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Statical shakedown analysis with temperature-dependent yield condition

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SUMMARY

This paper describes a general statical approach for shakedown analysis of structures of perfectly plastic material using non-linear optimization. The developed methods may be implemented with any displacement-based finite element code. The temperature-dependence of the yield limit is taken into account in shakedown analysis. Temperature-dependent shakedown analysis of a pipe-junction and a thick tube are performed by different methods.

1. INTRODUCTION

Plastic failure modes cannot be assessed from the state of stress, such that the plastic design has to consider the characteristic development of plastic strains towards structural failure instead:

- Instantaneous collapse by unrestricted plastic flow at limit load.
- Incremental collapse by accumulation of plastic deformations over subsequent load cycles (ratchetting).
- Low Cycle Fatigue (LCF) by alternating plasticity.

The possible structural failure may be predicted in a detailed incremental plastic analysis. In the case of variable loaded structures the prediction of failure is based on extrapolation because a detailed analysis is prohibitively time consuming. To avoid this drawback the direct computation of the load carrying capacity is the objective of shakedown analyses. Shakedown analyses are implemented into the FEM program PERMAS employing a subspace technique which can handle large models [5], [9].

Although in general the thermo-mechanical properties of a material are temperature dependent, the dependence is more significant for the yield stress than for other properties such as Young's modulus and coefficient of thermal expansion. For instance, an ordinary carbon steel has a reduction in the yield stress of about 45% due to a temperature increase from $20°C$ to $375°C$. The objective of the present paper is to complete the analysis of perfectly plastic behavior by deriving an approximative method to include the temperature dependence of the yield limit. The temperature-dependent shakedown analysis is performed by a post-processing of the results of a temperature independent shakedown analysis.

2. THEOREMS OF ELASTIC SHAKEDOWN FOR PERFECTLY PLASTIC MATERIAL

Static theorems are formulated in terms of stress and define safe structural states giving an optimization problem for safe loads. The maximum safe load is the limit load avoiding collapse and the shakedown load avoiding ratchetting and LCF. The presentation is restricted to perfectly plastic material and no elastic failure modes are considered (i.e. no elastic buckling or high cycle fatigue).

The given structure V is assumed to be composed of material points denoted by their coordinate vectors $\mathbf{x} \in V$. For an elastic, perfectly plastic material plastic strains occur if a yield function f reaches the yield stress σ_y. The elastic domain is defined with the von Mises yield function F by:

$$F(\boldsymbol{\sigma}) \leq \sigma_y^2 \tag{1}$$

with the actual elastic-plastic stress response $\boldsymbol{\sigma}$. Material stability in the sense of Drucker's postulate is assumed, i.e. the yield surface is convex. To avoid the possibility of plastic failure the maximum possible plastic energy dissipation must be bounded for all points and instances. The stresses $\boldsymbol{\sigma}(t)$ are decomposed into fictitious elastic stresses $\boldsymbol{\sigma}^E(t) = \mathbf{E} : \boldsymbol{\varepsilon}(t)$ and residual stresses $\boldsymbol{\rho}$ resulting from plastic deformations by $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^E(t) + \boldsymbol{\rho}$. All residual stresses $\boldsymbol{\rho}$ generate a linear vector space

$$\mathcal{B} = \{\boldsymbol{\rho} \mid \mathrm{div}\boldsymbol{\rho} = \mathbf{0} \text{ in } V, \boldsymbol{\rho}\,\mathbf{n} = 0 \text{ on } \partial V_\sigma\}. \tag{2}$$

The time history of a load $\mathbf{P}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ with volume $\mathbf{q}(t)$ and surface loads $\mathbf{p}(t)$ is often not well-known. It can however usually be stated that the loads vary only within a certain convex set $\mathcal{L}$. The following theorem holds:

> *Static shakedown theorem (Melan):*
> If there exists a factor $\alpha > 1$ and a time-independent residual stress field $\bar{\boldsymbol{\rho}}(\mathbf{x})$ with $\int_V \bar{\boldsymbol{\rho}} : \mathbf{E} : \bar{\boldsymbol{\rho}}\, dV < \infty$, such that for all loads $\mathbf{P}(t) \in \mathcal{L}$
>
> $$F[\alpha\boldsymbol{\sigma}^E(\mathbf{x}, t) + \bar{\boldsymbol{\rho}}(\mathbf{x})] \leq \sigma_y^2 \qquad \forall\, \mathbf{x} \in V \tag{3}$$
>
> is satisfied, then the structure will shake down elastically under the given load domain $\mathcal{L}$.

