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Frank Kalina

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On the quantum-field theoretical approach to the light-matter interaction in solid state physics: A critical analysis

Frank Kalina

Institut für Schichten und Grenzflächen (ISG 1), Forschungszentrum Jülich, 52425 Jülich, Germany

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Abstract

The following analysis has been done in part in collaboration with W. Schäfer (Forschungszentrum Jülich, Institut für Schichten und Grenzflächen (ISG1)) and consists of eleven parts. At the very beginning of our work, a detailed description of the optical properties of an $N-$level quantum dot coupled to an unlimited number of phonon modes was the most important topic. Later on, however, our involved investigations lead to a critical analysis of a basic element of the present theory of solid state physics, i.e., *the standard coupling of the electron-hole system to the quantized electromagnetic field.* Clearly, such a surprising and unexpected development requires an explanation. For this reason, a short summary of the comprehensive work and some introductory remarks are indispensable.

In section I, we are concerned with the general equations of motion of an electron-hole system, interacting with phonons and photons. Here, the light-matter interaction operator contains both the coupling of the electron-hole system to the *classical exciting light field* as well as the coupling to the *quantized emitted light field.* The time evolution of the photon number $\langle a^+a\rangle(t)$ is of particular interest, as it is directly related to the phenomenon of *photoluminescence.* It is well known that the corresponding equation of motion

leads to an infinite system of coupled equations for the *photon-assisted expectation values.* We discuss the mathematical structure of these equations up to an arbitrary order with respect to the particle-photon interaction. It should be emphasized that these general results are of basic importance for further conclusions and calculations, because both the linear part as well as the quadratic part of the light-matter interaction operator have been taken into account.

A *numerical approach* to quantum-field theoretical hierarchy equations is possible only in combination with some basic approximations. A complete list of those approximations which have been introduced previously in our own calculations can be found in section I.B. We stress explicitly that most of these approximations are *standard approximations* which are often used in the literature. Later on, however, it became clear that these approximations are in complete contradiction to the underlying theory itself (details will be explained below).

At the very beginning of our work, the application of the standard approximations mentioned above was crucial. It turned out that an *electron-hole two level system coupled to a single photon mode (with fixed energy* $\hbar\Omega$*) and interacting with an unlimited number of phonon modes* is well suited for numerical investigations under these circumstances. Here, the dispersion of the corresponding phonon branch was taken to be constant (Einstein approximation).

In section I.C, we derive the explicit equations of motion for the *phonon-assisted expectation values* in the framework of the two-level system under consideration. The calculation of the particle numbers, the polarization function and the optical absorption spectrum is of particular interest. If the power of the emitted light field is much smaller than the power of the exciting light field, one obtains *five infinite systems* of coupled equations to calculate the expectation values $\langle\hat{O}\rangle$, $\langle\hat{O}e^+e\rangle$, $\langle\hat{O}h^+h\rangle$, $\langle\hat{O}e^+h^+\rangle$ and $\langle\hat{O}e^+h^+he\rangle$ ($\hat{O}$ decribes the phononic part). It turns out that these equations can be reduced further provided that the initial state is the vacuum state and the damping constants are taken to be zero. Under these circumstances, only *three infinite systems* of coupled equations are necessary to calculate the expectation values $\langle \hat{O} e^+ e \rangle, \langle \hat{O} h h^+ \rangle$ and $\langle \hat{O} e^+ h^+ \rangle$. Moreover, we obtain the *exact* relation $\langle \hat{O} e^+ h^+ h e \rangle(t) = \langle \hat{O} e^+ e \rangle(t)$. These equations prove to be extremely useful for various numerical tests.

In order to solve the Heisenberg equations of motion, we have performed an involved numerical study up to $12-$th order in the particle-phonon coupling constant for different confinement parameters in a realistic range. The numerical calculation of the optical absorption spectrum $A(\omega)$ was based on the well-known formula $A(\omega) = Im[P(\omega)/E(\omega)]$ which is often used in the literature. Actually, we obtained excellent agreement with the *spectral properties* of the underlying Hamiltonian. At this point, however, we have to call the reader's attention to another remarkable fact: In the limit of vanishing (phenomenological) damping constants, a counter-example for the standard formalism does exist! Later on, we shall illustrate the apparent shortcoming of the quoted formula from above in more detail.

We have already mentioned above that the numerical calculation of the photon number $\langle a^+ a \rangle$ was mainly based on *standard approximations* (see again section I.B). These approximations lead to an infinite system of coupled equations for the photon-(and phonon-)assisted expectation values of type $\langle \hat{O} e^+ h^+ a \rangle$. Here, the time evolution is determined by the *phonon-assisted four-particle source terms* $\langle \hat{O} e^+ h^+ h e \rangle$. For a fixed time t_0 and for various excitation energies $\hbar\omega$, a numerical calculation of the photon number $\langle a^+ a \rangle(t_0, \omega)$ makes sense *even in the limit of vanishing (phenomenological) damping constants* and delivers an interesting result: If the system parameters (and also t_0) are chosen appropriately, the spectrum $\langle a^+ a \rangle(t_0, \omega)$ consists of highly resolved peaks with a finite line width! Clearly, such a result is in marked contrast to the basic features of the optical absorption spectrum $A(\omega)$: Under the conditions from above, $A(\omega)$ can only be represented by a *distribution*, i.e., we obtain *vanishing line widths* in the discrete part of the spectrum.

In view of our careful numerical analysis, there is no doubt that the obtained result for $\langle a^+a\rangle(t_0,\omega)$ is in fact correct; moreover, it is in excellent agreement with the *spectral properties* of the underlying Hamiltonian. During our further investigations, however, it became clear that neither the approximations mentioned above nor *the standard particle-photon interaction operator itself* can provide an appropriate approach to the light-matter interaction in solid state physics (details will be explained immediately). These surprising findings demonstrate that some of the basic intrinsic properties of the physical system under consideration are not yet well understood. Consequently, further investigations are necessary to obtain a satisfactory theory which is in fact missing until now.

The following part of our considerations contains a summary of the details of our critical analysis. We emphasize in advance that all statements are based on *exact results* - and not on perturbational approaches. Some of our results have already been published in Refs.[1,2]. In the present work, the reader will find further topics as well as detailed derivations of previous results.

In section II, we comment on the *standard approximations* which have been introduced in section I to calculate the desired photon number $\langle a^+a\rangle$ numerically. Interestingly enough, it turns out that the approximation $\langle hea\rangle = 0$ does *not* provide an appropriate approach to the Heisenberg equations of motion, and it is completely wrong in the case of a purely electronic two-level quantum dot model coupled to a single photon mode. How to understand this?

As far as the general case of an $N-$level quantum dot model (coupled to a phonon branch) is concerned, we obtain, for example, the result that there exists *a constant source term* in the equation of motion for any expectation value of type $\langle h_m e_n a \rangle$; here, any operator is in normal order. We emphasize that this result does not depend on the electronic part of the underlying Hamiltonian (see section I), but only on the *standard particle-photon interaction operator* itself. Obviously, the answer to our question from above is already contained in these general statements.

Our next conclusion is that there must exist a non-trivial solution for the complete infinite system of coupled equations for the expectation values *even in the case that the exciting light field vanishes.* If the initial state $|\Psi(t=0)\rangle = |0\rangle$ is chosen, such a non-trivial solution leads to dynamical vacuum fluctuations of the quantized electromagnetic field. *According to a basic physical experience, however, dynamical vacuum fluctuations do not exist!* Consequently, a particle-photon interaction operator of standard type is possible from a mathematical point of view, but can not be accepted for *physical* reasons.

During our studies we realized that the denotation "vacuum" is actually not well defined in the literature; instead, the situation is somewhat confused. Because of the fact that the *physical vacuum* plays an important role in the framework of our own argumentation, some further comments seem to be appropriate:

(i) Physical phenomena do only exist and can only be established in connection with matter (e.g. atoms, solids, accelerators or external fields). As far as the physical vacuum is concerned, no specific properties (like mass, charge, current or polarization) do exist.

(ii) The experimental evidence for the vacuum [1] *leads straightforwardly to the requirement that the vacuum state has to be a basic element of any physical theory.*

(iii) By a suitable unitary transformation, it is always possible to choose a specific representation of the underlying Hilbert space such that the physical vacuum is represented by the basic state $|0\rangle$*. In the framework of our present work, it is understood that such a unitary transformation has already been performed at the very beginning. Under these circumstances, any vacuum expectation value which is defined by operator sequences in normal order does vanish.*

(iv) We close our compilation of comments with a technical remark: If a semiclassical approach is preferred, the physical vacuum can be described by the quantum-field theoretical vacuum state $|0\rangle$ *in combination with the additional condition that any classical external field vanishes.*

Turning to the energy spectrum, the existence of dynamical vacuum fluctuations is nothing else but the statement that the vacuum state is *not* an eigenstate of the Hamiltonian under consideration, if the exciting light field vanishes. Consequently, the ground state of the standard Hamiltonian contains, in general, electrons, holes and photons. In solid state physics, however, it is well known that in many cases (e.g., in semiconductors) the ground state is characterized by an empty conduction band and a full valence band. Moreover, the existence of a *many-particle ground state* leads to an important consequence in connection with *optical properties* of semiconductors: Because of the fact that the *light-matter interaction operator of standard type does not*

[1]As an interesting example, we mention the recent technical progress of CERN[3]: On the inside of the available accelerators, the vacuum can be achieved with a high precision ($p \cong 10^{-9}$ Pascal).

conserve the particle numbers of the electron system and the hole system, it is possible, for example, *to create a biexciton just by a one-photon absorption process.* To the best of our knowledge, however, such a behaviour is in complete contradiction to the present experimental findings. Clearly, this is one of the most convincing arguments against the standard coupling.

A general analysis shows that a *constant source term* can only appear in the equations of motion for expectation values of type $\langle \ldots a\rangle$, $\langle \ldots a^+\rangle$, $\langle \ldots a^2\rangle$ and $\langle \ldots (a^+)^2\rangle$. Apart from this, however, it is a simple exercise to verify our general results for the special case of an electron-hole two-level system coupled to a single photon mode with energy $\hbar\Omega$. The explicit calculation at the end of section II, for example, leads to the relation $\langle a^+a\rangle(t) \sim t^2$; here, the following conditions are fulfilled: *(i)* $\hbar\Omega$ *is equal to the unrenormalized excitonic energy. (ii)* $|\Psi(t=0)\rangle = |0\rangle$. *(iii) The exciting light field vanishes. (iv) The approximation* $\langle hea\rangle = 0$ *is introduced.* An extension to a more general class of initial states can be found in section III.

In section IV, we have analyzed the dynamical vacuum fluctuations of the quantized electromagnetic field due to the standard particle-photon interaction in more detail. As indicated above, the model under consideration is now an electron-hole two-level system coupled to a single photon mode. *It turns out that no renormalizations are necessary in order to get finite results.* For a specific choice of the Coulomb matrix element, we obtain an analytical expression for the state $|\Psi(t)\rangle$ provided that the initial state is the vacuum state. Our result is based on a diagonalization of the complete Hamiltonian and enables one to calculate the time evolution of a comprehensive class of expectation values. The photon number $\langle a^+a\rangle(t)$, for example, is proportional to the absolute square of the particle-photon coupling constant, i.e., the dynamical vacuum fluctuations may become in fact comparatively large. The particle numbers $\langle e^+e\rangle(t)$ and $\langle h^+h\rangle(t)$ are bounded from above by 1/2.

