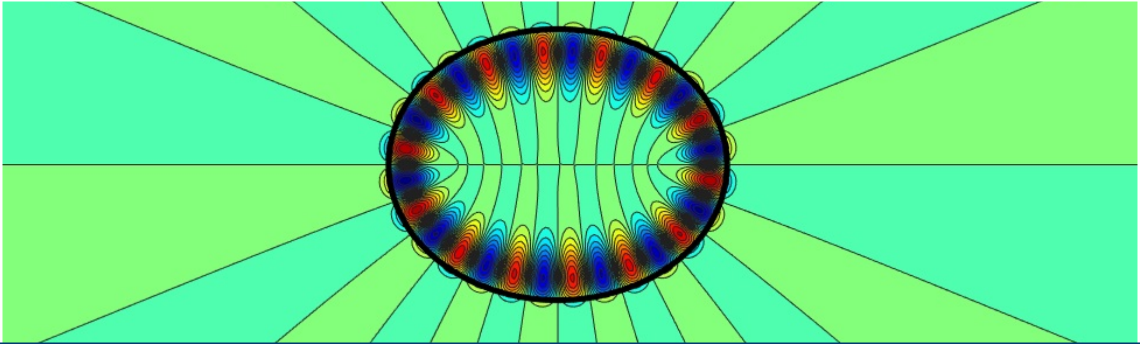


TRANSMISSION EIGENVALUES FOR BIHARMONIC PLATE SCATTERING

EXISTENCE, DISCRETENESS, FAR-FIELD RECOVERY, AND NUMERICAL COMPUTATIONS

(joint work with Isaac Harris & Heejin Lee)

WAVES2026 Montreal | June 25th, 2026 | Andreas Kleefeld | Jülich Supercomputing Centre

MOTIVATION

Scattering of an impenetrable clamped obstacle in a thin plate

- Dealing with a flat, (very) thin solid sheet (metal for example), modeled by the Kirchhoff–Love plate theory.
- Inside the plate there is a region D. Imagine either a cut-out hole (along the edge it is rigidly fixed) or a very stiff insert (does not move or bend) $\rightarrow$ clamped boundary condition.
- Impenetrable means that the wave cannot exist inside D; i.e. there is no displacement inside D. All energy is reflected.

Applications: Structural integrity and failure prevention, noise and vibration isolation, metamaterials and wave control.

INTRODUCTION

Biharmonic scattering in the exterior of a clamped cavity $D \subset \mathbb{R}^2$

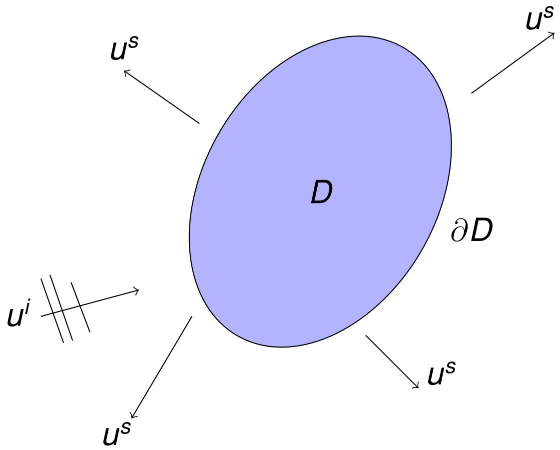
- Plane incident flexural wave u^i given.
- Scattered flexural wave u^s satisfies

$$\Delta^2 u^s - k^4 u^s = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}\,.$$

- Boundary conditions

$$\begin{aligned} u^s|_{\partial D} &= -u^i|_{\partial D} \\ \partial_\nu u^s|_{\partial D} &= -\partial_\nu u^i|_{\partial D}\,. \end{aligned}$$

- u^s satisfies Sommerfeld-type radiation condition.
- $k > 0$ is the wave number.



D is simply-connected.

INTRODUCTION

Decomposition and clamped transmission eigenvalue problem

- Use operator splitting: $u^s = u_{pr} + u_{ev}$.
- u_{pr} solves $\Delta u_{pr} + k^2 u_{pr} = 0$ and u_{ev} solves $\Delta u_{ev} - k^2 u_{ev} = 0$ in $\mathbb{R}^2 \setminus D$.
- Boundary condition gets $u_{pr} + u_{ev} = -u^i$ and $\partial_\nu(u_{pr} + u_{ev}) = -\partial_\nu u^i$.
- u_{pr} radiating (and u_{ev} exponentially decaying).

