Editors' Suggestion

Limitations of Caldeira-Leggett model for description of phase transitions in superconducting circuits

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The inherent complexity of system-bath interactions often requires making critical approximations, which we here show to have a radical influence on the renormalization group flow and the resulting phase diagram. Specifically, for the Caldeira-Leggett model Schmid and Bulgadaev (SB) predicted a phase transition, whose experimental verification in resistive superconducting circuits is currently hotly debated. For normal metal and Josephson junction array resistors, we show that the mapping to Caldeira-Leggett is only exact when applying approximations which decompactify the superconducting phase. We show that there exist treatments that retain phase compactness, which immediately lead to a phase diagram depending on four instead of two parameters. While we still find an SB-like transition in the transmon regime, the critical parameter is controlled exclusively by the capacitive coupling. In contrast, the Cooper pair box maps to the anisotropic Kondo model, where a pseudoferromagnetic phase is not allowed for regular electrostatic interactions.

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I. INTRODUCTION

A realistic depiction of any quantum system necessitates precise models to incorporate dissipation. For practical calculations, however, straightforward tracing out of bath degrees of freedom is often possible for noninteracting excitations only. For quantum circuits, there are two common types of resistors, either normal metals or superconducting transmission lines, both of which can under certain conditions be mapped to noninteracting bosons (via the Luttinger liquid paradigm [1,2], or by neglecting inductive nonlinearities [3–7], respectively). These mappings, however, break charge quantization—which is justified only when charge is defined with a fuzzy spatial resolution [1,8] and when conserving the size of the Hilbert space [9,10]. Generically, if an operator A has integer eigenvalues, the moment generating function $\langle e^{i\lambda A}\rangle$ is 2π-periodic in the counting field λ [11–15]. Crucially, the very same moment generating function structures are widely present in renormalization group schemes [16,17], where λ now parametrizes the coupling and enters in the critical exponent, and A represents the environment operator the system couples to [18]. We are unaware of existing works that address the impact of such spectral properties of A for renormalization group flow equations – as most charge-charge interactions indeed involve a finite length scale. In this work, we caution that depending on how the interaction is described, the spectral properties of A may change radically, and thus significantly impact the RG flow.

Moreover, since charge (in units of the number of Cooper pairs N) and the superconducting phase (ϕ) form a pair of canonically conjugate observables, $[N,\phi]=i$ [19–22], quantization of the former imposes compactness of the latter [23,24], such that the gate-induced offset charge $N\to N+N_g$ is *not* an irrelevant gauge term [25,26]. Indeed, while the superconducting phase must be fundamentally 2π-periodic at each point in time and space [3], for certain circuit elements, like linear inductive shunts, it is justified to apply a low-energy approximation that effectively breaks phase compactness, where N_g is gauged away. For the fluxonium, for instance, this approximation implies that one never returns to the actual charge qubit for infinite inductance [5]. A similar statement is true for resistive shunts, where it was likewise criticized very recently [27] that circuits with infinite resistance (which should essentially mean no dissipation) do not seem to return to the uncoupled version. This seemingly strange observation is best understood along the lines of Refs. [28,29], which argue that a noncompact phase arises due to entanglement of the inductor or environment with the number of windings in ϕ. Inspired by this understanding, Ref. [10] has recently developed a revised description of quantum phase slip junctions, where (among other results) it becomes directly evident, that already weak quantum phase slips immediately restore N_g as a physically relevant parameter.

In this work, we combine the above fundamental aspects regarding charge quantization, and show that they lead to a significantly modified understanding of dissipative phase transitions in superconducting circuits. For circuits coupled to a generic resistor, the Caldeira-Leggett model remains the most extensively studied and utilized to date. When applied to a circuit featuring a single Josephson junction (JJ), it is formulated as [30]

$$H=E_CN^2-E_J\cos(\phi)+\sum_j\left[\frac{2e^2}{C_j}N_j^2+\frac{(\phi_j-\lambda_j\phi)^2}{8e^2L_j}\right],\quad(1)$$

where the charging energy of the superconducting island is $E_C=2e^2/C$, while $[N_j,\phi_{j'}]=i\delta_{jj'}$ describe a fictitious ensemble of LC resonators (representing the resistive element).

The coupling to the phase difference ϕ across the junction (which we refer to as "electric" coupling [31]) via the parameter λ_j results in a resistance R experienced by the circuit [32].

Schmid [33] and Bulgadaev [34] discovered that this model undergoes a superconductor-insulator transition (hereafter referred to as the SB transition) at $R_Q/R = 1$ ($R_Q = \pi/2e^2$ denotes the resistance quantum in terms of the Cooper pair charge $2e$, with $\hbar = 1$). Although this finding has been theoretically substantiated in numerous subsequent studies [35–41], both experimental validation [27,42–46] as well as certain aspects of the theoretical treatment [47–51] are to this day highly controversial. We take a different direction: instead of exploring or revisiting the phenomenology predicted by Eq. (1), we critically reexamine its universal applicability, either for a circuit coupled to a piece of normal metal (the resistor used in the experiments of Refs. [27,52]), or to a transmission line made of Josephson junction arrays [44].

As a dissipative generalization of the fluxonium [7], the Caldeira-Leggett Hamiltonian likewise makes sense for a noncompact phase only. For superconductor-normal metal heterostructures, it has been shown [37] that there exists a mapping to Eq. (1), exploiting a bosonization akin to a Luttinger liquid [1,53,54]. We instead derive a low-energy Hamiltonian which integrates out the boson (photon) instead of the fermion degrees of freedom, and identify crucial differences with respect to Eq. (1). In our model, the phase remains compact even in presence of the resistive coupling. N_g, thus, enters as a tuning parameter, such that the transmon ($E_J > E_C$) [26,55] and the Cooper pair box regimes ($E_J < E_C$) [56,57] exhibit completely different phase diagrams. In addition, and contrary to the Caldeira-Leggett model, our treatment differentiates between two very different couplings to the resistor: a capacitive coupling and a direct electric coupling. Importantly, only the capacitive interaction couples to a fuzzy, and thus nonquantized, charge, whereas the electric contact exchanges integer numbers of Cooper pairs [at least withing the quite robust Bardeen's tunneling approximation in Eq. (2)]. Consequently, while we still find an SB-like transition in the transmon regime, the critical parameter depends *only* on the capacitive coupling, which is in large parts independent of the normal metal resistivity—the parameter modified in Ref. [27].