The greatest value α_{SD} which satisfies the theorem is called *shakedown*-factor. The objective of shakedown analyses is to find bounds to the shakedown–factor α_{SD}. This leads to the convex optimization problem with a continuous set of restrictions

$$\begin{array}{lll} \max & \alpha & \\ \text{s. t.} & F[\alpha\boldsymbol{\sigma}^E(\mathbf{x}, t) + \bar{\boldsymbol{\rho}}(\mathbf{x})] \leq \sigma_y^2 & \bar{\boldsymbol{\rho}} \in \mathcal{B},\ \mathbf{x} \in V, \forall t \end{array} \tag{4}$$

which is reduced to a finite problem by FEM discretization. The solution α_{SD} of the problem (4) is unique even if the residual stress field $\bar{\boldsymbol{\rho}}$ is not in general unique. It is supposed that the load domain $\mathcal{L}$ is a convex polyhedron with the vertices $\mathbf{P}(k),\ k = 1, \ldots, NV$ (*load vertices*). Consequently, any load $\mathbf{P}(t) \in \mathcal{L}$ is given by a convex combination of the $\mathbf{P}(j),\ j = 1, \ldots, NV$.

$$\mathbf{P}(t) = \lambda_1(t)\mathbf{P}_1 + \ldots + \lambda_{NV}(t)\mathbf{P}_{NV},\ \ 0 \leq \lambda_j(t) \leq 1. \tag{5}$$

For the FEM the structure V is decomposed in NE finite elements with the NG Gaussian points $\mathbf{x}_i$. The restrictions of the optimum problem are validated only in the Gaussian points. The discretized shakedown analysis is given by

$$\begin{array}{lll} \max & \alpha & \\ \text{s.t.} & F[\alpha\boldsymbol{\sigma}_i^E(j) + \bar{\boldsymbol{\rho}}_i] \leq \sigma_y^2 & i \in [1, NG], j \in [1, NV],\ \ \bar{\boldsymbol{\rho}} \in \mathcal{B}. \end{array} \tag{6}$$

Imposing the yield condition at the Gauss points only, instead of continuously everywhere in V leads to a maximum α value which is an upper bound to the theoretical value α_{SD}, because of the less strict optimization problem. The time discretization (5) is without effects on the optimal value α of (6) due to the convexity of $\mathcal{L}$. The unknowns of the problem are α and the residual stresses $\bar{\boldsymbol{\rho}}_i$. For structures with NG Gaussian points one has to handle $1 + NG \times NC$ unknowns where NC is the number of stress components and $NG \times NV$ constraints. A method for handling such large–scale optimization problems for perfect plasticity is called *basis reduction technique* or *subspace iteration* [5] for the linear space of residual stresses $\mathcal{B}$. The number of unknowns are reduced in a subspace $\mathcal{B}^d \subset \mathcal{B}$ such that the final iterative convex optimization problem has only a few unknowns.

If the structure V is subjected to a thermal load T, and the yield stress σ_y changes during the loading cycle being the yield stress temperature-dependent $\sigma_y = \sigma_y(T)$. For simplicity of presentation it is assumed, that the form of the yield condition does not vary with temperature and only the yield stress depends on it. When considering the yield stress temperature-dependent, the lower bound property of the shakedown solution is assured if the yield function f

$$f = F(\boldsymbol{\sigma}) - \sigma_y(T). \tag{7}$$

is convex in the stress-temperature space [2]. The yield function $f(\boldsymbol{\sigma})$ provides a potential for the plastic strain rates (associated flow rule) [8]

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \quad \text{with} \begin{cases} \dot{\lambda} \geq 0, \quad \text{if } f = 0 \text{ and } \frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{\partial \sigma_y(T)}{\partial T}\dot{T} = 0 & \text{(loading)} \\ \dot{\lambda} = 0, \quad \text{if } f < 0 \text{ or } f = 0 \text{ and } \frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{\partial \sigma_y(T)}{\partial T}\dot{T} < 0 & \text{(unloading)} \end{cases} \tag{8}$$

