INTRODUCTION TO DRIFT WAVE TURBULENCE MODELS

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ABSTRACT

This tutorial presents an introduction to the basic concepts of plasma turbulence models based on fluid theory. It is intended to elucidate basic features of drift fluid theory and drift wave turbulence. Theoretical methods widely used in tokamak transport and turbulence modelling are discussed briefly.

I. INTRODUCTION

The phenomenon of plasma turbulence is still an active field of research. It is accepted for years now that the turbulent transport is the main candidate to explain the degradation of plasma confinement known as the "anomalous" transport. Moreover, the theory of turbulent dynamics caused by the interplay of plasma transport and self-consistent electromagnetic fields, provides an explanation for the intermittent transport in the edge of tokamak plasmas, plasma oscillations and operational regimes found in experiments. In this tutorial we introduce a basic mathematical framework to study this kind of processes. Emphasis is put on illustrative examples of basic mechanisms driving turbulent processes in a plasma, e.g. linear instabilities, non-linear interaction and geometry effects. Of course, a comprehensive overview on all relevant aspects of turbulent transport and all varieties of theories would be too lengthy and confusing for this tutorial. Therefore, details on the physics of trapped particles, ITG and ETG modes [1], finite Larmor radius effects and elaborated kinetic theories have not been taken into account. However, the framework of fluid turbulence chosen here offers the opportunity to introduce systematically several terms and concepts relevant also for those methods and approaches not included here. The tutorial starts with a reminder of two-fluid plasma theory. By the use of the so-called drift approximation a very general set of model equations is established widely used in several variants in the literature to study a large number of plasma transport phenomena. By picking out certain ingredients of the general model presented basic processes like drift wave motion, drift wave coupling, linear instabilities and turbulent interaction are introduced. A second part is devoted to a discussion of problems and details on the numerical simulation of tokamak turbulence.

II. DRIFT FLUID MODELS

In this section the basic equations of fluid theory of magnetized plasmas - known as Braginskii equations - are recapitulated. The problem of solving the perpendicular momentum equation is moved to the derivation of a vorticity equation to compute the self-consistent electric field and therefore the perpendicular velocity of the particles. It is shown that the so-called drift waves represent an elementary ingredient of the plasma dynamics of this kind of drift fluid model. In subsequent sections the possibility of linear instability drive due to resistivity and geometry effects, non-linear interactions and zonal flow oscillations are discussed.

A. Fluid Models

We start with the general fluid equations for the particle, momentum and energy balance for each plasma species [2, 3] neglecting external sources and sinks

$$\frac{dn}{dt} = -n\boldsymbol{\nabla}\cdot\mathbf{V} \tag{1}$$

$$mn\frac{d\mathbf{V}}{dt} = -\boldsymbol{\nabla}\cdot\mathbf{P} + \mathbf{R} + Zen(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{2}$$

$$\frac{3}{2}n\frac{dT}{dt} = -\boldsymbol{\nabla}\cdot\mathbf{q} - \mathbf{P} : \boldsymbol{\nabla}\mathbf{V} - Q \tag{3}$$

where n denotes the density of the particular species, $\mathbf{V}$ is the flow velocity and T the particular temperature. The pressure tensor is denoted by $\mathbf{P}$, m is the particle mass, Z is the charge number, $\mathbf{q}$ is the heat flow and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field vectors, respectively. External sources for particle, momentum and energy are neglected here. Finally the quantity Q and the vector $\mathbf{R}$ denote the change of thermal energy and the force due to Coulomb collisions between the charged plasma particles. The total time derivative is defined by $d/dt{=}\partial/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla}$.

B. Drift Approximation

Using Eq. 2 one can write a formal solution for the perpendicular velocity $\mathbf{V}_\perp$ as

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times\boldsymbol{\nabla}p}{ZenB^2} + \frac{m}{ZeB^2}\mathbf{B}\times\frac{d\mathbf{V}}{dt} \tag{4}$$

Here the pressure tensor is approximated by $\mathbf{P}{=}p\,\mathbf{1}$ and the perpendicular part of the friction $\mathbf{R}$ has been neglected. The first term on the rhs of Eq. 4 defines the $E{\times}B$-velocity $\mathbf{V}_E$, the second the diamagnetic drift $\mathbf{V}_*$ and the third the polarization drift $\mathbf{V}_p$. Actually, Eq. 4 can be considered as an iteration scheme to find the perpendicular velocity $\mathbf{V}_\perp$. The iteration usually starts with the assumption that in zeroth order $\mathbf{V}_\perp^{(0)}{=}\mathbf{V}_E + \mathbf{V}_*$. Inserting this into Eq. 4 again gives in first order

$$\begin{aligned}\mathbf{V}_\perp^{(1)} &= \mathbf{V}_E + \mathbf{V}_* \\ &+ \frac{m}{ZeB^2}\,\mathbf{B} \times \left(\frac{\partial}{\partial t} + \mathbf{V}^{(1)} \cdot \boldsymbol{\nabla}\right)(\mathbf{V}_E + \mathbf{V}_*)\end{aligned} \tag{5}$$

where $\mathbf{V}^{(1)}{=}\mathbf{V}_\parallel + \mathbf{V}_\perp^{(1)}$ denotes the total velocity of first order and $\mathbf{V}_\parallel$ is the parallel velocity. Therefore, in first order a relation results between the perpendicular velocity $\mathbf{V}_\perp^{(1)}$ and the temporal evolution of the electric field $\dot{\mathbf{E}}_\perp{=}-\boldsymbol{\nabla}_\perp\dot{\phi}$.

C. The Vorticity Equation

To compute the electric field an additional equation is needed. This is found by employing the quasineutrality condition $n_e{=}n_i{\equiv}n$ in the form

$$\boldsymbol{\nabla} \cdot \mathbf{J} = \boldsymbol{\nabla} \cdot (en\mathbf{u}_\perp - en\mathbf{v}_\perp) + \nabla_\parallel J_\parallel = 0 \tag{6}$$

with $\mathbf{u}_\perp$ determined by

$$\begin{aligned}\mathbf{u}_\perp &= \mathbf{V}_E + \mathbf{u}_* \\ &+ \frac{m_i}{eB^2}\,\mathbf{B} \times \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\right)(\mathbf{V}_E + \mathbf{u}_*)\end{aligned} \tag{7}$$

and the electron drift approximated by

$$\mathbf{v}_\perp = \mathbf{V}_E + \mathbf{v}_* \tag{8}$$

i. e. due to their small mass the electron polarisation drift is neglected. This provides an equation - called vorticity equation or quasineutrality condition - for the time evolution of the electric potential ϕ and completes the model. Often the detailed vorticity equation is very diffult to derive, but as an example we will give an approximate expression widely used and valid for the case of cold ions, i. e. $T_i{\approx}0$

$$\begin{aligned}&\frac{m_i}{eB^2}\frac{\partial \nabla_\perp^2\phi}{\partial t} + \frac{m_i}{eB^2}\left(\mathbf{u}_\parallel + \mathbf{V}_E\right) \cdot \boldsymbol{\nabla}\nabla_\perp^2\phi \\ &= \frac{\nabla_\parallel J_\parallel}{e\,n} + \frac{\boldsymbol{\nabla} \cdot (n\,\mathbf{V}_E - n\,\mathbf{v}_\perp)}{n}\end{aligned} \tag{9}$$