In the strong-coupling limit, the maximum value of the particle numbers is $\approx 1/2$. Interestingly enough, the polarization function proves to be an exception: We have demonstrated that the expectation value $\langle e^+h^+\rangle(t)$ is equal to zero for any t. This is a surprising result, because the particle numbers exhibit a non-trivial time evolution. Last but not least, we mention the inequality $|\langle e^+h^+a\rangle(t)| \leq |\langle hea\rangle(t)|$: It demonstrates that the absolute value of the photon-assisted polarization never exceeds the absolute value of $\langle hea\rangle(t)$.

We close our compilation of exact relations with the remark that we have checked the validity of our analytical results for various equations of motion at the beginning of the infinite hierarchy (a list can be found in section III).

As far as the energy spectrum is concerned, the purely electronic two-level Hamiltonian in combination with the standard particle-photon interaction and the additional condition for the Coulomb matrix element does provide an example for *a degenerate ground-state energy.* Obviously, this property is in marked contrast to the basic features of, e.g., polaronic systems. As has been demonstrated in the literature, the ground-state energy of a large class of generalized Fröhlich models is in fact a simple eigenvalue. In the model under consideration, there are two eigenstates which belong to the ground-state energy E_0.

In section V, we have analyzed the dynamical behaviour of the two-level system from section IV *in the presence of an external light field.* Here, the initial state is a superposition of those eigenstates which belong to the ground-state energy E_0. We have determined the time evolution of an arbitrary expectation value of type $\langle\hat{E}(a^+)^k a^l\rangle(t)$, i.e., we have obtained the solution for the complete infinite system of coupled equations for the expectation values ($\hat{E}$ denotes an electron-hole operator sequence). It should be emphasized again that the validity of our solution has been checked for various equations of motion at the beginning of the infinite hierarchy (see section III again).

Surprisingly enough, it turns out that the photon number is just a constant. As far as the occupation numbers of the electron-hole system are concerned, our general formula shows that these expectation values are in fact not negative and bounded from above by 1. In the strong-coupling limit, we obtain $\langle e^+ e\rangle(t) = \langle h^+ h\rangle(t) \approx 1/2$.

If we split the complete Hamiltonian $H(t)$ into $H(t) = H_0 + V_{ehS}(t)$, where $V_{ehS}(t)$ decribes the coupling to the classical exciting light field, we obtain the equation $\langle\Psi(t)|H_0|\Psi(t)\rangle = \langle\Psi(0)|H_0|\Psi(0)\rangle$, i.e., the total energy of the particle-photon system is a constant, too. Consequently, if the amplitude $E(t)$ of the exciting light field vanishes in the limits $t \to -\infty$ and $t \to +\infty$, the optical absorption spectrum $A(\omega)$ of the particle-photon system is equal to zero: $A(\omega) = 0$. Do we obtain the same result from the formula $A(\omega) = Im[P(\omega)/E(\omega)]$? We expect that this relation is also applicable in the limit of vanishing (phenomenological) damping constants provided that $A(\omega)$ is introduced as a *distribution.* In the framework of the two-level system under consideration, however, a careful analysis of the *exact formula* for the polarization function $\langle e^+ h^+\rangle(t)$ leads to another surprising result: It turns out that *the optical absorption spectrum is - in general - different from zero; moreover, $A(\omega)$ may become negative.*

We emphasize that our statements are valid *even in the case that the particle-photon coupling constant vanishes.* Here, an additional numerical study demonstrates that the formula $Im(P(\omega)/E(\omega))$ leads to an incorrect result *also in a vicinity of the specific Coulomb matrix element mentioned above.*[2]

We thus conclude that the standard formalism leads to wrong results at least for sufficiently large values of the Coulomb matrix element W^{eh}. If W^{eh} is small enough, the implications of the standard formalism are in excellent agreement with the *spectral properties* of the underlying Hamiltonian (here, we refer again to our comments at the very beginning of the abstract). However, the question whether $A(\omega)$ shows the correct peak *heights* or not remains open. If we use the improved formula $A(\omega) = 2 \cdot (\omega/c_0) \cdot n_b \cdot Im\sqrt{1+\chi(\omega)/\varepsilon_b}$, the *exact expression* for $\langle e^+h^+\rangle(t)$ leads to a serious mathematical problem, the reason being that the square root contains a distribution. Here, the optical absorption spectrum $A(\omega)$ does in fact not exist.

An extension of the two-level Hamiltonian from section IV to a more realistic model can be found in section VI. Here, the electron-hole system is coupled to an *unlimited number* of photon modes. In order to ensure that the corresponding Hamiltonian is well-defined and bounded from below, we introduce two basic conditions for the particle-photon coupling function. Under these circumstances, *no renormalizations are necessary in order to get finite results.*

Because of the fact that the photon dispersion $\hbar\Omega_k$ depends explicitly on the $k-$vector, *relaxation phenomena* in the dynamics of the expectation values must exist. Surprisingly enough, such an intrinsic relaxation does not lead to a *damping* of the dynamical vacuum fluctuations: Direct inspection of the equations of motion for expectation values of type $\langle \ldots a\rangle$, $\langle \ldots a^+\rangle$, $\langle \ldots a^2\rangle$ and $\langle \ldots (a^+)^2\rangle$ shows that a long-time behaviour with the properties $\lim_{t\to\infty}\langle \ldots\rangle = 0$ and $\lim_{t\to\infty} \partial/\partial t\langle \ldots\rangle = 0$ for *any* expectation value can be excluded even in the most general case.

As far as the two-level system is concerned, we have demonstrated that the photon number $N_{Ph.}(t)$ approaches a *finite value* for sufficiently large t: $N_{Ph.}(t \to \infty) \equiv N_\infty > 0$. The time evolution of the occupation numbers $\langle e^+e\rangle(t)$ and $\langle h^+h\rangle(t)$ exhibits the same qualitative behaviour; here, we obtain the result $\langle e^+e\rangle(t\to\infty) = \langle h^+h\rangle(t\to\infty) = (1/2)\cdot(1-e^{-2\cdot N_\infty})$.

Last but not least, we have calculated the expectation values $Pa := \langle e^+h^+ \int d^3k \sum_\lambda f_{k\lambda}\cdot a_{k\lambda}\rangle(t\to\infty)$ and $P^+a := \langle he \int d^3k \sum_\lambda f_{k\lambda}\cdot a_{k\lambda}\rangle(t\to\infty)$

(here, the function $f_{k\lambda}$ has to be chosen such that the lemma of Riemann-Lebesgue can be applied to perform the long-time limit). Again, it turns out that the inequality $|Pa| \leq |P^{+}a|$ is valid, i.e., the expectation values $\langle hea_{k\lambda}\rangle$ are *not* comparatively small. This confirms our previous result in section IV and leads to the conclusion that *the well-known interpretation that an electron-hole pair and a photon are removed from the system under consideration is incorrect.* In fact, we only know that the expectation values are determined by the hierarchy equations and by the initial state: Any further assumption (which is often motivated by a so-called "physical interpretation") may cause intrinsic contradictions or contradictions to physical laws. We believe that these results are of basic importance in connection with the analysis of quantum-field theoretical hierarchy equations.

We have already emphasized that the standard particle-photon interaction operator does not provide an appropriate physical model in solid state physics. What about alternative approaches?

In a first step, it proves useful to analyze *the general relationship between the fully quantized and the semiclassical light-matter interaction operator.* In section VII, we are concerned with a particle-photon interaction operator V_{ehp} of general type $V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n$, where $\hat{A}_n$ are purely photonic operator sequences in normal order. We only require that the electron-hole operator sequences $\hat{E}\hat{H}_n$ fulfill the condition $\langle 0|\hat{E}\hat{H}_n|0\rangle = 0$ (i.e., the normal-ordered operators $\hat{E}\hat{H}_n$ should not contain a constituent $\sim \underline{1}_{eh}$). Under these circumstances, it turns out that the electronic constituents of V_{ehp} and the semiclassical light-matter interaction operator $V_{ehS}(t)$ must have the same mathematical structure, i.e., there exist time-dependent functions $S_n(t)$ such that $V_{ehS}(t) = \sum_n S_n(t) \cdot \hat{E}\hat{H}_n$ is true. Our proof is based on the physical requirement that the vacuum expectation value of $V_{ehS}(t)$ must vanish.

In a next step, we are concerned with the complete Hamiltonian $H(t) =$

$H_0 + V_{ehS}(t)$. Our analysis is based on two assumptions: (i) Apart from the vector potential $\hat{A}(r)$, the particle-photon interaction operator V_{ehp} does only contain *non-photonic* field operators (see section VII for an explicit definition). (ii) The vacuum state is an eigenstate of H_0. Additionally, we choose the initial state $|\Psi(t = 0)\rangle = |0\rangle$. Under these circumstances, we obtain the result $|\Psi(t)\rangle = |0\rangle$ for any $t \geq 0$. Consequently, neither a change of the particle numbers nor an emission of photons is possible. In other words: If we start from the initial state $|\Psi(t = 0)\rangle = |0\rangle$, a *non-trivial* time evolution is possible only if the quantum-field theoretical Hamiltonian under consideration leads to *dynamical vacuum fluctuations*.

Although the interaction operator V_{ehp} from above is quite general, it does not contain, for example, the "time derivative" $\hat{B}(r)$ of the vector potential $\hat{A}(r)$. By a suitable combination of $\hat{A}(r)$, $\hat{B}(r)$ and further Bose fields, it may be possible to create an interaction operator of type $V_{ehp} = \sum_n \hat{A}_n \hat{E} \hat{H}_n$, where $\hat{A}_n|0\rangle = 0$ is true; here, the photonic constituents $\hat{A}_n$ are of *density type*. If the latter condition is taken for granted, a non-trivial time evolution is in fact possible. However, if the initial state is an element of the particle-phonon subspace, we obtain the result $\langle\Psi(t)|\hat{A}|\Psi(t)\rangle = 0$ for any $t \geq 0$ ($\hat{A}$ is a purely photonic operator sequence in normal order which does not contain a constituent $\sim \underline{1}_{Phot.}$).

In section VIII, we have extended our analysis to a general particle-photon interaction operator of product type. As before, we require that the vacuum state is an eigenstate of H_0. Under these circumstances, the relation $\langle\Psi(t)|a^+a|\Psi(t)\rangle = 0$ is true provided that the initial state is the vacuum state. The latter statement thus delivers a necessary condition for any alternative interaction operator: If - for a specific physical system - a restriction to the weak-coupling limit and to only one mode for each particle is possible, a realistic particle-photon interaction operator can never be of product type under these circumstances.

