

Chiral versus deconfinement properties of the QCD crossover: Differences in the volume and chemical potential dependence from the lattice

Szabolcs Borsányi,¹ Zoltán Fodor,^{1,2,3,4,5} Jana N. Guenther¹, Ruben Kara¹, Paolo Parotto⁶,
Attila Pásztor^{3,7,*}, Ludovica Pirelli¹ and Chik Him Wong¹

¹Department of Physics, Wuppertal University, Gausstr. 20, D-42119, Wuppertal, Germany

²Pennsylvania State University, Department of Physics, State College, Pennsylvania 16801, USA

³Institute for Theoretical Physics, ELTE Eötvös Loránd University,

Pázmány P. sétány 1/A, H-1117 Budapest, Hungary

⁴Jülich Supercomputing Centre, Forschungszentrum Jülich, D-52425 Jülich, Germany

⁵Physics Department, UCSD, San Diego, California 92093, USA

⁶Dipartimento di Fisica, Università di Torino and INFN Torino, Via P. Giuria 1, I-10125 Torino, Italy

⁷HUN-REN-ELTE Theoretical Physics Research Group,

Pázmány Péter sétány 1/A, H-1117 Budapest, Hungary



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The crossover from hadronic to quark matter is understood to be both a deconfinement as well as a chiral symmetry–restoring transition. The precise definition of these aspects may be ambiguous. Here we use the static quark free energy and its temperature derivative as proxies for deconfinement and the chiral condensate and its mass derivatives for the chiral transition. At zero baryochemical potential, and infinite volume, the chiral and deconfinement crossover temperatures almost agree. However, as we vary the spatial volume of the simulation, we observe that chiral and deconfinement-related observables have a qualitatively different chemical potential and volume dependence. In general, deconfinement-related observables have a milder volume dependence. Furthermore, while the deconfinement transition appears to get broader with increasing μ_B , the width as well as the strength of the chiral transition is approximately constant. Our results are based on simulations at zero and imaginary chemical potentials using 4-stout-improved staggered fermions with $N_\tau = 12$ time slices and physical quark masses.

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I. INTRODUCTION

The study of the phases of strongly interacting matter is an active field of research. In particular, the phase diagram in the temperature T –baryochemical potential μ_B plane has garnered a lot of interest, from both the theoretical and experimental communities. First-principle lattice QCD calculations have shown that the transition from a hadron gas to a quark-gluon plasma is a smooth crossover at zero and small chemical potentials [1]. At larger baryochemical potentials, model and functional calculations [2–5] predict the existence of a critical end point, where the crossover line in the T – μ_B plane turns into a line of first-order transitions. The position of this critical end point is very different in different model calculations. Thus, a first-principle theory calculation, or an experimental discovery

of its location, would be a great achievement. Indeed, the experimental discovery of this critical end point is one of the major goals of relativistic heavy-ion physics [6].

First-principle lattice QCD calculations at nonzero real μ_B are hampered by a sign problem. Thus, most lattice results on physics at $\mu_B > 0$ come from analytic continuation, either using lattice calculations of Taylor expansions around $\mu_B = 0$ [7–16], or lattice calculations at purely imaginary chemical potentials [17–32]. While the ill-posedness of numerical analytic continuation limits the effectiveness of such methods, a lot has been learned by applying such methods to the study of the phase diagram at small enough μ_B . Nevertheless, due to the ill-posed nature of numerical analytic continuation, such methods have severe limitations. In particular, it is exceedingly difficult to reliably reach large chemical potentials with such methods.

A different set of methods is based on reweighting [33–40,40–44]. A subset of reweighting methods, sign and phase reweighting [39,40,43], have the advantage of giving more direct results. Thus, they have the advantage of better reliability, when compared to analytic continuation methods. However, they are, at least for the time being, restricted to smaller physical volumes, due to the high numerical

*Contact author: apasztor@bodri.elte.hu

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costs. One motivation of this current work is to identify physical observables with milder finite-volume effects, which can then later be calculated up to higher chemical potentials with such reweighting methods.

In this work, we use the imaginary chemical potential method to extrapolate certain physical observables to non-zero baryochemical potential μ_B . We have two main goals. First, we want to compare observables related to the two main theoretical aspects of the QCD transition: chiral symmetry, which is an $SU(2) \times SU(2)$ symmetry in the limit of light quarks, and confinement, which is related to a Z_3 symmetry in the limit of heavy quarks. To study chiral dynamics, we calculate the chiral condensate and chiral susceptibility. To study confinement/deconfinement mechanics we calculate the Polyakov loop and quantities derived from it: the static quark free energy and entropy. Second, we look at the 3-volume dependence of the phase diagram. Here volume means the simulation volume in physical units with periodic boundary conditions in spatial directions. In particular, we quantify the volume and μ_B dependence of the crossover temperature, using different inequivalent definitions, based on both chiral symmetry and deconfinement, and calculate parameters defining the strength and width of the transition.