This expression suffers from certain shortages concerning the energetic consistency of the entire model, this point has been discussed for cylindrical geometry in Ref. [4]. However, in practice, this is often a good approximation if $T_i{\ll}T_e$. This form also explains the name "vorticity equation", because the vorticity is given by $\nabla_\perp^2\phi{=}\mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{V}_E$ and is a measure for the local spinning of particles due to $E{\times}B$ motion. The vortices in plasmas are often called eddies. Now we have an equation for the perpendicular electric field $\mathbf{E}_\perp{=}-\boldsymbol{\nabla}_\perp\phi$. For the parallel electric field it is assumed that $E_\parallel{=}-\nabla_\parallel\phi - \partial A_\parallel/\partial t$, where $A_\parallel$ is the magnetic potential, which is related to a parallel current density via Ampére's law $\mu_0 J_\parallel{=}-\nabla_\perp^2 A_\parallel$. Inserting the resulting drift velocities into the model equations Eqs. 1-3 and project the momentum equations on the direction parallel to the magnetic field lines leads to a set of model equations describing the temporal evolution of density, parallel momentum and temperatures. These are the particle conservation

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} n = -n\,\boldsymbol{\nabla} \cdot \mathbf{v} \quad , \tag{10}$$

the conservation of parallel electron momentum (Ohm's law)

$$\begin{aligned}m_e\,n\left(\frac{\partial v_\parallel}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} v_\parallel\right) &= -\nabla_\parallel p_e + e\,n\,\eta_\parallel\,J_\parallel \\ &-\alpha\,n\,\nabla_\parallel T_e + e\,n\,\nabla_\parallel\phi + e\,n\,\frac{\partial A_\parallel}{\partial t} \quad ,\end{aligned} \tag{11}$$

the conservation of parallel ion momentum

$$\begin{aligned}m_i\,n\left(\frac{\partial u_\parallel}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} u_\parallel\right) &= -\nabla_\parallel p_i - e\,n\,\eta_\parallel\,J_\parallel \\ &+\alpha\,n\,\nabla_\parallel T_e - e\,n\,\nabla_\parallel\phi - e\,n\,\frac{\partial A_\parallel}{\partial t} \quad ,\end{aligned} \tag{12}$$

the heat transfer of electrons

$$\begin{aligned}\frac{3}{2}\,n\left(\frac{\partial T_e}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} T_e\right) &= -\boldsymbol{\nabla} \cdot \mathbf{q}_e \\ &-n\,T_e\,\boldsymbol{\nabla} \cdot \mathbf{v} - \alpha\,\frac{J_\parallel}{e}\,\nabla_\parallel T_e + \eta_\parallel\,J_\parallel^2 \quad ,\end{aligned} \tag{13}$$

and the heat transfer of ions

$$\begin{aligned}\frac{3}{2}\,n\left(\frac{\partial T_i}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T_i\right) &= -\boldsymbol{\nabla} \cdot \mathbf{q}_i \\ &-n\,T_i\,\boldsymbol{\nabla} \cdot \mathbf{u} \quad ,\end{aligned} \tag{14}$$

where $\mathbf{q}_e$ is the electron heat flux

$$\mathbf{q}_e = -\alpha\,T_e\,\mathbf{J}_\parallel/e + \kappa_\parallel^e\,\nabla_\parallel T_e - \kappa_\perp^e\,\boldsymbol{\nabla}_\perp T_e \tag{15}$$

and $\mathbf{q}_i$ is the ion heat flux

$$\mathbf{q}_i = \kappa_\parallel^i\,\nabla_\parallel T_i - \kappa_\perp^i\,\boldsymbol{\nabla}_\perp T_i \tag{16}$$

and $\kappa_\parallel$ and $\kappa_\perp$ denote the classical heat conductivities. The thermal force coefficient is set to $\alpha{=}0.71$ [2, 3] and the parallel current density is defined by $\mathbf{J}_\parallel{=}en\left(\mathbf{u}_\parallel - \mathbf{v}_\parallel\right)$. The parallel derivative is defined by $\nabla_\parallel{=}\mathbf{b}{\cdot}\boldsymbol{\nabla}$, where $\mathbf{B}{=}B\,\mathbf{b}$ denotes the total magnetic field (equilibrium field + fluctuations).

This forms the basis for the following discussions. Even though several physical effects are still missing (trapped particles, radiation, neutral physics etc.) this model contains a rich variety of physics such as the dynamics of drift waves, tearing modes, drift-Alfvén waves, ITG modes, sound waves, drift resistive ballooning modes etc. and it has been used as a starting point to study turbulent and intermittent plasma transport in the scrape-off-layer, the edge and the core plasma as well.

D. Drift Waves

To get an idea of the nature of drift waves we consider a subset of the model equations Eqs. 9-14 in cylindrical geometry. The cylinder geometry is not a severe restriction for the particular effects discussed here. We inspect the physical processes described by

$$\frac{\partial n}{\partial t} + \mathbf{V}_E \cdot \boldsymbol{\nabla} n = 0 \qquad (17)$$

$$\frac{T_e \nabla_{\|} n}{e\, n} - \nabla_{\|}\phi = 0 \qquad (18)$$

This means that the particles are advected by the $E{\times}B$-velocity and Ohm's law is reduced to a simple force balance between electron pressure and parallel electrostatic field. The temperature T_e is assumed to be constant and it follows $n{=}n_0\, \exp\left(e\phi/T_e\right)$ along the magnetic field line. To study the interplay of a poloidal perturbation of mode number m and the symmetric profiles ("background") we split the density n and the electric potential ϕ according to

$$n = n_0(r) + n_m(r)\, e^{im\theta} + n_m^*(r)\, e^{-im\theta} \qquad (19)$$

$$\phi = \phi_m(r)\, e^{im\theta} + \phi_m^*(r)\, e^{-im\theta} \qquad (20)$$

This gives the evolution equations

$$\frac{\partial n_0}{\partial t} = \frac{2\, m}{rB}\, \frac{\partial}{\partial r}\left(\mathrm{Im}\left\{n_m\, \phi_m^*\right\}\right) \qquad (21)$$

$$\frac{\partial n_m}{\partial t} = \frac{i\, m}{rB}\, \frac{\partial n_0}{\partial r}\, \phi_m \qquad (22)$$

According to $n_m{=}e\, n_0\, \phi_m/T_e$ the last equation gives

$$n_m(t) = n_m(0)\, \exp\left(i\, \frac{m}{rB}\, \frac{T_e}{en_0}\, \frac{\partial n_0}{\partial r}\, t\right) \qquad (23)$$

whereas $\partial n_0/\partial t{=}0$. This means for the simple system considered that the sinusoidal perturbation $n_m\, e^{im\theta}$ is traveling with the electron diamagnetic velocity in poloidal direction, i.e. $n_m(t)\, e^{im\theta}$ is moving with tangential velocity

$$r\, \frac{\partial \theta}{\partial t} = -\frac{T_e}{en_0B}\, \frac{\partial n_0}{\partial r} \qquad (24)$$

Notice that the velocity is independent of the mode number m. A sketch of the geometry of the drift wave motion is shown in Fig. 1

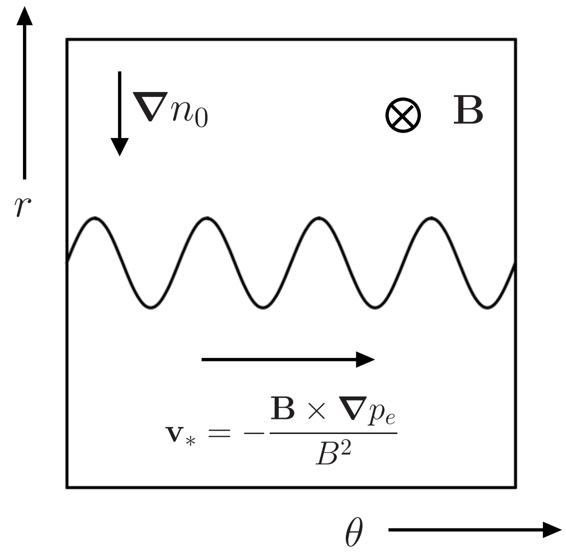

Figure 1: Geometry of a drift wave. The perturbation travels with the electron diamagnetic velocity.

