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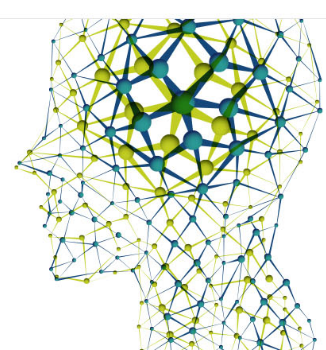

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Drifts of electron orbits induced by toroidal electric field in tokamaks

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The drifts of electron orbits induced by the toroidal electric field in tokamaks are analyzed. Based on the relativistic Hamiltonian equations for guiding centre motion, the formula for the drift velocity v_{dr} is derived. It describes the outward drift of passing particles as well as the inward drift (the Ware pinch) of trapped particles. Unlike the approximate formula for v_{dr} given in Guan *et al.* [Phys. Plasmas **17**, 092502 (2010)] for circular electron orbits, it describes qualitatively new features of the outward drift of electron orbits. Particularly, the new formula describes the evolution of the orbit's shape, the formation of X-point and the associated separatrix. It is shown that the outward drift velocity is proportional to the inverse aspect ratio of tokamaks.
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A consequence of plasma disruptions in tokamaks is the formation of high energy runaway electrons (REs) that may cause severe damages to the device wall (see, e.g., Ref. 1). Those REs are produced by the acceleration of electrons in large toroidal electric fields induced by the decay of the plasma current. The dynamics of the RE orbits during that acceleration is expected to play an important role in the evolution of the RE beam, especially of its decay.

It was shown in Refs. 2 and 3 that the RE orbits as whole drift continuously outward in the presence of a toroidal electric field in tokamaks. This effect takes place for electrons of arbitrary energy. For typical parameters during plasma disruptions, the outward drift velocity v_{dr} may reach values of the order of several m/s. One expects that this effect may therefore give a significant contribution to the decay of the RE beams formed during disruptions.

The formula $v_{dr} = qE_\varphi/B_0$ has been obtained in Refs. 2 and 3 for the outward drift velocity of the RE orbits. [Here B_0 is the toroidal magnetic field, E_φ is the toroidal electric field, and q is the safety factor.] It is however approximate and does not describe well the outward drift velocity in real situations during plasma disruptions.

In the present letter, we give a rigorous derivation of the formula for v_{dr}. It describes the outward drift velocity of passing particles as well as the inward drift (the Ware pinch) of trapped particles. The formula not only provides the quantitative values of v_{dr} but also describes the qualitatively new features that may occur during the decay phase of the RE beams. It is shown that drift of the guiding–centre (GC) orbits in the toroidal electric field is an *adiabatic dynamical process* that conserves the area encircled by the GC orbits in the poloidal plane.

Our considerations are based on the Hamiltonian formulation of relativistic GC motion in a toroidal system (see Ref. 4 for more details). For generality, we consider the motion of a charged particle of mass m_0 and a charge $q_a = Z_{qe}$, where e is the elementary charge, and $Z_q = -1$ for electrons, respectively, and $Z_q = 1$ for protons. We use the cylindrical coordinate system (R, Z, φ) where R, Z are, respectively, the radial and vertical coordinates and φ is the toroidal angle. The magnetic field is given by the vector potential $\mathbf{A}(R, Z, \varphi, \tilde{t}) = (A_R, A_Z, A_\varphi)$ we set $A_R = 0$ because of a gauge invariance. The electric field is given by the scalar potential $\Phi(R, Z, \varphi, \tilde{t})$ and the time denoted as $\tilde{t}$.

Furthermore, the following notations are introduced: $x = R/R_0$ and $z = Z/R_0$ are the normalized coordinates, $p_z = P_Z/m_0\omega_0 R_0$ and $p_\varphi = P_\varphi/m_0\omega_0 R_0^2$ are normalized momenta, $h = H/E_{ref}$, and $\varepsilon_0 = m_0c^2/E_{ref}$ are the normalized energy and rest energy, $t = \omega_0\tilde{t}$ is the normalized time. Here H is the full energy, $\omega_0 = eB_0/m_0$ is a reference gyrofrequency, B_0 is the toroidal magnetic field strength on the magnetic axis $R = R_0$, c is the speed of light in vacuum, $E_{ref} = m_0\omega_0^2 R_0^2$ is the reference energy.