In the Cooper pair box regime, we find that the system is described by an anisotropic Kondo model [58,59], where the two quasidegenerate charge states play the role of a pseudospin. This mapping enables us to predict the existence of a pseudoferromagnetic phase with suppressed Cooper-pair exchange. But this phase transition can only occur for a coupling with a negative capacitance. Electrostatics with ordinary (e.g., vacuum or dielectric) permittivity cannot give rise to negative capacitances, such that this transition seems forbidden. We note though that (partial) negative capacitances are a proven principle with ferroelectric materials [60–62], or can be engineered via coupling to a nearby system with special effective screening properties [63], see, e.g., attractive interactions in quantum dots via coupling to "polarizers" [64,65]. Very recently, it was proposed that a coupling to auxiliary transmons can even create capacitive interactions that are quasiperiodic in charge space [66,67].

Finally, we consider the Josephson junction array implementation of a resistor, where we argue that the same picture holds qualitatively. Adopting a path integral language, we show that if we explicitly keep quantum phase slip processes finite and insist on a compact phase, the dissipative part of the action likewise does not yield the Caldeira-Leggett result for the electric contact.

Our updated treatment of the system-bath interaction thus showcases the importance of charge quantization for dissipative quantum phase transitions—either in the form of nontrivial gauge aspects, or due to the special role of integer-valued observables in renormalization schemes. As we outline, there remains one potentially important detail concerning nonadiabatic contributions due to a coupling to higher energy states. While higher levels have to some degree been included into analytic renormalization schemes of the Kondo model [68], to the best of our knowledge there exist no treatments are tailored to our problem. We, therefore, expect that more precise predictions will necessitate numerical methods. Thus we note that the fermionic model (normal metal resistor) is readily amenable to a full numeric calculation of the phase diagram by means of numeric RG methods [69], envisaged for a follow-up research effort.

II. RESULTS

A. The normal metal resistor

Instead of bosonizing the Hamiltonian describing the superconducting-normal metal (SN) interface [37], we stick to a fermionic description which accounts for two types of interactions: a direct electric contact via the proximity effect and a capacitive coupling between the Cooper pair charge on the island. The Hamiltonian is [70]

$$H = E_C\left(N + \sum_{kk'\sigma}\lambda^z_{kk'}c^\dagger_{k\sigma}c_{k'\sigma}\right)^2 - E_J\cos(\phi) + \sum_{kk'}(e^{-i\phi}\lambda^\perp_{kk'}c^\dagger_{k\uparrow}c^\dagger_{k'\downarrow} + \text{h.c.}) + \sum_{k\sigma}\epsilon_k c^\dagger_{k\sigma}c_{k\sigma} + \dots, \tag{2}$$

where λ^z and $\lambda^\perp$ parametrize the capacitive coupling and electric contact, respectively. The notation is obviously borrowed from Kondo model physics, interpreting the quantized charge N as a type of generalized spin. The normal metal electrons with momentum k and spin σ are annihilated (created) with the fermionic operators $c^{(\dagger)}_{k\sigma}$. The normal metal shall further be in a grand canonical ensemble with a well-defined chemical potential (which we simply set to be zero for reference).

In Ref. [70], we explicitly show how to arrive at the different interactions in H (the capacitive coupling $\sim\lambda^z$ and the electric contact $\sim\lambda^\perp$). The capacitive coupling coefficient $\sim\lambda^z$ can be conveniently expressed in terms of a participation ratio of the capacitances between S and N and to ground [70] (a treatment that is valid in a pseudoelectrostatic regime, taking the speed of light to infinity). Specifically for the electric contact $\sim\lambda^\perp$, it is important to stress that the Hamiltonian in above form is valid at energies sufficiently below the superconducting gap Δ, *independent* of the quality of the SN interface (i.e., both for weak and strong tunneling across

SN). For concreteness, we deploy in Ref. [70] a perturbative Schrieffer-Wolff approximation, which is guaranteed to be a good approximation in the limit of weak tunneling across SN. We stress though, that the above form of H remains valid also for strong coupling: in that regime, the induced pairing ($\sim\lambda^\perp$) has to be determined self-consistently, which changes the value of the pairing strength, but still leads to the exact same pairing term, expressing the exchange of Cooper pairs at the SN interface (see also Ref. [71] and references therein). Note in particular that even for good SN contact, the induced pairing amplitude in the N region is always smaller than Δ (as any proximity-induced pairing can naturally never exceed the original superconducting gap). On the other hand, the coupling $\lambda^\perp$ does *not* have to be small with respect to the characteristic system energies like E_C or E_J. We do perform the expansion over it later [in Eqs. (9) and (11)], though we imply a small parameter $\nu\lambda^\perp$, where ν is the density of states of the fermionic bath.

The dots (...) at the end of Eq. (2) indicate that the normal metal may by all means include a whole host of other physics, such as disorder, electron-electron [72], and electron-phonon interactions. Many-body interactions will of course (as already stated in the introduction) render it difficult to make quantitative predictions. However, we note that we will be able to draw a number of very concrete qualitative conclusions that do not necessitate solving the bath Hamiltonian. For concrete quantitative predictions, we assume that screening allows for a description in terms of noninteracting excitations. Let us further emphasize that static disorder can (at least formally) be included in the single-particle picture, by simply diagonalizing the single-particle Hamiltonian of the uncoupled normal metal with a given impurity potential configuration. Consequently, the level index k in general no longer refers to a ballistically propagating wave, but to the set of eigenstates of the Hamiltonian in presence of disorder, leading to an updated k dependence of both energies ϵ_k as well as the coefficients $\lambda^z, \lambda^\perp$. Of course, our approach (which formally does not require averaging over ensembles of impurity configurations) has the disadvantage that it is no longer straightforward to cast the resulting resistance felt by the circuit in terms of the diffusive properties of the metal (e.g., mean free path). Indeed, diffusion is most conveniently captured by diagrammatic techniques involving artificial averaging over impurities which conveniently introduces a relaxation process, see Ref. [73] for an instructive review). The crucial advantage of our approach is however, that the indirect inclusion of a bulk resistance allows us to make statements that do not depend on the details of the metal bulk. In particular, while the coupling $\sim\lambda^\perp$ seems to nominally represent only the contact resistance, we can take it to effectively include the intrinsic normal metal resistivity. From a phenomenological point of view, we expect this approach to be sufficient, as in practice the actual resistance felt by the circuit is often obtained as a fitting parameter when experimentally characterizing the device under investigation.