Associated clamped transmission eigenvalue (CTE) problem: find $k > 0$ and non-trivial pair $(v = u^i, w = u_{ev})$ such that

$$\Delta v + k^2 v = 0 \text{ in } D, \tag{1}$$

$$\Delta w - k^2 w = 0 \text{ in } \mathbb{R}^2 \setminus D, \tag{2}$$

and the transmission conditions

$$v + w = 0 \text{ on } \partial D, \tag{3}$$

$$\partial_\nu (v + w) = 0 \text{ on } \partial D. \tag{4}$$

THEORETICAL RESULTS

Discreteness [4, Theorem 3.3]

Theorem

The set of CTEs $k \in \mathbb{C}_+$ satisfying

$$\begin{aligned} \Delta v + k^2 v = 0 \quad \text{in } D \quad &\text{and} \quad \Delta w - k^2 w = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ v + w = 0 \quad &\text{and} \quad \partial_\nu (v + w) = 0 \quad \text{on } \partial D \end{aligned}$$

is at most a discrete subset of the real line.

THEORETICAL RESULTS

Existence [3, Theorem 3.7]

Theorem

There exist infinitely many positive CTEs corresponding to

$$\begin{aligned} \Delta v + k^2 v = 0 \quad &\textit{in } D \quad \textit{and} \quad \Delta w - k^2 w = 0 \quad \textit{in } \mathbb{R}^2 \backslash \overline{D}\,, \\ v + w = 0 \quad &\textit{and} \qquad \partial_\nu (v + w) = 0 \quad \textit{on } \partial D. \end{aligned}$$

Do pure complex-valued CTEs exist? Why pure? (see [3, Lemma 3.2])

How to compute real CTEs numerically?

THEORETICAL RESULTS

Relation to first Dirichlet eigenvalue [3, Theorem 3.8]

Theorem

Let k_1 be the first clamped transmission eigenvalue and λ_1 be the first Dirichlet eigenvalue of the negative Laplacian in D. Then

$$k_1^2 \leq \lambda_1$$

holds.

What about such an inequality for higher CTEs?

THEORETICAL RESULTS

Recovery from far-field data [4, Theorem 4.1]

Let $\phi_z = \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}$ be the far field pattern of the fundamental solution $\Phi_k(\cdot\,, z)$ of the Helmholtz equation and F the far-field operator.

Theorem

If $k > 0$ is a clamped transmission eigenvalue and g_z^ε satisfy

$$\|Fg_z^\varepsilon - \phi_z\|_{L^2(\mathbb{S}^1)} \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

Then, for $z \in D$, except for possibly a nowhere dense set, $\|g_z^\varepsilon\|_{L^2(\mathbb{S}^1)}$ cannot be bounded as $\varepsilon \to 0$.

Can we use this to numerically compute CTEs?

NUMERICAL RESULTS

CTE of unit disk

For fixed $\ell \in \mathbb{N}_0$, find the zeros of the function

$$f_\ell(k) = \det \begin{pmatrix} H_\ell^{(1)}(\mathrm{i}k) & J_\ell(k) \\ \mathrm{i}H_\ell^{(1)\prime}(\mathrm{i}k) & J_\ell'(k) \end{pmatrix},$$

where J_ℓ and $H_\ell^{(1)}$ denote the first kind Bessel function and the first kind Hankel function of order ℓ, respectively.

NUMERICAL RESULTS

CTE of unit disk

TE	
k_1	1.6146349995639885158 ($\ell = 0$)
k_2	3.0516335028155405705 ($\ell = 1$)
k_3	3.0516335028155405705 ($\ell = 1$)
k_4	4.3645169097857215923 ($\ell = 2$)
k_5	4.3645169097857215923 ($\ell = 2$)

Table: First five CTEs for a unit disk (20 digits accuracy).

What about CTEs for general domains?

NUMERICAL RESULTS

Boundary integral equation

Ansatz

$$w(x) = \mathrm{SL}_{\mathrm{i}k}\varphi(x)\,, \quad x \in \mathbb{R}^2 \setminus \overline{D}\,, \tag{5}$$

$$v(x) = \mathrm{SL}_{k}\psi(x)\,, \quad x \in D\,, \tag{6}$$

where single–layer operator is given by

$$\mathrm{SL}_{\tau}\phi(x) = \int_{\partial D} \Phi_{\tau}(x,y)\phi(y)\,\mathrm{d}s(y)\,, \quad x \notin \partial D\,.$$

Here, $\Phi_{\tau}(x,y)$ is the radiating fundamental solution of Helmholtz equation for wave number τ in two dimensions.