The picture we arrive at is nuanced. At infinite volume the crossover temperatures defined with the help of chiral symmetry and deconfinement are close, but not identical, and the volume dependence of the two crossover temperatures is different. The definition related to deconfinement has much smaller finite-volume effects, and the sign of the finite-volume effects is also the opposite: while the chiral crossover temperature increases with increasing volume, the deconfinement temperature decreases. The width and strength of the transition also behave differently. Again, quantities related to chiral symmetry show much larger finite-volume effects. Furthermore, deconfinement-related quantities show a weakening of the crossover transition with increasing μ_B , both at small and large volumes. This is in contrast to chiral observables, which show a strengthening of the transition at smaller volumes and a roughly constant strength at large volumes.

The structure of the paper is as follows. First, we discuss the setup of our lattice simulations. Next, we discuss volume dependence at zero chemical potential. Then, we go on to volume dependence at nonzero imaginary and real chemical potentials. We conclude with a summary and some speculations on the possible source of our findings.

II. LATTICE SETUP

We use a tree-level Symanzik-improved gauge action and four steps of stout smearing in the staggered fermion action, with $N_f = 2 + 1 + 1$ flavors of dynamical quarks. The quark masses are tuned such that the pion and kaon masses are equal to 135 and 495 MeV. The scale is set with the pion decay constant $f_\pi = 130.41$ MeV. Details on the action, the scale setting and the line of constant physics can

be found in Ref. [11]. We use lattices with $N_\tau = 12$ time slices. We use several physical volumes, with the number of spatial sites $N_s = 20, 24, 28, 32, 40, 48$ and 64 for the $\mu_B = 0$ simulations. For three of the spatial volumes ($N_s = 32, 40$ and 48) we also simulate at a purely imaginary chemical potential, with $\text{Im} \frac{\mu_B}{T} \frac{8}{\pi} = 3, 4, 5, 6, 6.5$ and 7. Throughout this work we always use a strangeness chemical potential μ_S tuned in such a way that the expectation value of strangeness is equal to zero. This tuning of μ_S is done to match conditions in heavy-ion collisions, where the colliding nuclei in the initial state have zero strangeness. Details on setting the strangeness neutrality condition can be found in the supplemental material of Ref. [32]. In this work we only show statistical errors.

III. VANISHING CHEMICAL POTENTIAL

A. Chiral observables

In order to study the chiral aspects of the QCD crossover we calculate the chiral condensate $\langle \bar{\psi}\psi \rangle$ and the chiral susceptibility χ defined respectively as

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \frac{\partial \log Z}{\partial m_{ud}}, \quad \chi = \frac{T}{V} \frac{\partial^2 \log Z}{\partial m_{ud}^2}. \quad (1)$$

In the limit of zero quark masses $\langle \bar{\psi}\psi \rangle$ functions as the order parameter of the chiral phase transition. In terms of the discretized (massive) Dirac operator for the up and down quarks M_{ud} , the chiral condensate is written as

$$\langle \bar{\psi}\psi \rangle = \frac{1}{2} \frac{T}{V} \langle \text{Tr} M_{ud}^{-1} \rangle \quad (2)$$

where $\langle \dots \rangle$ means taking an expectation value over the gauge fields and the factor $\frac{1}{2}$ appears due to staggered rooting. In lattice language, the order parameter is then proportional to the trace $\text{Tr} M_{ud}^{-1}$. We also study the disconnected part of the chiral susceptibility, defined as

$$\chi_{\text{disc}} = \frac{1}{4} \frac{T}{V} (\langle \text{Tr} M_{ud}^{-1} \rangle^2) - \langle \text{Tr} M_{ud}^{-1} \rangle^2, \quad (3)$$

where the factor of 1/4 is present, again, due to staggered rooting. This is essentially the variance of $\text{Tr} M_{ud}^{-1}$ in lattice QCD language. The disconnected susceptibility can also be defined as a cross derivative:

$$\chi_{\text{disc}} = \frac{T}{V} \left(\frac{\partial^2}{\partial m_u \partial m_d} \log Z \right)_{m_u=m_d}; \quad (4)$$

thus, it is a legitimate physical observable. We study both the full and disconnected susceptibilities, because they may be sensitive in different ways to the distinct critical points present in the three-dimensional QCD phase diagram spanned by the variables T , μ_B^2 and m_{ud} .

The condensate and the susceptibilities contain both multiplicative and additive UV divergences. One possible definition of UV-finite condensate and susceptibility is given by the following renormalization prescription:

$$\begin{aligned}\langle\bar{\psi}\psi\rangle^R &= -\frac{m_{ud}}{f_\pi^4}[\langle\bar{\psi}\psi\rangle_T - \langle\bar{\psi}\psi\rangle_{T=0}], \\ \chi^R &= \frac{m_{ud}^4}{f_\pi^4}[\chi_T - \chi_{T=0}], \\ c_{\text{disc}}^R &= \frac{m_{ud}^4}{f_\pi^4}[\chi_{\text{disc},T} - \chi_{\text{disc},T=0}],\end{aligned}\quad (5)$$

where the division by f_π^4 is there to ensure the quantities $\langle\bar{\psi}\psi\rangle^R$, χ^R and χ_{disc}^R are dimensionless.