This implies that, in special cases, plastic flow may occur (i.e. $\dot{\lambda} \geq 0$) even if all the stress components decrease $(\partial f/\partial \sigma)\dot{\boldsymbol{\sigma}} < 0$ during the loading [8]. The energy dissipated depends thus not only on the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ but also on the instantaneous temperature. Since the von Mises function $F(\boldsymbol{\sigma})$ is convex, $\sigma_y(T)$ is required to be concave or linearized for an appropriate lower or upper bound statement [1], [8], [12]. This condition may be satisfied by many metal and alloys for a rather wide range of T [2], [8]. The shakedown analysis by which only the vertices of $\mathcal{L}$ need to be considered is restricted to yield function which are convex in the stress-temperature space [2]. As an approximation the Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α_t are considered as temperature-independent.

Borino [2] considered a material model with internal variables obeying thermo-plastic yielding laws to introduce a consistent formulation of the shakedown load factor. In [12] Yan and Nguyen used a nonlinear programming method to solve the kinematic shakedown problem. In this approach the yield function is updated during the iteration and the obtained upper bound becomes an approximation if the yield stress function is convex. Khoi [7] derived a dual formulation for a linearized temperature dependent yield stress function.

Let the yield stress $\sigma_y(T)$ be defined by a concave, monotone decreasing function $0 \leq g(T) \leq 1$ and with the abbreviation $\sigma_0 = \sigma_y(T_0)$ for the ambient temperature T_0 and $g(T_0) = 1$ it is defined

$$\sigma_y(T) = g(T)\sigma_0. \tag{9}$$

The following sections introduces methods to calculate the temperature at initial yielding and at shakedown which do not need an extra FEM calculation. The temperatures are approximated only with the results of the shakedown analysis for the temperature independent yield stress σ_0 and the structure of the function g, such that different yield stress functions need only different simple transformations.

3.1. Initial yielding

For temperature dependent yield stress plasticity starts at $\alpha_e \mathbf{T}$ where α_e solves

$$\begin{aligned} \max \quad & \alpha \\ \text{s. t.} \quad & F[\alpha \boldsymbol{\sigma}_i^E(j)] \leq \sigma_y^2(\alpha T_{i,j}) \qquad i \in [1, NG], j \in [1, NV] \end{aligned} \tag{10}$$

with the elastic stress field $\boldsymbol{\sigma}^E(j)$ corresponding to the temperature $\mathbf{T}(j)$ at initial yielding for the temperature independent yield stress σ_0. From optimization theory [3] follows, that in the solution of (10) equality holds for at least one constraint, such that an approximation of the solution is given by

$$\alpha_e = \min \left\{ \alpha_e^{i,j} \;\middle|\; \alpha_e^{i,j} = \frac{\sigma_0 g(\alpha_e^{i,j} T_{i,j})}{\sqrt{F(\boldsymbol{\sigma}_i^E(j))}} , i \in [1, NG], j \in [1, NV] \right\}. \tag{11}$$

In general there is no proportional relation between the temperature $T_{i,j}$ and the corresponding stress $F(\boldsymbol{\sigma}_i^E(j))$. For a linear function $\sigma_y(T) = \sigma_0(1 - \lambda T)$ condition (11) can be written for all i, j as:

$$\alpha_e = \min_{i,j} \alpha_e^{i,j} = \min_{i,j} \frac{\sigma_0}{\sigma_0 \lambda T_{i,j} + \sqrt{F(\boldsymbol{\sigma}_i^E(j))}} \tag{12}$$

such that $\alpha_m = 1/(1 + \lambda T_{max})$ is an upper bound. If the location m of the highest stresses $\boldsymbol{\sigma}_m^E$ corresponds to the highest temperatures T_{max} the bound is best possible and it holds $\alpha_m = \alpha_e$.

3.2. Shakedown condition

The static shakedown theorem for temperature-dependent yield stress is given in [8], such that the shakedown analysis is formulated by

$$\begin{aligned} &\max \quad \alpha \\ &\text{s.t.} \quad F[\alpha \boldsymbol{\sigma}_i^E(j) + \boldsymbol{\rho}_i] \le \sigma_y^2(T) \qquad i \in [1, NG], j \in [1, NV], \;\; \boldsymbol{\rho} \in \mathcal{B}. \end{aligned} \tag{13}$$

