When hadrons become unstable: a novel type of non-analyticity in chiral extrapolations

F.-K. Guo[1], C. Hanhart[2], F. J. Llanes-Estrada[3], U.-G. Meißner[1,2]

[1] *Helmholtz-Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany*
[2] *Institut für Kernphysik, Institute for Advanced Simulation and Jülich Center for Hadron Physics, Forschungszentrum Jülich, D-52425 Jülich, Germany*
[3] *Dept. Física Teórica I, Universidad Complutense, 28040 Madrid, Spain*

Abstract

Hadron masses show a specific dependence on the quark masses. Therefore, the variation of these masses can cause a resonance in a hadronic scattering amplitude to become a bound state. Consequently, the amplitude exhibits a non-analytic behavior at this transition. Crossed amplitudes, where the resonance can be exchanged in the t-channel, can be shown to exhibit the same phenomenon by $s \to t$ analytic continuation. This entails possible kinks in lattice quark mass extrapolations needed to compute hadronic observables.

Key words: Analyticity of S-matrix, Lattice QCD, chiral extrapolations
PACS: 11.55.Bq 12.38.Gc

1 Introduction

It is of current interest to obtain lattice Quantum Chromodynamics (QCD) predictions for hadronic observables, both to test QCD in the strong-coupling regime, and to compute QCD backgrounds to new physics searches. It is customary in these lattice gauge theory computations, due to the large numerical costs, to perform simulations with unphysically large masses of the light quarks. Then a smooth extrapolation formula to physical values, inspired by chiral perturbation theory is usually employed to obtain the physical results (for a recent review see Ref. [1]). [1] However, there are no theorems of S-matrix theory guaranteeing the analyticity of such an extrapolation for larger quark

[1] We are well aware that first simulations with physical quark masses or even less [2,3] become available, but these are still exceptions.

masses, denoted as m_q in what follows, beyond the regime where chiral perturbation theory is applicable. [2] Note that one can equally well talk of the pion mass instead of the quark mass, since they are related by the Gell-Mann-Oakes-Renner relation [5] $m_\pi^2 f_\pi^2 = 2m_q\langle\bar{q}q\rangle + \mathcal{O}(m_q^2)$, where the corrections $\sim m_q^2$ are known to be very small. On the contrary, in this letter we expose non-analyticities (kinks in the m_q-extrapolation) that may arise when a resonance becomes bound upon varying m_q. For example, in the pion form factor or the $B \to \pi\pi$ decay amplitude, the relevant resonance is the $\rho(770)$. For the weak $K \to \pi$ transition form factor, the K^* matters. Let us stress that the value of s is of no concern, the non-analyticity in the variable m_q if present, affects the entire amplitude or form factor. We will exemplify this with both time-like and space-like pion form factors. So far the analysis has been carried out for the m_q dependence of the resonance mass itself, $m_\rho(m_q)$ for example [6,7]. Here, we demonstrate the generality of the phenomenon affecting the computation of most hadron observables, which completes the preliminary results reported in [8].

Our results are relevant because there is much active lattice research in form factor determinations, see e.g. [9,10,11] and spectroscopy [12,13,14,15]. Extrapolation formulae are available for both form factors [16] and spectroscopy, e.g. [17]. Typically these extrapolations are smooth except for the usual chiral logarithms of the pion mass, $\log(m_\pi^2/\mu^2)$, that present a non-analiticity at $m_\pi = 0$.

2 Illustration: a simple model

To expose the feature in the simplest possible physical manner, we now focus on amplitudes with two pions, and the role of the ρ-resonance. In the next section it will be shown, however, that the results are general. In a simple field theory where the two pions are coupled to the resonance and the latter is represented as an additional field, the threshold effect appears through the vacuum polarization of the resonance, whose imaginary part controls the decay width. Therefore, the size of the possible non-analyticities in any amplitude has to be proportional to the width of the resonance and appear only in, at least, one-loop calculations, such as depicted in Fig. 1 for the time-like pion form factor.

We therefore proceed to study the π-ρ case at one loop. The bare ρ-meson propagator is $1/(s - m_{0\rho}^2)$ — the one-loop vacuum polarization will renor-

[2] It is well-known that certain non-analyticities in the quark masses can be shown to hold for arbitrary momenta (see Ref. [4] and references therein), but these are not the effects we are dealing with.