The Hamiltonian equations of the GC motion are

$$\frac{dq_i}{d\tau} = \frac{\partial \tilde{H}}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial \tilde{H}}{\partial q_i}, \quad (i = 1, 2), \tag{1}$$

where $(q_1,\ q_2,\ p_1,\ p_2)$ are the canonical GC variables, $\tilde{H}$ is the Hamiltonian, and τ is the independent variable. If the time t is chosen as the independent variable, i.e., $\tau = t$, then $(q_1, q_2, p_1, p_2) = (z, \varphi, p_z, p_\varphi)$ and the Hamiltonian $\tilde{H} = h(z, \varphi, p_z, p_\varphi, t)$ is given by

$$h = \varepsilon_0\gamma_t + Z_q\phi, \tag{2}$$

where $\gamma_t = [1 + (2\omega_x I_x + u_\varphi^2)/\varepsilon_0]^{1/2}$ is the relativistic factor, and $u_\varphi = (p_\varphi + Z_q\psi)/x$. Here $\psi = -RA_\varphi/B_0R_0^2$ and $\phi = e\Phi/E_{ref}$ are the normalized vector and electric potentials, respectively, $\omega_x = |Z_q|\exp(p_z/Z_q)$ is the normalized radial gyrofrequency, I_x is the normalized adiabatic invariant associated with the radial gyro–oscillations.[4] The coordinate x is related to the canonical momentum p_z by $x = \exp(-p_z/Z_q)$.

For passing particles one can choose the toroidal angle φ as the independent variable τ and the corresponding canonical momentum p_φ as a new Hamiltonian $K = -p_\varphi$. Then the pair $(q_2, p_2) = (t, p_t = -h)$ are the canonical variables. The corresponding Hamiltonian function $\tilde{H} = K \equiv K(z, t, p_z, p_t, \varphi)$ is given by

$$K = Z_q\psi(x, z, \varphi, t) - \sigma x u_\varphi(x, z, \varphi, t, p_t), \tag{3}$$

where $u_\varphi(x,z,\varphi,t,p_t) = [\varepsilon_0(\gamma_t^2-1) - 2\omega_x I_x]^{1/2}$, $\gamma_t = (-p_t - Z_q\phi)/\varepsilon_0$. In this case the Hamiltonian equations (1) can be rewritten as

$$\frac{dz}{d\varphi} = -x\frac{\partial\psi}{\partial x} + \frac{x\sigma}{Z_q}\left(u_\varphi + \frac{\omega_x I_x}{u_\varphi} - \frac{x\gamma_t Z_q}{u_\varphi}\frac{\partial\phi}{\partial x}\right),$$
$$\frac{dp_z}{d\varphi} = -Z_q\left(\frac{\partial\psi}{\partial z} + \frac{\sigma\gamma_t x}{u_\varphi}\frac{\partial\phi}{\partial z}\right), \quad \frac{dt}{d\varphi} = \frac{\sigma\gamma_t x}{u_\varphi}, \tag{4}$$
$$\frac{dp_t}{d\varphi} = -Z_q\left(\frac{\partial\psi}{\partial t} + \frac{\sigma\gamma_t x}{u_\varphi}\frac{\partial\phi}{\partial t}\right).$$

Here, the parameter $\sigma = \pm 1$ stands for the direction of motion along the toroidal angle φ.

The system of Eq. (4) for the GC motion can be presented in a form similar to the equations for the magnetic field lines in the cylindrical coordinate system (R, Z, φ). We consider the case when the electric field potential to vanish: $\Phi \equiv 0$. Using the definitions, $B_R = B_\varphi \partial\psi/\partial z$ and $B_Z = -(B_0/x)\partial\psi/\partial x$ for the poloidal components of the magnetic field ($B_\varphi = R_0 B_0/R$ is the toroidal magnetic field), one can reduce the system of Eq. (4) to the form

$$\frac{dZ}{d\varphi} = \frac{RB_Z^*}{B_\varphi}, \quad \frac{dR}{d\varphi} = \frac{RB_R}{B_\varphi},$$
$$\frac{d\tilde{t}}{d\varphi} = \frac{\sigma R}{v_\varphi}, \quad \frac{dH}{d\varphi} = -Z_q\frac{\partial(RA_\varphi)}{\partial\tilde{t}}, \tag{5}$$

where $v_\varphi = u_\varphi R_0\omega_0/\gamma_t$ is the toroidal velocity, B_Z^* is the effective poloidal field defined as

$$B_Z^* = B_Z + \frac{\sigma B_\varphi}{Z_q}\left(u_\varphi + \frac{\omega_x I_x}{u_\varphi}\right). \tag{6}$$

Consider the dynamics of electrons in the presence of the toroidal electric field. The latter can be represented by means of the toroidal component of the vector potential $A_\varphi^{(ind)}(R, Z, \tilde{t})$

$$E_\varphi(R,Z,\tilde{t}) = -\frac{\partial A_\varphi^{(ind)}(R,Z,\tilde{t})}{\partial\tilde{t}}. \tag{7}$$

For simplicity, we suppose that the electric field is determined by the loop voltage V: $E_\varphi = V/2\pi R_0$.