In practical applications $\sigma_y(T)$ corresponds to the highest possible temperature T_{max} in $\mathcal{L}$ to guarantee a conservative formulation of (13). In this case all constraints of (13) have an unique right hand side, and the left hand side is similar to the temperature-independent formulation. Therefore it is reasonable to transform (13) with a concave, monotone decreasing function $0 \le g(T) \le 1$ into

$$\begin{aligned} &\max \quad \alpha \\ &\text{s.t.} \quad F[\alpha(\hat{\boldsymbol{\sigma}}_i^E(j) + \hat{\boldsymbol{\rho}}_i)] \le g^2(\alpha T_{i,j}^s)\sigma_0^2 \qquad i \in [1, NG], j \in [1, NV], \hat{\boldsymbol{\rho}} \in \mathcal{B} \end{aligned} \tag{14}$$

The stress fields $\hat{\boldsymbol{\sigma}}^E$ and $\hat{\boldsymbol{\rho}}$ are the solutions of the temperature-independent problem with the yield stress σ_0 and the temperature $\mathbf{T}^s$. For temperature dependent yield stress shakedown is limited to the temperature $\alpha_s \mathbf{T}^s$ where α_s solves (14). From optimization theory [3] follows, that in the solution of (14) equality holds for at least one constraint, such that an approximation of the solution is given by

$$\alpha_s = \min \left\{ \alpha_s^{i,j} \;\middle|\; \alpha_s^{i,j} = \frac{\sigma_0 g(\alpha_s^{i,j} T_{i,j})}{\sqrt{F(\hat{\boldsymbol{\sigma}}_i^E(j) + \hat{\boldsymbol{\rho}}_i)}} , i \in [1, NG], j \in [1, NV] \right\}. \tag{15}$$

This is solved using the FORTRAN subroutine RPOLY for finding the zeros of a real polynomial [6].

3.3. Example

To demonstrate the thermal actions in structural safety an elementary example is analyzed [8]. A straight bar, clamped at both ends at the temperature $T = T_0$ is heated and cooled cyclically within the limits $T_0 \le T \le T_1$. It is assumed that the bar is sufficiently thick to prevent buckling and that the temperature dependence of the yield stress $\sigma_y(T)$ is a monotone decreasing function. The temperature $T_0 = 0$ is fixed and the task is to evaluate the highest temperature T_1 to prevent alternating plasticity. As clamping makes the bar inextensible the total strain must vanish and with Young's modulus E, the thermal expansion coefficient α_t and $\Delta T = T - T_0 = T$ it holds $\sigma = -E\alpha_t \Delta T - E\varepsilon^p = \sigma^E(T) + \rho$.

3.4. Analytical solution

The maximum temperature T_E which does not cause plastic deformations (initial yielding) is the solution of $|\sigma^E(T)| = \sigma_y(T)$. If shakedown takes place there must exist a time-independent residual stress $\bar{\rho}$ satisfying the condition $|\sigma(T)| \leq \sigma_y(T)$ for all temperatures $0 \leq T \leq T_S$. An appropriate $\bar{\rho}$ exists only if the following condition holds:

$$\max\left[-\sigma_y(T) + E\alpha_t T\right] \leq \min\left[\sigma_y(T) + E\alpha_t T\right], \qquad 0 \leq T \leq T_S. \tag{16}$$

The maximum is reached at T_S and the minimum at 0, such that $E\alpha_t T_S - \sigma_y(T_S) \leq \sigma_0$ and the following equations have to be solved depending on $\sigma_y(T)$

$$T_E = \frac{\sigma_y(T_E)}{E\alpha_t} \qquad \text{and} \qquad T_S = \frac{\sigma_0 + \sigma_y(T_S)}{E\alpha_t}. \tag{17}$$

For a linear function $\sigma_y(T) = \sigma_0(1 - \lambda T)$ from (17) follows

$$T_E = \frac{\sigma_0}{E\alpha_t + \lambda\sigma_0} \quad \text{and} \quad T_S = \frac{2\sigma_0}{E\alpha_t + \lambda\sigma_0} = 2T_E. \tag{18}$$