E. "Anomalous" Transport in Drift Wave Models

Reconsidering Eq. 21 one finds that the homogeneous component n_0 will change in time if n_m and ϕ_m are not in phase anymore. This can be regarded as a transport mechanism because for the radial component v_m of the $E{\times}B$ velocity due to the perturbation ϕ_m one finds $v_m{=}{-}im\, \phi_m/rB$ and therefore a radial flux Γ_r shows up giving

$$\frac{\partial n_0}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma} \quad , \quad \boldsymbol{\Gamma} = 2\ \mathrm{Re}\left\{n_m\, v_m^*\right\}\, \mathbf{e}_r \qquad (25)$$

This relation is illustrated by Fig. 2 showing the effect on the net transport due to a phase shift σ between perturbations $\widetilde{n} \sim \sin(m\theta)$ and $\widetilde{\phi} \sim \sin(m\theta + \sigma)$. For $\sigma{=}0$, the net transport $\int \widetilde{\Gamma}\, d\theta$ is zero and for $\sigma{\neq}0$ the total particle transport is finite.

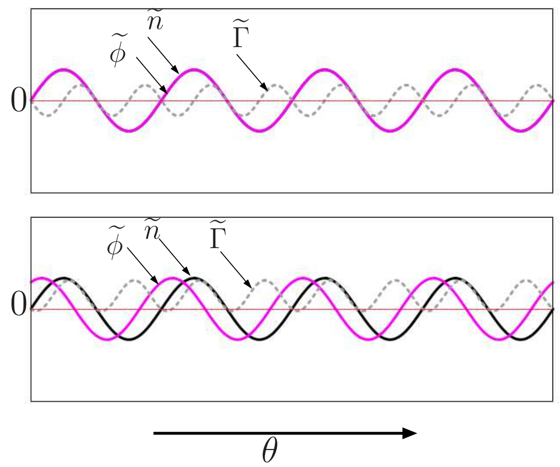

Figure 2: Effective transport $\widetilde{\Gamma}$ for perturbations $\widetilde{n}$ and $\widetilde{\phi}$ with $m{=}4$ with phase shift $\sigma{=}0$ (top) and $\sigma{=}\pi/3$ (bottom). The net flux is (dotted lines) is zero if $\widetilde{n}$ and $\widetilde{\phi}$ are in phase.

F. Linear Instabilities

As has been shown in the last section a phase shift between perturbations in density and potential is able to modify the background profile of the density by means of an effective flux which can be much larger than the collisional diffusive flux [5]. On the other hand the Eq. 22 tells us that such a phase shift will also modify the profile of the perturbation density. Instead of a stable drift wave as before, when n_m and ϕ_m were in phase, it is then possible that the perturbation is damped or amplified. Of particular interest are of course the perturbations which grow in time and reach an amplitude level such that they might affect the global plasma profile. The question of possible candidates for a significant impact on plasma transport can be answered to a certain extent by a linear analysis. This approach is based on the linearization of the model equations and inspection of the temporal behaviour of Fourier decomposed perturbations. This will be illustrated by a simple example, the resistive instability. We start with a subset of model equations similar to the one of Section II. D but also including resistivity, parallel particle flow and a simplified vorticity equation.

$$\frac{\partial n}{\partial t} + \mathbf{V}_E \cdot \boldsymbol{\nabla} n = -\nabla_\| \left(n\, v_\|\right) \tag{26}$$

$$\eta_\|\, J_\| = \frac{T_e\,\nabla_\| n}{e\,n} - \nabla_\| \phi \tag{27}$$

$$\frac{m_i}{eB^2}\left(\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \mathbf{V}_E \cdot \boldsymbol{\nabla}\nabla_\perp^2\phi\right) = \frac{\nabla_\| J_\|}{en} \tag{28}$$

Neglecting the parallel ion motion, i. e. $J_\|\approx -env_\|$, and replace $\nabla_\|^2$ by a constant $-k_\|^2$ leads to set of model equations known as the Hasegawa-Wakatani model or dissipative coupling model [6]. One finds for perturbations with mode number m as defined in Eqs. 19 and 20 the evolution equations

$$\frac{\partial n_m}{\partial t} = \frac{i\,m}{rB}\,\frac{\partial n_0}{\partial r}\,\phi_m + \frac{k_\|^2}{e\,\eta_\|}\,\phi_m - \frac{k_\|^2\,T_e}{e^2\,n_0\,\eta_\|}\,n_m \tag{29}$$

$$\frac{m_i}{eB^2}\,\frac{\partial \nabla_\perp^2\phi_m}{\partial t} = \frac{k_\|^2}{e\,n_0\,\eta_\|}\,\phi_m - \frac{k_\|^2\,T_e}{e^2\,n_0^2\,\eta_\|}\,n_m \tag{30}$$

Replacing now also $\partial/\partial t \to -i\,\omega$ and $\nabla_\perp^2 \to -k_\perp^2$ via Fourier decomposition one finds the algebraic equations

$$(D - i\,\omega)\,\frac{n_m}{n_0} - (D - i\,\omega_*)\,\frac{e\,\phi_m}{T_e} = 0 \tag{31}$$

$$D\,\frac{n_m}{n_0} - \left(D - i\,\omega\,k_\perp^2\,\rho_s^2\right)\,\frac{e\,\phi_m}{T_e} = 0 \tag{32}$$

where the coupling parameter D and the diamagnetic frequency ω_* are defined by

$$D = \frac{k_\|^2\,T_e}{e^2\,n_0\,\eta_\|} \quad , \quad \omega_* = -\frac{m\,T_e}{e\,n_0\,r\,B}\,\frac{\partial n_0}{\partial r} \tag{33}$$

and $\rho_s = c_s/\omega_i$, $c_s^2 = T_e/m_i$, $\omega_i = eB/m_i$, are the drift scale, the sound speed and the gyro-frequency, respectively. Finally, this leads to the dispersion relation for the frequency ω

$$\omega^2 + i\,\frac{1 + k_\perp^2\rho_s^2}{k_\perp^2\rho_s^2}\,D\,\omega - i\,\frac{D}{k_\perp^2\rho_s^2}\,\omega_* = 0 \tag{34}$$

If $k_\perp^2\rho_s^2 \ll 1$ this reduces to

$$\omega^2 + i\,\alpha\,(\omega - \omega_*) = 0 \quad , \quad \alpha = \frac{D}{k_\perp^2\rho_s^2} \tag{35}$$

and for $\eta_\| \to 0$ the stable drift wave is recovered ($\omega = \omega_*$). For finite resistivity the frequency ω contains a non-zero imaginary part, which represents an unstable branch. If $\eta_\|$ is small but finite, the solution can be expressed approximately by

$$\omega \approx \omega_* + i\,\frac{\omega_*^2}{\alpha} \tag{36}$$

Therefore it is possible that the drift wave perturbation becomes unstable, i. e. it starts to grow exponentially with a growth rate $\gamma \approx \omega_*^2/\alpha$.