Let us now point out the main differences between the models of Eqs. (2) and (1). As foreshadowed, Eq. (2) allows for a compact representation of the phase ϕ even in the presence of the bath (that is, we can impose 2π-periodicity on the wave functions of the full Hamiltonian). Consequently, and contrary to Eq. (1), the system is no longer insensitive to offset charges $N \to N + N_g$, and the phase diagram depends on N_g. Since the N_g-sensitivity of the qubit depends very strongly on the ratio E_J/E_C [25], we will discuss the different regimes separately.

Phase compactness plays a second important role. We cast Eqs. (1) and (2) into a more comparable form by transforming the couplings to the phase ϕ into an effective capacitive coupling. Applying respective unitary transformations $U^{(\mathrm{CL})} = \prod_j e^{i\lambda_j N_j \phi}$ and $U^{(\mathrm{SN})} = \prod_{q\sigma} e^{ic^\dagger_{q\sigma}c_{q\sigma}\phi/2}$, we get

$$H^{(X)} = E_C\big[N + N_{\mathrm{env}}^{(X)}\big]^2 - E_J\cos(\phi) + H_{\mathrm{env}}^{(X)}, \tag{3}$$

with $X = \mathrm{CL}, \mathrm{SN}$ for either the Caldeira-Leggett model [74,75], or the fermionic model. In this representation, the interaction term universally appears inside the charging energy $\sim E_C$ as a bath-induced shift of the charge, the respective charges being $N_{\mathrm{env}}^{(\mathrm{CL})} = \sum_j \lambda_j N_j$ and

$$N_{\mathrm{env}}^{(\mathrm{SN})} = \underbrace{\sum_{kk'\sigma} \lambda^z_{kk'} c^\dagger_{k\sigma} c_{k'\sigma}}_{N_{\mathrm{loc}}} + \frac{1}{2}\underbrace{\sum_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma}}_{N_{\mathrm{tot}}}. \tag{4}$$

Note that the latter terms are equivalent to the exponents in the unitary transformation. The (uncoupled) environment Hamiltonians read

$$H_{\mathrm{env}}^{(\mathrm{CL})} = \sum_j \left[\frac{2e^2}{C_j}N_j^2 + \frac{\phi_j^2}{8e^2L_j}\right], \tag{5}$$

$$H_{\mathrm{env}}^{(\mathrm{SN})} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \ldots + \sum_{kk'} \lambda^\perp_{kk'}(c^\dagger_{k\uparrow} c^\dagger_{k'\downarrow} + \mathrm{H.c.}). \tag{6}$$

We note that in this basis choice, the Caldeira-Leggett Hamiltonian is likewise 2π-periodic in ϕ—which naively seems to render our previous concerns moot. In particular, for a closed system, a 2π-periodic Hamiltonian and a 2π-periodic ϕ-space are straightforwardly related, in that the former allows for the qubit basis (the system without the reservoir) to be expressed in terms of a Bloch wave vector k, whereas for the latter, Bloch wave vector is fixed to the value $k = N_g$, with N_g being the equivalent of the vector potential along the compact ϕ [10,76].

But, if we cast ϕ on the circle for the transformed Caldeira-Leggett model, there is no way to get back to the original representation in Eq. (1), because the above introduced unitary transformation $U^{(\mathrm{CL})}$ is not periodic in ϕ and thus ill-defined for compact ϕ. That is, even though the Caldeira-Leggett Hamiltonian in the basis choice given in Eq. (3) is 2π-periodic in ϕ, we need to include all (periodic as well as nonperiodic) Bloch states k, in order to guarantee the equivalence of Eqs. (1) and (3). The Bloch state argument will reappear further below for the Kondo model discussion.

Conversely, the Caldeira-Leggett model, Eq. (3), with *compact* ϕ explicitly includes only a capacitive coupling to the charge (for pure capacitive coupling, see also Ref. [77]). If the bath is both capacitively *and* electrically coupled, however, only the version with extended ϕ is applicable. Here, there is only one meaningful total resistance R that does not depend on the physical origin. In contrast, ϕ is always compact in fermionic model, such that the different couplings are distinguishable: while we can transform the electric contact into an effective capacitive coupling (it is up to us how to keep track of the number of exchanged Cooper pairs), we cannot simply transform away the entire capacitive coupling into an effective electric contact. At this step, it is not only the quantized Cooper pair charge (expressed in terms of a compact phase), which is important. The two environment charges in Eq. (4) are of very different nature: $N_{\rm loc}$ a local portion of the charge within the normal metal, which partakes in the capacitive coupling to the Cooper pair charge. This coupling occurs in general with a finite spatial resolution (a fuzziness, given by, e.g., fringe effects of the electric fields). Consequently [8,9], the distribution of eigenvalues of $N_{\rm loc}$ can by all means be assumed to be (quasi)continuous. $N_{\rm tot}$ on the other hand corresponds to the total charge on the normal metal (in units of the Cooper pair charge), which changes due to dissipative transport across SN. Within the here considered low-energy approximation, the SN interface has an infinitely small length scale, such that, in the absence of quasiparticle processes (more on that later) $N_{\rm tot}$ must be integer quantized. Therefore $N_{\rm tot}$, and only $N_{\rm tot}$, can be subject to unitary transformations defined on a compact ϕ. While both types of couplings contribute to the total resistance, they are nonetheless physically distinguishable—particularly regarding poor man's scaling, as we show below. In particular, we can in a way distinguish these two resistances as either providing an Ohmic or a non-Ohmic dissipation mechanism in the Caldeira-Leggett sense. In this context, we note that Ref. [78,79] have considered the dissipative impact of Bogoliubov quasiparticles (above the superconducting gap) tunneling across the Josephson junction, and have concluded that under certain conditions, they may likewise not provide a true Ohmic dissipation mechanism. In contrast, the coupling to a normal metal resistor (with a continuum of states below the gap) was up to know regarded as purely Ohmic [37].