3.5. Application of the proposed method

With condition (11) and the initial yielding temperature T_E^0 for temperature independent yield stress σ_0 plasticity starts at $T_E = \alpha_e T_E^0$, such that with (16) it follows $T_S^0 = \frac{2\sigma_0}{E\alpha_t} = 2T_E^0$. The solution of (15) is given by

$$\begin{aligned} \alpha_s &= \min\left\{\alpha_j = \frac{\sigma_0 g(\alpha_j T_j)}{|-E\alpha_t T_j + \bar{\rho}|}, j = 1, 2\right\} \\ &= \min\left\{\alpha_1 = g(\alpha_1 T_0), \alpha_2 = g(\alpha_2 T_S^0)\right\} = \min\left\{1, \alpha_2 = g(\alpha_2 T_S^0)\right\}. \end{aligned} \tag{19}$$

such that the proposed method gives a lower bound to the analytic shakedown factor

$$T_S^l = \alpha_s T_S^0 = \frac{2\sigma_0 g(\alpha_s T_S^0)}{E\alpha_t} = \frac{2\sigma_y(T_S)}{E\alpha_t} \leq \frac{\sigma_0 + \sigma_y(T_S)}{E\alpha_t} = T_S. \tag{20}$$

For the linear case $\sigma_y(T) = \sigma_0(1 - \lambda T)$ it is

$$T_E = \frac{\sigma_0}{E\alpha_t + \lambda\sigma_0}, \quad \alpha_s = \frac{1}{\lambda T_S^0 + 1}, \quad T_S^l = \frac{2\sigma_0}{E\alpha_t + 2\lambda\sigma_0} < \frac{2\sigma_0}{E\alpha_t + \lambda\sigma_0} = T_S = 2T_E. \tag{21}$$

This shows, that the proposed method gives a lower bound to the analytic shakedown temperature. For example for a mild steel the corresponding material data are $\sigma_0 = 235MPa, E = 20.6 \cdot 10^4 MPa, \alpha_t = 2 \cdot 10^{-5} T^{-1}$ and $\lambda = 0.0004 T^{-1}$ [8], such that the proposed approximation results in an error of 2 %.

4. APPLICATIONS

The shakedown analyses are performed for perfectly plastic material with a temperature dependent yield stress. Two different yield stress functions are investigated. For a kind of 316L(N) steel the yield stress is written in the following explicit nonlinear form [12]

$$\sigma_y^a(T) = 230.65 - 0.5599T + 0.00096T^2 - 6 \times 10^{-7} T^3 \quad [N/mm^2]. \tag{22}$$

The yield stress function $f = F(\boldsymbol{\sigma}) - \sigma_y^a(T)$ is concave in the temperature range $0° - 533°$, such that the methods give approximations instead of strict bounds of the solution [12]. For a mild steel the yield stress function is written in the following explicit linear form [8]

$$\sigma_y^b(T) = 235(1 - 0.0004T) \quad [N/mm^2]. \tag{23}$$

The nonlinear function $\sigma_y^a(T)$ decreases much faster then the linear function $\sigma_y^b(T)$ in temperature. The other material parameters are chosen temperature-independent $E = 210000[N/mm^2]$, $\alpha_t = 2 \cdot 10^{-6} deg^{-1}$, $\nu = 0.3$. Two different applications are performed which show different failure modes, i.e. alternating plasticity and ratchetting.

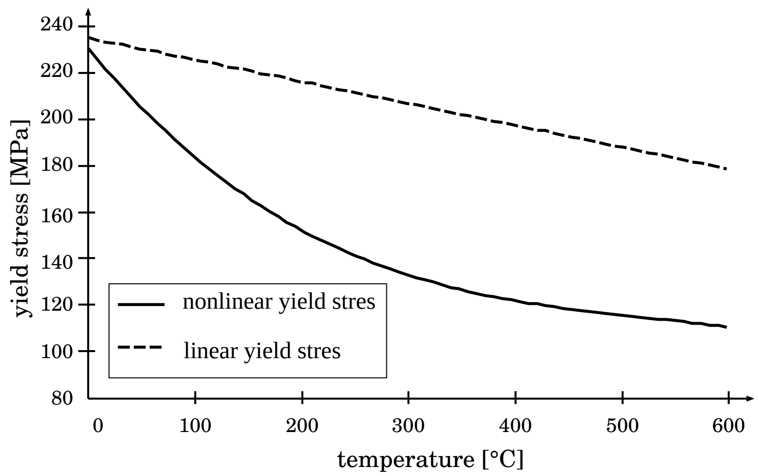

Figure 1: Different temperature dependent yield functions

4.1. Pipe-junction subjected to internal pressure and temperature loads

The dimensions are based on a pipe benchmark problem of PERMAS [9], the FE-mesh and the dimensions are given in Fig. 2. The pipe-junction is subjected to an internal reference pressure P_0 and inner reference temperature T_0 with a linear distribution to the zero outer temperature. P_0 and T_0 vary independently (two-parameter loading), i.e. the reference load domain $\mathcal{L}$ has four load vertices $\mathbf{P}(1) = (0, T_0),\ \mathbf{P}(2) = (P_0, T_0),\ \mathbf{P}(3) = (P_0, 0),\ \mathbf{P}(4) = (0, 0)$.