The peak position of either the full or disconnected susceptibilities can be used to define the chiral crossover temperature. Similarly, the maximal value of either of these susceptibilities can be used to study the strength of the crossover transition, as they both diverge in the presence of true critical behavior.

The lattice results for the chiral condensate as well as the full and disconnected susceptibilities as functions of the temperature are shown in Fig. 1 for several lattice volumes.

At zero temperature, chiral perturbation theory predicts [45] the condensate to have finite-volume effects of the form $\propto Ae^{-m_\pi L}L^{-3/2}$. We have checked that this ansatz describes our data for temperatures $T \leq 150$ MeV [46]. The prefactor A is related to the chiral condensate in the chiral limit, i.e., to the topological susceptibility—according to the Leutwyler-Smilga relation [47]—which is known to have very large cutoff effects with staggered fermions. We will thus not present it here, from these $N_\tau = 12$ lattices.

The left panel of Fig. 1 shows the temperature dependence of the chiral condensate for different volumes, while the right panel shows the temperature dependence of both

the full and disconnected susceptibilities. While there are strong volume effects, both the slope of the condensate and the maximum value of the susceptibility converge to a finite value at large volume, consistent with a crossover. We clearly see that the finite-volume effects on both the condensate and the two susceptibilities decrease with increasing temperature. This can be intuitively understood, because the screening mass of the pseudoscalar channel (i.e., the smallest of the screening masses) is increasing at temperatures above the crossover transition, or equivalently, the largest spatial correlation length relevant for the finite-volume effects is decreasing.

Near the transition temperature, both the full and disconnected susceptibilities show a behavior that is characteristic of a crossover transition: their maximal value tends to a finite number in the thermodynamic limit. Later in this manuscript, we will study the chemical potential dependence of this maximal value, to infer whether the crossover transition gets stronger with increasing chemical potential, i.e., in some sense closer to a second-order transition, as expected in the presence of a critical end point.

B. Deconfinement observables

We now look at properties of static test quarks, which is a common way to characterize confinement.

The static quark free energy [48] F_Q and the static quark entropy S_Q are defined as [49,50]

$$\begin{aligned}F_Q &= -T \log \left(\frac{1}{V} \sum_{\vec{x}} |\langle P(\vec{x}) \rangle_T| \right) \\ &\quad + T_0 \log \left(\frac{1}{V} \sum_{\vec{x}} |\langle P(\vec{x}) \rangle_{T_0}| \right), \\ S_Q &= -\frac{\partial F_Q}{\partial T},\end{aligned}\quad (6)$$

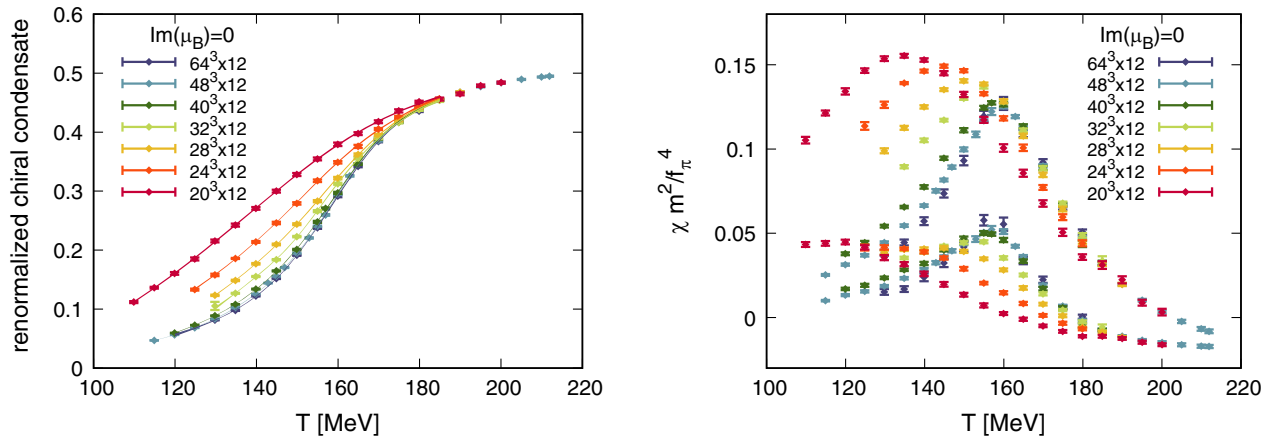


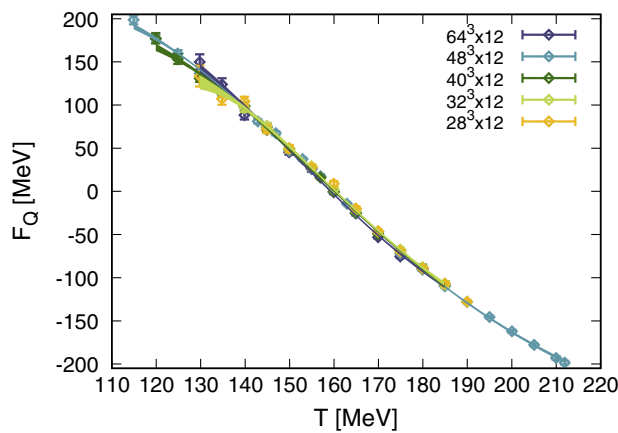
FIG. 1. Left panel: the renormalized chiral condensate [see Eq. (5)] as a function of temperature for different spatial volumes. The leftmost curves correspond to the smallest volume. Right panel: the renormalized full and disconnected chiral susceptibility as a function of temperature for different spatial volumes. Lower peak positions correspond to smaller volumes.