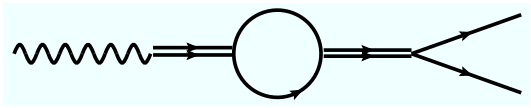

Fig. 1. ρ-meson contribution to pion time-like form factor with one-loop vacuum polarization. Solid, double and wiggly lines denote pions, ρ-mesons and photons, respectively.

malize the bare mass $m_{0\rho}$ to m_ρ. The s-channel on-shell unitarization for the scattering amplitude reads

$$T^{11}(s) = \frac{V^{11}(s)}{1 - G(s)V^{11}(s)} = \frac{-\frac{4}{3}g^2_{\rho\pi\pi}|\mathbf{p}|^2}{z_0^\rho(s - m_{0\rho}^2) + \frac{4}{3}g^2_{\rho\pi\pi}|\mathbf{p}|^2 G(s)} \tag{1}$$

in terms of the Born amplitude for $\rho \to \pi\pi$ (with $J = I = 1$),

$$V^{11}(s) = -\frac{4}{3}\frac{g^2_{\rho\pi\pi}}{z_0^\rho}\frac{|\mathbf{p}|^2}{s - m_{0\rho}^2} \ , \tag{2}$$

where the ρ wave function renormalization constant z_0^ρ comes from the relation between the bare coupling constant and the renormalized one. The factor $|\mathbf{p}|^2 = s/4 - m_\pi^2$ stems from the on-shell p-wave derivative coupling. With this on-shell factorization, the denominator in Eq. (1) contains the unregularized scalar one-loop function

$$G(s) = \frac{1}{16\pi^2}\left(R + \log\left(\frac{m_\pi^2}{\mu^2}\right) + 1 - \bar{J}(s)\right) \ . \tag{3}$$

We use the convention

$$\bar{J}(s) = 2 + \sigma \log\left(\frac{\sigma - 1}{\sigma + 1}\right) \tag{4}$$

with $\sigma = 2|\mathbf{p}|/\sqrt{s} = \sqrt{1 - 4m_\pi^2/s}$ for the relativistic phase space. The divergence in dimensional regularization appears in

$$R = \frac{2}{d - 4} - \log(4\pi) - \Gamma'(1) - 1 \ .$$

with the number of space-time dimensions $d \to 4$.

Returning to our main issue, consider the imaginary part of the vacuum polarization in the denominator of Eq. (1). It is given by

$$\text{Im } \Pi = \frac{\pi\sigma}{16\pi^2}\frac{4}{3}g^2_{\rho\pi\pi}|\mathbf{p}|^2\theta(s - 4m_\pi^2) \tag{5}$$

and leads to the well known non-analyticity in s (branch point at $s = 4m_\pi^2$). A trivial observation is that, reciprocally, there is a non-analyticity in m_π for

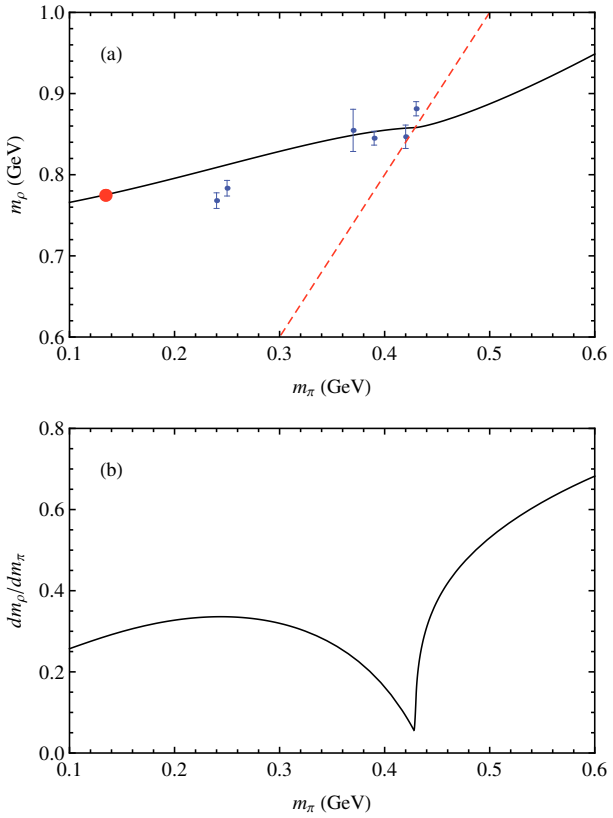

Fig. 2. Dependence of the rho mass m_ρ (a) and its derivative dm_ρ/dm_π (b) on the pion mass obtained from Eq. (12). The dashed line in (a) denotes the motion of the $\pi\pi$ threshold. Data are from a lattice calculation by the QCDSF Collaboration [18], and the physical ρ mass is represented by a thick circle.

fixed s at $m_\pi = \sqrt{s/4}$. More subtle is to notice that when $2m_\pi = m_\rho(m_\pi)$ (in App. 2, we show generally that such a situation will occur for the ρ), a similar non-analyticity affects the amplitude for all s due to the dependence of the amplitude on the renormalized m_ρ (physical pole position) that suddenly changes from a bound state on the real axis to an unbound resonance.