The poloidal flux ψ in the Hamiltonian function (3) is given by

$$\psi = \psi^{(0)}(x,z) + \psi^{(ind)}(x,z,t),$$
$$\psi^{(ind)}(x,z,t) = -\frac{RA_\varphi^{(ind)}(R,Z,\tilde{t})}{B_0R_0^2}. \tag{8}$$

Here $\psi^{(0)}(x, z)$ is the poloidal flux of the equilibrium plasma. In the normalized variables, the inductive poloidal flux $\psi^{(ind)}(x, z, t)$ can be represented as

$$\psi^{(ind)}(x,z,t) = \int^t \mathcal{E}_\varphi(t')dt', \tag{9}$$

where $\mathcal{E}_\varphi(t)$ is the normalized toroidal electric field

$$\mathcal{E}_\varphi(t) = \frac{RE_\varphi(R,Z,t)}{B_0R_0^2\omega_0} = \frac{V}{2\pi B_0R_0^2\omega_0}. \tag{10}$$

The variation of energy with time is given by

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} = Z_q\frac{\partial h}{\partial p_\varphi}\frac{\partial\psi}{\partial t} = Z_q\frac{d\varphi}{dt}\frac{\partial\psi}{\partial t} = \frac{Z_q u_\varphi}{x_c\gamma_t}\mathcal{E}_\varphi(t). \tag{11}$$

The energy grows if $Z_q u_\varphi\mathcal{E}_\varphi(t) > 0$. Furthermore, we assume that the loop voltage V and thus $\mathcal{E}_\varphi(t)$ are constants in the poloidal cross section. Then the increment of the particle energy in one poloidal turn is given by

$$\Delta E = E_{ref}\Delta h = E_{ref}\int_t^{t+T}\frac{dh}{dt}dt$$
$$= E_{ref}\sigma 2\pi q_{eff}Z_q\mathcal{E}_\varphi(t) = \sigma q_{eff}V, \tag{12}$$

where q_{eff} is the effective safety factor defined as $q_{eff} = |\Delta\varphi|/2\pi$, $\Delta\varphi$ is the increment of the toroidal angle φ in one poloidal turn, and T is the normalized transition time.

We now estimate the drift velocity v_{dr} of the RE orbit induced by the toroidal electric field. We assume axisymmetry, i.e., $\psi = \psi(x, z, t)$ and $u_\varphi = u_\varphi(x,z,p_t)$. The toroidal momentum p_φ is then a constant of motion. According to (3), the drift surface at time t is determined by

$$p_\varphi = -Z_q\psi(x,z,t) + \sigma x u_\varphi(x,z,p_t) = \text{const}, \tag{13}$$

where the poloidal flux $\psi(x, z, t)$ is given by (8). According to the latter and (12), the poloidal flux ψ and the energy $h = -p_t$ get, respectively, increments $\Delta\psi = TZ_q\mathcal{E}_\varphi$ and Δh in one poloidal turn. Since the increment $\Delta p_\varphi = 0$, the drift surface is shifted along the radial direction by the distance Δx, the expression of which can be obtained from (13)

$$\Delta x = \frac{Z_q\mathcal{E}_\varphi\left(x2\pi q_{eff}\gamma_t/u_\varphi - T\right)}{\left(Z_q\frac{\partial\psi}{\partial x} - \sigma\left[u_\varphi + \omega_x I_x/u_\varphi\right]\right)}. \tag{14}$$

The expression of the orbit's drift velocity is obtained from (14)

$$v_{dr} = \omega_0\frac{\Delta R}{T} = \frac{R_0E_\varphi}{RB_Z^*}\left(1 - \frac{RT_{av}}{R_0\tilde{T}}\right), \tag{15}$$

where $\tilde{T} = \omega_0^{-1}T$ is the actual transition time, and $\Delta R = R_0\Delta x$ is the radial shift of the orbit. The quantity,

$$T_{av} = \frac{2\pi q_{eff}\gamma_t}{\omega_0 u_\varphi} = \frac{2\pi q_{eff}R_0}{v_\varphi}, \tag{16}$$

is the average transition time.