Clearly, our surprising results from above do provide a serious argument against the existing quantum-field theoretical approach itself: *It is already the general mathematical structure of a comprehensive class of interaction operators which leads to the conclusion that the corresponding Hamiltonians have to be excluded.*

If we want to ensure that the vacuum state is an eigenstate of H_0, the introduction of *electronic densities* is sufficient. This can be achieved by introducing *two electronic field operators* $\hat{\Psi}(r)$ and $\hat{\Phi}(r)$ which fulfill the usual anticommutation relations: $\hat{\Psi}(r) = \sum_n \Psi_n(r) \cdot e_n$, $\hat{\Phi}(r) = \sum_m \Phi_m(r) \cdot h_m$. It is obvious that these expansions can no longer be compared with the standard band structure expansion $\hat{\Psi}(r) = \sum_n \chi_n(r) \cdot e_n + \sum_m \Lambda_m(r) \cdot h_m^+$, i.e., the denotations *electron* and *hole* have a completely different meaning under these circumstances. In Ref.[1], we have analyzed a model Hamiltonian which is based on such an alternative approach. Here, we have introduced the additional condition that the basic substitution $p \rightarrow p \pm e \cdot \hat{A}(r)$ *remains* an essential element of the modified theory. It turns out that the complete Hamiltonian H_0 commutes with the particle-number operators of the electron system and the hole system, i.e., a change of the particle numbers due to external excitation or radiative recombination is impossible [2].

Having excluded a comprehensive class of alternative interaction operators, we are now concerned again with a particle-photon interaction operator of standard type. Our starting point in section IX is the remark that our previous investigations *in sections III, IV, V and VI* are essentially based on a Hamiltonian which is restricted to *interband transitions*. Strictly speak-

[2] Note that the ground state is - in general - a many-particle state (see Ref.[1] for further details).

ing, however, the complete interaction operator leads to further contributions which are not contained in such an approximation. For this reason, we have analyzed some extended versions of the two-level Hamiltonians from sections III, IV, V and VI. Here, operator products of *density type* have also been taken into account to describe the coupling of the electron-hole system to the quantized electromagnetic field. Surprisingly enough, it turns out that most of our analytical results obtained in sections IV, V and VI remain valid. Under the initial conditions from above, the time evolution of the expectation values is determined *by the same solution* $|\Psi(t)\rangle$ *as before*, i.e., a particle-photon interaction operator of *polarization type* is in fact a reasonable approximation to calculate, for example, the dynamical vacuum fluctuations in the framework of the standard model.

Last but not least, we generalize our analysis of the *hierarchy equations* in section III; again, the standard approximation $\langle hea\rangle = 0$ is of particular interest. In the framework of the *extended* two-level system (coupled to a single photon mode), the explicit calculation in section IX.D leads to the result that the photon number $\langle a^+a\rangle(t)$ is dominated again by an expression $\sim t^2$; here, the following conditions are fulfilled: *(i)* $\hbar\Omega$ *is equal to the unrenormalized excitonic energy. (ii)* $|\Psi(t=0)\rangle = \alpha_1\cdot|0\rangle + \alpha_2\cdot e^+|0\rangle + \alpha_3\cdot h^+|0\rangle + \alpha_4\cdot e^+h^+|0\rangle$, *where* $(\alpha_1, \alpha_4) \neq (0,0)$. *(iii) The exciting light field vanishes. (iv) The approximation* $\langle hea\rangle = 0$ *is introduced.* We stress explicitly that the latter statement is true for *arbitrary* Coulomb matrix elements.

Under the additional condition $\alpha_2 = \alpha_3 = 0$, our previous result $\langle a^+a\rangle(t) \sim t^2$ (see section III again) remains even vaild.

The basic topic of all previous investigations was the analysis of various aspects of the particle-photon interaction. In fact, the semiclassical interaction operator which describes the coupling of the classical exciting light field to the electron-hole system can only be justified if the fully quantized theory is already known. For this reason, a microscopic theory of the light-matter interaction is indispensable and should be regarded as a basic element of any theoretical approach.

We have already seen that the standard coupling of the electron-hole system to the quantized electromagnetic field leads to serious and fundamental difficulties, the reason being that *constant source terms* do appear in the equations of motion for a specific class of expectation values. At this point, however, it is important to realize that the problem of dynamical vacuum fluctuations is by no means restricted to the standard particle-photon interaction operator. As a further and quiet important example, we mention the Coulomb interaction between electrons and holes in a purely electronic system. In view of the anticommutation relations for Fermi operators, it is obvious that dynamical vacuum fluctuations can only appear in an *$N-$level system with $N \geq 3$.*

In section X, we have analyzed the dynamical behaviour of a purely electronic three-level system. In order to gain some insight into possible mathematical structures of exact solutions, we have introduced two additional conditions for the Coulomb matrix elements of the underlying Hamiltonian. If these conditions are fulfilled, the time evolution of the expectation values can be calculated analytically provided that the initial state is the *vacuum state* $|0\rangle$. Interestingly enough, it turns out that the mathematical structure of the obtained solutions is in marked contrast to the basic features of the *two-level systems* from sections IV, V and VI: The formula for the state $|\Psi(t)\rangle$ contains an expression of type *coupling times t* ! As an impressable illustration, we mention the occupation numbers of the electron-hole system: $\langle e_N^+ e_N\rangle(t) = \langle h_M^+ h_M\rangle(t) = \sin^2\left(\frac{V}{\hbar}\cdot t\right)$ (V: Coulomb matrix element). The latter equation demonstrates, for example, that the maximum value of the particle numbers is actually 1 - *even if V is arbitrarily small.*

It is possible to introduce a third condition for the Coulomb matrix elements which is often used in numerical calculations. If this additional condition is fulfilled, too, the Hamiltonian under consideration can be diagonalized analytically. Moreover, it can always be achieved that the corresponding three-level system has a *unique ground state.* At this point, the following question immediately arises: What does happen in the *presence* of an external light field? In order to analyze this problem in more detail, we have introduced realistic conditions for the transition matrix elements which are contained in the semiclassical interaction operator. Under these circumstances, it is in fact possible to calculate the time evolution of the state $|\Psi(t)\rangle$ analytically provided that the initial state is the *ground state* $|\Psi_0\rangle$. The unexpected and surprising result, however, is the trivial time evolution $|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \cdot E_0 \cdot t\right) \cdot |\Psi_0\rangle$ (E_0: ground state energy). It is remarkable that such a solution is actually possible in the framework of the standard model.

In section XI, we generalize our analysis in section IV. Here, we are concerned with an electron-hole two-level system coupled to a single photon mode *in the presence of an external light field.* As before, the initial state is the *vacuum state* $|0\rangle$; moreover, the Coulomb matrix element is fixed again. Under these circumstances, we obtain exact analytical expressions for a comprehensive class of expectation values; these results are valid for arbitrary electric field amplitudes.

It turns out that the photon number $\langle a^+ a\rangle(t)$ has the same time evolution as before, i.e., it does not depend on the external light field; additionally, the semiclassical interaction energy $\langle V_{ehS}(t)\rangle$ vanishes for any $t \geq 0$. *Under the initial condition from above, there is actually no energy transfer from the external field to the rest of the system.* It is therefore highly surprising that there are many expectation values which depend *explicitly* on $E(t)$. As important examples, we mention the particle numbers of the electron-hole system, the polarization function and the photon-assisted expectation values $\langle e^+h^+a\rangle(t)$ and $\langle hea\rangle(t)$. In the presence of an external light field, the upper bound 1/2 for the particle numbers is no longer valid; here, the maximum value 1 can be achieved under suitable conditions. The polarization function $\langle e^+h^+\rangle(t)$ has a non-trivial time evolution, too. If the particle-photon coupling constant vanishes, the explicit formula is in agreement with our previous findings in section V (here, the initial state was a superposition of those eigenstates which belong to the ground state energy of H_0). As far as the calculation of the optical absorption spectrum is concerned, we have already discussed the implications of our analytical results. Last but not least, we turn to the expectation value $\langle hea\rangle(t)$ which is a particular element of the quantum-field theoretical hierarchy equations: Our generalized formula in section XI demonstrates again that this expectation value is - in almost all cases - not comparatively small.

We close our compilation of comments and exact results with our essential statements (remember that $H(t) = H_0 + V_{ehS}(t)$):

(i) The standard particle-photon interaction operator violates the basic condition of physics that the vacuum state has to be an eigenstate of H_0.

(ii) The standard approximation $\langle hea\rangle(t) = 0$ *and related approximations are in complete contradiction to the underlying theory itself.*

(iii) As far as the calculation of the optical absorption spectrum in the framework of the semiclassical approach is concerned, a counter-example does exist in the limit of vanishing (phenomenological) damping constants.

(iv) A comprehensive class of alternative particle-photon interaction operators can already be excluded. Moreover, there is a necessary condition for any alternative approach: If - for a specific physical system - a restriction to the weak-coupling limit and to only one mode for each particle is possible, a realistic particle-photon interaction operator can never be of product type

under these circumstances.

Apart from the standard approximations already mentioned above, there are further approximations which have been introduced in the literature to calculate the photon number $\langle a^+a\rangle(t)$. One of the most prominent examples is the so-called *rotating wave approximation* [3]. Here, the Heisenberg equations of motion are replaced by the equations $\partial/\partial t\langle\hat{A}\rangle = (i/\hbar)\cdot\langle[H',\hat{A}]\rangle(+\langle\partial\hat{A}/\partial t\rangle)$, where H' is obtained from H by a cancellation of any operator product of type $e^+h^+a^+, hea$ etc. It is obvious that H' is actually not a quantum-field theoretical Hamiltonian, the reason being that H' can not be represented by a functional form of the vector potential $\hat{A}(r)$ and the electronic field operators $\hat{\Psi}^+(r), \hat{\Psi}(r)$. A further remark is concerned with the time evolution of the expectation values: Because of the fact that $[H', H] \neq 0$, the total energy $\langle H\rangle$ is - in the absence of an external light field - no longer a constant of motion, i.e., a fundamental symmetry property of the system under consideration is violated, too.

At the end of our introductory remarks, we should also comment on the well-known *renormalization procedures* of standard quantum field theory. These procedures have been originally introduced in the framework of relativistic quantum electrodynamics (see, e.g., Refs.[5–7]) and can be regarded as an answer to the failure of perturbation theory[7]. Because of the fact that all statements in our present work are based on *exact results* and not on perturbational approaches, a "renormalization" in the spirit of Refs.[5–7] makes definitely no sense. If the physical vacuum is represented by the basic state $|0\rangle$, the introduction of a modified Hamiltonian seems to be the only possibility to overcome the difficulties caused by the existence of *constant source*

[3]See, e.g., Ref.[4]. Note that these authors do not use electron-hole operators.

terms. Until now, however, such a modified Hamiltonian was not found.