We present in Fig. 2 the behavior of $m_\rho(m_\pi)$. The results agree qualitatively with those from the more sophisticated treatment in, e.g., Refs. [6,7]. Further details of the computation within the simple model are given in the App. 1. Eq. (12) contains the combination

$$\frac{1}{z_0^\rho}\mathrm{Re}\ \bar{J}(m_\rho^2)\left(\frac{m_\rho^2}{4}-m_\pi^2\right)$$

that is non-analytic for the value of the pion mass where the ρ becomes bound — c.f. Eqs. (4,18). This is seen especially as a kink in the derivative dm_ρ/dm_π, shown in Fig. 2(b). Therefore, the position of the pole in the pion scattering amplitude appears not to be an analytic function of the pion mass. The same phenomenon will appear in other amplitudes, such as the time-like electromag-

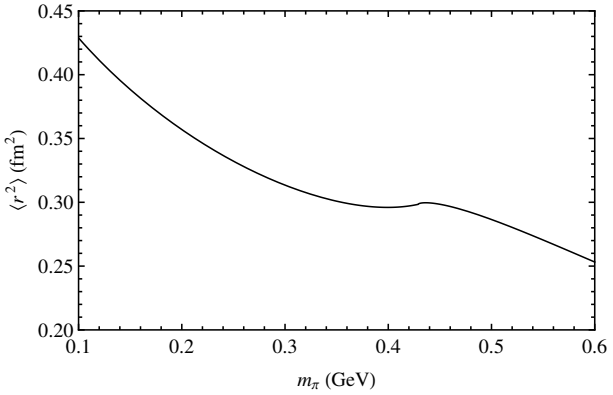

Fig. 3. The squared pion charge radius also presents a kink in its first derivative as a function of m_π, at the point where the ρ resonance becomes bound.

netic form-factor. This can then be analytically continued (in s) to the space-like side and the same phenomenon will appear for, say, the radius squared (which appears naturally in the low-energy expansion of the space-like form-factor). To illustrate this effect we now use the simplest realization of vector meson dominance (VMD), where the direct coupling of the photon to the pion is neglected and the form factor is entirely given by the photon-ρ-meson coupling to the intermediate resonance (with the strength $g_{\rho\gamma}$) [19]. The tree level formula

$$F^{\rm tree}(s) = \frac{m_\rho^2}{m_\rho^2 - s} \tag{6}$$

already suggests a kink if one substitutes the dependence of the pole mass $m_\rho(m_\pi)$ that we have argued to be non-analytic. This lack of analyticity appears then in the squared charge radius in the Breit frame $\langle r^2 \rangle = 6/m_\rho^2$. Although the mentioned VMD description is too simplistic to exhibit all pertinent features of the pion vector form factor, it is very useful for illustrative purposes. In Appendix 1 it is demonstrated that the full one loop amplitude exhibits the same features and in the next section we demonstrate that our findings are indeed model independent.

The one-loop result for the charge radius squared as a function of the pion mass is displayed in Fig. 3. We have assumed that $g_{\rho\pi\pi}$ is independent of the pion mass.[3] Thus, the threshold non-analyticity when the ρ–resonance becomes bound is inherited by the space-like form factor. This result may not seem intuitive, since the argument t of the space-like form factor is apparently very far from any thresholds associated with s, so perhaps it is helpful to think of the form factor as a function of two variables $F(s, m_q)$. The non-analyticity enters because of the implicit pion–mass dependence through $m_\rho(m_q)$ and is not affected by the analytic continuation in the other variable $s \to t$.

[3] The pion mass dependence of $g_{\rho\pi\pi}$ is very moderate from both the unitarized chiral perturbation theory [6] and very recent lattice simulations [20].

