$\Omega_l^0(s) = \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dx \delta_l^0(x)}{x(x-s)} \right\}. \quad (9)$$

For the  $S$  wave, we use the phase of the nonstrange pion scalar form factor as determined in Ref. [34]. For the  $D$  wave, we employ the parametrization for  $\delta_2^0$  given in Ref. [35]. A diagrammatic representation of all contributions is given in Fig. 1.

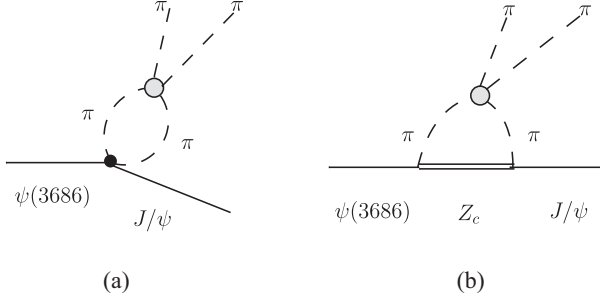


FIG. 1. Diagrams considered for  $\psi' \rightarrow J/\psi\pi^+\pi^-$ . Diagram (a) denotes the contributions of the chiral contact terms, while diagram (b) corresponds to the contributions of the  $Z_c$  exchange. The gray blob denotes pion-pion rescattering.

### III. PHENOMENOLOGICAL DISCUSSION

In this work, we perform fits taking into account the experimental datasets of the  $\pi\pi$  invariant-mass distributions and the helicity angular distribution for  $\psi' \rightarrow J/\psi\pi^+\pi^-$  measured by the BESIII Collaboration [12].

To illustrate the effect of the  $Z_c$ -exchange term, we perform three fits. In Fit I, we only consider the contribution of the chiral contact terms, while in Fits II and III, the  $Z_c$  exchange is taken into account in addition. In Fit II, the mass and width of  $Z_c$  are fixed at the central values determined in our previous work [36]:  $M_{Z_c} = 3.880$  GeV and  $\Gamma_{Z_c} = 0.036$  GeV. In contrast, in Fit III we treat the mass and width of  $Z_c$  as free parameters within the range obtained in Ref. [36]:  $M_{Z_c} \in 3.880(24)$  GeV and  $\Gamma_{Z_c} \in 0.036(17)$  GeV. The fitted results of Fits I, II, and III are shown as the blue dotted, red solid, and green dashed lines in Fig. 2, respectively. The fitted parameters as well as the  $\chi^2$  per degree of freedom ( $\chi^2/\text{dof}$ ) are shown in Table I. As can be seen from Fig. 2 and Table I, all the Fits I, II, and III agree with the experimental data in most regions by eye, while their  $\chi^2/\text{dof}$  are much larger than 1. Note that the BESIII paper [12] alternatively presents the folded  $\pi\pi$  spectra data in the low-energy range of  $M(\pi^+\pi^-) \in (0.28, 0.32)$  GeV, which were fitted using a Breit-Wigner function convolved with a Gaussian energy resolution function and multiplied by the detection efficiency. Therefore, we perform three further Fits Ib, IIb, and IIIb for the folded  $\pi\pi$  spectra dataset, neglecting the detection efficiency corrections, using the same strategy as in Fits I, II, and III, respectively. The result is shown in Fig. 3, and the  $\chi^2/\text{dof}$  values for Fits Ib, IIb, and IIIb are 1.04, 1.03, and 1.08, respectively.

In our Fits I and Ib, the enhancement near the  $\pi\pi$  threshold can be described by the chiral contact terms and FSI. In particular, no additional  $2\pi$  resonance is required. Note that Ref. [12] claims that chiral perturbation theory supplemented with a leading-order unitarization as used in Ref. [15] fails to reproduce this threshold enhancement. We find that in Ref. [12], the  $E_{\pi^+}E_{\pi^-}$  and  $(p_{\pi^+}^\mu p_{\pi^-}^\nu + p_{\pi^+}^\nu p_{\pi^-}^\mu)$  parts are considered as pure  $\pi\pi$   $S$  and  $D$  waves,