Expression (15) is obtained by expansion of (13) with respect to the radial shift Δx in one poloidal turn, keeping only the first term. For realistic plasma parameters the shift Δx is extremely small ($\sim 10^{-7}$) and the procedure is well justified. (That has been confirmed by numerical calculations.)

In particular, when $\tilde{T} = T_{av}$ and for low–energy electrons, we have $B_Z^* \approx B_Z$ and (15) can be reduced to the expression obtained in Refs. 2 and 3 for the circular GC orbits

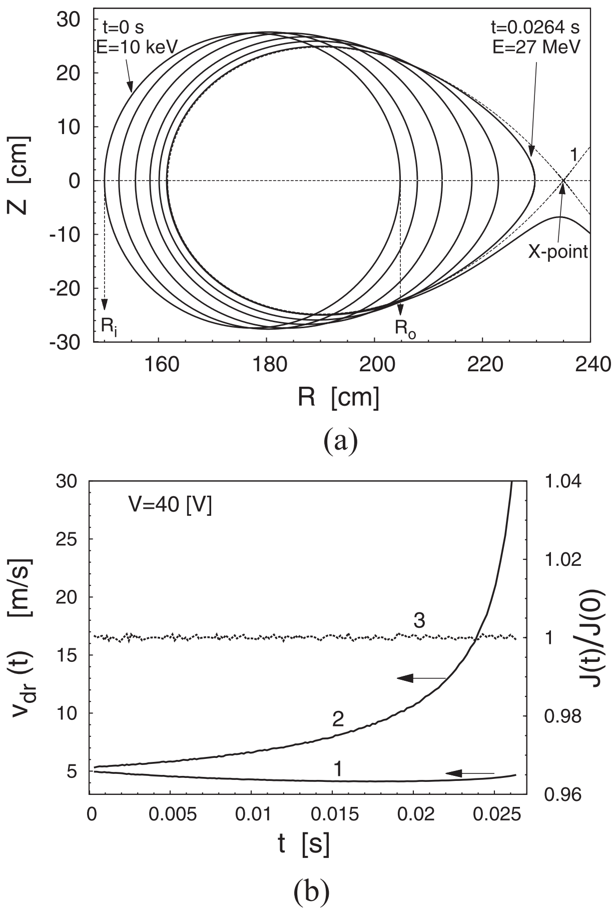

FIG. 1. (a) Evolution of the GC orbit in the presence of the toroidal electric field. Curve 1 corresponds to the separatrix of the GC orbit of energy $E = 27$ MeV. The plasma current $I_p = 150$ kA, the toroidal field $B_t = 2.5$ T, the major and minor radii $R_0 = 175$ cm and $a = 46$ cm, and the loop voltage $V = 40$ V; (b) Time evolution of the outward drift velocity v_{dr} corresponding to the orbit in (a). Curve 1 corresponds to $v_{dr}(R_i)$, curve 2—to $v_{dr}(R_o)$, and curve 3 (the right hand axis) describes the adiabatic invariant $J(t)$ normalized to its initial value $J(0)$.

$$v_{dr} = -\frac{E_\varphi(R - R_0)}{RB_Z} = \frac{qE_\varphi}{B_0}, \tag{17}$$

where $q = (R - R_0)B_0/RB_Z$ is the safety factor.

The drift velocity v_{dr} is proportional to the strength of the toroidal electric field E_φ and to the inverse of the plasma current I_p, i.e., $v_{dr} \propto I_p^{-1}$. The dependence of v_{dr} on the particle energy E is implicitly given through the effective poloidal magnetic field B_Z^* (6), the average transition time T_{av} (16) and the transition time $\tilde{T}$. For a given orbit, the quantities B_Z^* and T_{av} depend on the radial position R on the orbit, i.e., v_{dr} is a local function of R. Furthermore, we consider the drift velocities $v_{df}(R_i)$ and $v_{df}(R_o)$ at the orbit's two radial positions in the equatorial plane $Z = 0$, i.e., its innermost R_i and outermost R_o points. For REs, the outermost point drifts faster than the innermost point, i.e., $v_{df}(R_o) > v_{df}(R_i)$. This leads to an elongation of the orbit along the radial direction. In particular, an initially circular orbit evolves into an oval shaped one owing to the electron acceleration. Figures 1(a) and 1(b) illustrate the evolution of the GC orbit and the outward drift velocities $v_{df}(R_i)$, $v_{df}(R_o)$ in a tokamak plasma.[8]

At a certain critical energy E_{cr}, the GC orbit bifurcates by creating an unstable fixed point (or X-point) inside the plasma region. With the further increase of the RE energy, the orbit crosses the separatrix (a homoclinic orbit associated the X-point) and hits the wall.