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I. SUMMARY OF THE BASIC DEFINITIONS AND EQUATIONS

A. General part

In our first section, we are going to derive the whole set of equations which determines the *photon number* in the framework of an electron-hole system, interacting with phonons and photons. Setting up the Heisenberg equations of motion, the desired photon number is obtained from an infinite system of coupled equations for the corresponding *photon-assisted expectation values*. The detailed structure of these equations will be discussed now. We should emphasize in advance that the following considerations are essentially based on commutation relations and on the Heisenberg equation of motion:

$$\frac{\partial}{\partial t}\langle\hat{A}\rangle = \frac{i}{\hbar}\cdot\langle\big[H,\hat{A}\big]\rangle\left(+\left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle\right) \ . \tag{1}$$

Our model Hamiltonian H reads as follows[8,9]:

$$H = H_1 + H_2 \ , \tag{2}$$

where

$$H_1 = H_1(e_{i_1}, e^+_{i_2}, h_{j_1}, h^+_{j_2}, b_{q_1}, b^+_{q_2}) \tag{3}$$

depends on the electron operators $e_{i_1}, e^+_{i_2}$, the hole operators $h_{j_1}, h^+_{j_2}$ and the phonon operators $b_{q_1}, b^+_{q_2}$:

$$\begin{aligned} H_1 = &\sum_n \varepsilon_n e^+_n e_n + \sum_m \varphi_m h^+_m h_m + \sum_q \hbar\omega(q) b^+_q b_q + \\ &+\sum_{qnn'} g_q M_{qnn'} b_q e^+_n e_{n'} - \sum_{qmm'} g_q \bar{M}_{qm'm} b_q h^+_m h_{m'} + h.c. + \\ &+\sum_{nm} S_{nm}(t) e^+_n h^+_m + \sum_{nm} S^*_{nm}(t) h_m e_n + \\ &+\sum_{n_1n_2n_3n_4} W^{ee}_{n_1n_2n_3n_4}\cdot e^+_{n_1} e^+_{n_2} e_{n_3} e_{n_4} + \sum_{m_1m_2m_3m_4} W^{hh}_{m_1m_2m_3m_4}\cdot h^+_{m_1} h^+_{m_2} h_{m_3} h_{m_4} + \\ &-2\cdot\sum_{nmm'n'} W^{eh}_{nmm'n'} e^+_n h^+_m h_{m'} e_{n'} \ . \end{aligned} \tag{4}$$

Here, $S_{nm}(t)$ represents the coupling of the *classical exciting light field* to the electron-hole system. The second constituent

$$H_2 = \int d^3k \sum_\lambda \hbar\Omega_k a^+_{k\lambda} a_{k\lambda} + V_1 + V_2 \tag{5}$$

corresponds to the *emitted light field* and contains the linear part V_1 and the quadratic part V_2 of the light-matter interaction operator as well as the photon dispersion $\hbar\Omega_k$; in particular, $a_{k\lambda}$ and $a^+_{k\lambda}$ are annihilation- and creation-operator of the photons with wave vector k and polarization λ:

$$[a_{k\lambda}, a^+_{k'\lambda'}] = \delta(k - k') \cdot \delta_{\lambda\lambda'} \ , \ [a_{k\lambda}, a_{k'\lambda'}] = 0 \ , \ [a^+_{k\lambda}, a^+_{k'\lambda'}] = 0 \ . \tag{6}$$

The vector potential $A(r)$ reads as follows:

$$A(r) = \int d^3k \sum_\lambda B_{k\lambda}(r) a_{k\lambda} + \int d^3k \sum_\lambda B^*_{k\lambda}(r) a^+_{k\lambda} \ , \tag{7}$$

where

$$B_{k\lambda}(r) = G_k \varepsilon_{k\lambda} e^{ik\cdot r} \ , \ G_k = \sqrt{\frac{\hbar}{2\cdot(2\pi)^3\cdot\varepsilon_0\cdot\Omega_k}} \ . \tag{8}$$

The interaction operator $V_1 \sim A \cdot p$ is given by

$$V_1 = \int d^3k \sum_\lambda \hat{P}_{k\lambda} a_{k\lambda} + \int d^3k \sum_\lambda \hat{P}^+_{k\lambda} a^+_{k\lambda} \ ; \tag{9}$$

here, the field operator $\hat{P}_{k\lambda}$ has the structure

$$\hat{P}_{k\lambda} = \sum_{ij} U_{ijk\lambda} \cdot e^+_i h^+_j + \sum_{ij} V^*_{ijk\lambda} \cdot h_j e_i \ . \tag{10}$$

As far as the contribution $V_2 \sim A^2$ is concerned, *we have to apply the basic rule of quantum field theory that any operator sequence has to be in normal order.*[7] Under these circumstances, the quadratic part V_2 of the light-matter interaction operator has the structure

$$V_2 = \sum_{nm} \hat{\Sigma}_{nm} e^+_n h^+_m + \sum_{nm} \hat{\Sigma}^+_{nm} h_m e_n \ , \tag{11}$$

where

$$\hat{\Sigma}_{nm} := \int d^3k \int d^3k' \sum_{\lambda\lambda'} P_{nmkk'\lambda\lambda'} a_{k\lambda} a_{k'\lambda'} + \int d^3k \int d^3k' \sum_{\lambda\lambda'} Q_{nmkk'\lambda\lambda'} a^+_{k\lambda} a^+_{k'\lambda'} + \\ + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R_{nmkk'\lambda\lambda'} a^+_{k\lambda} a_{k'\lambda'} \ . \tag{12}$$

Now, let

$$\hat{O} = \hat{O}(b_{q_1}, b_{q_2}^+) \tag{13}$$

be a functional form of the phonon operators $b_{q_1}, b_{q_2}^+$ and let

$$\hat{E} = \hat{E}(e_{i_1}, e_{i_2}^+, h_{j_1}, h_{j_2}^+) \tag{14}$$

be a functional form of the electron and hole operators $e_{i_1}, e_{i_2}^+$ and $h_{j_1}, h_{j_2}^+$, respectively. We choose a fixed quantum number $(k_0\lambda_0)$ and introduce the abbreviations

$$a := a_{k_0\lambda_0}\ ,\ a^+ := a^+_{k_0\lambda_0}\ ,\ \Omega := \Omega_{k_0\lambda_0}\ ,\ \hat{P} := \hat{P}_{k_0\lambda_0}\ ,\ \hat{P}^+ := \hat{P}^+_{k_0\lambda_0}\ . \tag{15}$$

Under these circumstances, one obtains by a straightforward calculation the basic commutation relation

$$\begin{aligned}
[H, \hat{O}\hat{E}(a^+)^i a^j] &= [H_1, \hat{O}\hat{E}](a^+)^i a^j + \hat{O}\hat{L}_{ij} + \\
&+\hat{O}\left(\sum_{nm} \hat{E} e_n^+ h_m^+ \hat{\Pi}^{(1)}_{ijnm} + \sum_{nm} \hat{E} h_m e_n \hat{\Pi}^{(2)}_{ijnm}\right) + \\
&+\hat{O}\left(\sum_{nm} \left[e_n^+ h_m^+, \hat{E}\right] \hat{\Pi}^{(3)}_{ijnm} + \sum_{nm} \left[h_m e_n, \hat{E}\right] \hat{\Pi}^{(4)}_{ijnm}\right)\ ,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\hat{L}_{ij} &= (i-j)\hbar\Omega\hat{E}(a^+)^i a^j + i\cdot\hat{P}\hat{E}(a^+)^{i-1}a^j - j\cdot\hat{E}\hat{P}^+(a^+)^i a^{j-1} + \\
&+\int d^3k \sum_\lambda [\hat{P}_{k\lambda}, \hat{E}](a^+)^i a^j a_{k\lambda} + \int d^3k \sum_\lambda [\hat{P}^+_{k\lambda}, \hat{E}] a^+_{k\lambda}(a^+)^i a^j\ ,
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\hat{\Pi}^{(1)}_{ijnm} := \left[\hat{\Sigma}_{nm}, (a^+)^i a^j\right] = \\
&= i\cdot\int d^3k \sum_\lambda R_{nmkk_0\lambda\lambda_0}\cdot a^+_{k\lambda}(a^+)^{i-1}a^j - j\cdot\int d^3k \sum_\lambda R_{nmk_0k\lambda_0\lambda}\cdot(a^+)^i a^{j-1}a_{k\lambda} + \\
&+i\cdot\int d^3k \sum_\lambda \left(P_{nmk_0k\lambda_0\lambda} + P_{nmkk_0\lambda\lambda_0}\right)(a^+)^{i-1}a^j a_{k\lambda} + \\
&-j\cdot\int d^3k \sum_\lambda \left(Q_{nmk_0k\lambda_0\lambda} + Q_{nmkk_0\lambda\lambda_0}\right) a^+_{k\lambda}(a^+)^i a^{j-1} + \\
&+i\cdot(i-1)\cdot P_{nmk_0k_0\lambda_0\lambda_0}\cdot(a^+)^{i-2}a^j - j\cdot(j-1)\cdot Q_{nmk_0k_0\lambda_0\lambda_0}\cdot(a^+)^i a^{j-2}\ ,
\end{aligned} \tag{18}$$

$$\hat{\Pi}^{(2)}_{ijnm} := \left[\hat{\Sigma}^{+}_{nm}, (a^{+})^{i} a^{j}\right] = \tag{19}$$

$$= i \cdot \int d^{3}k \sum_{\lambda} R^{*}_{nmk_0k\lambda_0\lambda} \cdot a^{+}_{k\lambda} (a^{+})^{i-1} a^{j} - j \cdot \int d^{3}k \sum_{\lambda} R^{*}_{nmkk_0\lambda\lambda_0} \cdot (a^{+})^{i} a^{j-1} a_{k\lambda} +$$

$$+ i \cdot \int d^{3}k \sum_{\lambda} \left(Q^{*}_{nmk_0k\lambda_0\lambda} + Q^{*}_{nmkk_0\lambda\lambda_0}\right) (a^{+})^{i-1} a^{j} a_{k\lambda} +$$

$$- j \cdot \int d^{3}k \sum_{\lambda} \left(P^{*}_{nmk_0k\lambda_0\lambda} + P^{*}_{nmkk_0\lambda\lambda_0}\right) a^{+}_{k\lambda} (a^{+})^{i} a^{j-1} +$$

$$+ i \cdot (i-1) \cdot Q^{*}_{nmk_0k_0\lambda_0\lambda_0} \cdot (a^{+})^{i-2} a^{j} - j \cdot (j-1) \cdot P^{*}_{nmk_0k_0\lambda_0\lambda_0} \cdot (a^{+})^{i} a^{j-2} \ ,$$

$$\hat{\Pi}^{(3)}_{ijnm} := \hat{\Sigma}_{nm} (a^{+})^{i} a^{j} = \tag{20}$$

$$= \int d^{3}k \int d^{3}k' \sum_{\lambda\lambda'} \left(P_{nmkk'\lambda\lambda'} \cdot (a^{+})^{i} a^{j} a_{k\lambda} a_{k'\lambda'} + Q_{nmkk'\lambda\lambda'} \cdot a^{+}_{k\lambda} a^{+}_{k'\lambda'} (a^{+})^{i} a^{j}\right) +$$

$$+ \int d^{3}k \int d^{3}k' \sum_{\lambda\lambda'} R_{nmkk'\lambda\lambda'} \cdot a^{+}_{k\lambda} (a^{+})^{i} a^{j} a_{k'\lambda'} +$$