where $P(\vec{x})$ is the Polyakov loop

$$P(\vec{x}) = \frac{1}{3} \text{Tr} \prod_{x_4=0}^{N_t-1} U(\vec{x}, x_4), \quad (7)$$

the $U_\mu(\vec{x}, x_4)$ are the usual link variables in lattice gauge theory, T_0 is a reference temperature, and the second term in the definition of F_Q is needed to remove the additive divergence in the free energy. The value of the reference temperature T_0 , together with Eq. (6) defines a renormalization scheme for the static quark free energy. The subtraction has to be done at the same value of the lattice spacing. The quantity F_Q is then a UV-finite quantity, with a well-defined continuum limit. Note that taking the absolute value of the (complex) Polyakov loop expectation value in Eq. (6) is not important at vanishing or real chemical potential, as in both cases this expectation value is real. We include it in the formula, in order to keep the same definition for arbitrary complex chemical potential, including a purely imaginary chemical potential, where the expectation value is no longer real and the absolute value is needed. The entropy S_Q is estimated by interpolating the lattice results for $F_Q(T)$ and differentiating the interpolating function. For the interpolation we use a second-order-by-second-order rational function fit. The chi squares of all of the fits in T are acceptable.

For pure gauge theory, where a true first-order deconfinement phase transition is present, the static quark free energy is infinite in the confined phase (i.e., the Polyakov loop is zero). For full QCD, this is no longer the case. Nevertheless, one can use the peak of $S_Q(T)$ [the inflection point of $F_Q(T)$] to define a crossover temperature. In the limit of infinitely heavy quarks, the peak position of $S_Q(T)$ coincides with the deconfinement phase transition temperature. The height of the peak of $S_Q(T)$ is inversely proportional to the thermal width of the deconfinement crossover. Our results for $F_Q(T)$ and $S_Q(T)$ are shown in Fig. 2 for several different spatial volumes.



Another popular proxy to the deconfinement transition has been the Polyakov loop susceptibility, especially in the pure gluonic theory, where there is a genuine first-order transition. The signal in full QCD turns out to be weaker, and, more importantly, it is renormalization scheme dependent, as opposed to $S_Q(T)$ [50].

Yet another observable was used to define the confinement crossover temperature in Ref. [51], where the authors introduced the susceptibility $\chi_{Q,\mu_B} \equiv -(\partial(F_Q/T)/\partial(\mu_B/T)^2)_{\mu_B=0}$, which also shows a peak at the crossover transition. We, however, have found that we get statistically more precise results for S_Q , and therefore use this quantity to define the deconfinement crossover temperature in what follows.

We are now in a position to compare the deconfinement- and chiral-symmetry-related definitions of the crossover temperature. We show the different values of T_c defined as the peak of $\chi^R(T)$, the peak of $\chi_{\text{disc}}^R(T)$ as well as the peak of $S_Q(T)$ in Fig. 3. For comparison, we also show a curve where the chiral condensate, or the static quark free energy is constant, with the constant chosen to be the infinite-volume value at the crossover temperature defined via the full chiral susceptibility.

Figure 3 establishes an ordering of the different crossover temperatures in the infinite-volume limit:

$$T_c^{(S_Q)} < T_c^{(\chi_{\text{disc}}^R)} < T_c^{\chi^R}. \quad (8)$$

The differences between definitions are small, amounting only to a few MeV. The identification of such an ordering had never been possible in existing literature, due to larger error bars. Thus, our work is an improvement on previous work in Ref. [50] where the chiral and deconfinement transition temperatures were in agreement within error bars. Furthermore, we see a different volume dependence of the different crossover temperatures. Namely, the two chiral definitions lead to a T_c that is monotonically increasing

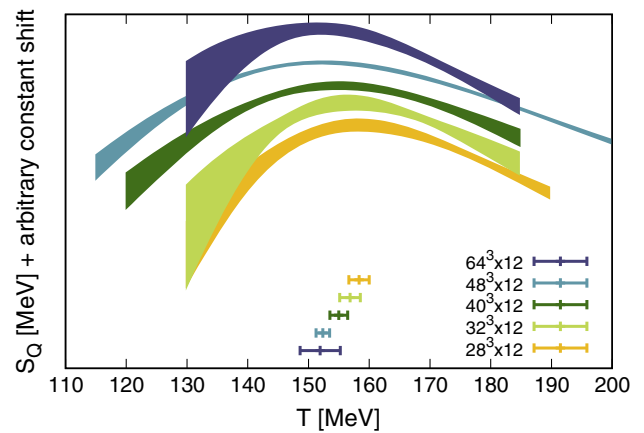


FIG. 2. Left panel: the static quark free energy [see Eq. (6)] as a function of temperature for different spatial volumes. Right panel: the static quark entropy as a function of temperature for different spatial volumes. In order for the curves to not overlap, they are shifted by arbitrary amounts in the vertical direction. Larger peak positions correspond to the smaller volumes.