3 Generalisation of the results

We now turn to a model-independent discussion of the effect, introduced so far within a particular model, for a fixed $\pi\pi$ partial wave. In a model-independent way this effect can be studied by employing an Omnès representation for the form-factor as given e.g. in Refs. [21,22,23,24]. This renowned relation expresses the form factor as an integral over the scattering phase shift. In once-subtracted form it reads in the absence of bound states

$$F(t,m_\pi^2) = \Omega(t,m_\pi^2) = \exp\left(\frac{t}{\pi}\int\limits_{4m_\pi^2}^{\infty} ds \frac{\delta_{11}(s,m_\pi^2)}{s(s-t-i\epsilon)}\right) \ . \tag{7}$$

Then the charge radius in the Omnès representation is expressed in terms of the $\pi\pi$ scattering phase shift as [23]

$$\langle r^2\rangle = \frac{6}{\pi}\int\limits_{4m_\pi^2}^{\infty} ds \frac{\delta_{11}(s,m_\pi^2)}{s^2} \ . \tag{8}$$

In the presence of a bound state there is an additional singularity on the first sheet and thus the dispersion integral needs to be modified. It now reads [25],

$$F_b(t,m_\pi^2) = \left(1 + \frac{t g_{\gamma\rho} g_{\rho\pi\pi}}{m_\rho^2(m_\rho^2-t)}\frac{1}{\Omega(m_\rho^2,m_\pi^2)}\right)\Omega(t,m_\pi^2) \ . \tag{9}$$

Eq. (8) needs to be adapted accordingly

$$\langle r^2\rangle_b = \frac{6}{\Omega(m_\rho^2,m_\pi^2)}\frac{g_{\gamma\rho}g_{\rho\pi\pi}}{m_\rho^4} + \frac{6}{\pi}\int\limits_{4m_\pi^2}^{\infty} ds \frac{\delta_{11}(s,m_\pi^2)}{s^2} \ . \tag{10}$$

Here, we introduced the subscript b to distinguish the quantities defined in the presence of a bound state from those given in Eqs. (7,8). It should be stressed that the form factor is continuous at the value of the pion mass, where the ρ becomes a stable state, for all $t \neq m_\rho^2$. To see this we first observe that the integrals over the phases in Eqs. (8,10) converge towards each other, as the ρ–mass approaches the two–pion threshold. This follows directly from the behaviour of the phases shown in Fig. 4 — as soon as the ρ appears as a stable state at $m_\pi = 430\,\mathrm{MeV}$, according to Levinson's theorem the phase shift starts from π. In addition, when approaching the point $m_\rho(m_\pi) = 2m_\pi$ from larger pion masses one finds

$$\log\Omega(t,m_\pi^2) \simeq \frac{t}{\pi}\delta_{11}\int\limits_{4m_\pi^2}^{\infty} ds \frac{1}{s(s-t-i\epsilon)} = \frac{\delta_{11}}{\pi}\log\frac{4m_\pi^2}{|4m_\pi^2-t|}, \tag{11}$$

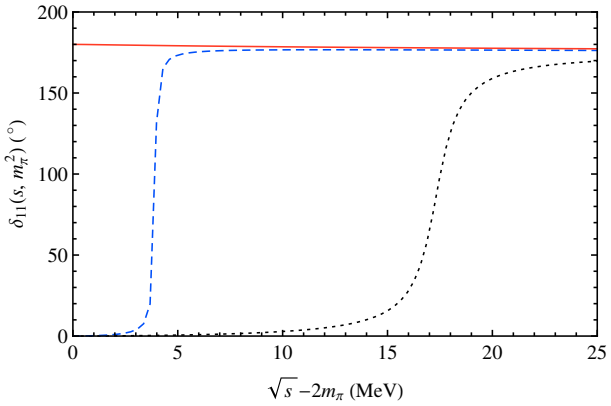

Fig. 4. Pion mass dependence of $\delta_{11}(s, m_\pi^2)$ from the one–loop model, as the pion mass approaches the value where the ρ becomes stable. Shown are the phases for m_π =420 MeV (dotted line), 427 MeV (dashed line) and 431 MeV (solid line).

where we used that in this limit δ_{11} is a slowly varying function of s in the energy range of interest. Evidently, $1/\Omega(t, m_\pi^2)$ vanishes, when $t = 4m_\pi^2$. However, as becomes apparent in Eq. (10), the non–analyticity in, e.g., m_ρ directly influences the quark mass dependence of the squared radius [26].

4 Summary and conclusions

In this work we discussed a non–analyticity in the chiral extrapolation of physical quantities that emerges when due to a change in the quark mass a state transforms from a resonance — poles on the second sheet — to a physical state with a pole on the first sheet. We established that the analytic continuation in the kinematic variable $s \to t$ does carry over the non-analyticity in m_π from the time-like to the space-like domain. This kind of behavior is model-independent.

We have analyzed in this letter the case of the pion form factor, but the same phenomenon should appear in other form factors. For example, in the $K \to \pi$ weak vector transition form factor [27,28], when $m_\pi \simeq 350$ MeV the K^* resonance should also become bound, and develop a non-analyticity.