G. Mixing Length Estimate

The linear instabilities discussed in the last section give important information on the plasma dynamics. Using the linear theory allows to draw conclusions on risky plasma configurations and typical time scales of plasma dynamics. However, it does not capture the important non-linear interaction in a plasma leading to non-linear saturation and the "anomalous" transport due to collective effects. To obtain more insight into this kind of effects requires substantial effort in development and analysis of non-linear models, theoretically and numerically as well. Nevertheless, many estimates widely used in theories on radial turbulent fluxes and plasma confinement use the so-called mixing length approach to draw conclusions just from linear growth rates of plasma instabilities. This is based on the idea that the linear instability with wave number $k_\perp$ dominates the spatial structure of the turbulent field. On the other hand a rough estimate for the diffusion of particles is given by the ratio two basic statistical quantities describing the turbulent dynamics, i. e. $D \sim \lambda_c^2/\tau_c$, where λ_c denotes the correlation length and τ_c the correlation time. If one equates now the correlation length λ_c with the inverse wave number $k_\perp$ (see Fig. 3 for an illustration) and considers the correlation time τ_c to be of the order of the inverse growth rate γ (the imaginary part of the frequency ω in the standard linear theory), one finds

$$D_\perp \sim \frac{\gamma}{k_\perp^2} \tag{37}$$

This is the standard mixing length diffusion coefficient used often in the literature. A refined version of this has been proposed by Connor and Pogutse [7].

However, it must be noted that this estimate can be quite useful for a qualitative analysis of trends, but very often the estimate is not useful for quantitative results.

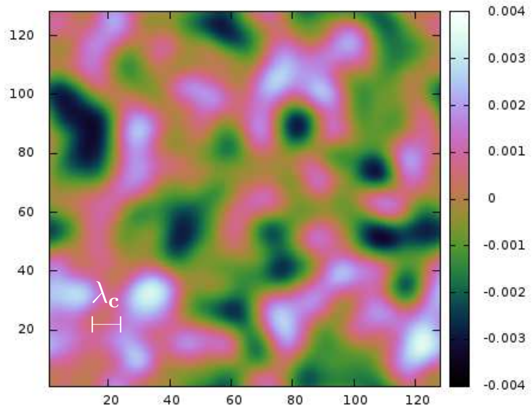

Figure 3: Snapshot of a turbulent structure of the electric potential ϕ. Lengths and amplitudes are given in a.u. The correlation length λ_c is indicated by the scale at the bottom left.

H. Non-Linear Interaction via $E\times B$-Advection, Three-Wave-Coupling

So far we have considered the $E\times B$ advection just for the quasilinear model, where this mechanism leads to rotation and destabilization of perturbations and introduces an additional "anomalous" transport. If more perturbation modes are present another effect comes into play: the energy conserving interaction between modes leading to an energy cascade. To study this we reconsider the change in particle density due to the $E\times B$-advection.

$$\frac{\partial n}{\partial t} = -\mathbf{V}_E \cdot \boldsymbol{\nabla} n \tag{38}$$

Again a Fourier decomposition of the density n and the electric potential ϕ is useful.

$$n = \sum_{\mathbf{k}} n_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \quad , \quad \phi = \sum_{\mathbf{k}} \phi_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{39}$$

Here $\mathbf{x}$ is the three-dimensional position vector and $\mathbf{k}$ the wave vector of the particular Fourier component. This gives

$$\mathbf{V}_E\cdot\boldsymbol{\nabla} n = -\sum_{\mathbf{k}',\mathbf{k}''} \frac{\mathbf{B}\times\mathbf{k}'}{B^2}\cdot\mathbf{k}''\,\phi_{\mathbf{k}'}\, n_{\mathbf{k}''}\, e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{x}} \tag{40}$$

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\mathbf{B}}{B^2}\cdot\sum_{\mathbf{k}'}(\mathbf{k}'\times\mathbf{k})\;\phi_{\mathbf{k}'}\, n_{\mathbf{k}-\mathbf{k}'} \tag{41}$$

This means that the change of $n_{\mathbf{k}}$ can be considered as a (in general infinite) sum of interactions where three different wave vectors are involved, namely $\mathbf{k}$, $\mathbf{k}'$ and their difference $\mathbf{k}-\mathbf{k}'$. Using this relation it can be proved that the following relation holds for a each triplet in Fourier space $\{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\}$ which fulfills $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|n_{\mathbf{k}_1}|^2}{2} &= \frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_3\times\mathbf{k}_1)\ \mathrm{Re}\left\{\phi_{\mathbf{k}_3}\, n^*_{\mathbf{k}_2}\, n^*_{\mathbf{k}_1}\right\}\\ &+\frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_1\times\mathbf{k}_2)\ \mathrm{Re}\left\{\phi^*_{\mathbf{k}_2}\, n_{\mathbf{k}_3}\, n^*_{\mathbf{k}_1}\right\}\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|n_{\mathbf{k}_2}|^2}{2} &= \frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_2\times\mathbf{k}_1)\ \mathrm{Re}\left\{\phi^*_{\mathbf{k}_1}\, n_{\mathbf{k}_3}\, n^*_{\mathbf{k}_2}\right\}\\ &+\frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_3\times\mathbf{k}_2)\ \mathrm{Re}\left\{\phi_{\mathbf{k}_3}\, n^*_{\mathbf{k}_1}\, n^*_{\mathbf{k}_2}\right\}\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{|n_{\mathbf{k}_3}|^2}{2} &= \frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_1\times\mathbf{k}_3)\ \mathrm{Re}\left\{\phi_{\mathbf{k}_1}\, n_{\mathbf{k}_2}\, n^*_{\mathbf{k}_3}\right\}\\ &+\frac{\mathbf{B}}{B^2}\cdot(\mathbf{k}_2\times\mathbf{k}_3)\ \mathrm{Re}\left\{\phi_{\mathbf{k}_2}\, n_{\mathbf{k}_1}\, n^*_{\mathbf{k}_3}\right\}\end{aligned} \tag{44}$$

Here the reality condition $n_{-\mathbf{k}} = n^*_{\mathbf{k}}$ has been used. Therefore, the sum of the three squared amplitudes is conserved

$$\frac{\partial}{\partial t}\left(\frac{|n_{\mathbf{k}_1}|^2}{2}+\frac{|n_{\mathbf{k}_2}|^2}{2}+\frac{|n_{\mathbf{k}_3}|^2}{2}\right) = 0 \tag{45}$$

This result allows the conclusion that any non-

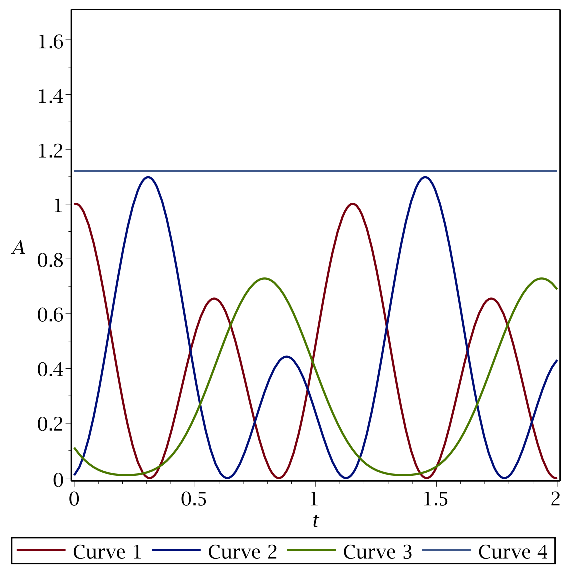

Figure 4: Example of the temporal evolution of three wave amplitudes $|n_{\mathbf{k}_1}|^2$, $|n_{\mathbf{k}_2}|^2$, $|n_{\mathbf{k}_3}|^2$ (curve 1, 2 and 3) interacting via $E\times B$-advection. The sum of the three contributions is also shown (curve 4).

zero mode $\mathbf{k}_1$ in the system interacts with all pairs $\{\mathbf{k}_2,\mathbf{k}_3\}$ of modes fulfilling the relation $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$ in a way that fluctuations are permanently exchanged and distributed in Fourier space, but always conserved. This is the non-linear mechanism behind the breaking of structures into smaller entities and vice versa.