To proceed, the resistance is computed via the quantum Langevin equation [32,70]

$$\frac{C}{4e^2}\ddot{\phi} = -E_J\sin(\phi) - J(t) + \int_{t_0}^{t} dt' Y(t-t')\dot{\phi}(t'), \tag{7}$$

where $J(t) = i[H_{\rm env}(t), N_{\rm env}(t)]_-$ and dissipation is described by $Y(t-t') = [N_{\rm env}(t'), J(t)]_-$. Assuming weak time dependence, we can simplify the last term to $R^{-1}\dot{\phi}(t)$. The resistance for CL and SN take the form

$$R_Q/R^{\rm (CL)} = \pi\sum_k \left|\lambda_k^z\right|^2 \omega_k\delta(\omega_k), \tag{8}$$

$$R_Q/R^{\rm (SN)} = \underbrace{\pi|\lambda^\perp(0)|^2\nu^2}_{R_Q/R_\perp} + \underbrace{\pi\nu^2\lim_{\omega\to 0}(\lambda^z(\omega)\omega)^2}_{R_Q/R_z}, \tag{9}$$

respectively. Here for the CL model we use the designation $\lambda_k^z = \lambda_k\sqrt[4]{\frac{C_k}{4L_k}}$, and for SN $\lambda^{z/\perp}(\omega)$ defined as $\lambda_{kk'}^{z/\perp} = \lambda^{z/\perp}(\omega_k-\omega_{k'})$. Note that while $U^{\rm (SN)}$ eliminates the explicit coupling to ϕ, it leads to pairing fluctuations in the metal, which gives rise to the first term in Eq. (9).

B. Transmon regime

To illustrate the different roles of both resistance contributions in Eq. (9), we consider $E_J > E_C$ [80]. Here the low-energy behavior of the charge qubit reduces to a simple quantum phase slip process, leading to the system-bath Hamiltonian

$$H^{(X)} \approx -E_S\cos\left(2\pi\left[N + N_{\rm env}^{(X)}\right]\right) + H_{\rm env}^{(X)}, \tag{10}$$

where $E_S = 16[E_C E_J^3/(8\pi^2)]^{1/4}e^{-\sqrt{32E_J/E_C}}$ and regime $E_S \ll \sqrt{E_C E_J}$ is easy to reach [25]. In accordance with the above discussion, we have two possible types of quantized charges, either the charge on the superconducting island (compact ϕ), or the possibility of integer-valued contributions in $N_{\rm env}^{(X)}$. For extended ϕ, we replace the island charge operator N with the Bloch wave vector k, representing the fluxonium basis for infinite inductance [5,7]. The resulting energy $-E_S\cos(2\pi k)$ represents quantum tunneling between different minima in the $\cos(\phi)$ potential. For compact ϕ, we replace $k \to N_g$, providing the charge dispersion of the transmon ground state, see also inset in Fig. 1(b).

To estimate the renormalization of the E_S due to the fluctuations of the $N_{\rm env}$, we resort to the adiabatic approximation implying that the bath is effectively slow comparing to the superconducting island dynamics [16,81]. We stress that this approximation scheme is exact in the limit where $\sqrt{E_J E_C} \gg E_S$ (note that it is always possible to choose the values of E_J and E_C such that E_S remains constant, and the energy gap $\sqrt{E_J E_C}$ is sent to infinity), though the weakening of this constraint introduces only minor quantitative changes to the general statements, see also the discussion below.

Integrating out a given high-energy slice of the bath degrees of freedom (from frequency Λ to $\Lambda_0 > \Lambda$), the coupling in Eq. (10) renormalizes the amplitude E_S (see also Ref. [30]),

$$E_S \to \left\langle e^{i2\pi N_{\rm env}^{(X)}}\right\rangle E_S, \tag{11}$$

where $\langle\ldots\rangle \equiv {\rm tr}[\ldots]_{\{\Lambda,\Lambda_0\}}$. This procedure is also referred to as adiabatic renormalization [16]. For the Caldeira-Leggett model, we get the standard renormalized quantum phase slip amplitude

$$E_S \to e^{-R_Q\int_\Lambda^{\Lambda_0} d\omega\frac{Y(\omega)}{\omega}} E_S, \tag{12}$$

where $Y(\omega) = \frac{\pi}{2}\sum_j \frac{\lambda_j^2}{L_j}\delta(\omega-\omega_j)$ (with $\omega_j = 1/\sqrt{L_j C_j}$) is the abovementioned admittance of the dissipative bath. For an Ohmic bath, $Y(\omega) = 1/R^{\rm (CL)}$, we arrive at

$$E_S \to \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{R_Q}{R^{\rm (CL)}}} E_S, \tag{13}$$

with the dissipative phase transition marked at the point where the critical exponent $R_Q/R^{\rm (X)} = 1$, in accordance with Refs. [33,34]. Here, we would like to stress that despite obvious equivalence of the capacitive and electric couplings demonstrated in Eq. (3), the implementation of the physical capacitive ohmic coupling is much more sophisticated than the electric one [82,83].

For the fermionic model, we similarly integrate out a finite frequency slice from Λ to Λ_0 of the metal's electron degrees of freedom. Due to $[N_{\rm loc}, N_{\rm tot}] = 0$ and the eigenvalues of $N_{\rm tot}$ being integer, since the Hamiltonian changes the number of electrons by two, we find $e^{i2\pi N_{\rm tot}} = 1$, such that $\langle e^{2\pi i(N_{\rm loc}+N_{\rm tot})}\rangle = \langle e^{2\pi i N_{\rm loc}}\rangle$. Hence, here the coupling to ϕ does *not* directly contribute to the renormalization of E_S. Of course, the finite proximity effect term $\sim\lambda^\perp$ will have an indirect