Equation (15) describes not only the outward drift of passing particles but also the inward drift of trapped particles known as the *Ware pinch*[7] (see, also Ref. 1). Indeed, for trapped particles the quantity T_{av} (16) is much smaller than the transit time $\tilde{T}$ and the expression of the drift velocity v_{dr} (15) reduces to

$$v_{dr} = \frac{R_0E_\varphi}{RB_Z^*} \approx \frac{R_0E_\varphi}{RB_Z}. \tag{18}$$

The average value of v_{dr} (18) over the radial coordinate R coincides the standard formula for the Ware pinch $v_p = -E_\varphi/|B_Z|$.

It is important to note that the drift of the electron GC orbit in the toroidal electric field is an *adiabatic process*. The area encircled by the GC orbit in the poloidal section is thus conserved, i.e., the integral $J = (2\pi)^{-1}\oint_C p_z dz$ is an *adiabatic invariant* (C is the closed contour along the GC orbit). The time-dependence of J for passing electrons is shown by curve 3 in Fig. 1(b). Figure 2 illustrates the evolution of an initially trapped electron orbit into a passing one: at a certain time, due to its inward drift, the banana orbit turns into a

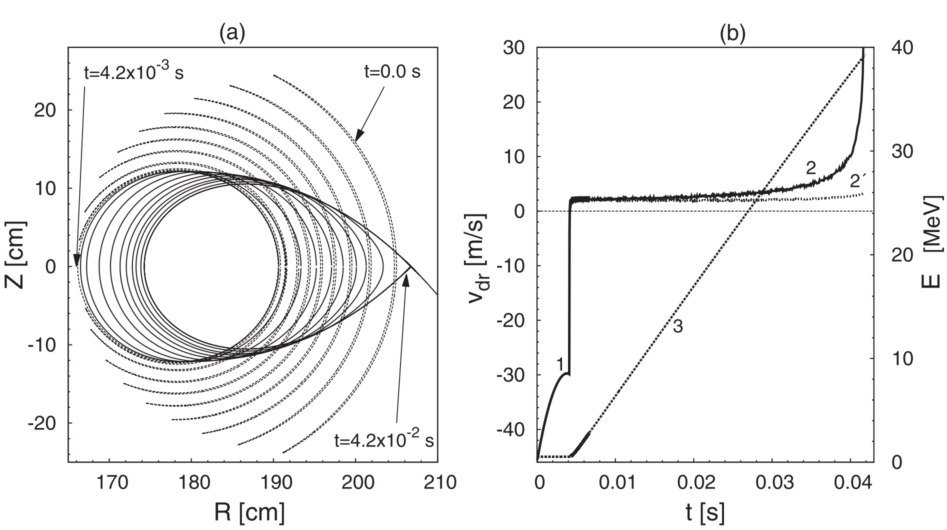

FIG. 2. (a) Evolution of the GC orbit of the initially trapped electron in the presence of the toroidal electric field. (b) Time dependence of the drift velocity v_{dr} (15) of the orbit: curve 1 corresponds to the inward drift of trapped electrons, curves 2 and 2′ correspond to the outermost and innermost points of the orbit, curve 3 (the right hand side axis) describes the energy growth. The plasma parameters are the same as in Fig. 1, but the plasma current $I_p = 100$ kA.

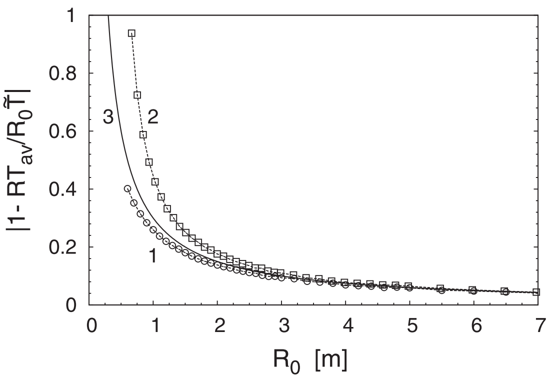

FIG. 3. Dependence of the factor $|1 - RT_{av}/R_0\tilde{T}|$ in Eq. (15) on the major radius of plasma R_0: curve 1 corresponds to the innermost point of orbit, curve 2—to its outermost point, curve 3 describes the ratio $|R_0 - R|/R_0$. The difference $|R - R_0|$ is fixed and taken equal to 0.3 m. The plasma current $I_p = 0.5$ MA, the toroidal field $B_0 = 2.5$ T, and the minor radius $a = 0.46$ m.

circular orbit and starts drifting outwards with the growth of the electron energy.