I. Electromagnetic Effects, Shear Alfvén Waves, Tearing Modes

In this section the role of the electromagnetic pieces in the momentum equations are discussed. For this purpose we consider the following subset relating the electric potential ϕ and the magnetic potential $A_\parallel$.

$$\nabla_\parallel \phi + \frac{\partial A_\parallel}{\partial t} = 0 \tag{46}$$

$$\frac{m_i}{eB^2} \frac{\partial \nabla_\perp^2 \phi}{\partial t} = \frac{\nabla_\parallel J_\parallel}{en} \tag{47}$$

The first equation is a part of the parallel momentum balances and simply expresses that the parallel electric field is zero, i. e. $E_\parallel$=0. The second represents the response of the perpendicular electric field on a change in the parallel current to ensure quasineutrality. Now we consider small perturbations in the electric and magnetic field (labeled by a tilde) such that $\nabla_\parallel$ is undisturbed and governed by the dominant equilibrium field. Also the density n is assumed to be constant. Due to Ampére's law $\mu_0 J_\parallel = -\nabla_\perp^2 A_\parallel$ one obtains for the first equation Eq. 46

$$\mu_0 \frac{\partial \widetilde{J}_\parallel}{\partial t} = \nabla_\parallel \nabla_\perp^2 \widetilde{\phi} \tag{48}$$

and together with the second equation Eq. 47 it follows

$$\frac{\partial^2}{\partial t^2} \nabla_\perp^2 \widetilde{\phi} = V_A^2 \ \nabla_\parallel^2 \nabla_\perp^2 \widetilde{\phi} \tag{49}$$

$$\frac{\partial^2}{\partial t^2} \widetilde{J}_\parallel = V_A^2 \ \nabla_\parallel^2 \widetilde{J}_\parallel \tag{50}$$

where

$$V_A = \sqrt{\frac{B^2}{\mu_0 \, m_i \, n}} \tag{51}$$

These are wave equations for the perturbations in vorticity and current density, describing the traveling of the perturbations along the magnetic field with Alfvén speed V_A. If also a response of particle density via electron velocity is included and back-reaction via pressure gradient in the parallel electron momentum balance is taken into account, the resulting (linear) perturbations are called drift-Alfvén waves. It is also to be noted that by adding the resistive term in Eq. 46, i. e. considering $E_\parallel = -\eta_\parallel J_\parallel$ instead of $E_\parallel$=0, and taking into account the magnetic field fluctuations in the parallel gradient $\nabla_\parallel$ the two equations Eq. 46 and Eq. 47 describe a tearing mode, an important resistive instability [8]

J. Sound Waves

If one considers now a particular situation where the dynamics is governed by parallel motion

$$\frac{\partial n}{\partial t} = -\nabla_\parallel (n v_\parallel) \tag{52}$$

and - by neglect of electron mass - Ohm's law can be reduced to

$$\begin{aligned} -\nabla_\parallel p_e + e\, n\, \eta_\parallel \, J_\parallel - \alpha\, n\, \nabla_\parallel T_e \\ + e\, n\, \nabla_\parallel \phi + e\, n\, \frac{\partial A_\parallel}{\partial t} = 0 \end{aligned} \tag{53}$$

Then one finds for the linearized parallel ion momentum equation

$$m_i\, n \left(\frac{\partial u_\parallel}{\partial t} + u_\parallel \, \nabla_\parallel u_\parallel \right) = -\nabla_\parallel \left(p_i + p_e \right) \tag{54}$$

If one also assumes that the parallel current density is zero, i. e. $J_\parallel = e\, n\, \left(u_\parallel - v_\parallel\right)$=0, the continuity equation becomes

$$\frac{\partial n}{\partial t} = -\nabla_\parallel (n u_\parallel) \tag{55}$$

and linearizing the equations about a stationary equilibrium with constant temperatures T_e and T_i, i. e. $n = \bar{n} + \widetilde{n}$ and $u_\parallel = \widetilde{u}_\parallel$ gives

$$\frac{\partial \widetilde{n}}{\partial t} = -\bar{n}\, \nabla_\parallel \widetilde{u}_\parallel \ , \quad \frac{\partial \widetilde{u}_\parallel}{\partial t} = -\frac{T_e + T_i}{m_i\, \bar{n}} \nabla_\parallel \widetilde{n} \tag{56}$$

Therefore the perturbations $\widetilde{n}$ and $\widetilde{u}_\parallel$ fulfill

$$\frac{\partial^2 \widetilde{n}}{\partial t^2} = c_s^2 \, \nabla_\parallel^2 \widetilde{n} \ , \quad \frac{\partial^2 \widetilde{u}_\parallel}{\partial t^2} = c_s^2 \, \nabla_\parallel^2 \widetilde{u}_\parallel \tag{57}$$

where $c_s = \sqrt{(T_e + T_i)/m_i}$ is the sound speed. Like the Alfvén waves this is a wave like motion of perturbations along the magnetic field lines.

K. Curvature Effects

In the vorticity equation Eq. 9 and the continuity equation Eq. 10 terms containing the divergence of the perpendicular electron velocity $\mathbf{v}_\perp$ appear. They can introduce an important kind of dynamics which we want to study by considering the subset

$$\frac{\partial n}{\partial t} = -n\, \boldsymbol{\nabla} \cdot \mathbf{v}_\perp \tag{58}$$

$$\frac{m_i}{eB^2} \frac{\partial \nabla_\perp^2 \phi}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{v}_\perp + \boldsymbol{\nabla} \cdot \mathbf{V}_E \tag{59}$$

According to the discussions above it is assumed that the dominant contributions in $\mathbf{v}_\perp$ are given by the $E \times B$-velocity and the diamagnetic velocity, i.e.

$$\mathbf{v}_\perp = \frac{\mathbf{B} \times \boldsymbol{\nabla} \phi}{B^2} - \frac{T_e\, \mathbf{B} \times \boldsymbol{\nabla} n}{enB^2} \tag{60}$$

Here T_e=const. has been assumed, but this does not restrict the conclusions to be drawn from this illustrative example. It can be seen that for $\mathbf{B}$=const., i. e. for a homogeneous magnetic field, the divergence $\boldsymbol{\nabla} \cdot \mathbf{v}_\perp$ vanishes. This means that the perpendicular electron motion is incompressible for a homogeneous magnetic field. But in a curved magnetic field the motion becomes compressible and a coupling between the density n and the vorticity $\nabla_\perp^2 \phi$ shows up. To continue the exercise we have to assume a concrete magnetic field structure. A reasonable choice is the standard magnetic field

$$\mathbf{B} = \frac{B_0}{qR}\,\mathbf{e}_\theta + \frac{B_0 R_0}{R^2}\,\mathbf{e}_\varphi \tag{61}$$

where θ is the poloidal angle, φ the toroidal angle, $R{=}R_0 + r\,\cos\theta$, with major radius R_0 and minor radius r. The factor $q{=}q(r)$ is the so-called pitch parameter. This represents a twisted toroidal magnetic field with nested concentric circular flux surfaces and a magnetic field strength B_0 at the magnetic axes. Evaluating the divergences is now straightforward but cumbersome. We will just quote the result for the limit of high aspect ratio ($r/R_0{\to}0$) and define the curvature operator $\mathcal{K}$.