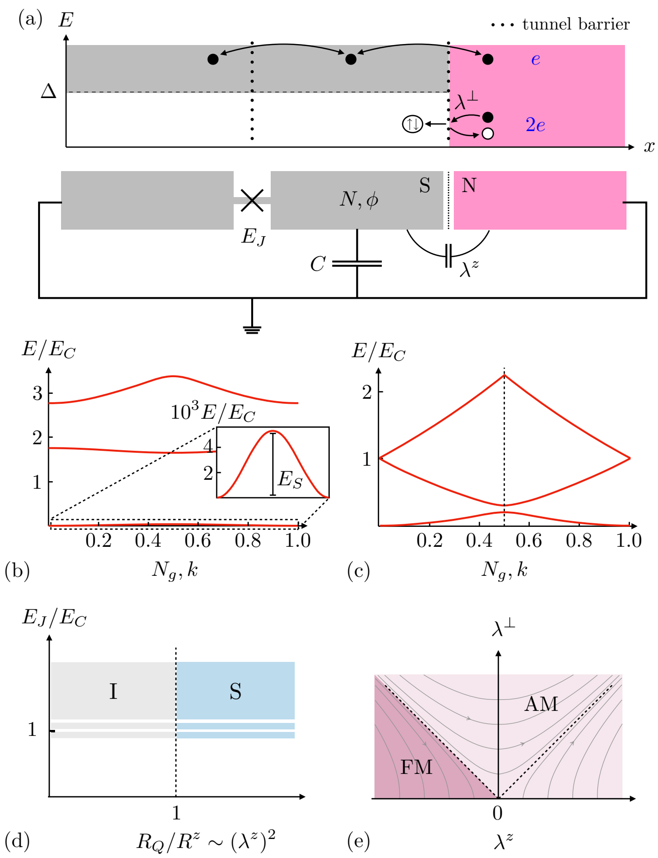

FIG. 1. An overview of main accomplishments of this work. (a) JJ-based charge qubit coupled to a normal metal with illustration of two main couplings: a capacitive one ($\sim\lambda^z$) between the superconducting island and the normal metal, and an electric contact ($\sim\lambda^\perp$) leading to a lossy Cooper pair exchange via Andreev reflection. Here, the additional high energy quasiparticle tunneling processes carrying a single elementary charge e are ignored. (b) and (c) show the energy regimes $E_J > E_C$ ($E_J = 2E_C$) and $E_J < E_C$ ($E_J = 0.1E_C$), respectively, as a function of the Bloch wave vector k (for extended ϕ) or the offset charge N_g (compact ϕ). (d) In the regime $E_J > E_C$, we still find an insulator-superconductor phase transition (which is insensitive to N_G), but with the critical parameter solely determined by the capacitive contribution to the total the resistance [$\sim (\lambda^z)^2$]. The more E_C approaches to E_J, the less reliable is the result, what is indicated by the shading. We did not study the opposite case for general N_G (that is why it is left blank), but can cite well-known result for certain values: (e) for $E_J < E_C$, and tuning N_g close to the charge degeneracy point (N_g close to half-integer), the system may instead exhibit a Kondo-like transition with a pseudoferromagnetic phase. We stress that the full phase diagram depends on four parameters (E_J/E_C, λ^z, $\lambda^\perp$, and N_g) and is scarcely studied, which is why we are able to represent it in easy accessible way only by splitting in separate panels (d) and (e).

influence on $N_{\rm loc}$, but this is a higher order effect. In leading order, we can take the trace with respect to the unperturbed normal metal state. This is our first central result: while both capacitive coupling and electric contact contribute to the total resistance, Eq. (9), only the capacitive contribution $\sim\lambda^z$ is relevant for the RG and thus for the SB phase transition, see also Fig. 1(d). To summarize, the capacitive coupling essentially provides a noninteger scrambling of the offset charge, and thus leads to a suppression of the observed E_S by destructive interference. The electric contact cannot accomplish the same effect, as it provides only integer-fluctuations at low energies, which leave the $\cos(2\pi N)$ term unchanged.

Let us reiterate, that this finding is a consequence of standard adiabatic renormalization [16]. A more precise treatment will likely provide a finite contribution of the $\lambda^\perp$ term to the renormalization of E_S, e.g., due to coupling to the transmon excited state (which has a charge dispersion with opposite sign). Namely, even though the integer contribution to the environment charge $N_{\rm tot}$ cannot directly suppress E_S, the transmon eigenbasis nonetheless depends on the fluctuating $N_{\rm tot}$ such that transitions to higher lying states are possible. As already stated, there is a well-defined limit within the theory (E_S finite, while $\sqrt{E_J E_C}$ large), where this effect can be safely neglected, and our finding is precise. For more moderate ratios of E_J and E_C, our finding is simply that on the lowest perturbative level of the RG, the Caldeira-Leggett model massively overestimates the contribution of the electric contact. On a similar token, we comment on Bogoliubov quasiparticle excitations, see Fig. 1(a). In the course of the low-energy approximation (energies below the superconducting gap Δ), quasiparticles have been discarded from the theory. These excitations lead to half-integer (instead of integer) fluctuations of $N_{\rm tot}$, and thus likewise contribute to a renormalization of E_S. However, at the very least for weak tunneling, these fluctuations are highly suppressed compared to the fluctuations of $N_{\rm loc}$, by the fact that the quasiparticles need to traverse the tunnel barriers. Also here, we therefore expect that the renormalization by means of the capacitive coupling is dominant. The quantitative statements on this issue require the approaches lying beyond Born-Oppenheimer approximation and, thus, also beyond the scope of this paper. A numerical treatment for arbitrary parameter regimes is envisaged in the future.

Crucially, Eq. (11) allows to draw a deep connection between the Ohmic nature of the fermionic system and the full-counting statistics (FCS) problem. For concreteness, consider the SN junction as a parallel plate capacitor. The charge $N_{\rm loc}$ can then be approximated as $\lambda_0^z n$, where n is the local charge at the interface, i.e., the charge on the interval of length l. The moment generating function $\langle e^{i\lambda n}\rangle$ with the counting field λ is well-known in 1D [11–15,84,85], $\langle e^{i\lambda n}\rangle \approx e^{i\lambda\langle n\rangle-\lambda^2(\langle n^2\rangle-\langle n\rangle^2)/2} \propto e^{i\lambda\langle n\rangle-\lambda^2(K/2\pi^2)\ln\langle n\rangle}$, where K is the Luttinger parameter. The average charge is proportional to the large ratio of the ultraviolet (Fermi momentum) and infrared cutoffs. Importantly, only the second moment leads to a renormalization of E_S, such that the logarithmic dependence on $\langle n\rangle$ guarantees an Ohmic behavior with respect to the running frequency cutoff Λ, i.e., the constant exponent in Eq. (13). Comparing $\lambda_0^z n$ with $N_{\rm loc}$ in Eq. (4) allows us to connect the critical exponent of the SN model with the resistive contribution due solely to capacitive coupling, R_Q/R^z, in Eq. (9), via $\lambda^z_{kk'} = \lambda^z_{k-k'}$, where $\lambda^z_k = \lambda^z_0 \sin(\pi kl/L)/\pi k \approx \lambda^z_0/\nu\omega$. Moreover, the similar calculation at $\lambda^z_{kk'} \to \delta_{kk'}$ shows that $N_{\rm tot}$ can be neglected even for general counting field [70]. Both calculations give the scaling for the Hamiltonian (10) as $E_S \to E_S(\Lambda/\Lambda_0)^{a|\lambda^z_0|^2}$, where a is simply a numeric coefficient.