NUMERICAL RESULTS

Boundary integral equation

On the boundary, we obtain

$$w(x) = \mathrm{S}_{\mathrm{i}k}\varphi(x)\,, \quad x \in \partial D\,, \tag{7}$$

$$v(x) = \mathrm{S}_{k}\psi(x)\,, \quad x \in \partial D\,, \tag{8}$$

where single–layer boundary integral operator is given by

$$\mathrm{S}_{\tau}\phi(x) = \int_{\partial D} \Phi_{\tau}(x,y)\phi(y)\,\mathrm{d}s\,, \quad x \in \partial D\,.$$

NUMERICAL RESULTS

Boundary integral equation

Taking normal derivative and letting x approach boundary along with jump conditions, yields

$$\partial_{\nu(x)} w(x) = \left(-\frac{1}{2}I + \mathrm{D}_{\mathrm{i}k}^{\top}\right)\varphi(x)\,, \quad x \in \partial D\,, \tag{9}$$

$$\partial_{\nu(x)} v(x) = \left(\frac{1}{2}I + \mathrm{D}_{k}^{\top}\right)\psi(x)\,, \quad x \in \partial D\,, \tag{10}$$

where normal derivative of single–layer boundary integral operator is defined by

$$\mathrm{D}_{\tau}^{\top}\phi(x) = \int_{\partial D} \partial_{\nu(x)}\Phi_{\tau}(x,y)\phi(y)\,\mathrm{d}s\,, \qquad x \in \partial D$$

for wave number τ. We solve (7) for φ and plug this into (9). Likewise, we solve (8) for ψ and insert this into (10).

NUMERICAL RESULTS

Boundary integral equation

This yields

$$\partial_{\nu(x)} w(x) = \left(-\frac{1}{2} I + \mathrm{D}_{\mathrm{i}k}^{\top}\right) \mathrm{S}_{\mathrm{i}k}^{-1} w(x)\,, \quad x \in \partial D\,, \tag{11}$$

$$\partial_{\nu(x)} v(x) = \left(\frac{1}{2} I + \mathrm{D}_{k}^{\top}\right) \mathrm{S}_{k}^{-1} v(x)\,, \quad x \in \partial D\,. \tag{12}$$

Using boundary condition $w = -v$ and $\partial_\nu w + \partial_\nu v = 0$ within (11) and (12) ultimately gives non-linear eigenvalue problem

$$\left(\left(-\frac{1}{2} I + \mathrm{D}_{\mathrm{i}k}^{\top}\right) \mathrm{S}_{\mathrm{i}k}^{-1} - \left(\frac{1}{2} I + \mathrm{D}_{k}^{\top}\right) \mathrm{S}_{k}^{-1}\right) v(x) = 0\,, \quad x \in \partial D\,.$$

NUMERICAL RESULTS

Discretization via boundary element collocation method

- Discretization via boundary element collocation method.
- Yields non-linear eigenvalue problem which is solved with Beyn's algorithm [1].
- Solution is approximation for k and approximation of eigenfunction v on the boundary.
- We than can compute density φ in (7) using $w = -v$.
- With this, the solution of w within $\mathbb{R}^2 \backslash \overline{D}$ using (5).
- Likewise, we can compute density ψ using (9) and with this, the solution v within D using (6).

NUMERICAL RESULTS
Unit disk revisited

With 120 collocation nodes, we can obtain five digits accuracy.

CTE	determinant	BEM
k_1	1.6146349995639885158	1.61464
k_2	3.0516335028155405705	3.05164
k_3	3.0516335028155405705	3.05164
k_4	4.3645169097857215923	4.36453
k_5	4.3645169097857215923	4.36453

Table: First five CTEs for a unit disk via determinant and BEM.