$$\begin{aligned}\mathcal{K}(f) &\equiv \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{B}\times\boldsymbol{\nabla} f}{B^2}\right)\\ &\approx -\frac{2}{B_0 R_0}\left(\cos\theta\,\frac{1}{r}\frac{\partial f}{\partial\theta} + \sin\theta\,\frac{\partial f}{\partial r}\right)\end{aligned} \tag{62}$$

One obtains

$$\frac{\partial n}{\partial t} = -n\,\mathcal{K}(\phi) + \frac{T_e}{e}\,\mathcal{K}(n) \tag{63}$$

$$\frac{m_i}{eB^2}\,\frac{\partial \nabla_\perp^2 \phi}{\partial t} = \frac{T_e}{en}\,\mathcal{K}(n) \tag{64}$$

Now consider small perturbations of the form

$$\widetilde{n} = \sum_m \widetilde{n}_m\, e^{i\,m\theta} \quad , \quad \widetilde{\phi} = \sum_m \widetilde{\phi}_m\, e^{i\,m\theta} \tag{65}$$

and linearize the equations (also the term $\sim\mathcal{K}(n)$ in the density equation is neglected here, because it is not relevant for our considerations)

$$\begin{aligned}\frac{\partial \widetilde{n}_m}{\partial t} = \frac{i\,n}{R_0 B_0}&\left(\frac{m-1}{r}\,\widetilde{\phi}_{m-1} + \frac{m+1}{r}\,\widetilde{\phi}_{m+1}\right.\\ &\left.-\frac{\partial\widetilde{\phi}_{m-1}}{\partial r} + \frac{\partial\widetilde{\phi}_{m+1}}{\partial r}\right)\end{aligned} \tag{66}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial^2\widetilde{\phi}_m}{\partial r^2} + \frac{1}{r}\,\frac{\partial\widetilde{\phi}_m}{\partial r} - \frac{m^2}{r^2}\,\widetilde{\phi}_m\right)\\ &= -\frac{i\,T_e B_0}{m_i n R_0}\left(\frac{m-1}{r}\,\widetilde{n}_{m-1} + \frac{m+1}{r}\,\widetilde{n}_{m+1}\right.\\ &\qquad\left.-\frac{\partial\widetilde{n}_{m-1}}{\partial r} + \frac{\partial\widetilde{n}_{m+1}}{\partial r}\right)\end{aligned} \tag{67}$$

The result is that the curvature of the magnetic field introduces an additional coupling between a mode $\widetilde{n}_m$ and the side bands $\widetilde{\phi}_{m\pm1}$ and vice versa. Actually this leads to the so-called ballooning instability [9] which is located at the low field side of the tokamak, reflecting that a toroidal configuration is not symmetric anymore with respect to the poloidal angle. An important special case of this interaction is to be mentioned. If one assumes that the only perturbations present are

$$\widetilde{n} = \widetilde{n}_*(r)\,\sin\theta \quad , \quad \widetilde{\phi} = \widetilde{\phi}_0(r) \tag{68}$$

one obtains for the sinusoidal density fluctuation

$$\frac{\partial\widetilde{n}_*}{\partial t} = \frac{2n}{B_0 R_0}\,\frac{\partial\widetilde{\phi}_0}{\partial r} \tag{69}$$

and for axisymmetric component of the electric potential

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial\widetilde{\phi}_0}{\partial r}\right)\right] = -\frac{T_e m_i B_0}{n R_0}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\widetilde{n}_*)\right] \tag{70}$$

Assuming $n{\approx}$const. one can integrate the second equation with respect to r and it follows

$$\frac{\partial}{\partial t}\,\frac{\partial\widetilde{\phi}_0}{\partial r} = -\frac{T_e B_0}{m_i n R_0}\,\widetilde{n}_* \tag{71}$$

By inserting Eqs. 69 and 71 into each other one obtains

$$\frac{\partial^2\widetilde{n}_*}{\partial t^2} = -\omega_{\mathrm{GAM}}^2\,\widetilde{n}_* \tag{72}$$

$$\frac{\partial^2}{\partial t^2}\,\frac{\partial\widetilde{\phi}_0}{\partial r} = -\omega_{\mathrm{GAM}}^2\,\frac{\partial\widetilde{\phi}_0}{\partial r} \tag{73}$$

The solutions are oscillations with frequency

$$\omega_{\mathrm{GAM}} = \sqrt{\frac{2T_e}{m_i R_0^2}} \tag{74}$$

This oscillation of the sinusoidal component of the density and the radial derivative $\partial\widetilde{\phi}_0/\partial r$ is called the Geodesic Acoustic Mode (GAM) and has been observed in many tokamak experiments [10]. The derivative $\partial\widetilde{\phi}_0/\partial r$ actually represents a homogeneous poloidal flow in the plasma which is known as the Zonal Flow.

L. Résumé I

In the last sections we have paid attention to particular pieces of the basic set of model equations Eqs. 9-14. These pieces represent limiting cases of the full dynamics represented by the complete model. By this separation several isolated processes could be identified like drift waves (Sec. II. D), the dissipative instability (Sec. II. F), Alfvén waves (Sec. II. I), sound waves (Sec. II. J) and GAM oscillations (Sec. II. K). In the full model and in reality all these effects appear simultaneously and interact in a complicated way. Sometimes a single effect might be dominant, but usually they have to be considered all together to obtain a consistent picture. Of course there are even more important effects hidden in the set of model equations or even more complete models not elucidated here. But the discussion of such effects can be conducted by similar reduction techniques presented here. We conclude this section by a rough estimate of the time scales related to the mechanisms mentioned above. For this purpose we compare the time scales of the drift wave motion $\tau_*^{-1}\sim k_\perp v_*$, the dissipative instability $\tau_\alpha^{-1}\sim \omega_*^2/\alpha$, the Alfvén waves $\tau_A^{-1}\sim k_\| V_A$, the sound waves $\tau_s^{-1}\sim k_\| c_s$ and the GAM oscillations $\tau_{\rm GAM}\sim \sqrt{2}\, c_s/R_0$ and the diffusive time scale according to the mixing length estimate $\tau_D\sim k_\perp^2\, a^2\, \tau_\alpha$ for the dissipative instability. Using realistic estimates $k_\perp \rho_s\sim 1$, $k_\| R_0\sim 1$, $\partial n_0/\partial r \sim n_0/a$, where a is the minor radius of the tokamak and R_0 its major radius this gives for T_e=T_i=100 eV, n_0=10^{19} m^{-3}, a=0.5 m, R_0=1.75 m, m_i=2 m_p and B=1 T

$$\tau_* \sim 7.2\cdot 10^{-6}\,\mathrm{s} \quad , \quad \tau_\alpha \sim 8.6\cdot 10^{-4}\,\mathrm{s} \tag{75}$$

$$\tau_A \sim 3.6\cdot 10^{-7}\,\mathrm{s} \quad , \quad \tau_s \sim 2.5\cdot 10^{-5}\,\mathrm{s} \tag{76}$$

$$\tau_{\rm GAM} \sim 1.7\cdot 10^{-5}\,\mathrm{s} \quad , \quad \tau_D \sim 1.0\cdot 10^{2}\,\mathrm{s} \tag{77}$$

Of course these are rough estimates and the precise values can differ strongly for different plasma parameters. But it is typical that the Alfvén wave motion is the fastest process and that the time scales of the different effects cover a range of several orders of magnitude. The same can be concluded for the spatial scales hidden in the complete plasma transport model. This has important consequences for the practical computation of plasma transport and turbulence. The necessity to resolve very short and very large temporal and spatial in a single model scales makes it an enormous challenge to develop appropriate numerical methods for an efficient use of computers available.