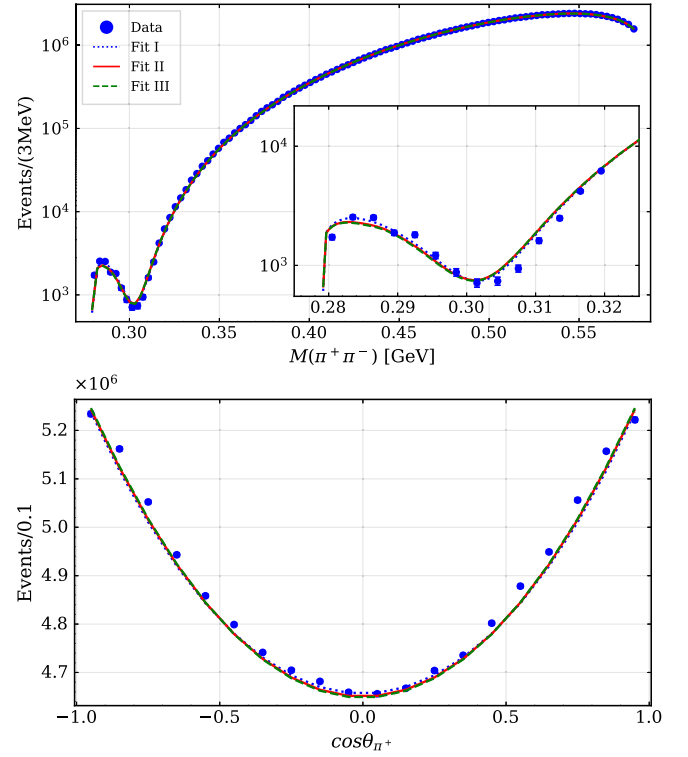


FIG. 2. Fit results of the  $\pi\pi$  invariant-mass spectra (top) and the  $\cos\theta$  distributions (bottom) in  $\psi' \rightarrow J/\psi\pi^+\pi^-$  of Fits I (blue dotted), II (red solid), and III (green dashed). The enlarged inset (top) shows the fit results in the region of  $(0.28, 0.32)$  GeV. In the  $\cos\theta$  distributions (bottom), the zero is suppressed to make differences in the fits visible. The data are taken from Ref. [12].

TABLE I. Fit parameters from the simultaneous fit of the efficiency-corrected unfolded  $\pi\pi$  invariant-mass distributions and the helicity angular distribution for the  $\psi' \rightarrow J/\psi\pi^+\pi^-$  process. Fit I: contributions of the chiral contact terms only. Fits II and III: contributions of the chiral contact terms and the  $Z_c$  exchange. In Fit II, the mass and width of the  $Z_c$  are fixed at  $M_{Z_c} = 3.880$  GeV and  $\Gamma_{Z_c} = 0.036$  GeV, while in Fit III, they are treated as free parameters within the ranges  $M_{Z_c} \in 3.880(24)$  GeV and  $\Gamma_{Z_c} \in 0.036(17)$  GeV [36]. The  $\chi^2$  values are explicitly presented as the sum of contributions from the  $\pi\pi$  invariant-mass distribution (the first number) and the helicity angular distribution (the second number).

	Fit I	Fit II	Fit III
$g_1(\text{GeV}^{-1})$	2.972(1)	3.368(1)	3.456(1)
$h_1(\text{GeV}^{-1})$	-1.680(6)	-2.435(2)	-2.571(3)
$j_1(\text{GeV}^{-1})$	-0.281(3)	-0.260(3)	-0.274(3)
$c_m(\text{GeV}^{-1})$	2.112(23)	0.424(7)	-0.012(7)
$C_{\psi'\psi}^{Z_c}$	*	-1.217(3)	-1.443(5)
$M_{Z_c}$ (GeV)	*	3.880(fixed)	3.873(2)
$\Gamma_{Z_c}$ (GeV)	*	0.036(fixed)	0.019(1)
$\chi^2/\text{dof}$	$\frac{1105+479}{(121-4)}$ = 13.5	$\frac{1095+327}{(121-5)}$ = 12.3	$\frac{1086+289}{(121-7)}$ = 12.1