NUMERICAL RESULTS

Eigenfunctions *v* and *w* for a unit disk

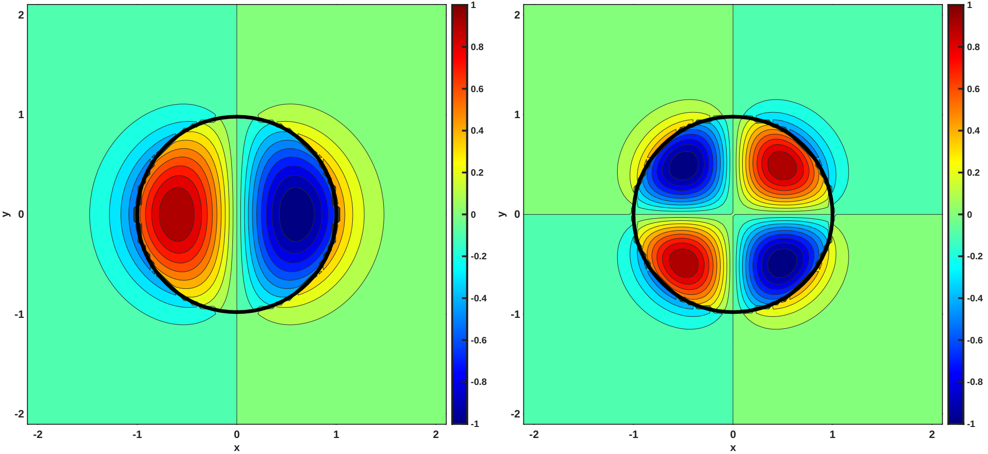


Figure: Real part of eigenfunctions *w* within $\mathbb{R}^2 \setminus \overline{D}$ and *v* within *D* for the unit disk corresponding to $k_2 \approx 3.05164$ (left) and $k_4 \approx 4.36453$ (right).

NUMERICAL RESULTS

Ellipse with various half-axis

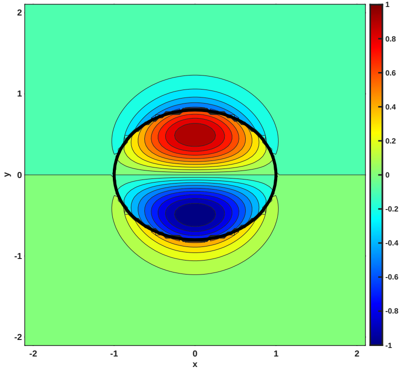
TE	$(1,1)$	$(1,0.9)$	$(1,0.8)$	$(1,0.7)$	$(1,0.6)$	$(1,0.5)$
k_1	1.61464	1.70401	1.81492	1.95646	2.14377	2.40418
k_2	3.05164	3.13673	3.24382	3.38233	3.56787	3.82845
k_3	3.05164	3.30629	3.62554	4.03737	4.58853	5.26231
k_4	4.36453	4.54060	4.67571	4.82125	5.00599	5.36324

Table: First four CTEs for various ellipses with half-axis (a, b) using BEM.

Interestingly, we observe a monotonicity behavior with respect to all transmission eigenvalues, when the area decreases. Is this true in general?

NUMERICAL RESULTS

Eigenfunctions v and w for a $(1, 0.8)$-ellipse



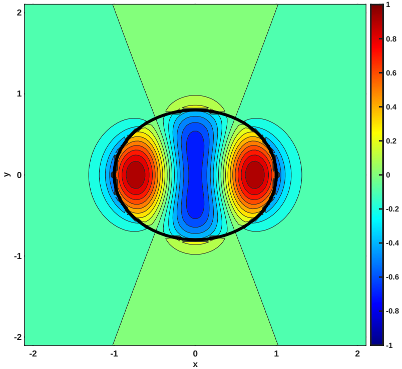


Figure: Real part of eigenfunctions w within $\mathbb{R}^2 \backslash \overline{D}$ and v within D for the $(1, 0.8)$-ellipse corresponding to $k_3 \approx 3.62554$ (left) and $k_4 \approx 4.67571$ (right).

Do we have similar nodal line properties as for Dirichlet eigenvalues?

NUMERICAL RESULTS

Deformed ellipse

Given by $\partial D = \big(0.75\cos(t) + \epsilon\cdot\cos(2t), \sin(t)\big)^\top$ for $t \in [0, 2\pi]$ with $\epsilon > 0$.

TE	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.3$
k_1	1.88515	1.89716	1.91665
k_2	3.31667	3.34174	3.38373
k_3	3.80574	3.77859	3.75151
k_4	4.77845	4.86338	4.96416

Table: First four CTEs for various deformed ellipses with parameter ϵ using BEM.

Monotone increasing behavior of all CTEs with respect to parameter ϵ. Area is constant and given by $3\pi/4 \approx 2.3562$ for all $\epsilon > 0$. Hence, we conclude that monotonicity of (first) CTE is not only given through monotonicity of area.

What are the exact properties?