III. TURBULENT TRANSPORT MODELLING

Even though the basic model defined by Eqs. 9-14 is not complete and misses certain important physical effects, it would be desirable to solve at least this reduced set in detail and without any approximation. Unfortunately the large range of temporal and spatial scales mentioned in the last section makes it very difficult to obtain results for realistic tokamak device parameters and operational regimes in an acceptable time. High resolution grids and a huge number of small enough time steps would be needed in a numerical computation. To make it worse, the implementation of complicated magnetic field geometries, e. g. including X-points, is an additional challenge for analytical and numerical methods. A first order workaround often used is the splitting of time scales and the restriction of the model to a certain range of dynamics. Quite often it is useful to consider turbulent fluctuations only and to consider the large scale and slow dynamics as quasistationary. This reduces simulation run time because the slow processes do not have to taken into account. The need to cope with such requirements led to the derivation of several models with different content especially designed for particular scenarios, plasma devices and parameter regimes. It is not possible to compare them all in a short tutorial. But in the next sections we present an example of how a model reduction can be conducted and checked in a systematic way using a scale separation and an appropriate energy theorem.

A. The Problem of Setting up a Consistent Turbulence Model

In developing a model suitable for numerical solution we require that it is

- reasonably appropriate for the physics problem to be studied with respect to dominant transport mechanisms and geometry
- numerically tractable
- allowing a simulation in acceptable run time
- meeting the requirement of energetic consistency

In particular the last point is sometimes missed and model equations which might be reasonable and beautifully simple suffer from artificial effects due to inconsistencies. In the next section a practical example is presented. The four-field-model has been used often in turbulence studies and for this model the principles of energetic consistency can be illustrated by simple means.

B. Example: Four-Field-Model in Toroidal Geometry

In a first step we simplify the model Eqs. 9-14 by assuming T_i=0, T_e=const., and by neglecting most of the parallel advection. Then one obtains for the continuity equation

$$\begin{aligned}\frac{\partial n}{\partial t} &+ \mathbf{V}_E\cdot\boldsymbol{\nabla} n = -\nabla_\| \left(n\, v_\|\right)\\ &-n\,\boldsymbol{\nabla}\cdot\mathbf{V}_E - n\,\boldsymbol{\nabla}\cdot\mathbf{v}_* \quad ,\end{aligned} \tag{78}$$

and for Ohm's law

$$\frac{\partial A_\|}{\partial t} = \frac{T_e}{en}\,\nabla_\| n - \nabla_\|\phi - \eta_\|\, J_\| \tag{79}$$

The equation for the conservation of parallel ion momentum reduced by the use of Ohm's law Eq. 79 and reads

$$\frac{\partial u_\|}{\partial t} + \mathbf{V}_E\cdot\boldsymbol{\nabla} u_\| = -\frac{T_e}{m_i\, n}\,\nabla_\| n \tag{80}$$

and the vorticity equation is

$$\frac{m_i}{eB^2}\left(\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \mathbf{V}_E \cdot \boldsymbol{\nabla} \nabla_\perp^2 \phi\right) = \frac{\nabla_\parallel J_\parallel}{e\,n} - \frac{\boldsymbol{\nabla}\cdot(n\,\mathbf{v}_*)}{n} \tag{81}$$

Now a partly linearization with respect to fluctuations is performed. Except for n the background pieces of ϕ, $u_\parallel$, and $A_\parallel$ are assumed to be zero, i.e.

$$n = \bar{n} + \widetilde{n}\,,\ \phi = \widetilde{\phi}\,,\ u_\parallel = \widetilde{u}_\parallel\,,\ A_\parallel = \widetilde{A}_\parallel \tag{82}$$

where $\bar{n}$ is a constant density. This defines a four-field-model for fluctuations

$$\frac{d\widetilde{n}}{dt} = -\widetilde{\mathbf{V}}_E \cdot \boldsymbol{\nabla}\bar{n} + \frac{\nabla_\parallel \widetilde{J}_\parallel}{e} - \bar{n}\,\nabla_\parallel \widetilde{u}_\parallel - \bar{n}\,\mathcal{K}(\widetilde{\phi}) + \frac{T_e}{e}\,\mathcal{K}(\widetilde{n}) \tag{83}$$

$$\frac{\partial \widetilde{A}_\parallel}{\partial t} = \frac{T_e}{e\,\bar{n}}\,\nabla_\parallel \widetilde{n} - \nabla_\parallel \widetilde{\phi} - \eta_\parallel\,\widetilde{J}_\parallel \tag{84}$$

$$\frac{d\widetilde{u}_\parallel}{dt} = -\frac{T_e}{m_i\,\bar{n}}\,\nabla_\parallel \widetilde{n} \tag{85}$$

$$\frac{m_i\,\bar{n}}{B^2}\,\frac{d\widetilde{w}}{dt} = \nabla_\parallel \widetilde{J}_\parallel + T_e\,\mathcal{K}(\widetilde{n}) \tag{86}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \widetilde{\mathbf{V}}_E \cdot \boldsymbol{\nabla} \quad , \quad \nabla_\parallel = \frac{\mathbf{B} + \widetilde{\mathbf{B}}}{B} \cdot \boldsymbol{\nabla} \tag{87}$$

$$\widetilde{w} = \nabla_\perp^2 \widetilde{\phi} \quad , \quad \mu_0\,\widetilde{J}_\parallel = -\nabla_\perp^2 \widetilde{A}_\parallel \tag{88}$$

Notice that a radial derivative of $\bar{n}$ is taken into account. This is a second parameter of the model in addition to the density $\bar{n}$ itself. Both quantities are taken into account as constants to keep the framework as simple as possible (of course $\bar{n}$=const. excludes a finite gradient $\partial\bar{n}/\partial r$ when taken accurately). Notice also that the parallel derivative contains the fluctuating magnetic field $\widetilde{\mathbf{B}}$ related to the magnetic potential $\widetilde{A}_\parallel$. This is approximately given by

$$\widetilde{\mathbf{B}} = -\frac{\mathbf{B} \times \boldsymbol{\nabla}\widetilde{A}_\parallel}{B} \tag{89}$$

Despite the approximations needed to derive the four-field-model it still contains enough physics to describe reasonably the drift-Alfvén turbulence in the edge region of tokamak plasmas. Applications of this model and similar or even more reduced variants have been reported in Refs. [11–20], to mention only a few and without claiming to be exhaustive.

C. Energetics of the Four-Field-Model

The difficulties in the derivation of the four-field-model of the last section gives rise to the question to what extent the resulting model equations are still realistic and appropriate for the problem to be studied. A very powerful and useful method to get some insight into the particular features of the simplified model found by certain manipulations is the analysis of an energy theorem. This means that the desirable property of energy conservation is still present in the simplified model. For this purpose an appropriate energy functional has to be found. For the example of the four-field-model this is the energy density

$$\widetilde{U} = \frac{m_i\,\bar{n}}{2}\,\widetilde{V}_E^2 + \frac{m_i\,\bar{n}}{2}\,\widetilde{u}_\parallel^2 + \frac{\widetilde{B}^2}{2\mu_0} + \frac{\bar{n}T_e}{2}\,\frac{\widetilde{n}^2}{\bar{n}^2} \tag{90}$$