Through simple dimensional analysis it can be seen that the logarithmic dependence on the ultraviolet cutoff persists 2D and 3D metals. Likewise, interactions or disorder [implied in form of ... in Eq. (2)] do not change this result, indicating the fundamental ohmicity of the considered fermionic system. A potential exception could be due to electrostatic counter terms $\sim N_{\text{loc}}^2$ in the Hamiltonian [70,72], which have recently been shown to lead to a non-Ohmic bath, as seen from the charge island [83]. We note however, that Ref. [83] considers a bosonic bath, which connects to a normal metal only for 1D, where screening is well-known to be less effective.

C. Cooper-pair-box regime

For $E_C > E_J$ and $N_g \approx 1/2$, the fermionic model (with compact ϕ) maps onto the anisotropic Kondo model. The island's Fock space is then reduced to $N = -1, 0$ and can be represented as a local pseudospin, while λ^z and $\lambda^\perp$ play the role of generalized parallel and perpendicular Kondo couplings [59] obeying the phase diagram in Fig. 1(e) [58]. Note that the transition to the pseudoferromagnetic phase (where tunneling across the SN interface is suppressed) takes place only for $\lambda^z < 0$, i.e., a *negative* capacitive shunt across the SN junction. In a recent work, it was already noted that the sign of the capacitive coupling is important for a purely capacitively coupled charge qubit (where the resistor was modeled as a transmission line) [77]. We here extend this result to the combined presence of capacitive and electric coupling. It is therefore interesting to note that while this phase transition seems inaccessible for conventional (vacuum or dielectric) electrostatics, the engineering of negative capacitances (which, as mentioned in the introduction is an actively pursued topic in various works [60–67]) could potentially render it realizable.

The Cooper pair box regime cannot be reached for the Caldeira-Leggett model for the simple reason that ϕ is not compact. Here, the tuning parameter N_g is replaced by a Bloch vector k. If we then assume the system to be in a Gibbs ensemble state, the Bloch vector will invariably relax to the ground state, which, for the closed system is simply $k \approx 0$. Overall, we thus see the following difference between the phase diagram resulting from Eqs. (1) and (2). For the standard Caldeira-Leggett model, E_J/E_C and the total resistance R/R_Q [Eq. (8)] are the only two relevant tuning parameters. For the fermionic model here, we have in total four relevant tuning parameters. In addition to E_J/E_C, the outcome depends strongly on N_g, and likewise, the two couplings $\lambda^\perp$ and λ^z assume very different roles, in both the Cooper-pair box regime, and the transmon regime [where they contribute to two physically distinct resistive contributions, see Eq. (9)].

D. Bosons versus fermions

For the above normal metal resistor, we obtained a Hamiltonian description and resulting phase diagram that is markedly different from the Caldeira-Leggett model. As pointed out in the introduction, a mapping from the SN interface to the Caldeira-Leggett model has been demonstrated [37] through integrating out fermionic degrees of freedom. When considering the SN interface in a standard bosonized language (see also Ref. [70]), we can identify the crucial step, where charge quantization gets lost. Qualitatively speaking, charges transported across the interface are no longer integer quantized, as bosonization blurs the charge density operator over a characteristic length scale (for a homogeneous 1D metal, this would be Fermi wave length [1]). In this final results section, we want to provide an equivalent, but alternative perspective on this aspect in terms of quantum phase slip processes in the bath, respectively entanglement of phase windings between system and bath. We will do so by comparing normal metal resistors with Josephson junction array resistors, where we detect some formal similarities with regards to charge quantization, phase compactness and bosonization.

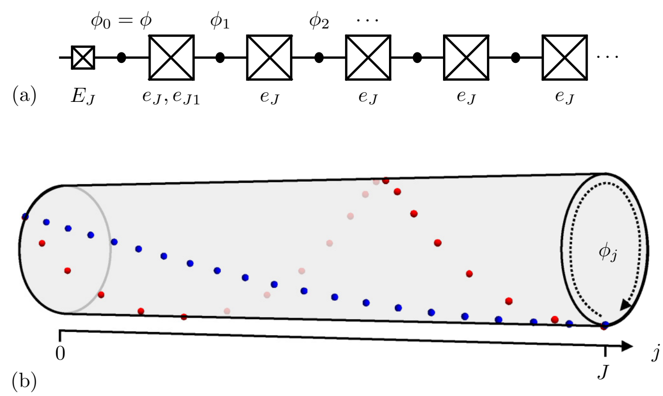

FIG. 2. Low-energy treatment of Josephson junction array resistor compatible with phase compactness. (a) The charge island ϕ is electrically coupled to the array with nodes ϕ_j with $j = 1, \ldots, J$ (for notational simplicity, we extend the index j to 0 to include the charge island as $\phi_{j=0} = \phi$). (b) Even when e_J is large, such that the phase differences $\phi_{j+1} - \phi_j$ are small, it is important to consider the configuration space of the total system on a compact manifold (cylinder for the 1D array). An extended ϕ is only meaningful when the array has a coupling to ground at the far end $j = J$ (such that the phase ϕ_J is strongly coupled to ground, $\phi_J \approx 0$). Here, configurations with different number of windings in the phase profile within the array (blue and red) count the number of times ϕ has revolved around the circle (see also arguments in Refs. [10,28,29]).

Let us begin with modeling an environment by means of a Josephson junction array [see also Fig. 2(a)],

$$H = E_C N^2 - E_J \cos(\phi) - e_J \cos(\phi_1 - \phi) + e_C \sum_{j=1}^{J} N_j^2 - e_J \sum_{j=1}^{J-1} \cos(\phi_{j+1} - \phi_j) , \quad (14)$$

where $j = 1, \ldots, J$ indexes the nodes of the array. The array can be regarded as a bath in the thermodynamic limit $J \to \infty$, and assuming (in analogy to the normal metal resistor) it to be in a grand canonical ensemble. We note however, that our line of reasoning below is also valid for finite J. There is an interesting alternative case for finite J, where in addition the last node is connected back to ground via a term $-e_J \cos(\phi_J)$. The charge qubit and array now form a loop (the actual fluxonium). As we briefly comment at the end of this work, this system works differently, as here, the bath is in a certain regime capable of storing the information of the number of windings that the charge qubit performed in ϕ-space.