NUMERICAL RESULTS

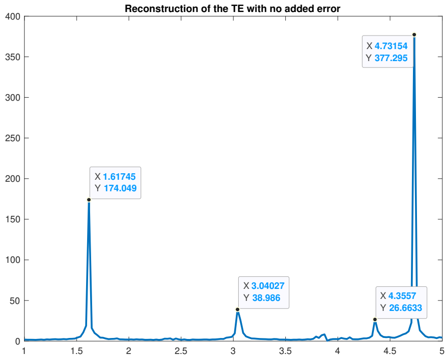
Interlacing property with Dirichlet eigenvalues and Neumann eigenvalues of the negative Laplacian

Unit disk	NE	0.00000	1.84119	1.84119	3.05424	3.05424
	CTE	1.61464	3.05164	3.05164	4.36453	4.36453
	DE	2.40483	3.83171	3.83171	5.13563	5.13562
Kite–shaped	NE	0.00000	1.77091	2.18272	3.39190	3.52298
	CTE	1.91665	3.38373	3.75151	4.96416	5.03581
	DE	2.95502	4.37204	4.78839	6.03903	6.66370
Peanut–shaped	NE	0.00000	1.72126	3.02611	3.45854	3.66118
	CTE	2.13093	3.41900	4.70289	4.89266	5.55246
	DE	3.36058	4.38535	5.79406	6.08287	6.68797
Star–shaped	NE	0.00000	3.51176	3.51176	5.32657	5.32657
	CTE	3.26716	6.18638	6.18638	8.70290	8.70290
	DE	5.06979	8.00314	8.00314	10.45544	10.45544

Table: First five Dirichlet eigenvalues (DE), Neumann eigenvalues (NE), and CTEs for various scatterers. Monotonicity of the (first) CTE w.r.t. area?

NUMERICAL RESULTS

Reconstructions from far-field data



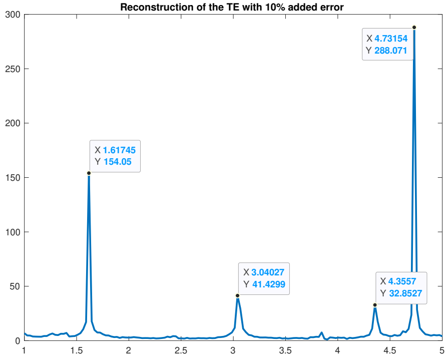


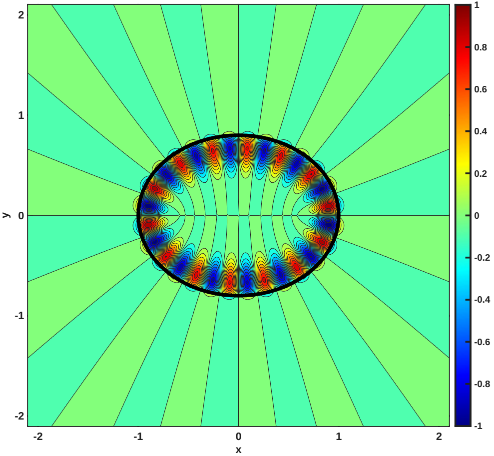
Figure: Recovered CTEs for unit disk from far-field data. Recall that first three distinct eigenvalues are given by $k \approx 1.61464, 3.05164, 4.36453$.

SUMMARY

- Have considered new transmission eigenvalue problem.
- Existence and discreteness is shown.
- As well as a relation to the first Dirichlet eigenvalue including recovery from far-field data.
- Extensive numerical results for the computation of real CTEs and its eigenfunctions via BEM are given for various scatterers.
- Reconstruction from far-field data is possible, too.

OUTLOOK AND FUTURE WORK

- Do pure complex-valued CTEs exists. None were (yet) found numerically.
- Do eigenfunctions show interesting spectral patterns? Yes, some show localization close to the boundary (see for example [2] and references therein).
- Can the interlacing property w.r.t. DEs and NEs be shown for all CTEs? There is numerical evidence that this is true.
- What about giving a proof of monotonicity w.r.t. to the area? What else besides area is needed?
- Can one compute CTEs for given near-field data and/or limited data?
- What about nodal line properties?



$k \approx 20.066$

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CONTACT INFORMATION

E-mail addresses:

Isaac Harris: **harri814@purdue.edu**

Andreas Kleefeld: **a.kleefeld@fz-juelich.de**

Heejin Lee: **heejin.lee@dukekunshan.edu.cn**

Looking forward to your ideas, comments, and inspiring conversations.