$$+ i \cdot \int d^{3}k \sum_{\lambda} \left(P_{nmk_0k\lambda_0\lambda} + P_{nmkk_0\lambda\lambda_0}\right) \cdot (a^{+})^{i-1} a^{j} a_{k\lambda} +$$

$$+ i \cdot \int d^{3}k \sum_{\lambda} R_{nmkk_0\lambda\lambda_0} \cdot a^{+}_{k\lambda} (a^{+})^{i-1} a^{j} + i \cdot (i-1) \cdot P_{nmk_0k_0\lambda_0\lambda_0} \cdot (a^{+})^{i-2} a^{j}$$

and

$$\hat{\Pi}^{(4)}_{ijnm} := \hat{\Sigma}^{+}_{nm} (a^{+})^{i} a^{j} = \tag{21}$$

$$= \int d^{3}k \int d^{3}k' \sum_{\lambda\lambda'} \left(Q^{*}_{nmkk'\lambda\lambda'} \cdot (a^{+})^{i} a^{j} a_{k\lambda} a_{k'\lambda'} + P^{*}_{nmkk'\lambda\lambda'} \cdot a^{+}_{k\lambda} a^{+}_{k'\lambda'} (a^{+})^{i} a^{j}\right) +$$

$$+ \int d^{3}k \int d^{3}k' \sum_{\lambda\lambda'} R^{*}_{nmk'k\lambda'\lambda} \cdot a^{+}_{k\lambda} (a^{+})^{i} a^{j} a_{k'\lambda'} +$$

$$+ i \cdot \int d^{3}k \sum_{\lambda} \left(Q^{*}_{nmk_0k\lambda_0\lambda} + Q^{*}_{nmkk_0\lambda\lambda_0}\right) \cdot (a^{+})^{i-1} a^{j} a_{k\lambda} +$$

$$+ i \cdot \int d^{3}k \sum_{\lambda} R^{*}_{nmk_0k\lambda_0\lambda} \cdot a^{+}_{k\lambda} (a^{+})^{i-1} a^{j} + i \cdot (i-1) \cdot Q^{*}_{nmk_0k_0\lambda_0\lambda_0} \cdot (a^{+})^{i-2} a^{j} \ .$$

By taking $\hat{O} = 1, \hat{E} = 1$ and $i = 1, j = 1$, it follows immediately that

$$[H, a^{+}a] = \hat{P} a - \hat{P}^{+} a^{+} + \sum_{nm} e^{+}_{n} h^{+}_{m} \hat{\Pi}^{(1)}_{11nm} + \sum_{nm} h_{m} e_{n} \hat{\Pi}^{(2)}_{11nm} \ , \tag{22}$$

$$\hat{\Pi}^{(1)}_{11nm} = \int d^{3}k \sum_{\lambda} R_{nmkk_0\lambda\lambda_0} \cdot a^{+}_{k\lambda} a - \int d^{3}k \sum_{\lambda} R_{nmk_0k\lambda_0\lambda} \cdot a^{+} a_{k\lambda} + \tag{23}$$

$$+ \int d^{3}k \sum_{\lambda} \left(P_{nmk_0k\lambda_0\lambda} + P_{nmkk_0\lambda\lambda_0}\right) a a_{k\lambda} - \int d^{3}k \sum_{\lambda} \left(Q_{nmk_0k\lambda_0\lambda} + Q_{nmkk_0\lambda\lambda_0}\right) a^{+}_{k\lambda} a^{+} \ ,$$

$$\hat{\Pi}^{(2)}_{11nm} = \int d^3k \sum_\lambda R^*_{nmk_0k\lambda_0\lambda} \cdot a^+_{k\lambda} a - \int d^3k \sum_\lambda R^*_{nmkk_0\lambda\lambda_0} \cdot a^+ a_{k\lambda} + \tag{24}$$
$$+ \int d^3k \sum_\lambda \left(Q^*_{nmk_0k\lambda_0\lambda} + Q^*_{nmkk_0\lambda\lambda_0} \right) a a_{k\lambda} - \int d^3k \sum_\lambda \left(P^*_{nmk_0k\lambda_0\lambda} + P^*_{nmkk_0\lambda\lambda_0} \right) a^+_{k\lambda} a^+$$

is true. Consequently, the determination of the expectation value $\langle a^+ a \rangle$ requires the knowledge of the expectation value $\langle \hat{P} a \rangle$. It is due to the particle-phonon interaction that the Heisenberg equation of motion for an expectation value of general type $\langle \hat{E} a \rangle$ leads to an infinite system of coupled equations for the corresponding *phonon-assisted expectation values.* In fact, by choosing $i = 0$ and $j = 1$, one obtains from Eqs. (16) and (17) the commutation relation

$$[H, \hat{O}\hat{E}a] = [H_1, \hat{O}\hat{E}]a + \hat{O}\hat{L}_{01} + \tag{25}$$
$$+\hat{O}\left(\sum_{nm} \hat{E} e^+_n h^+_m \hat{\Pi}^{(1)}_{01nm} + \sum_{nm} \hat{E} h_m e_n \hat{\Pi}^{(2)}_{01nm} \right) +$$
$$+\hat{O}\left(\sum_{nm} \left[e^+_n h^+_m, \hat{E} \right] \hat{\Pi}^{(3)}_{01nm} + \sum_{nm} \left[h_m e_n, \hat{E} \right] \hat{\Pi}^{(4)}_{01nm} \right) ,$$

$$\hat{L}_{01} = -\hbar\Omega \hat{E} a - \hat{E}\hat{P}^+ + \int d^3k \sum_\lambda [\hat{P}_{k\lambda}, \hat{E}] a a_{k\lambda} + \int d^3k \sum_\lambda [\hat{P}^+_{k\lambda}, \hat{E}] a^+_{k\lambda} a , \tag{26}$$

$$\hat{\Pi}^{(1)}_{01nm} = -\int d^3k \sum_\lambda R_{nmk_0k\lambda_0\lambda} a_{k\lambda} - \int d^3k \sum_\lambda \left(Q_{nmk_0k\lambda_0\lambda} + Q_{nmkk_0\lambda\lambda_0} \right) a^+_{k\lambda} , \tag{27}$$

$$\hat{\Pi}^{(2)}_{01nm} = -\int d^3k \sum_\lambda R^*_{nmkk_0\lambda\lambda_0} a_{k\lambda} - \int d^3k \sum_\lambda \left(P^*_{nmk_0k\lambda_0\lambda} + P^*_{nmkk_0\lambda\lambda_0} \right) a^+_{k\lambda} , \tag{28}$$

$$\hat{\Pi}^{(3)}_{01nm} = \int d^3k \int d^3k' \sum_{\lambda\lambda'} P_{nmkk'\lambda\lambda'} \cdot a a_{k\lambda} a_{k'\lambda'} + \tag{29}$$
$$+ \int d^3k \int d^3k' \sum_{\lambda\lambda'} Q_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda} a^+_{k'\lambda'} a + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda} a a_{k'\lambda'} ,$$

$$\hat{\Pi}^{(4)}_{01nm} = \int d^3k \int d^3k' \sum_{\lambda\lambda'} Q^*_{nmkk'\lambda\lambda'} \cdot a a_{k\lambda} a_{k'\lambda'} + \tag{30}$$
$$+ \int d^3k \int d^3k' \sum_{\lambda\lambda'} P^*_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda} a^+_{k'\lambda'} a + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R^*_{nmk'k\lambda'\lambda} \cdot a^+_{k\lambda} a a_{k'\lambda'} .$$

The hierarchy of equations for a phonon-assisted expectation value of general type $\langle \hat{O}\hat{E} \rangle$ is provided by Eq. (16) by taking $i = 0$ and $j = 0$:

$$[H,\hat{O}\hat{E}] = [H_1,\hat{O}\hat{E}] + \hat{O}\hat{L}_{00} + \tag{31}$$
$$+\hat{O}\left(\sum_{nm}\left[e_n^+ h_m^+,\hat{E}\right]\hat{\Pi}^{(3)}_{00nm} + \sum_{nm}\left[h_m e_n,\hat{E}\right]\hat{\Pi}^{(4)}_{00nm}\right) \ ,$$

$$\hat{L}_{00} = \int d^3k \sum_{\lambda}\left[\hat{P}_{k\lambda},\hat{E}\right] a_{k\lambda} + \int d^3k \sum_{\lambda}\left[\hat{P}^+_{k\lambda},\hat{E}\right] a^+_{k\lambda} \ , \tag{32}$$

$$\hat{\Pi}^{(3)}_{00nm} = \int d^3k \int d^3k' \sum_{\lambda\lambda'} P_{nmkk'\lambda\lambda'} \cdot a_{k\lambda}a_{k'\lambda'} + \tag{33}$$
$$+\int d^3k \int d^3k' \sum_{\lambda\lambda'} Q_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda}a^+_{k'\lambda'} + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda}a_{k'\lambda'} \ ,$$

$$\hat{\Pi}^{(4)}_{00nm} = \int d^3k \int d^3k' \sum_{\lambda\lambda'} Q^*_{nmkk'\lambda\lambda'} \cdot a_{k\lambda}a_{k'\lambda'} + \tag{34}$$
$$+\int d^3k \int d^3k' \sum_{\lambda\lambda'} P^*_{nmkk'\lambda\lambda'} \cdot a^+_{k\lambda}a^+_{k'\lambda'} + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R^*_{nmk'k\lambda'\lambda} \cdot a^+_{k\lambda}a_{k'\lambda'} \ .$$

Obviously, Eqs. (22), (25) and (31) can be used to determine the photon number $\langle a^+a\rangle$ in the framework of our model Hamiltonian (2).

B. Approximations for numerical calculations

In this subsection, we present a list of the approximations which have been introduced to calculate the photon number $\langle a^+a\rangle$ numerically. Making use of the *enveloppe approximation*, we first obtain:

$$U_{ijk\lambda} = -\frac{e}{m}\cdot F_{ij}\cdot U_{k\lambda} \ , \ V_{ijk\lambda} = -\frac{e}{m}\cdot F_{ij}\cdot V_{k\lambda} \ , \tag{35}$$

$$P_{ijkk'\lambda\lambda'} = \frac{e^2}{2m}\cdot F_{ij}\cdot X_{kk'\lambda\lambda'} \ , \ Q_{ijkk'\lambda\lambda'} = \frac{e^2}{2m}\cdot F_{ij}\cdot Y_{kk'\lambda\lambda'} \tag{36}$$

and

$$R_{ijkk'\lambda\lambda'} = \frac{e^2}{m}\cdot F_{ij}\cdot Z_{kk'\lambda\lambda'} \ , \tag{37}$$

where $U_{k\lambda}, V_{k\lambda}, X_{kk'\lambda\lambda'}, Y_{kk'\lambda\lambda'}$ and $Z_{kk'\lambda\lambda'}$ contain, for example, *interband transition matrix elements.* The expressions

$$F_{ij} := \int_\Omega d^3r \Psi_i^*(r) \cdot \Phi_j(r) \tag{38}$$

contain the eigenfunctions $\Psi_i(r)$ and $\Phi_j(r)$ of an appropriate one-particle Hamiltonian for the electron and the hole system, respectively; Ω denotes the domain of the unit cell. More explicitly, $\Psi_i(r)$ and $\Phi_j(r)$ are determined by the equations

$$\left[\frac{p^2}{2m_e^*} + V_e(r)\right] \Psi_i(r) = \varepsilon_i \Psi_i(r) \ , \ \left[\frac{p^2}{2m_h^*} + V_h(r)\right] \Phi_j(r) = \varphi_j \Phi_j(r) \ . \tag{39}$$

As far as the overlap integral (38) is concerned, the further approximation $F_{ij} \cong \delta_{ij}$ is often used and leads to the well-known selection rule that optical transitions can only take place between states having the same quantum numbers. Strictly speaking, however, this conclusion is correct only if the electron and hole eigenfunctions are *identical* (see Ref.[10] for further comments). In the framework of our own approach, there is no technical difficulty to maintain the general expression F_{ij} in our equations.