To proceed, while the original array Hamiltonian, Eq. (14), preserves charge quantization (and thus phase compactness) on each node (both for the central charge island, and the nodes that make up the array), one can make an approximation which is in its essence very similar to the bosonization procedure for fermionic degrees of freedom. Assuming that the Josephson energies in the array are very large ($e_J \gg E_J, e_C$), we arrive at the Caldeira-Leggett model, Eq. (1), by approximating the cosine energy-phase relationship of each Josephson junction $\sim e_J$ as $\cos(\delta\phi) \approx 1 - \delta\phi^2/2$ [7,86]. What we have essentially done is to neglect quantum phase slip processes within the array, and extending the phase difference $\delta\phi$ to the real line. This is by all means a reasonable procedure for computing eigenenergies of the bath (array) Hamiltonian, but, crucially, it again fails to conserve the global properties of the moment generating function $\langle e^{i\lambda N_j}\rangle$, which we have already shown to be relevant for RG, see Eq. (11).

In particular, we note that while compactness of the phase *differences* is to some degree irrelevant (since the large e_J guarantees small phase differences) we should still think of the individual phases ϕ_j to be compact. That is, an accurate analogy for the system is that of a chain of coupled pendulums [such that the phase profile ϕ_j as a function of j can be mapped to a cylinder, see Fig. 2(b)]. Consequently, the phase of the charge qubit ϕ (and of each array node) is and always has been compact.

Let us now argue, that this proper consideration of phase compactness is of utmost importance for phase transitions also for JJ arrays. We again follow the same procedure as for the normal metal resistor, but importantly, we refrain from decompactifying the local phases. Instead, we take the exact Hamiltonian as given in Eq. (14), and again perform a unitary transformation, here $U = \prod_j e^{-iN_j\phi}$, to transform the coupling to the ϕ into an effective charge-charge coupling, which modifies the charging term again as $N^2 \to (N + N_{\text{tot}})^2$, where $N_{\text{tot}} = \sum_j N_j$, and the renormalization prefactor is predicted to be $\langle e^{2\pi i \sum_j N_j}\rangle$. Since the canonical charge N_j is quantized, all eigenvalues of the charge operators are integer, and the average of the exponent $e^{2\pi i N_j} = 1$, yet again over *any* ensemble (that is, we can make this conclusion without even solving for the bath Hamiltonian). This means that in the transmon limit, the renormalization of E_S is absent in the same way it already was for the normal metal. Note that we may include an actual capacitive coupling to the JJ array, just as we did for the normal metal, which again provides an SB-like phase transition in the transmon regime, as we would here obtain a coupling of the form $N^2 \to (N + \sum_j \alpha_j N_j)^2$, where α_j can assume any real value.

To complete this section, we corroborate the above qualitative discussion of the JJ array by means of more quantitative arguments within the path integral formalism. Following Ref. [3], the ohmic bath in the Caldeira-Leggett model contributes with a term $\eta \sum_{\omega_n} |\omega_n| \phi(\omega_n)\phi(-\omega_n)$ to the action (where $\eta = R_Q/R^{\text{CL}}$, which dominates in between ultraviolet (Λ_0) and infrared (Λ) cutoffs, and can be written in time domain as

$$\delta L_{\text{bath}}^{(\text{CL})} \approx \iint d\tau d\tau' K(\tau - \tau')\phi(\tau)\phi(\tau'), \tag{15}$$

where $K(\tau) = \eta/4\pi^2\tau^2$. For small $E_J \ll E_C$, one can perform perturbation theory in E_J, and one finds that the Josephson energy is renormalized as $E_{J,\text{eff}} = b^{-1/\eta} E_J$, where $b = \Lambda_0/\Lambda \gg 1$ [70]. In order to see the SB transition, we have to be able to vary η on the scales from $\eta \gg 1$ to $\eta \ll 1$ and still remain in the frequency window, when the dynamics of the field ϕ is governed by the dominating bath term $\eta|\omega_n|$. In this regime, the average fluctuation of the field is equal to $\phi \sim \sqrt{\eta^{-1} \ln b}$. Thus the contribution of the bath term can be estimated as $\ln b$, guaranteeing the strong renormalization. For the opposite, transmon case $E_J \gg E_C$, the characteristic energy E_S of lowest level coming from the phase slip processes. In order to estimate the contribution from the bath term (15) we exploit the instanton solution for $\phi(\tau)$, obtaining the value of order of $\sim\eta \ln b$ [70], which, in principle, can be large.

To study of the decoupling of the system from the environment and the importance of the closed manifold, we analyze the bath contribution for the JJ array model introduced in Fig. 2 with a special parameter choice. We implement a weak coupling to the bath with a Josephson junction with energy e_{J1} [which may be small compared to e_J see Fig. 2(a)], such that its cosine can certainly not be approximated by a parabola. Nonetheless, for the sake of the argument, we assume a parabolic approximation for all other Josephson junctions within the array (treating it as a simple transmission line, with a reservation, see below). Integrating out the phase ϕ_1 we get the bath contribution in the form

$$\delta L_{\text{bath}}^{(\text{JJa})} \approx \iint d\tau d\tau' K(\tau - \tau') \sin\phi(\tau) \sin\phi(\tau'), \tag{16}$$

with new $\tilde{\eta} = \eta(e_{J1,\text{eff}}/\Lambda)^2$ (see derivation in Ref. [70]). This term *conserves* phase compactness and for small $\tilde{\eta}$ the field in delocalized over the whole closed manifold (as long as the bath contribution is dominating), so that it's value can be estimated as $\phi \sim 1$. Thus the total contribution to the action vanished as $\sim\tilde{\eta}$ an therefore cannot affect the island's action changing the value of the E_J. For the transmon case, the same instantonic solution returns the estimation of the bath term (16) of order of $\sim\eta$, up to numerical coefficient of order of 1 [70], and therefore likewise cannot invoke the phase transition. We additionally have to stipulate the large bath coupling element $e_{J1} \to e_J$. Naively, this case may seem equivalent to the CL model with all cosines expanded. However, as we have shown above, the phase transition implies a strong delocalization of the phase, and the bath cannot be adequately described by the $\eta|\omega_n|$ term. We hope to resolve this issue in the upcoming numerical treatment.

Overall, while the Caldeira-Leggett model is suitable to describe some traits of Josephson junction array resistors, it does not seem to be adequate to accurately predict phase transitions, at least in the here considered perturbative electric coupling. Importantly, we expect that this reasoning can be extended for all perturbation orders, while the JJ array model will retain its main property that all fields are entering in the expression as continuous trigonometric functions. Hence, phase transitions, which in CL model happen due to delocalization of the wave function, cannot overcome the obvious limit of $|(\cos)\sin\phi| < 1$.