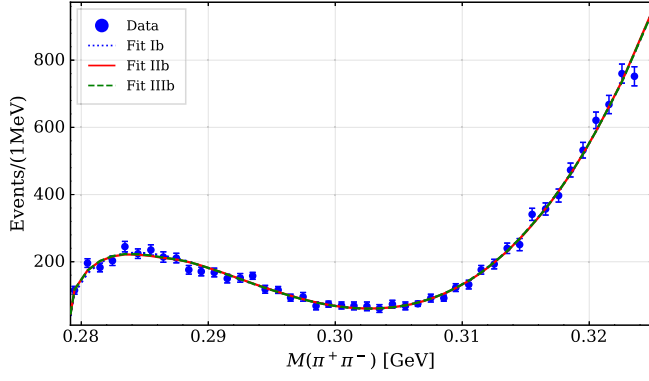


FIG. 3. Fit results of the folded  $\pi\pi$  spectra data without efficiency correction in the range of  $M(\pi^+\pi^-) \in (0.28, 0.32)$  GeV in  $\psi' \rightarrow J/\psi\pi\pi$  of Fits Ib (blue dotted), IIb (red solid), and IIIb (green dashed).

respectively, where  $E_{\pi(\pm)}$  and  $p_{\pi(\pm)}$  are the energies and the four-momenta of the pions, while both of these actually contribute to both  $S$  and  $D$  waves [15,37]. Also note that Ref. [12] considers the FSI within the chiral unitary approach [23]. While it already yields a dynamical generation of the scalar mesons, the very precise information available on pion-pion phase shifts is not strictly implemented there.

Comparing Fit I with Fits II and III, one finds that including the  $Z_c$  exchange improves the fit quality, especially for the helicity angular distribution. Furthermore, comparing Fits II and III, one observes that although a smaller  $Z_c$  mass leads to a smaller  $\chi^2$ , the change of fit quality is tiny. Therefore we conclude that a virtual intermediate  $Z_c$  state indeed can play a relevant role in  $\psi' \rightarrow J/\psi\pi^+\pi^-$ .

In Fig. 4, we plot the moduli of the  $S$ - and  $D$ -wave amplitudes from the chiral contact terms proportional to  $g_1$ ,  $h_1$ ,  $j_1$ ,  $c_m$ , and the  $Z_c$ -exchange term for Fits I and II. We observe that the  $D$ -wave amplitudes are about two orders of magnitude smaller than the corresponding  $S$ -wave amplitudes. Therefore, a flat angular distribution is expected in our scheme, which agrees with the BESIII measurement. For the  $S$ -wave amplitudes, the contribution from the helicity-flip term is much smaller than the others in most regions as expected. While the contribution from the  $j_1$  term has a zero in the energy region of interest, its effect is non-negligible for the low-energy dip in the  $\pi\pi$  invariant-mass distribution. Also note that for the  $S$ -wave amplitudes, the shapes of the curves corresponding to the  $c_m$  term as shown in Fit I (upper left panel of Fig. 4) and the  $Z_c$ -exchange term as shown in Fit II (lower left panel of Fig. 4) are too similar to be disentangled by a fit to the  $\pi\pi$  invariant mass distribution. Nevertheless, we find that either the  $c_m$  term or the  $Z_c$ -exchange term is indispensable for describing the experimental data well. For the  $D$ -wave amplitudes, the helicity-flip contribution dominates by far. This means that the curved behavior of the observed