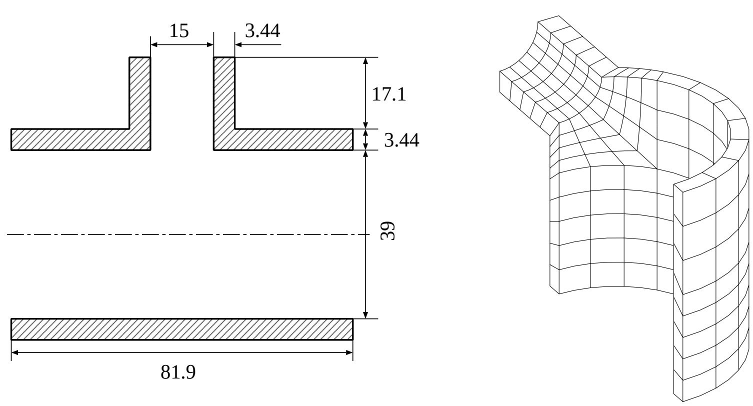

Figure 2: FE-mesh and dimensions of the pipe-junction

4.2. Thick tube subjected to constant internal pressure and temperature loads

The second application is a closed thick tube with inner radius $R_i = 10mm$ and outer radius $R_o = 20mm$ which is subjected to a constant inner pressure P_0 and a temperature field $T(r)$ with

$$T(r) = T_i \frac{\ln(R_o) - \ln(r)}{\ln(R_o) - \ln(R_i)}. \tag{24}$$

The tube is discretized by 5 axisymmetric 9-noded ring elements (QUAX9) over the thickness of the tube. The inner reference temperature T_0 varies, i.e. the load domain $\mathcal{L}$ has two load vertices $\mathbf{P}(1) = (P_0, 0),\ \mathbf{P}(2) = (P_0, T_0)$.

4.3. Comparison of direct and proposed method

In the direct method the yield stress function $\sigma_y(T)$ is updated during the iteration with the load factor α^k. All load vertices use the updated yield stress $\sigma_y(T_{max})$ with the highest temperature $T_{max} = \alpha^k T_0$ in the corresponding load domain $\alpha\mathcal{L}$. For the approximative solution the shakedown analysis is performed with the unique yield stress $\sigma_0 = \sigma_y(T_0)$. The results are post-processed with the proposed method.

Fig. 3 and 4 show the comparison of the direct and the proposed method for both applications. The temperature independent shakedown results are significantly reduced in the temperature dependent case, e.g. for $\sigma_y^a(T)$ the shakedown range for pure temperature load is halved. The shakedown limit of the pipe-junction is determined by alternating plasticity due to local stress concentrations at the inner nozzle corner [10]. The shakedown limit of the thick tube is determined by ratchetting if the pressure is predominant and by alternating plasticity elsewhere (horizontal part of the diagram) [11]. For both failure modes the results of the proposed method correspond well with the results of the direct method.