The situation is even more interesting for scalar form factors, where one has a kink directly in the function (be it the mass or the squared radius) and not in its derivative with respect to the pion mass. This is simply because the factor $|\mathbf{p}|^2$ from the $J = 1$ derivative coupling is absent (as already discussed in detail in Ref. [21]). Then it will be easier for lattice data to isolate such a structure (that is not yet visible in existing simulations, see e.g. [11]). If a relative drop $(\delta F^{(n)})/F^{(n)}$ in the n^{th} derivative of a function is to be identified visually, the error acceptable in the lattice computation of F itself is, as a rule

of thumb, $(\delta F)/F \leq (\delta F^{(n)})/(2^n F^{(n)})$ since each derivative with a good mid-point numerical method requires two evaluations. Hence, we would propose that the scalar pion form factor be computed with smaller statistical error bars and smaller t-intervals, as a favorable system to try to find the non-analyticity, given that there is no phase-space suppression and that the coupling $g_{\sigma\pi\pi}$ is large. Calculations using unitarized chiral perturbation theory predict that the σ meson becomes bound at about $m_\pi \simeq 350$ MeV [6]. Full QCD simulations for scalar quantities at sufficiently low pion masses will, however, not appear in the near future for those are a lot more computer time intensive compared to the ones discussed due to the presence of disconnected diagrams.

We now examine to what extent non-analyticities have been stressed in earlier studies. Very old work focused on the particle virtuality for fixed mass, be it in perturbation theory or with the Lehman representation [29], or for scaling deeply-inelastic scattering functions [30]. In both cases the phenomenon of a resonance becoming a bound state during the particle mass variation is absent, and those authors found analyticities in the transferred momentum plane with the physical values of the particle masses. Closer in spirit to our work, features in quark–mass extrapolations due to presence of a threshold have already been discussed in Ref. [31], which focuses on the avoided level crossing in a finite volume. A small cusp in the pion mass dependence of the mass of the Δ resonance [32] is produced. In Ref. [33] it was stressed that the kind of non-analyticity discussed in Ref. [32] also shows up in electro-magnetic properties such as the magnetic moment. The effect we discussed introduces an *additional* non-analyticity in the radii. Finally, another kind of non-analyticity — possible discontinuities — in the chiral extrapolation of hadron masses was proposed in [34].

The non-analiticity that we uncover is a feature of continuum field-theory, and not an artifact of lattice–quantization. It is possible that accurate lattice data should be able to isolate these non-analyticities, provided the volume is large enough that the resonance is not bound by the minimum momentum possible on the lattice (though the alternative non-analiticities of Ref. [32] mask the effect). They should be taken into account when attempting to extrapolate lattice data to physical pion masses when high precision is expected.

We thank Stephan Dürr and Akaki Rusetsky for useful discussions. This work was supported in part by grants provided by the HGF to the virtual institute "Spin and strong QCD" (VH-VI-231), the DFG (SFB/TR 16) and the EU I3HP "Study of Strongly Interacting Matter" under the Seventh Framework Program of the EU, FPA 2008-00592, 2007-29115-E, FIS2008-01323 (Spain) and 227431, HadronPhysics2 (EU). FJLE thanks the members of the Nuclear Theory Center at Forschungszentrum Jülich for their hospitality during this work. UGM also thanks the BMBF for support (Grant No. 06BN9006).

Appendix 1: One-loop renormalization of the $\rho\pi\pi$ model.

Our choice of renormalization is meant to expose the pole mass and decay coupling constant in the amplitude, so this one is expressed in terms of directly measurable quantities. To achieve this, we add and subtract to the denominator of Eq. (1) the vacuum polarization evaluated at the (still unknown) ρ pole mass, that is,

$$\frac{4}{3}g^2_{\rho\pi\pi}G(m^2_\rho)\left(\frac{m^2_\rho}{4}-m^2_\pi\right) .$$

Imposing now the renormalization condition that the position of the pole in the denominator be at m_ρ yields the equation

$$m^2_\rho = m^2_{0\rho} - \frac{4}{3}\frac{g^2_{\rho\pi\pi}}{z^\rho_0}G(m^2_\rho)\left(\frac{m^2_\rho}{4}-m^2_\pi\right) . \tag{12}$$

Above the two-pion threshold, m_ρ is complex. But in the numerical calculations, for simplicity, we take m_ρ to be real. This amounts to neglecting Im G — which is a very good approximation near the kink where phase space closes. Note that G contains an infinity that needs to be absorbed into the bare mass. Since the divergence is multiplied by $(m^2_\rho/4 - m^2_\pi)/z^\rho_0$ the subtraction procedure calls for introducing a pion–mass dependent mass term. Thus, since we want to keep the pion mass dependences explicit, the Lagrangian density for the model needs to contain a counterterm proportional to $m^2_\pi\rho^\dagger\rho$.