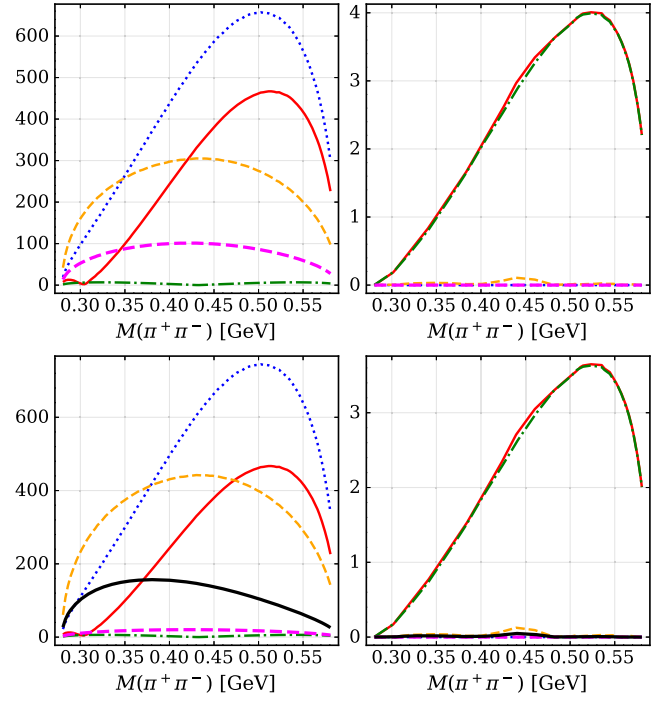


FIG. 4. Moduli of the  $S$ - (left) and  $D$ -wave (right) amplitudes for  $\psi' \rightarrow J/\psi\pi^+\pi^-$  in Fits I (top) and II (bottom). The red solid lines represent our best fit results, while the blue dotted, orange dashed, dark green dot-dashed, thick magenta dashed, and thick black solid lines correspond to the contributions from the chiral contact terms proportional to  $g_1$ ,  $h_1$ ,  $j_1$ , and  $c_m$ , as well as the  $Z_c$ -exchange term, respectively.

angular distribution is mainly due to the  $j_1$  term. We find that the  $D$ -wave contribution from the  $Z_c$  exchange is much smaller than its  $S$ -wave amplitude, which is due to the fact that the  $Z_c$  is virtual in this process.

#### IV. CONCLUSIONS

We used dispersion theory to study the FSI in the decay  $\psi' \rightarrow J/\psi\pi\pi$ . Through fitting the data of the  $\pi\pi$  mass spectra and the angular  $\cos\theta$  distributions, we find that the structure near the  $\pi^+\pi^-$  mass threshold observed by [12] can be described well without introducing a new resonance. We find that the helicity-flip amplitude plays an important role in the low-energy  $\pi\pi$  mass spectrum and the angular distributions. Also we find that the virtual  $Z_c(3900)$ -exchange mechanism improves the agreement between the theoretical prediction and the experimental data.

*Note added.* During completion of this manuscript we became aware of Ref. [38] addressing the same problem and explaining the structure near threshold as the result from the interference and superposition of two decay mechanisms. We point out that the second decay mechanism in Fig. 1 of Ref. [38] in practice refers to a

contribution for Goldstone-boson singlets, i.e., it exclusively gives a contribution to the  $\eta\eta$  channel, for which they need to include nonoctet parts of the  $\eta$ . In every other coupled-channel analysis of the  $I=0$   $S$  wave, the  $\eta\eta$  channel is tiny compared to the  $\pi\pi-K\bar{K}$  channel coupling [35,39,40]; indeed, from the fit results in Table I of Ref. [38], in order for this mechanism to have any effect, the corresponding coupling  $V_2$  is found to be larger by five orders of magnitude than that for the octet meson pair. Also, the complex phase  $\phi$ , although very small, should not exist required by the quantum field theory.