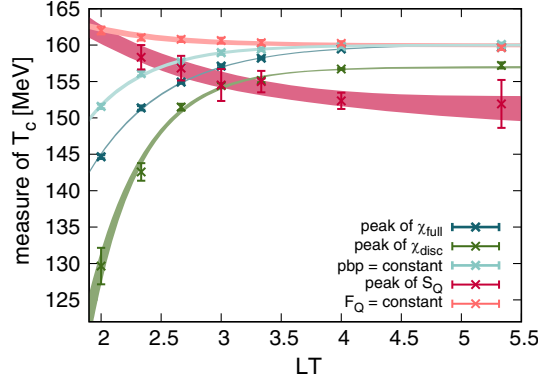


FIG. 3. Five different definitions of the transition temperature as functions of the simulation volume, labeled by the aspect ratio $LT = N_x/N_t$. The monotonically increasing curves, from top to bottom correspond to a fixed condensate value, the peak of the full susceptibility and the peak of the disconnected susceptibility. The monotonically decreasing curves, from top to bottom correspond to a constant static quark free energy and the peak of the static quark entropy. The blue and green points refer to the peak position of the full and disconnected chiral susceptibility, respectively. In addition, we show the temperature where the chiral condensate is constant (light blue). In all cases, the infinite-volume value is approached from below. The two other measures of T_c come from the Polyakov loop, either keeping a constant value for F_Q (pink), or determining the maximum of the static quark entropy S_Q (purple). In both cases, the infinite-volume value is approached from above. The fixed values of the chiral condensate and F_Q have been set such that these match the infinite-volume limit at the maximum of the full chiral susceptibility.

with the physical volume, while the deconfinement definition leads to a T_c that is monotonically decreasing with the volume. This means that at small enough volumes, the ordering changes, and the definition of T_c based on the static quark entropy becomes the largest. We also note that finite-volume effects on the deconfinement-based definition of T_c appear to be smaller. Even on the smallest lattice, with aspect ratio $LT = 2$, the deconfinement-based definition gives a results of around 160 MeV, which is quite close to the infinite-volume value, and is much larger than the chiral-symmetry-based definitions.

IV. NONVANISHING CHEMICAL POTENTIAL

We now go on to discuss the phase diagram for $\mu_B > 0$. First, we briefly explain the analysis method, then we discuss the phase diagram, that is the volume and chemical potential dependence of the crossover temperature, and finally, the volume and chemical potential dependence of the strength and width of the crossover transition. We use the same quantities that we used in the $\mu_B = 0$ analysis.

A. Observables at imaginary chemical potential

Our extrapolation method to real baryochemical potential is based on the observation of an approximate data

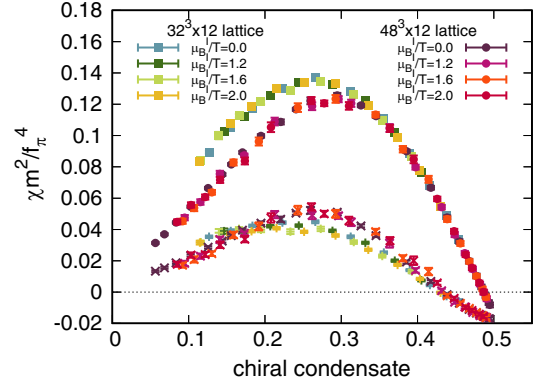


FIG. 4. The full (filled symbols) and disconnected (crosses) chiral susceptibilities as a function of the condensate for two different volumes and several values of the imaginary chemical potential. The curves for different imaginary chemical potentials approximately collapse, while the lower peak position corresponds to the smaller volume.

collapse [32]. The chiral susceptibility as a function of the chiral condensate is a curve that is almost independent of the imaginary chemical potential. In Ref. [32] we demonstrated this for the full susceptibility. Here, we show it for both χ_{full} and χ_{disc} in Fig. 4 where both susceptibilities are shown as functions of the condensate for several imaginary chemical potentials, for two different volumes. While the collapse curves for both susceptibilities are approximately independent of the imaginary chemical potential, they significantly depend on the volume. Also note that the susceptibilities as functions of the condensate are considerably simpler functions than the susceptibilities as functions of the temperature, in the sense that they are well described by low-order polynomials.

In light of this collapse, the extraction of the crossover temperature follows the following steps:

- (1) Determine the renormalized (full or disconnected) susceptibility as a function of the condensate for several values of the imaginary baryochemical potential.
- (2) Find the peak position in the susceptibility as a function of the condensate for each value of $\text{Im } \mu_B/T$ with a low-order polynomial fit.
- (3) Use an interpolation of the condensate as a function of T to convert the peak position from the condensate value to the temperature for each $\text{Im } \mu_B/T$.
- (4) Perform a fit of $T_c(\text{Im } \mu_B/T_c)$, and use the fit to extrapolate the crossover temperature from $\mu_B^2 \leq 0$ to $\mu_B^2 > 0$.