We therefore define the renormalized mass and its mass derivative with respect to m^2_π through

$$\bar{m}^2_{0\rho} + \bar{m}^{'\,2}_{0\rho}m^2_\pi = m^2_{0\rho} - \frac{4}{3}\frac{g^2_{\rho\pi\pi}}{z^\rho_0}\left(\frac{m^2_\rho}{4}-m^2_\pi\right)\frac{R+1}{16\pi^2} \tag{13}$$

and demand that the two constants $\bar{m}^2_{0\rho}$ and $\bar{m}^{'\,2}_{0\rho}$ be pion-mass independent. These two parameters can be fixed using the physical ρ meson mass at the physical point for m_π and lattice data. We choose as renormalization scale, which enters Eq. (12)) through $G(m^2_\rho)$ (c.f. Eq. (3)), $\mu = m_\rho$, the pole mass itself. Then z^ρ_0 becomes known (see below), and Eq. (12) can be solved. The best fit to the lattice data from the QCDSF Collaboration [18] with the constraint from the physical rho mass gives $\bar{m}_{0\rho} = 0.707$ GeV, and $\bar{m}'_{0\rho} = 1.13$.

At this point we have guaranteed that the pion-pion scattering amplitudes has a pole at physical m_ρ for the physical pion mass, and we can compute the variation of the pole position with the pion mass if this dependence is known for z^ρ_0, so we also need to solve for it.

The second renormalization condition we impose is that $g_{\rho\pi\pi}$ be the physical coupling at the ρ-pole, obtainable from the residue of the pion scattering

amplitude

$$\text{Res } T^{11}(s) = \lim_{s\to m_\rho^2} (s - m_\rho^2)\, T^{11}(s) \ . \tag{14}$$

Imposing that the residue be

$$\text{Res } T^{11}(s) = -\frac{4}{3} g_{\rho\pi\pi}^2 \left(\frac{m_\rho^2}{4} - m_\pi^2 \right) \tag{15}$$

and taking into account that, in terms of the pole mass, we have

$$T^{11}(s) = \frac{-\frac{4}{3} g_{\rho\pi\pi}^2 |\mathbf{p}|^2}{z_0^\rho (s - m_\rho^2) + \Delta(s) + i\text{Im } \Pi(s)} \tag{16}$$

with

$$\Delta(s) \equiv \text{Re } \Pi(s) - \text{Re } \Pi(m_\rho^2) \ , \tag{17}$$

where

$$\Pi(s) = \frac{1}{12\pi^2} g_{\rho\pi\pi}^2 \bar{J}(s) \left(\frac{s}{4} - m_\pi^2 \right) .$$

We find

$$z_0^\rho = 1 - \frac{d\Delta(s = m_\rho^2)}{ds} \ . \tag{18}$$

Note the m_ρ in the last expression is not the physical value but $m_\rho(m_\pi)$, to guarantee that $m_\rho(m_\pi)$ is always defined as the pole mass in the propagator. As discussed below Eq. (12), although m_ρ is complex, for simplicity we take m_ρ to be real.

Resumming the Dyson series originating from the vacuum polarization of the ρ-meson one obtains for the form factor

$$F(s) = \frac{-g_{\rho\gamma} g_{\rho\pi\pi} \sqrt{z_0^A / z_0^\rho}}{z_0^\rho (s - m_\rho^2) + \Delta(s) + i\text{Im}\Pi(s)} \ . \tag{19}$$

The denominator is of course the same as in Eq. (16), and since the numerator is real this guarantees the same phase for form factor and scattering amplitude. In order to get the proper normalization in this most simple formulation of VMD one needs to impose

$$g_{\rho\gamma} g_{\rho\pi\pi} \sqrt{z_0^A / z_0^\rho} = z_0^\rho m_\rho^2 - \Delta(0) \tag{20}$$

on the photon-rho coupling. The resulting form factor

$$F(s) = \frac{-z_0^\rho m_\rho^2 + \Delta(0)}{z_0^\rho (s - m_\rho^2) + \Delta(s) + i\text{Im}\Pi(s)} \tag{21}$$

satisfies now $F(0) = 1$ and has the correct unitarity cut. Through all the one-loop quantities m_ρ, z_0^ρ and Δ, as shown in Fig. 2 and Fig. 5, the form-factor acquires a non-analyticity in m_π. This non-analyticity appears at one loop