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### DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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- [1] M. B. Voloshin and V. I. Zakharov, Measuring QCD anomalies in hadronic transitions between onium states, *Phys. Rev. Lett.* **45**, 688 (1980).
- [2] V. A. Novikov and M. A. Shifman, Comment on the  $\psi' \rightarrow J/\psi\pi\pi$  Decay, *Z. Phys. C* **8**, 43 (1981).
- [3] Y.-P. Kuang and T.-M. Yan, Predictions for hadronic transitions in the  $b\bar{b}$  system, *Phys. Rev. D* **24**, 2874 (1981).
- [4] L. S. Brown and R. N. Cahn, Chiral symmetry and  $\psi' \rightarrow \psi\pi\pi$  decay, *Phys. Rev. Lett.* **35**, 1 (1975).
- [5] T. Mannel and R. Urech, Hadronic decays of excited heavy quarkonia, *Z. Phys. C* **73**, 541 (1997).
- [6] Y.-H. Chen, J. T. Daub, F.-K. Guo, B. Kubis, U.-G. Meißner, and B.-S. Zou, Effect of  $Z_b$  states on  $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$  decays, *Phys. Rev. D* **93**, 034030 (2016).
- [7] Y.-H. Chen, M. Cleven, J. T. Daub, F.-K. Guo, C. Hanhart, B. Kubis, U.-G. Meißner, and B.-S. Zou, Effects of  $Z_b$  states and bottom meson loops on  $\Upsilon(4S) \rightarrow \Upsilon(1S, 2S)\pi^+\pi^-$  transitions, *Phys. Rev. D* **95**, 034022 (2017).
- [8] Y.-H. Chen and F.-K. Guo, Chromopolarizabilities of bottomonia from the  $\Upsilon(2S, 3S, 4S) \rightarrow \Upsilon(1S, 2S)\pi\pi$  transitions, *Phys. Rev. D* **100**, 054035 (2019).
- [9] V. Baru, E. Epelbaum, A. A. Filin, C. Hanhart, R. V. Mizuk, A. V. Nefediev, and S. Ropertz, Insights into  $Z_b(10610)$  and  $Z_b(10650)$  from dipion transitions from  $\Upsilon(10860)$ , *Phys. Rev. D* **103**, 034016 (2021).
- [10] Y.-H. Chen, Chromopolarizability of charmonium and  $\pi\pi$  final state interaction revisited, *Adv. High Energy Phys.* **2019**, 7650678 (2019).
- [11] X.-K. Dong, V. Baru, F.-K. Guo, C. Hanhart, A. Nefediev, and B.-S. Zou, Is the existence of a  $J/\psi J/\psi$  bound state plausible?, *Sci. Bull.* **66**, 2462 (2021).
- [12] M. Ablikim *et al.* (BESIII Collaboration), Observation of a resonance-like structure near the  $\pi^+\pi^-$  mass threshold in  $\psi(3686) \rightarrow \pi^+\pi^- J/\psi$ , arXiv:2509.23761.
- [13] J. R. Peláez, From controversy to precision on the sigma meson: A review on the status of the non-ordinary  $f_0(500)$  resonance, *Phys. Rep.* **658**, 1 (2016).
- [14] V. V. Anisovich, D. V. Bugg, A. V. Sarantsev, and B.-S. Zou,  $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$  decay: Is the  $\pi\pi$  spectrum puzzle an indication of a  $b\bar{b}q\bar{q}$  resonance?, *Phys. Rev. D* **51**, R4619 (1995).
- [15] F.-K. Guo, P.-N. Shen, H.-C. Chiang, and R.-G. Ping, Heavy quarkonium  $\pi^+\pi^-$  transitions and a possible  $b\bar{b}q\bar{q}$  state, *Nucl. Phys.* **A761**, 269 (2005).
- [16] F.-K. Guo, P.-N. Shen, H.-C. Chiang, and R.-G. Ping, On the structure of the  $\pi\pi$  invariant mass spectra of the  $\Upsilon(4S) \rightarrow \Upsilon(1S, 2S)\pi\pi$  decays, *Phys. Lett. B* **658**, 27 (2007).
- [17] M. Ablikim *et al.* (BESIII Collaboration), Observation of a charged charmoniumlike structure in  $e^+e^- \rightarrow \pi^+\pi^- J/\psi$  at  $\sqrt{s} = 4.26$  GeV, *Phys. Rev. Lett.* **110**, 252001 (2013).
- [18] Z. Q. Liu *et al.* (Belle Collaboration), Study of  $e^+e^- \rightarrow \pi^+\pi^- J/\psi$  and observation of a charged charmoniumlike state at Belle, *Phys. Rev. Lett.* **110**, 252002 (2013); **111**, 019901(E) (2013).
- [19] R. García-Martín and B. Moussallam, MO analysis of the high statistics Belle results on  $\gamma\gamma \rightarrow \pi^+\pi^-, \pi^0\pi^0$  with chiral constraints, *Eur. Phys. J. C* **70**, 155 (2010).
- [20] B. Kubis and J. Plenter, Anomalous decay and scattering processes of the  $\eta$  meson, *Eur. Phys. J. C* **75**, 283 (2015).
- [21] X.-W. Kang, B. Kubis, C. Hanhart, and U.-G. Meißner,  $B_{14}$  decays and the extraction of  $|V_{ub}|$ , *Phys. Rev. D* **89**, 053015 (2014).
- [22] L.-Y. Dai and M. R. Pennington, Two photon couplings of the lightest isoscalars from BELLE data, *Phys. Lett. B* **736**, 11 (2014).
- [23] J. A. Oller and E. Oset, Chiral symmetry amplitudes in the  $S$  wave isoscalar and isovector channels and the  $\sigma, f_0(980), a_0(980)$  scalar mesons, *Nucl. Phys.* **A620**, 438 (1997); **A652**, 407(E) (1999).
- [24] F.-K. Guo, P.-N. Shen, and H.-C. Chiang, Chromopolarizability and  $\pi\pi$  final state interaction, *Phys. Rev. D* **74**, 014011 (2006).