We note that this extrapolation method was also cross-checked in a calculation with truncated Dyson-Schwinger equations, where the approximation allows for direct calculations at $\mu_B > 0$. The authors of Ref. [52] found good agreement between the direct calculations and the extrapolated results.

If instead of the chiral condensate, one attempts to show the chiral susceptibility as a function of the static quark free

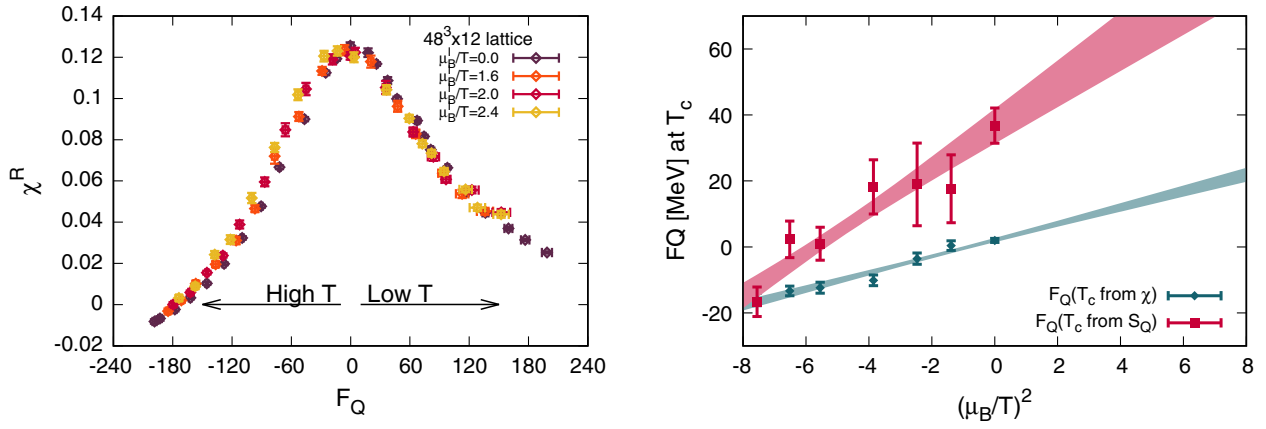


FIG. 5. Left: the full chiral susceptibility as a function of the static quark free energy. Right: the value of F_Q at the chiral crossover temperature (defined via the peak of the full chiral susceptibility or the static quark entropy) as a function of μ_B^2/T^2 . A linear extrapolation is also shown. The top curve corresponds to the crossover temperature defined by the static quark entropy.

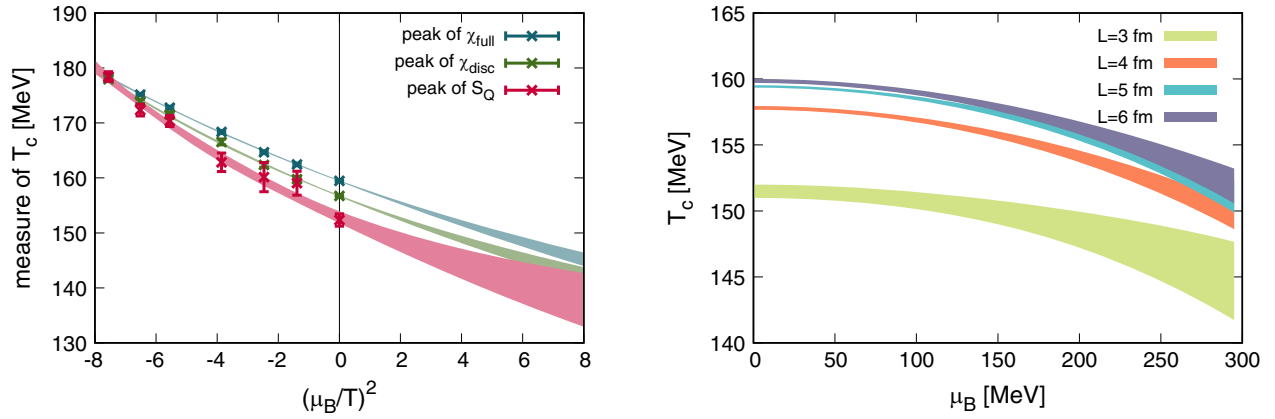


FIG. 6. Left panel: extrapolation of the crossover temperature as a function of μ_B^2/T^2 using the three different definitions (from top to bottom: using χ_{full} , χ_{disc} and S_Q) discussed in the main text on our $48^3 \times 12$ lattices. Right panel: crossover temperature defined via the peak of χ_{full} for different lattice volumes, fixed in fermis. The curve on the top corresponds to the largest volume.

energy F_Q , the collapse is less accurate. This is shown in Fig. 5, where the left panel shows χ_{full} as a function of F_Q for several imaginary chemical potentials on our $48^3 \times 12$ lattices. To quantify the inaccuracy of the collapse plot we show the value of F_Q at the crossover temperature (defined via either χ_{full} or S_Q) as a function of μ_B^2/T^2 in the right panel of the same figure, where a linear extrapolation of the value of F_Q at the crossover temperature is also shown. While it is not shown in the plot, to avoid clutter, we note that the disconnected susceptibility also has a similar (nonperfect) collapse feature as a function of F_Q .