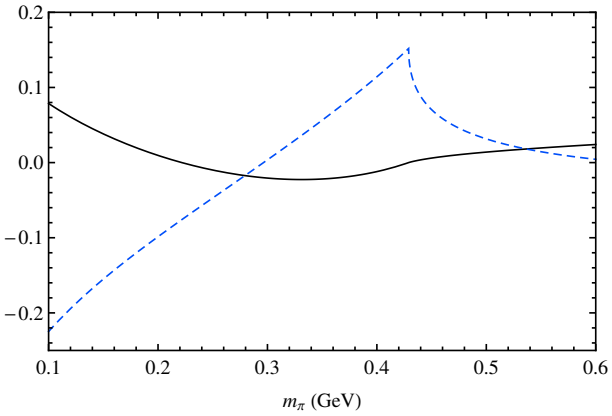

Fig. 5. Pion mass dependence of auxiliary $\Delta(0)$ (solid) and $\Delta'(m_\rho^2) = 1 - z_0^\rho$ (dashed).

and is therefore proportional to $g^2_{\rho\pi\pi}$ and hence the physical resonance width. Using Eq. (18) the squared charge radius becomes now

$$\langle r^2 \rangle = 6 \frac{z_0^\rho + \Delta'(0)}{z_0^\rho m_\rho^2 - \Delta(0)} \ . \tag{22}$$

The derivative of $\Delta(s)$ at $s = 0$ is a constant $\Delta'(0) = -g^2_{\rho\pi\pi}/(72\pi^2)$.

Appendix 2: Position of the point $m_\rho(m_\pi) = 2m_\pi$

In this appendix, we will show that the ρ mass grows slower than the two-pion threshold when increasing the pion mass, and hence there must be a certain point after which the ρ will be below the two-pion threshold.

Expanding the ρ mass in terms of m_π, to the order $\mathcal{O}(p^2)$ (it is sufficient for our purpose to work to this order; for the expansion to higher orders, see [17]), one has

$$m_\rho(m_\pi) = m_{\rho 0} + c_1 m_\pi^2 = m_{\rho 0} + 2 c_1 B_0 \hat{m}, \tag{23}$$

where $m_{\rho 0}$ is the rho mass in the chiral limit, c_1 is a low-energy constant related to the quark mass term in the chiral expansion, $B_0 = -\langle 0|\bar{q}q|0\rangle / f_\pi^2$ and $\hat{m} = (m_u + m_d)/2$. Generally, since the rho has a non-vanishing (and not small) mass even in the chiral limit, one has $m_{\rho 0} > 2m_\pi$ for small values of the pion mass. The points where m_ρ coincides with $2m_\pi$ are then simply given by the solutions of $m_{\rho 0} + c_1 m_\pi^2 = 2m_\pi$, i.e.

$$m_\pi = \frac{1}{c_1}(1 \pm \sqrt{1 - c_1 m_{\rho 0}}). \tag{24}$$

Hence a crossing happens if and only if

$$c_1 \leq \frac{1}{m_{\rho 0}} \ . \tag{25}$$

To determine c_1 we resort to quark-mass controlled $SU(3)$ breaking, and expand the mass of the K^* in analogy withEq. (23)

$$m_{K^*}(m_\pi) = m_{\rho 0} + c_1 B_0 (m_s + \hat{m}) = m_{\rho 0} + c_1 m_K^2. \tag{26}$$

Because c_1 is independent of the quark mass by definition, it can be used for unphysical pion masses after determining it using physical meson masses by

$$c_1 = \frac{m_{K^*} - m_\rho}{m_K^2 - m_\rho^2} = 0.51 \text{ GeV}^{-1}. \tag{27}$$

Therefore, as long as $m_{\rho 0} \leq 1960$ meV, the inequality $c_1 \leq 1/m_{\rho 0}$ can be fulfilled. It is believed that $m_{\rho 0}$ is not far from the physical mass $m_\rho = 770\,\text{MeV}$, so that the rho mass will coincides with $2m_\pi$ at some value(s) of m_π. One can even estimate that value. Taking, e.g., $m_{\rho 0} \approx 700$ meV, the crossing point will be at around $m_\pi \approx 400$ meV. Finally, we note that the second solution of Eq. (24) is far beyond the applicability region of the chiral expansion.

References

[1] V. Bernard, Prog. Part. Nucl. Phys. **60** (2008) 82 [arXiv:0706.0312 [hep-ph]].

[2] S. Aoki *et al.* [PACS-CS Collaboration], Phys. Rev. D **81** (2010) 074503 [arXiv:0911.2561 [hep-lat]]; S. Aoki *et al.* [PACS-CS Collaboration], Phys. Rev. **D79** (2009) 034503.