Introducing the approximation $e^{ik\cdot r} \cong 1$, it follows immediately that

$$U_{k_0\lambda_0} = V_{k_0\lambda_0} =: M \tag{40}$$

is true.

Further approximations: We neglect the expectation values

$$\langle h_j e_i a\rangle \ , \ \langle e_i^+ h_j^+ a^+\rangle \ , \ \langle e_i^+ h_j^+ e_k^+ h_l^+\rangle \tag{41}$$

and any expectation value of type

$$\int d^3k \ldots \langle \ldots \rangle \ , \ \int d^3k \int d^3k' \ldots \langle \ldots \rangle \ . \tag{42}$$

Under these circumstances, we obtain the equations

$$\begin{aligned} \frac{\partial}{\partial t}\langle a^+ a\rangle &= \frac{i}{\hbar} \cdot \langle [H, a^+ a]\rangle = \\ &= \frac{i}{\hbar} \cdot M \cdot \sum_{ij} F_{ij} \cdot \langle e_i^+ h_j^+ a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \sum_{ij} F_{ij}^* \cdot \langle h_j e_i a^+\rangle \ , \end{aligned} \tag{43}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\langle \hat{O} e_i^+ h_j^+ a\rangle = \frac{i}{\hbar} \cdot \langle [H, \hat{O} e_i^+ h_j^+ a]\rangle = \\ &= \frac{i}{\hbar} \cdot \langle [H_1, \hat{O} e_i^+ h_j^+] a\rangle - \frac{i}{\hbar} \cdot \hbar\Omega \cdot \langle \hat{O} e_i^+ h_j^+ a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \sum_{kl} F_{kl}^* \cdot \langle \hat{O} e_i^+ h_j^+ h_l e_k\rangle \ , \end{aligned} \tag{44}$$

$$\frac{\partial}{\partial t}\langle\hat{O}\rangle = \frac{i}{\hbar}\cdot\langle[H_1,\hat{O}]\rangle\ ,\ \frac{\partial}{\partial t}\langle\hat{O}e_m^+h_n^+\rangle = \frac{i}{\hbar}\cdot\langle[H_1,\hat{O}e_m^+h_n^+]\rangle\ , \tag{45}$$

$$\frac{\partial}{\partial t}\langle\hat{O}e_m^+e_n\rangle = \frac{i}{\hbar}\cdot\langle[H_1,\hat{O}e_m^+e_n]\rangle\ ,\ \frac{\partial}{\partial t}\langle\hat{O}h_m^+h_n\rangle = \frac{i}{\hbar}\cdot\langle[H_1,\hat{O}h_m^+h_n]\rangle \tag{46}$$

and

$$\frac{\partial}{\partial t}\langle\hat{O}e_k^+h_l^+h_me_n\rangle = \frac{i}{\hbar}\cdot\langle[H_1,\hat{O}e_k^+h_l^+h_me_n]\rangle\ . \tag{47}$$

Eqs. (43)-(47) constitute the general part of our theory. It is obvious that the operator $[H_1,\hat{O}e_i^+h_j^+]a$ in Eq. (44) has the same mathematical structure as the commutator $[H_1,\hat{O}e_m^+h_n^+]$ in Eq. (45), i.e., it is sufficient to calculate the expressions $[H_1,\hat{O}]$, $[H_1,\hat{O}e_m^+h_n^+]$, $[H_1,\hat{O}e_m^+e_n]$, $[H_1,\hat{O}h_m^+h_n]$ and $[H_1,\hat{O}e_k^+h_l^+h_me_n]$ in Eqs. (45) - (47).

C. Analysis of a two-level quantum dot model

In the following part of our considerations, we restrict ourselves to a two-level quantum dot model and to optical phonons, i.e., we take $i=j=0$ and $\omega(q)=\omega$. Here, only the factor F_{00} appears in our equations. We replace $M\cdot F_{00}\to M$ and define the phonon-assisted expectation values[11]

$$O_{mn}^{kl} := \langle(A^+)^k(B^+)^lA^mB^n\rangle\ ,\ E_{mn}^{kl} := \langle(A^+)^k(B^+)^lA^mB^ne^+e\rangle\ , \tag{48}$$

$$H_{mn}^{kl} := \langle(A^+)^k(B^+)^lA^mB^nh^+h\rangle\ ,\ P_{mn}^{kl} := \langle(A^+)^k(B^+)^lA^mB^ne^+h^+\rangle\ , \tag{49}$$

$$V_{mn}^{kl} := \langle(A^+)^k(B^+)^lA^mB^ne^+h^+he\rangle \tag{50}$$

and

$$PA_{mn}^{kl} := \frac{i}{\hbar}\cdot M\cdot\langle(A^+)^k(B^+)^lA^mB^ne^+h^+a\rangle\ , \tag{51}$$

where

$$A := \sum_q g_q \cdot M^e_{q00} b_q \ , \ B := \sum_q g_q \cdot M^h_{q00} b_q \ . \tag{52}$$

In view of our definitions from above, Eqs. (43) and (44) now read as follows:

$$\frac{\partial}{\partial t} \langle a^+ a \rangle = PA^{00}_{00} + \left(PA^{00}_{00} \right)^* \ , \tag{53}$$

$$\begin{aligned} \frac{\partial}{\partial t} PA^{kl}_{mn} = & \frac{i}{\hbar} \cdot (\varepsilon_0 + \varphi_0 - \hbar\Omega + (k+l-m-n)\hbar\omega) \cdot PA^{kl}_{mn} - 2 \cdot \frac{i}{\hbar} \cdot W^{eh} \cdot PA^{kl}_{mn} + \\ & + \frac{i}{\hbar} \cdot \left(PA^{kl}_{m+1n} - PA^{kl}_{mn+1} \right) + \frac{i}{\hbar} \cdot \left(PA^{k+1l}_{mn} - PA^{kl+1}_{mn} \right) + \\ & + \frac{i}{\hbar} \cdot k \cdot \left(\tilde{W}^{ee} - \tilde{W}^{eh} \right) \cdot PA^{k-1l}_{mn} + \frac{i}{\hbar} \cdot l \cdot \left(\tilde{W}^{eh} - \tilde{W}^{hh} \right) \cdot PA^{kl-1}_{mn} + \\ & + \frac{|M|^2}{\hbar^2} \cdot V^{kl}_{mn} \ . \end{aligned} \tag{54}$$

The phonon-induced Coulomb matrix elements $\tilde{W}$ will be discussed below. We are now going to derive the explicit equations for the expectation values O^{kl}_{mn}, E^{kl}_{lm}, H^{kl}_{mn}, V^{kl}_{mn} and P^{kl}_{mn}. A straightforward calculation shows that the functions O^{kl}_{mn}, E^{kl}_{mn}, H^{kl}_{mn}, V^{kl}_{mn} and P^{kl}_{mn} are determined by the equations

$$\frac{\partial}{\partial t} O^{kl}_{mn} = \frac{i}{\hbar} \cdot (k+l-m-n)\hbar\omega \cdot O^{kl}_{mn} + EM1^{kl}_{mn} + HM2^{kl}_{mn} \ , \tag{55}$$

$$\begin{aligned} \frac{\partial}{\partial t} E^{kl}_{mn} = & \frac{i}{\hbar} \cdot (k+l-m-n)\hbar\omega \cdot E^{kl}_{mn} - \frac{i}{\hbar} \cdot S \cdot P^{kl}_{mn} + \frac{i}{\hbar} \cdot S^* \cdot (P^{mn}_{kl})^* + \\ & + EM1^{kl}_{mn} + VM2^{kl}_{mn} \ , \end{aligned} \tag{56}$$

$$\begin{aligned} \frac{\partial}{\partial t} H^{kl}_{mn} = & \frac{i}{\hbar} \cdot (k+l-m-n)\hbar\omega \cdot H^{kl}_{mn} - \frac{i}{\hbar} \cdot S \cdot P^{kl}_{mn} + \frac{i}{\hbar} \cdot S^* \cdot (P^{mn}_{kl})^* + \\ & + VM1^{kl}_{mn} + HM2^{kl}_{mn} \ , \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{\partial}{\partial t} V^{kl}_{mn} = & \frac{i}{\hbar} \cdot (k+l-m-n)\hbar\omega \cdot V^{kl}_{mn} - \frac{i}{\hbar} \cdot S \cdot P^{kl}_{mn} + \frac{i}{\hbar} \cdot S^* \cdot (P^{mn}_{kl})^* + \\ & + VM1^{kl}_{mn} + VM2^{kl}_{mn} \end{aligned} \tag{58}$$

and

$$\frac{\partial}{\partial t}P^{kl}_{mn} = \frac{i}{\hbar}\cdot(\varepsilon_0+\varphi_0+(k+l-m-n)\hbar\omega)\cdot P^{kl}_{mn} + \qquad (59)$$
$$+\frac{i}{\hbar}\cdot S^*\cdot\left(O^{kl}_{mn}-E^{kl}_{mn}-H^{kl}_{mn}\right) - 2\cdot\frac{i}{\hbar}\cdot W^{eh}\cdot P^{kl}_{mn} +$$
$$+\frac{i}{\hbar}\cdot\left(P^{k+1l}_{mn}-P^{kl+1}_{mn}\right) + \frac{i}{\hbar}\cdot\left(P^{kl}_{m+1n}-P^{kl}_{mn+1}\right) +$$
$$+\frac{i}{\hbar}\cdot k\cdot\left(\tilde W^{ee}-\tilde W^{eh}\right)\cdot P^{k-1l}_{mn} + \frac{i}{\hbar}\cdot l\cdot\left(\tilde W^{eh}-\tilde W^{hh}\right)\cdot P^{kl-1}_{mn} .$$