For the calculation of the crossover temperature defined with S_Q , we simply use the same rational function fit of $F_Q(T)$ that we used for the $\mu_B = 0$ case.

B. The phase diagram for real $\mu_B > 0$

We are now in a position to discuss the volume dependence of the phase diagram. We show the crossover temperature, defined via the peak position of the full and

disconnected chiral condensates, as well as the static quark entropy as a function of μ_B^2/T^2 in the left panel of Fig. 6 in a fixed volume, on our $48^3 \times 12$ lattices. We see that not only $T_c(\mu_B = 0)$, but also the chemical potential dependence is different for the different definitions. At larger imaginary chemical potentials, the three definitions come closer to each other, which might be due to the presence of the Roberge-Weiss critical end point. In the near vicinity of such a critical point, the crossover transition should get narrower, and we thus expect different definitions of the crossover temperature to converge towards each other. This is exactly what we observe.

Here, we note that in the strangeness neutral setting employed in the present paper, the Roberge-Weiss transition is not located at $\text{Im } \mu_B/T = \pi$, but at a slightly larger value [53].

We also show contours of constants values of the static quark free energy and the chiral condensate for two different 3-volumes in the $T - \mu_B^2$ plane in Fig. 7. It is apparent that F_Q has much milder finite-volume effects

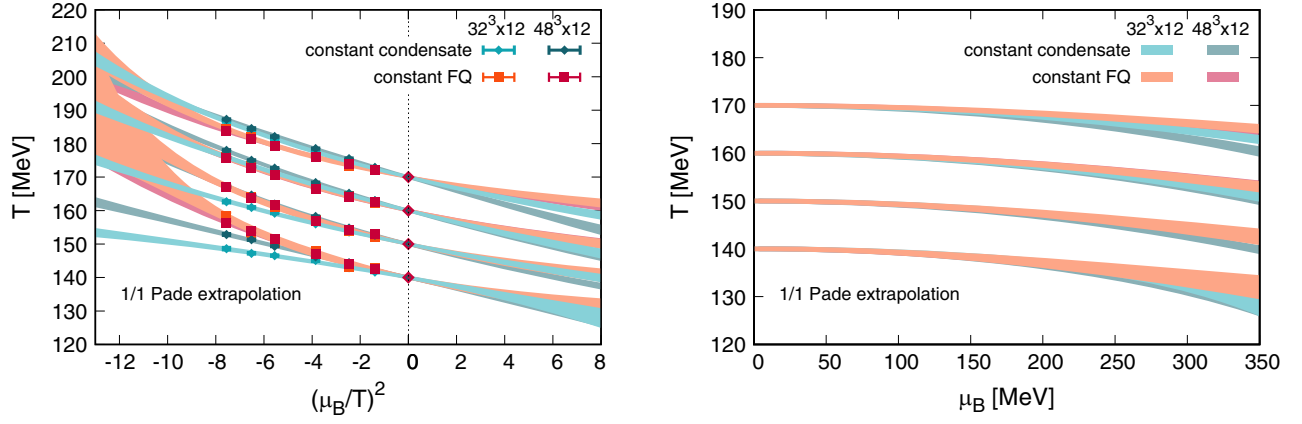


FIG. 7. Extrapolated contours with a constant value of the static quark free energy F_Q and the renormalized chiral condensate $\langle \bar{\psi}\psi \rangle^R$ in the $T - \mu_B^2$ plane (left) and in the $T - \mu_B$ plane (right) for two different volumes. The curves with a constant F_Q overlap. At a fixed real chemical potential, curves of constant chiral condensate at larger volume are below the smaller volume curves, and both constant chiral condensate curves are below the constant F_Q curves.

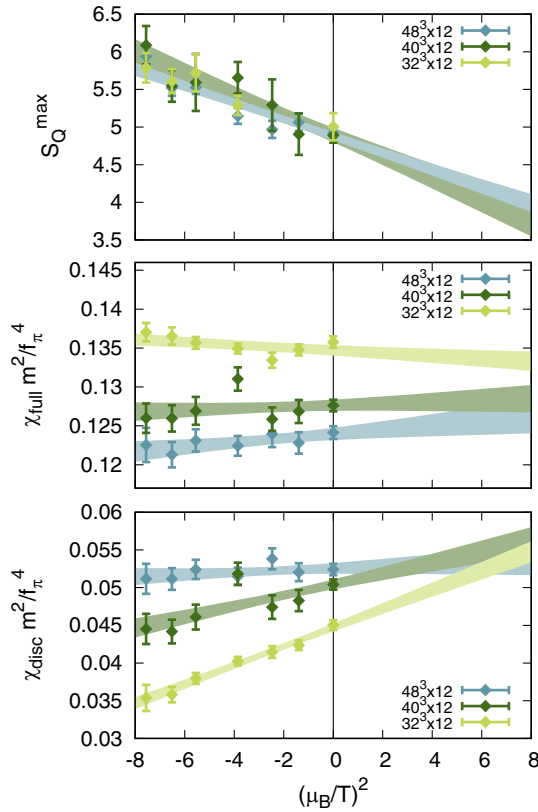


FIG. 8. Extrapolation of the maximal value of the full and disconnected chiral susceptibilities as a function of μ_B^2 for three different volumes. In the middle panel smaller values at a fixed chemical potential correspond to larger volumes, while in the bottom panel they correspond to smaller volumes.