The strong growth of the drift velocity at outermost point of a GC orbit shown by curve 2 in Fig. 1(b) is related to the creation of the X-point and the separatrix at a certain critical energy. This effect is described by the effective poloidal field B_Z^* in Eq. (15): B_Z^* decreases with the growth of the electron energy E and at the critical value E_{cr}, the field B_Z^* zeroes at the radial distance R_s within the plasma region. The coordinates (R_s, Z_s) of the X-point are determined by the equations, $dR/d\varphi = 0, dZ/d\varphi = 0$, or according to (5) by the zeros of the effective poloidal field,

$$B_Z^*(R_s, Z_s) = 0, \quad B_R(R_s, Z_s) = 0. \tag{19}$$

The critical energy E_{cr} for the creation of the X-point can be estimated using the definition of the effective magnetic field B_Z^* (6). For sufficiently large energy with the relativistic factor $\gamma_t \gg 1$, one can assume the longitudinal velocity to be close to the speed of light, $v_\varphi \approx c$. Neglecting the last term in (6), Eq. (19) can be written as

$$B_Z^{(s)} + \frac{\sigma B_\varphi}{Z_q}\frac{c}{R_0\omega_0}\gamma_t \approx 0, \quad B_Z^{(s)} = B_Z(R_s, Z_s). \tag{20}$$

From (20) one obtains the critical energy $E_{cr} = m_0c^2\gamma_t$

$$\begin{aligned} E_{cr} &\approx m_0c^2\left|\frac{B_Z}{B_\varphi}\right|\frac{R_0\omega_0}{c} = ceR_s|B_Z| \\ &= 2.998 \times 10^2 R_s|B_Z^{(s)}|\,\text{MeV}, \end{aligned} \tag{21}$$

where we set $\sigma = 1$, $Z_q = -1$, and $B_Z^{(s)}$ is expressed in Tesla (T) and R_s is given in meter (m). The critical energy E_{cr} is therefore determined by the product of the poloidal field B_Z at the X-point to it's radial position R_s.

We should note that the formation of the separatrix of the GC orbit in the toroidal electric field has first been predicted three decades ago in Ref. 5. This phenomenon has been only recently confirmed by numerical simulations in realistic tokamak conditions during plasma disruptions.[4,6]

The outward drift velocity v_{dr} (15) is proportional to the factor $|1 - RT_{av}/R_0\tilde{T}|$. This factor is only weekly sensitive to the toroidal magnetic field B_0 and plasma current I_p. However, it strongly depends on the tokamak aspect ratio R_0/a: at the given energy E, it decreases as R_0/a increases. For large aspect ratios $R_0/a \gg 1$, the transit time $\tilde{T}$ approaches T_{av} so that $|1 - RT_{av}/R_0\tilde{T}| \to |R_0 - R|/R_0 \sim a/R_0$, i.e., the outward drift velocity v_{dr} is proportional to the inverse aspect ratio: $v_{dr} \propto a/R_0$. Numerical calculations of $|1 - RT_{av}/R_0\tilde{T}|$ presented in Fig. 3 confirms such a dependence. Therefore, one expects the outward drift velocity in spherical tokamaks is larger than in standard tokamaks.

We have derived the formula for the radial drift velocity of electron orbits induced by the toroidal electric field in tokamaks. It describes the outward drift of passing electrons as well as the inward drift of trapped electrons. The outward drift of electrons may give a significant contribution to the decay of the relativistic electron current created during plasma disruptions in tokamaks.

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[8]For calculations we have used the model of a tokamak magnetic field with a circular cross section described in Ref. 1 (Sec. 3.4) and in Ref. 4 with the toroidal corrections. We supposed that the directions of the plasma current I_p and the toroidal electric field E_φ are opposite to the toroidal field B_φ as in the TEXTOR tokamak. Then $\sigma = 1$ for the REs. The evolution of the trapped electrons is obtained by the integration of Eq. (1) with Hamiltonian (2).