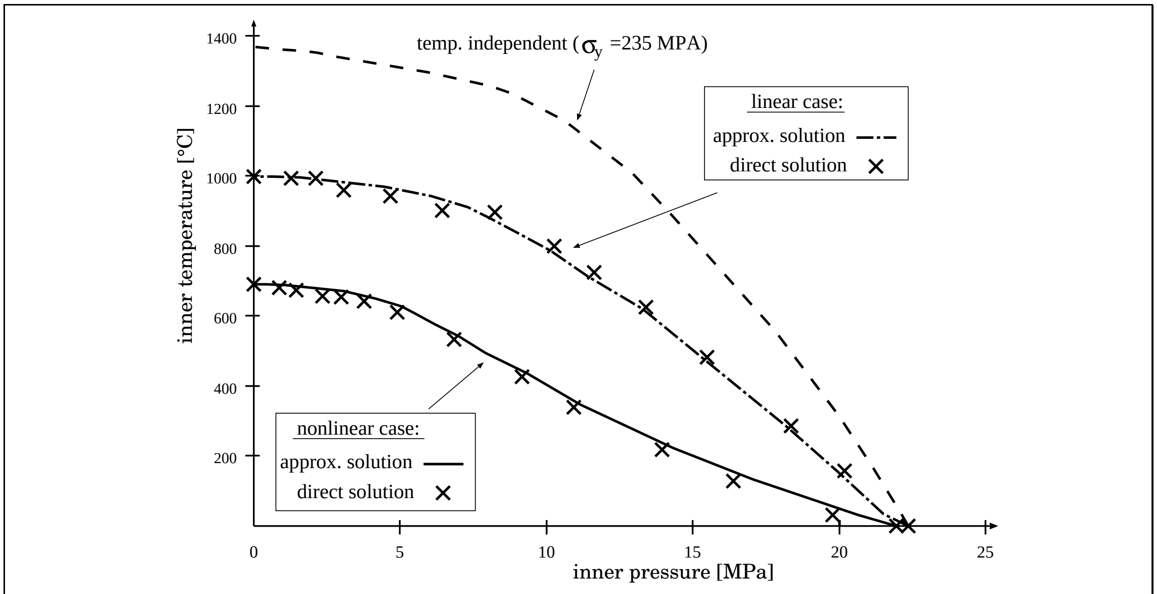

Figure 3: Comparison of direct and approximative solutions for the pipe-junction

5. CONCLUSION

The paper describes a general non-linear statical approach for shakedown analysis of structures of perfectly plastic material subjected to thermal and mechanical loading. The influence of the temperature dependence of the yield stress of the material on the shakedown behaviour is investigated. As

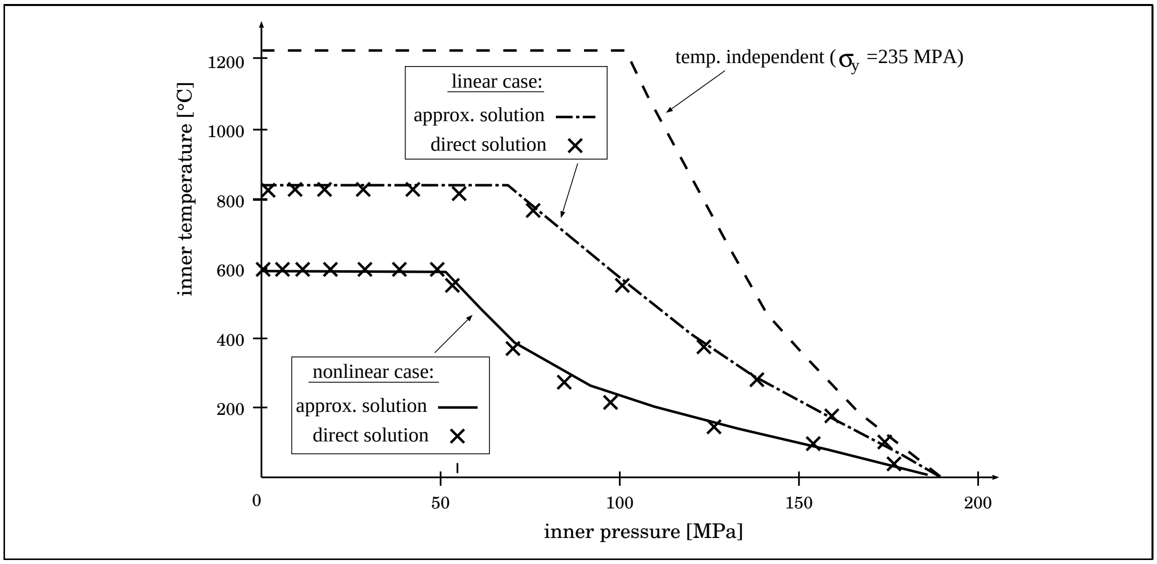

Figure 4: Comparison of direct and approximative solutions for the tube

the yield stress depends on the current temperature different possible updating schemes are tested in the shakedown analysis of a pipe junction. These schemes are compared to the results of a direct post-processing of the temperature independent shakedown analysis and the results are similar. Obviously the proposed method has the advantage, that only one temperature independent shakedown analysis has to be performed, for all possible temperature-dependencies.

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