than the chiral condensate, especially at imaginary chemical potentials $\mu_B^2 < 0$. This is somewhat surprising, since at imaginary chemical potentials, one might expect F_Q to have strong finite-volume effects, being closely related to

the Polyakov loop, which is the order parameter for the Roberge-Weiss transition at imaginary chemical potentials.

C. The strength of the crossover for real $\mu_B \neq 0$

A very interesting question, with more phenomenological implications, is whether the crossover line turns into a line of first-order transitions at a critical end point at some real value of the baryochemical potential μ_B . If it does, it is expected that the crossover transition becomes narrower and stronger at larger μ_B , at least in the vicinity of the critical end point. Here, we discuss some measures of the width or strength of the transition as a function of μ_B for small chemical potentials.

In Fig. 8 we show the values of S_Q , χ^R and χ_{disc}^R at the crossover temperature as functions of μ_B^2/T^2 , for two different physical 3-volumes. The points with error bars are our direct lattice results at zero and purely imaginary chemical potentials, while the bands show linear extrapolations in μ_B^2/T^2 . All three quantities should diverge at the critical end point in the infinite-volume limit. Thus, with increasing volume, one expects these quantities to grow in the vicinity of the critical end point. The only one of these quantities showing a rise at larger μ_B^2 is χ_{disc}^R , while χ^R remains approximately constant up to our largest volume, and S_Q decreases with increasing μ_B^2 . From the behavior of χ_{disc}^R alone, one might be tempted to conclude that the crossover transition gets stronger at larger μ_B , which would be a signal of the coveted critical end point. However, the lack of a similar behavior in the other two quantities, S_Q and χ^R , makes the interpretation of the physical picture uncertain.

Furthermore, we once again see that the deconfinement-related quantity (S_Q) has milder finite-volume effects than the quantities related to chiral symmetry restoration (χ^R and χ_{disc}^R). Just like in the case of F_Q above, the small volume

dependence of S_Q at imaginary chemical potentials is surprising, since one expects large finite-volume effects due to the Roberge-Weiss end point. Apparently the imaginary chemical potentials considered in our study are not close enough to the Roberge-Weiss critical end point for this to be the case.

The disconnected susceptibility shows the most interesting behavior: an increasing trend with increasing μ_B^2 , especially for the two lower volumes. If the coveted QCD critical end point exists, such a behavior could be due to the critical region shrinking with increasing physical volume. (I.e., a smaller volume is more tolerant to the mistuning of the parameters of a system, and criticality can be observed from farther away.) Due to the lack of such a signal in χ^R and S_Q , we would be cautious about making such an interpretation, but the behavior of χ_{disc}^R certainly is suggestive and warrants further investigation.

V. DISCUSSION

In this work we investigated the volume and chemical potential dependence of the location and the strength of the QCD transition, by studying observables related to either deconfinement or chiral symmetry restoration.

At vanishing chemical potential, we found—in the thermodynamic limit—the following ordering of the crossover temperatures: $T_c^{(S_Q)} < T_c^{(\chi_{\text{disc}}^R)} < T_c^{(\chi^R)}$. Hence, we found the deconfinement crossover temperatures to be slightly below the chiral crossover temperature. Furthermore, we found that the deconfinement crossover temperature and the chiral crossover temperatures have a different sign for their volume dependence. While with increasing volume $T_c^{(S_Q)}$ decreases, $T_c^{(\chi_{\text{disc}}^R)}$ and $T_c^{(\chi^R)}$ both increase.

Curves of constant heavy quark free energy (related to confinement) and constant chiral condensate (related to chiral symmetry breaking) in the $T - \mu_B^2/T^2$ plane are similar, but not identical. In particular, the curvature of

these lines is about the same at $T_c(\mu_B = 0)$, but not below or above.

In general, we found that physical quantities related to confinement dynamics appear to have much milder finite-volume effects, when compared to quantities related to chiral symmetry. This is true at both zero and nonzero baryochemical potential, be it imaginary (simulated) or real (extrapolated). Thus, confinement-deconfinement mechanics will be easier to study in settings where larger volumes are too expensive, which is the case, e.g., with reweighting techniques.

Finally, we found that the volume and μ_B dependence of the maximal value of χ^R , χ_{disc}^R and S_Q are nontrivial, and a clear physical picture cannot be drawn at the moment. However, the disconnected susceptibility χ_{disc}^R in particular shows suggestive behavior, which is promising in light of future theoretical searches for the QCD critical point.

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