Here, we have introduced the abbreviations

$$EM1^{kl}_{mn} := +\frac{i}{\hbar}\cdot k\cdot\tilde W^{ee}\cdot E^{k-1l}_{mn} + \frac{i}{\hbar}\cdot l\cdot\tilde W^{eh}\cdot E^{kl-1}_{mn} + \qquad (60)$$
$$-\frac{i}{\hbar}\cdot m\cdot\tilde W^{ee}\cdot E^{kl}_{m-1n} - \frac{i}{\hbar}\cdot n\cdot\tilde W^{eh}\cdot E^{kl}_{mn-1} ,$$

$$HM2^{kl}_{mn} := -\frac{i}{\hbar}\cdot k\cdot\tilde W^{eh}\cdot H^{k-1l}_{mn} - \frac{i}{\hbar}\cdot l\cdot\tilde W^{hh}\cdot H^{kl-1}_{mn} + \qquad (61)$$
$$+\frac{i}{\hbar}\cdot m\cdot\tilde W^{eh}\cdot H^{kl}_{m-1n} + \frac{i}{\hbar}\cdot n\cdot\tilde W^{hh}\cdot H^{kl}_{mn-1} ,$$

$$VM1^{kl}_{mn} := +\frac{i}{\hbar}\cdot k\cdot\tilde W^{ee}\cdot V^{k-1l}_{mn} + \frac{i}{\hbar}\cdot l\cdot\tilde W^{eh}\cdot V^{kl-1}_{mn} + \qquad (62)$$
$$-\frac{i}{\hbar}\cdot m\cdot\tilde W^{ee}\cdot V^{kl}_{m-1n} - \frac{i}{\hbar}\cdot n\cdot\tilde W^{eh}\cdot V^{kl}_{mn-1}$$

and

$$VM2^{kl}_{mn} := -\frac{i}{\hbar}\cdot k\cdot\tilde W^{eh}\cdot V^{k-1l}_{mn} - \frac{i}{\hbar}\cdot l\cdot\tilde W^{hh}\cdot V^{kl-1}_{mn} + \qquad (63)$$
$$+\frac{i}{\hbar}\cdot m\cdot\tilde W^{eh}\cdot V^{kl}_{m-1n} + \frac{i}{\hbar}\cdot n\cdot\tilde W^{hh}\cdot V^{kl}_{mn-1} .$$

It turns out that the infinite systems (55) - (59) can be reduced further by introducing the expectation values

$$\bar O^{kl}_{mn} := O^{kl}_{mn}\ ,\ \bar E^{kl}_{mn} := E^{kl}_{mn}\ ,\ \bar H^{kl}_{mn} := \langle (A^+)^k(B^+)^l A^m B^n h h^+\rangle\ , \qquad (64)$$

$$\bar P^{kl}_{mn} := P^{kl}_{mn}\ ,\ \bar V^{kl}_{mn} := \langle (A^+)^k(B^+)^l A^m B^n e^+ h h^+ e\rangle\ ,\ \bar{P}A^{kl}_{mn} = PA^{kl}_{mn}\ . \qquad (65)$$

Making use of these definitions, we obtain from (55) - (59) the equations

$$\frac{\partial}{\partial t}\bar O^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar O^{kl}_{mn} + \bar{E}M1^{kl}_{mn} - \bar{H}M2^{kl}_{mn} + \bar{O}M2^{kl}_{mn}\ , \qquad (66)$$

$$\frac{\partial}{\partial t}\bar{E}^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar{E}^{kl}_{mn} - \frac{i}{\hbar}\cdot S\cdot\bar{P}^{kl}_{mn} + \frac{i}{\hbar}\cdot S^*\cdot\left(\bar{P}^{mn}_{kl}\right)^* + \\ +\bar{E}M1^{kl}_{mn} + \bar{E}M2^{kl}_{mn} - \bar{V}M2^{kl}_{mn} \ , \tag{67}$$

$$\frac{\partial}{\partial t}\bar{H}^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar{H}^{kl}_{mn} + \frac{i}{\hbar}\cdot S\cdot\bar{P}^{kl}_{mn} - \frac{i}{\hbar}\cdot S^*\cdot\left(\bar{P}^{mn}_{kl}\right)^* + \\ +\bar{V}M1^{kl}_{mn} \ , \tag{68}$$

$$\frac{\partial}{\partial t}\bar{V}^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar{V}^{kl}_{mn} + \bar{V}M1^{kl}_{mn} \tag{69}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\bar{P}^{kl}_{mn} = & \frac{i}{\hbar}\cdot(\varepsilon_0+\varphi_0+(k+l-m-n)\hbar\omega)\cdot\bar{P}^{kl}_{mn} + \\ & +\frac{i}{\hbar}\cdot S^*\cdot\left(\bar{H}^{kl}_{mn}-\bar{E}^{kl}_{mn}\right) - 2\cdot\frac{i}{\hbar}\cdot W^{eh}\cdot\bar{P}^{kl}_{mn} + \\ & +\frac{i}{\hbar}\cdot\left(\bar{P}^{k+1l}_{mn}-\bar{P}^{kl+1}_{mn}\right) + \frac{i}{\hbar}\cdot\left(\bar{P}^{kl}_{m+1n}-\bar{P}^{kl}_{mn+1}\right) + \\ & +\frac{i}{\hbar}\cdot k\cdot\left(\tilde{W}^{ee}-\tilde{W}^{eh}\right)\cdot\bar{P}^{k-1l}_{mn} + \frac{i}{\hbar}\cdot l\cdot\left(\tilde{W}^{eh}-\tilde{W}^{hh}\right)\cdot\bar{P}^{kl-1}_{mn} \ .\end{aligned} \tag{70}$$

Eq. (69) exhibits an interesting feature of our two-level quantum dot model: In combination with definition (65), we conclude that

$$\bar{V}^{kl}_{mn}(t) = 0 \ , \ V^{kl}_{mn}(t) = E^{kl}_{mn}(t) \tag{71}$$

is true provided that the initial conditions

$$V^{kl}_{mn}(t=0) = 0 \ , \ E^{kl}_{mn}(t=0) = 0 \tag{72}$$

are fulfilled. Under these circumstances, we obtain from Eqs. (67), (68) and (70):

$$\frac{\partial}{\partial t}\bar{E}^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar{E}^{kl}_{mn} - \frac{i}{\hbar}\cdot S\cdot\bar{P}^{kl}_{mn} + \frac{i}{\hbar}\cdot S^*\cdot\left(\bar{P}^{mn}_{kl}\right)^* + \\ +\bar{E}M1^{kl}_{mn} + \bar{E}M2^{kl}_{mn} \ , \tag{73}$$

$$\frac{\partial}{\partial t}\bar{H}^{kl}_{mn} = \frac{i}{\hbar}\cdot(k+l-m-n)\hbar\omega\cdot\bar{H}^{kl}_{mn} + \frac{i}{\hbar}\cdot S\cdot\bar{P}^{kl}_{mn} - \frac{i}{\hbar}\cdot S^*\cdot\left(\bar{P}^{mn}_{kl}\right)^* \ , \tag{74}$$

$$\frac{\partial}{\partial t}\bar{P}^{kl}_{mn} = \frac{i}{\hbar}\cdot(\varepsilon_0+\varphi_0+(k+l-m-n)\hbar\omega)\cdot\bar{P}^{kl}_{mn} + \\ +\frac{i}{\hbar}\cdot S^*\cdot\left(\bar{H}^{kl}_{mn}-\bar{E}^{kl}_{mn}\right) - 2\cdot\frac{i}{\hbar}\cdot W^{eh}\cdot\bar{P}^{kl}_{mn} + \\ +\frac{i}{\hbar}\cdot\left(\bar{P}^{k+1l}_{mn}-\bar{P}^{kl+1}_{mn}\right)+\frac{i}{\hbar}\cdot\left(\bar{P}^{kl}_{m+1n}-\bar{P}^{kl}_{mn+1}\right) + \\ +\frac{i}{\hbar}\cdot k\cdot\left(\tilde{W}^{ee}-\tilde{W}^{eh}\right)\cdot\bar{P}^{k-1l}_{mn}+\frac{i}{\hbar}\cdot l\cdot\left(\tilde{W}^{eh}-\tilde{W}^{hh}\right)\cdot\bar{P}^{kl-1}_{mn}\ . \tag{75}$$

Consequently, if the initial conditions (72) are fulfilled, only three infinite systems of coupled equations are necessary to calculate the expectation values $\bar{E}^{kl}_{mn}$, $\bar{H}^{kl}_{mn}$ and $\bar{P}^{kl}_{mn}$. Moreover, the source term in Eq. (54) is nothing but the phonon-assisted electronic particle number E^{kl}_{mn}.

D. Relationship between the phonon-induced and the pure Coulomb matrix elements

Any phonon-induced Coulomb matrix element $\tilde{W}$ is related to the corresponding Coulomb matrix element W by the relation

$$\tilde{W} = \frac{\hbar\omega}{2}\cdot\left(1-\frac{\epsilon(\infty)}{\epsilon(0)}\right)\cdot W\ ; \tag{76}$$

here, $\epsilon(0)$ and $\epsilon(\infty)$ are the static and high-frequency dielectric constants, respectively. In order to derive Eq. (76), we start (without loss of generality) from the phonon-induced Coulomb matrix element

$$\tilde{W}^{eh} \equiv \sum_q |g_q|^2\cdot M^e_{q00}\cdot\left(M^h_{q00}\right)^* \rightarrow \frac{V}{(2\pi)^3}\cdot\int d^3q|g_q|^2\cdot M^e_{q00}\cdot\left(M^h_{q00}\right)^*\ , \tag{77}$$

which is generated by the commutator $[H_1,\hat{O}\hat{E}]$. The corresponding Coulomb matrix element W^{eh} is given by

$$W^{eh} = \int d^3r d^3r' \Psi^*_0(r)\Phi^*_0(r')V(r-r')\Phi_0(r')\Psi_0(r)\ , \tag{78}$$

where

$$V(r-r') = \frac{e^2}{4\pi\epsilon_0\epsilon(\infty)|r-r'|}\ . \tag{79}$$

Performing the Fourier transform

$$V(r) = \int d^3q \tilde{V}(q) e^{-iq\cdot r} \ , \ \tilde{V}(q) = \frac{e^2}{8\pi^3 \epsilon_0 \epsilon(\infty)} \cdot \frac{1}{q^2} \ , \tag{80}$$

one obtains:

$$W^{eh} = \frac{1}{\pi^2} \cdot E_R \cdot a_B \cdot \int d^3q \frac{1}{q^2} \cdot M^e_{q00} \cdot \left(M^h_{q00} \right)^* \ . \tag{81}$$

Here, we have introduced the *excitonic Rydberg energy*

$$E_R = \frac{e^2}{8\pi\epsilon_0 \cdot \epsilon(\infty) \cdot a_B} \tag{82}$$

and the *excitonic Bohr radius* a_B. The particle-phonon coupling function g_q (Fröhlich coupling) is defined by the equation

$$g_q = \sqrt{\frac{4\pi\alpha}{V}} \cdot \sqrt{a} \cdot \hbar\omega \cdot \frac{1}{|q|} \ , \tag{83}$$

where $a \equiv \sqrt{\hbar/2m\omega}$ is the so-called polaron radius; the particle-phonon coupling constant α is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar} \cdot \sqrt{\frac{m}{2\hbar\omega}} \cdot \left(\frac{1}{\epsilon(\infty)} - \frac{1}{\epsilon(0)} \right) = \frac{E_R}{\hbar\omega} \cdot \frac{1}{a} \cdot \left(1 - \frac{\epsilon(\infty)}{\epsilon(0)} \right) \cdot a_B \ . \tag{84}$$

Note that g_q does not depend on m. From Eqs. (83) and (84), we obtain:

$$\frac{V}{(2\pi)^3} \cdot |g_q|^2 = \frac{1}{2\pi^2} \cdot E_R \cdot \hbar\omega \cdot \left(1 - \frac{\epsilon(\infty)}{\epsilon(0)} \right) \cdot a_B \cdot \frac{1}{q^2} \ . \tag{85}$$

Eq. (76) now follows immediately from Eqs. (77), (81) and (85).