- [25] J. Yan, X.-H. Cao, M.-L. Du, and F.-K. Guo, Scattering lengths of the  $J/\psi\pi$  and  $J/\psi K$  systems, [arXiv:2601.18103](#).
- [26] J. Gasser and H. Leutwyler, Chiral perturbation theory to one loop, *Ann. Phys. (N.Y.)* **158**, 142 (1984).
- [27] J. Gasser and H. Leutwyler, Chiral perturbation theory: Expansions in the mass of the strange quark, *Nucl. Phys.* **B250**, 465 (1985).
- [28] M. Cleven, F.-K. Guo, C. Hanhart, and U.-G. Meißner, Bound state nature of the exotic  $Z_b$  states, *Eur. Phys. J. A* **47**, 120 (2011).
- [29] A. Sommerfeld, Über die Beugung und Bremsung der Elektronen, *Ann. Phys. (Berlin)* **403**, 257 (1931).
- [30] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1964).
- [31] S. Cassel, Sommerfeld factor for arbitrary partial wave processes, *J. Phys. G* **37**, 105009 (2010).
- [32] A. V. Anisovich and H. Leutwyler, Dispersive analysis of the decay  $\eta \rightarrow 3\pi$ , *Phys. Lett. B* **375**, 335 (1996).
- [33] R. Omnès, On the solution of certain singular integral equations of quantum field theory, *Nuovo Cimento* **8**, 316 (1958).
- [34] M. Hoferichter, C. Ditsche, B. Kubis, and U.-G. Meißner, Dispersive analysis of the scalar form factor of the nucleon, *J. High Energy Phys.* **06** (2012) 063.
- [35] R. García-Martín, R. Kamiński, J. R. Peláez, J. Ruiz de Elvira, and F. J. Ynduráin, The pion–pion scattering amplitude. IV: Improved analysis with once subtracted Roy-like equations up to 1100 MeV, *Phys. Rev. D* **83**, 074004 (2011).
- [36] Y.-H. Chen, M.-L. Du, and F.-K. Guo, Precise determination of the pole position of the exotic  $Z_c(3900)$ , *Sci. China Phys. Mech. Astron.* **67**, 291011 (2024).
- [37] M.-L. Yan, Y. Wei, and T.-L. Zhuang, Comment on the hadronic decay of excited heavy quarkonia, *Eur. Phys. J. C* **7**, 61 (1999).
- [38] Z.-Y. Wang, Z. Liu, and X. Liu, Revealing di-pion correlations for the observed substructure near the  $\pi^+\pi^-$  mass threshold in  $\psi(3686) \rightarrow J/\psi\pi^+\pi^-$ , *Phys. Rev. D* **113**, L031502 (2026).
- [39] N. Kaiser,  $\pi\pi$  S-wave phase shifts and nonperturbative chiral approach, *Eur. Phys. J. A* **3**, 307 (1998).
- [40] R. A. Briceño, J. J. Dudek, R. G. Edwards, and D. J. Wilson, Isoscalar  $\pi\pi, K\bar{K}, \eta\eta$  scattering and the  $\sigma, f_0, f_2$  mesons from QCD, *Phys. Rev. D* **97**, 054513 (2018).