We conclude this section with a final remark on Eq. (16). Had we formulated the normal metal resistor physics likewise within the path integral language, the electric contribution to resistance ($\sim\lambda_\perp$) would have yielded a similar functional ϕ-dependence with geometric functions ($\sim$ sin, cos), and thus would have resulted in the same boundedness of the action. We can thus relate the presence or absence of the phase transition to whether or not a certain physical process can be cast into a true Ohmic form, Eq. (15), or not. We again refer to Refs. [78,79], who noticed the possibility of similar geometric functions of ϕ for the related, but distinct case of dissipation via Bogoliubov quasiparticles.

E. Decompactification through entanglement

In all above cases (normal metal and JJ array resistor), the electric coupling (coupling to ϕ) could be mapped to a capacitive coupling, where the corresponding unitary transformation yielded a shift of the charge as $N \rightarrow N + N_{\text{tot}}$, where N_{tot} had integer quantized eigenvalues, thus resulting in a coupling that is irrelevant for the RG flow. We here show with the example of the Josephson junction array a possible exception.

Namely, we pick up on the already briefly mentioned variant of the Hamiltonian in Eq. (14), where we keep J finite and add a Josephson coupling to ground at the end of the chain, $H \rightarrow H - e_J \cos(\phi_J)$. Importantly, note that here, the same unitary as before ($U = \prod_j e^{-iN_j\phi}$) cannot eliminate the coupling to ϕ, as it simply maps from a coupling $\cos(\phi_1 - \phi)$ to a coupling $\cos(\phi_J - \phi)$. There is however a different transformation that can eliminate the coupling to ϕ (see also Refs. [9,10]), with $U = \prod_j e^{-i(1-j/[J+1])N_j\phi}$. Note that this transformation is strictly speaking illegal, as it does not conserve compactness of ϕ. But as shown in Ref. [10], this procedure can be justified by considering the JJ array part of the Hamiltonian separately from the charge qubit, and assume the phase ϕ to be a classically fixed parameter. In the limit of large e_J, the phases at both ends of the array are thus pinned close to the values $\phi_1 \approx \phi$ and $\phi_J \approx 0$. This pinning however still allows many low-energy solutions, with different number of windings around the cylinder, see Fig. 2(b). After appropriately solving the low-energy physics of the JJ array with fixed ϕ, the system can subsequently reintegrated into the circuit including the charge qubit, and the phase ϕ quantized [10].

In accordance with Refs. [10,28,29], it is here reasonable to think of ϕ as extended, but only in a very specific sense. Configurations that have the same (compact) value of ϕ, but different winding number, can be mapped to a different representation wherein ϕ is now extended. If we start from a configuration without windings, and let the system evolve (importantly, while neglecting phase slips within the bath), we know that any winding of the system final state contains the precise information of how many times the phase of the charge island ϕ has revolved around the circle on which it is defined.

In this regime, the bath perfectly entangles with the number of times the phase ϕ revolves around the circle. Consequently, ϕ can indeed be regarded as effectively extended (noncompact) and as a consequence, N_g is here an irrelevant gauge term. Note that the intrinsic strain within the array provides in addition a (small) linear inductive energy term $\sim E_L\phi^2$. Moreover, the above unitary creates a different type of environment charge $N \rightarrow N + \sum_j (1 - j/[J + 1])N_j$, which, crucially, has *no* integer quantized values. Thus the system where the JJ array and charge qubit form a loop maps in this regime precisely onto the Caldeira-Legget model, in accordance with the findings of Ref. [87].

Let us conclude by presenting some final caveats. First of all, the loop description is only valid for a finite size array (J not infinitely large). It must therefore be questioned, whether such an array (at least within the here presented minimal description) actually serves as a true dissipative reservoir. Moreover, as remarked in Ref. [10], the presence of quantum phase slips (the array may be subject to quantum tunneling between configurations with different winding number) immediately restores phase compactness, and thus revives N_g as a real physical parameter. In terms of quantum information, the bath now has a finite probability to "forget" the number of revolutions of ϕ (such that the entanglement is subject to some decay). In the transmon regime ($E_J \gg E_C$), there are thus essentially two quantum phase slip processes, one coming from the charge qubit, the other from the intrinsic tunneling of different winding configurations within the array. These two phase slip amplitudes can be regarded as interfering due to N_g (also known as the cQED version of the Aharonov-Casher effect [88,89]). While the former gets renormalized as per the SB transition, the latter grows with system size (either with $\sim J$ for clean systems, or $\sim\sqrt{J}$ when the charge islands of the array have fluctuating gate offset charges). It was recently predicted [86] that for very long arrays, quantum phase slips within the array are themselves subject to renormalization according to the RG scheme by Giamarchi and Schulz [17].

At any rate, the above exception shows that topology enters in more than one way. While the superconducting phase ϕ is in principle always compact, closing the JJ array system to a loop effectively allows for its decompactification. With this counterexample, we show that an exact mapping to Caldeira-Leggett is possible if (and only if) the bath is capable of exactly storing the information of the number of times the (compact) phase ϕ has revolved around the circle.

III. DISCUSSION

We demonstrate that physically distinct dissipation mechanisms (which all lead to a contribution to the resistance) do not have the same impact on the predicted phase diagram of dissipative quantum circuits. By deploying a fermionic model describing capacitive coupling as well as the proximity effect with a normal metal resistor, we find that the mapping to the standard Caleira-Leggett model is not exact. In particular, capacitive coupling turns out to be the dominant factor for phase transitions, rendering the normal metal resistivity (the parameter controlled, e.g., by Ref. [27]) irrelevant. Our results suggest that future experiments could specifically address parts of the device geometry that change the SN capacitance, but not the metal resistance, in order to separate both contributions. We also identified a Cooper-pair box regime where the dissipative system maps to the anisotropic Kondo model. Also here, capacitive and electric coupling play a distinct role, and in particular, the transition to a pseudoferromagnetic phase is forbidden for regular (repulsive) electrostatic coupling. We further analyze an alternative realization of a resistor element by means of Josephson junction arrays, and demonstrate that the same conclusions hold qualitatively. Overall, for both types of resistor realizations, we link the decompactification of the superconducting phase with the approximation of the bath degrees of freedom in terms of noninteracting bosons. Without this approximation, we show that the electric contact is irrelevant for the RG flow. The only identifiable exception is when the bath is capable of exactly entangling with the number of times the superconducting phase ϕ revolves around itself. Finally, we identify a nontrivial open aspect with respect to nonadiabatic coupling to higher energy states. In this context, we point out that the fermionic (normal metal) model lends itself to future research endeavors by means of numerical RG methods [69], where the currently presented predictions can be refined.

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