[3] S. Dürr *et al.*, [arXiv:1011.2403 [hep-lat]]; S. Dürr *et al.*, [arXiv:1011.2711 [hep-lat]].

[4] J. Gasser and A. Zepeda, Nucl. Phys. B **174** (1980) 445.

[5] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. **175** (1968) 2195.

[6] C. Hanhart, J. R. Pelaez, G. Rios, Phys. Rev. Lett. **100** (2008) 152001.

[7] J. Nebreda, J. R. Pelaez, Contributed to the 12th Intnal. Conference on Meson-Nucleon Physics and the Structure of the Nucleon [arXiv:1010.3982 [hep-ph]].

[8] F. Llanes-Estrada, F. K. Guo, C. Hanhart and U.-G. Meißner, PoS **EFT09** (2009) 054.

[9] P. A. Boyle *et al.*, arXiv:0804.3971 [hep-lat].

[10] R. Frezzotti, V. Lubicz and S. Simula [ETM Collaboration], Phys. Rev. D **79** (2009) 074506 [arXiv:0812.4042 [hep-lat]].

[11] S. Aoki *et al.* [JLQCD Collaboration and TWQCD Collaboration], Phys. Rev. D **80** (2009) 034508 [arXiv:0905.2465 [hep-lat]].

[12] S. Durr, Z. Fodor, J. Frison, C. Hoelbling, R. Hoffmann, S. D. Katz, S. Krieg, T. Kurth *et al.*, Science **322** (2008) 1224. [arXiv:0906.3599 [hep-lat]].

[13] H. W. Lin *et al.* [Hadron Spectrum Collaboration], Phys. Rev. D **79** (2009) 034502 [arXiv:0810.3588].

[14] R. Baron *et al.*, JHEP **1006** (2010) 111 [arXiv:1004.5284].

[15] G. P. Engel *et al.* [BGR Collaboration], Phys. Rev. **D82** (2010) 034505. [arXiv:1005.1748 [hep-lat]].

[16] J. Bijnens, G. Colangelo and P. Talavera, JHEP **9805** (1998) 014 [arXiv:hep-ph/9805389].

[17] P. C. Bruns and U.-G. Meißner, Eur. Phys. J. C **40** (2005) 97 [arXiv:hep-ph/0411223].

[18] M. Göckeler *et al.* [QCDSF Collaboration], PoS **LATTICE2008** (2008) 136. [arXiv:0810.5337 [hep-lat]].

[19] J. J. Sakurai, *Currents and Mesons*, University of Chicago Press, Chicago, 1969.

[20] X. Feng, K. Jansen, D. B. Renner, arXiv:1011.5288 [hep-lat].

[21] J. Gasser and U.-G. Meißner, Nucl. Phys. B **357** (1991) 90.

[22] J. F. De Troconiz, F. J. Yndurain, Phys. Rev. D **65** (2002) 093001 [hep-ph/0106025].

[23] J. A. Oller and L. Roca, Phys. Lett. B **651** (2007) 139 [arXiv:0704.0039 [hep-ph]].

[24] B. Ananthanarayan, I. Caprini and I. S. Imsong, arXiv:1102.3299 [hep-ph].

[25] G. Barton, *Introduction to Dispersion Techniques in Field Theory*, W. A. Benjamin Inc., Amsterdam, 1965.

[26] F.-K. Guo, C. Hanhart, F. J. Llanes-Estrada, U.-G. Meißner, Phys. Lett. B **678** (2009) 90 [arXiv:0812.3270 [hep-ph]].

[27] D. J. Antonio *et al.*, PoS **LAT2006** (2006) 101 [arXiv:hep-lat/0610080].

[28] D. Becirevic *et al.*, Nucl. Phys. B **705** (2005) 339 [arXiv:hep-ph/0403217].

[29] I. M. Dremin, Zh. Eksp. Teor. Fiz. **41** (1961) 821.

[30] B. V. Geshkenbein and B. L. Ioffe, Pisma Zh. Eksp. Teor. Fiz. **18** (1973) 646 [JETP Lett. **70** (1999) 161].

[31] V. Bernard, U.-G. Meißner, A. Rusetsky, Nucl. Phys. B **788** (2008) 1 [arXiv:hep-lat/0702012].

[32] V. Bernard, D. Hoja, U.-G. Meißner, A. Rusetsky, JHEP **0906** (2009) 061 [arXiv:0902.2346 [hep-lat]].

[33] T. Ledwig *et al.* arXiv:1104.3877 [hep-ph].

[34] A. Semke, M. F. M. Lutz, Nucl. Phys. A **789** (2007) 251 [arXiv:nucl-th/0606027].