Using appropriate boundary conditions the temporal change of the different contributions integrated over the entire computational volume is given by

$$\frac{\partial}{\partial t}\int \frac{m_i\,\bar{n}}{2}\,\widetilde{V}_E^2\,dV = -\frac{m_i\,\bar{n}}{B^2}\int \widetilde{\phi}\,\frac{\partial \nabla_\perp^2 \widetilde{\phi}}{\partial t}\,dV = -\int \widetilde{\phi}\,\nabla_\parallel \widetilde{J}_\parallel\,dV - \int T_e\,\widetilde{\phi}\,\mathcal{K}(\widetilde{n})\,dV \tag{91}$$

$$\frac{\partial}{\partial t}\int \frac{m_i\,\bar{n}}{2}\,\widetilde{u}_\parallel^2\,dV = m_i\bar{n}\int \widetilde{u}_\parallel\,\frac{\partial \widetilde{u}_\parallel}{\partial t}\,dV = -\int T_e\,\widetilde{u}_\parallel\,\nabla_\parallel \widetilde{n}\,dV \tag{92}$$

$$\frac{\partial}{\partial t}\int \frac{\widetilde{B}^2}{2\mu_0}\,dV = \int \widetilde{J}_\parallel\,\frac{\partial \widetilde{A}_\parallel}{\partial t}\,dV = \int \frac{T_e}{e\,\bar{n}}\,\widetilde{J}_\parallel \nabla_\parallel \widetilde{n}\,dV - \int \widetilde{J}_\parallel \nabla_\parallel \widetilde{\phi}\,dV - \int \eta_\parallel\,\widetilde{J}_\parallel^2\,dV \tag{93}$$

$$\frac{\partial}{\partial t}\int \frac{\bar{n}T_e}{2}\,\frac{\widetilde{n}^2}{\bar{n}^2}\,dV = \int \frac{T_e}{\bar{n}}\,\widetilde{n}\frac{\partial \widetilde{n}}{\partial t}\,dV = -\int T_e\,\widetilde{n}\,\widetilde{\mathbf{V}}_E \cdot \frac{\boldsymbol{\nabla}\bar{n}}{\bar{n}}\,dV + \int \frac{T_e}{e\bar{n}}\,\widetilde{n}\,\nabla_\parallel \widetilde{J}_\parallel\,dV - \int T_e\,\widetilde{n}\,\nabla_\parallel \widetilde{u}_\parallel\,dV - \int T_e\,\widetilde{n}\,\mathcal{K}(\widetilde{\phi})\,dV \tag{94}$$

All contributions where the fluctuations are advected with the $E{\times}B$-velocity $\widetilde{V}_E$ vanish. Strictly speaking, these contributions vanish exactly only if $\boldsymbol{\nabla}\cdot\widetilde{\mathbf{V}}_E$=0, i. e. if $\widetilde{V}_E$ is incompressible. But one can consider the corrections due to compressibility for these terms to be small if the fluctuations are small. For the other contributions it can be proved that the following relations hold

$$\int \widetilde{f}\,\nabla_\parallel \widetilde{g}\,dV = -\int \widetilde{g}\,\nabla_\parallel \widetilde{f}\,dV \tag{95}$$

$$\int \widetilde{f}\,\mathcal{K}(\widetilde{g})\,dV = -\int \widetilde{g}\,\mathcal{K}(\widetilde{f})\,dV \tag{96}$$

Therefore, one gets for the temporal evolution of the total energy

$$\begin{aligned}\frac{\partial E}{\partial t} &= \int \frac{\partial U}{\partial t}\, dV \\ &= -\int \frac{T_e\, \widetilde{n}\, \widetilde{\mathbf{V}}_E \cdot \boldsymbol{\nabla} \bar{n}}{n}\, dV - \int \eta_\parallel\, \widetilde{J}_\parallel^2\, dV\end{aligned} \tag{97}$$

The second term in the second line is negative definite and represents a sink for the energy due to resistivity. The first term of the second line is usually positive and represents the source for the turbulence due to the gradient $\partial\bar{n}/\partial r$ in the background density. All other contributions cancel each other and this represents an energy conserving exchange of energy. For example the piece $\int T_e\, \widetilde{u}_\parallel\, \nabla_\parallel \widetilde{n}\, dV$ gives the channel of energy exchange between $\widetilde{n}$ and $\widetilde{u}_\parallel$ due to sound waves. Such a kind of energy theorem helps to check if a certain set of model equations is consistent and to get some insight into its dynamics. Also it is obvious that any modifications in the model should preserve the energetic consistency. If, e. g., the curvature term $\sim\mathcal{K}(\widetilde{\phi})$ in the continuity equation is removed from the model, the corresponding term $\sim\mathcal{K}(\widetilde{n})$ should be removed too. Otherwise an artifical sink/source of energy disturbs the dynamics of the model and leads to unpredictable results. The same is valid if some manipulations are done to a single equation of the model. Usually this needs also modifications in other equations to keep it consistent.

D. Résumé II

It has been shown that even a simplified model with limited applicability needs a careful derivation and inspection of the consistencies of approximations used. This has to be kept in mind as long as one is restricted to simplified approaches due to the lack of computational power or appropriate numerical methods for more general problems. To overcome the limitations several activities are still ongoing to improve the physics content and the numerical treatment, e. g. the extension to gyro-fluid models and the development of gyro-kinetic models and simulations (see, e .g. , Refs. [21–24] and references therein). Also the increase of computational power available, in particular the use of parallel computers, offers the improvement of accuracy in the modelling of turbulent plasma dynamics. Nevertheless, the basic concepts presented in this tutorial remain of importance also in interpretation and analysis of improved models and techniques.

REFERENCES

1. J. Weiland, “Collective Modes in Inhomogeneous Plasmas”, IoP Publishing, 2000.
2. S. I. Braginskii, *Rev. Plasma Phys.*, **1**, 205 (1965).
3. K. Miyamoto, “Plasma Physics for Nuclear Fusion”, MIT Press, Cambridge 1980, p. 128-133.
4. D. Reiser, *Phys. Plasmas*, **19**, 072317 (2012).
5. F. F. Chen, “Introduction to Plasma Physics and Controlled Fusion”, Plenum Press, New York 1984, p. 186.
6. A. Hasegawa and M. Wakatani, *Phys. Rev. Lett.*, **50**, 682 (1983).
7. J. W. Connor and O. P. Pogutse, *Plasma Phys. Control. Fusion*, **43**, 155 (2001).
8. G. Schmidt, “Physics of High Temperature Plasmas”, 2nd edition, Academic Press, New York 1979, p. 146.
9. R. J. Hastie *et al. Phys. Plasmas*, **10**, 4405 (2003).
10. Y. Xu *et al. Phys. Rev. Lett.*, **97**, 165003 (2006).
11. M. Wakatani, A. Hasegawa, *Phys. Fluids* **27**, 611 (1984).
12. B. D. Scott, *Phys. Fluids B* **4**, 2468 (1992).
13. F. L. Hinton and C. W. Horton, Jr., *Phys. Fluids* **14**, 116 (1971).
14. B. D. Scott, *Plasma Phys. Contr. Fusion* **39**, 1635 (1997).
15. D. Reiser and B. Scott, *Phys. Plasmas* **12**, 122308 (2005).
16. D. Reiser, *Phys. Plasmas* **14**, 082314 (2007).
17. P. Beyer *et al.*, *Phys. Plasmas* **5**, 4271 (1998).
18. V. Naulin, *Phys. Plasmas* **10**, 4016 (2003).
19. B. D. Scott, *New J. Phys.* **4**, 52 (2002).
20. B. Scott, *Phys. Plasmas* **8**, 447 (2001).
21. F. Jenko *et al.*, *Phys. Plasmas*, **7**, 1904 (2000).
22. B. D. Scott, *Phys. Plasmas* **14**, 102318 (2007)
23. B. D. Scott, *Phys. Plasmas* **17**, 112302 (2010)
24. J. Madsen , *Phys. Plasmas* **20**, 072301 (2013)