II. COMMENTS

In our next section, we comment on some approximations which have been introduced in section I in the framework of *a numerical calculation of the photon number* $\langle a^+ a \rangle(t)$. To do so, we refer again to the general commutation relation (16), to Eqs. (17)-(21) and to Eq. (25).

By taking $\hat{O} = 1$ and $\hat{E} = h_m e_n$, one obtains from Eq. (25) the equation of motion for the expectation value $\langle h_m e_n a \rangle$:

$$\frac{\partial}{\partial t}\langle h_m e_n a\rangle = -\frac{i}{\hbar} \cdot V_{nmk_0\lambda_0} + \ldots \tag{86}$$
$$\ldots + (expectation\ values\ of\ normal - ordered\ operators)\ .$$

Relation (86) demonstrates that *the complete infinite system* of coupled equations for the expectation values is *inhomogeneous*, the reason being that at least one overlap integral is different from zero (otherwise, the emission of photons would be impossible). Consequently, even for $S = 0$, i.e., in the absence of an exciting light field, the infinite system of equations must have a non-trivial solution. If the initial state $|\Psi(t = 0)\rangle = |0\rangle$ is chosen, such a non-trivial solution leads to dynamical vacuum fluctuations of the quantized electromagnetic field. It is therefore obvious that the vacuum state is *not* an eigenstate of the Hamiltonian under consideration, i.e., the standard particle-photon interaction operator does in fact *not* provide an appropriate physical model in solid state physics.

While the source term in Eq. (86) is due to the interaction operator $\sim A \cdot p$, there are further source terms *in other equations* which can be derived from the contribution $\sim A^2$. Direct inspection of the general commutation relation (16) and Eqs. (17)-(21) shows that a source term of type (86) (i.e., a constant) can only appear in the equations of motion for expectation values of type

$$\langle \ldots a^+\rangle\ ,\ \langle \ldots a\rangle\ ,\ \langle \ldots (a^+)^2\rangle\ ,\ \langle \ldots a^2\rangle\ . \tag{87}$$

By taking $i = 2, j = 0, \hat{O} = 1$ and $\hat{E} = e_i^+ h_j^+$, one obtains, for example, from Eq. (16) the equation of motion for the expectation value $\langle e_i^+ h_j^+ (a^+)^2\rangle$:

$$\frac{\partial}{\partial t}\langle e_i^+ h_j^+ (a^+)^2\rangle = 2 \cdot \frac{i}{\hbar} \cdot Q^*_{ijk_0k_0\lambda_0\lambda_0} + \cdots \tag{88}$$
$$\ldots + (expectation\ values\ of\ normal - ordered\ operators)\ .$$

The corresponding equation for the expectation value $\langle h_j e_i a^2\rangle$ is immediately obtained from Eq. (88) or from Eq. (16) by taking $i = 0, j = 2, \hat{O} = 1$ and $\hat{E} = h_j e_i$ (if the second alternative is chosen, Eq. (88) and the following equation do provide an additional check for the validity of Eqs. (16)-(21)):

$$\frac{\partial}{\partial t}\langle h_j e_i a^2\rangle = -2\cdot\frac{i}{\hbar}\cdot Q_{ijk_0k_0\lambda_0\lambda_0} + \dots \tag{89}$$
$$\dots + (expectation\ values\ of\ normal-ordered\ operators)\ .$$

It is an interesting task to investigate the standard particle-photon interaction further. Intuitively, one might expect that $\langle hea\rangle = 0$ does provide a good approximation for the analysis of the Heisenberg equations of motion. This seems to be supported by the physical interpretation that an electron-hole pair *and* a photon are removed from the system under consideration[8]. But is this really true? Apart from the statement that the expectation value $\langle hea\rangle$ has a *constant source term* (see again Eq. (86)), there is in fact a more serious problem which can be illustrated in more detail in the framework of a simplified model.

Let

$$H = H_1 + H_2\ , \tag{90}$$

where

$$H_1 = H_1(e, e^+, h, h^+)\ ,\ H_2 = \hbar\Omega a^+ a + (M\cdot a + M^*\cdot a^+)(e^+h^+ + he)\ . \tag{91}$$

Hamiltonian (90) describes a purely electronic two-level quantum dot model coupled to a single photon mode.

Now, let

$$\hat{E} = \hat{E}(e, e^+, h, h^+) \tag{92}$$

be a functional form of the electron and hole operators e, e^+ and h, h^+, respectively. A straightforward calculation shows that the general commutation relation

$$[H, \hat{E}(a^+)^i a^j] = [H_1, \hat{E}](a^+)^i a^j + (i-j)\cdot\hbar\Omega\cdot\hat{E}(a^+)^i a^j + \tag{93}$$
$$+ i\cdot M\cdot(e^+h^+ + he)\hat{E}(a^+)^{i-1}a^j - j\cdot M^*\cdot\hat{E}(e^+h^+ + he)(a^+)^i a^{j-1} +$$
$$+[(e^+h^+ + he), \hat{E}](M\cdot(a^+)^i a^{j+1} + M^*\cdot(a^+)^{i+1}a^j)$$

is true. Introducing the operator

$$\hat{P} := e^+h^+ + he \ , \tag{94}$$

one obtains from Eq. (93) the commutation relations

$$[H, a^+a] = M \cdot \hat{P}a - M^* \cdot \hat{P}a^+ \ , \tag{95}$$

$$[H, \hat{P}^n a] = [H_1, \hat{P}^n]a - \hbar\Omega \cdot \hat{P}^n a - M^* \cdot \hat{P}^{n+1} \tag{96}$$

and

$$[H, \hat{P}^n] = [H_1, \hat{P}^n] \ . \tag{97}$$

Additionally, we have

$$\hat{P}^2 = 2 \cdot e^+h^+he + 1 - e^+e - h^+h \ , \ \hat{P}^3 = \hat{P} \ . \tag{98}$$

To be explicit, we now choose

$$H_1 = \varepsilon_0 e^+e + \varphi_0 h^+h + S \cdot e^+h^+ + S^* \cdot he - 2 \cdot W^{eh} \cdot e^+h^+he \ . \tag{99}$$

From Eqs. (95)-(98), we thus obtain the following equations of motion:

$$\frac{\partial}{\partial t}\langle a^+a\rangle = \frac{i}{\hbar} \cdot M \cdot \langle \hat{P}a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{P}a^+\rangle \ , \tag{100}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\langle \hat{P}a\rangle = -\frac{i}{\hbar} \cdot \hbar\Omega \cdot \langle \hat{P}a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{P}^2\rangle + \\ &+\frac{i}{\hbar} \cdot (\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh}) \cdot \langle e^+h^+a\rangle + \frac{i}{\hbar} \cdot (-\varepsilon_0 - \varphi_0 + 2 \cdot W^{eh}) \cdot \langle hea\rangle + \\ &+\frac{i}{\hbar} \cdot (S^* - S) \cdot (\langle a\rangle - \langle e^+ea\rangle - \langle h^+ha\rangle) \end{aligned} \tag{101}$$

and

$$\frac{\partial}{\partial t}\langle \hat{P}^2\rangle = 0 \ . \tag{102}$$

The latter equation demonstrates in combination with relation (98) that

$$\langle \hat{P}^2 \rangle = 1 \tag{103}$$

is true provided that the usual initial conditions are fulfilled.

So far everything is exact. Now, we introduce the approximation

$$\langle hea \rangle = 0 \; . \tag{104}$$

What are the consequences? Making use of the definitions

$$\Omega_E := \frac{1}{\hbar} \cdot (\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh}) \; , \tag{105}$$

$$PA := \frac{i}{\hbar} \cdot M \cdot \langle e^+ h^+ a \rangle \; , \; A := \frac{i}{\hbar} \cdot M \cdot \langle a \rangle \; , \tag{106}$$

$$EA := \frac{i}{\hbar} \cdot M \cdot \langle e^+ ea \rangle \; , \; HA := \frac{i}{\hbar} \cdot M \cdot \langle h^+ ha \rangle \; , \tag{107}$$

we first obtain the equations

$$\frac{\partial}{\partial t} \langle a^+ a \rangle = PA + (PA)^* \tag{108}$$

and

$$\frac{\partial}{\partial t} PA = i \cdot (\Omega_E - \Omega) \cdot PA + \frac{|M|^2}{\hbar^2} + \frac{i}{\hbar} \cdot (S^* - S) \cdot (A - EA - HA) \; . \tag{109}$$

In a next step, we choose $\Omega = \Omega_E$ (i.e., the photon energy is equal to the (unrenormalized) excitonic energy) and assume $S = 0$ (i.e., there is no external light field). Under these circumstances, we obtain from Eqs. (108) and (109) *the exact solution*

$$\langle a^+ a \rangle = \frac{|M|^2}{\hbar^2} \cdot t^2 \; , \tag{110}$$

provided that the usual initial conditions are fulfilled. Although we have to keep in mind that vacuum fluctuations of the quantized electromagnetic field do exist in our model, solution (110) can not be accepted for obvious physical reasons. Consequently, relation (104) does not provide an appropriate approximation to calculate the photon number $\langle a^+ a \rangle$ from the Heisenberg equations of motion. We emphasize that these results can be extended to a more general class of initial states; details can be found in section III.

III. THE BEGINNING OF THE INFINITE HIERARCHY OF EQUATIONS OF MOTION FOR THE PHOTON-ASSISTED EXPECTATION VALUES

In our third section, we present a list of exact equations which appear at the beginning of the infinite hierarchy of photon-assisted expectation values in the framework of the two-level quantum dot model (90), (91) and (99).

First of all, the photon number $\langle a^+a\rangle$ is determined by the equation

$$\frac{\partial}{\partial t}\langle a^+a\rangle = \frac{i}{\hbar}\cdot M\cdot\langle\hat{P}a\rangle - \frac{i}{\hbar}\cdot M^*\cdot\langle\hat{P}a^+\rangle\ . \tag{111}$$

Here, the expectation value $\langle\hat{P}a\rangle$ fulfills the equation

$$\begin{aligned}\frac{\partial}{\partial t}\langle\hat{P}a\rangle &= -\frac{i}{\hbar}\cdot\hbar\Omega\cdot\langle\hat{P}a\rangle - \frac{i}{\hbar}\cdot M^*\cdot\langle\hat{P}^2\rangle + \\ &+\frac{i}{\hbar}\cdot(\varepsilon_0+\varphi_0-2\cdot W^{eh})\cdot\langle e^+h^+a\rangle + \frac{i}{\hbar}\cdot(-\varepsilon_0-\varphi_0+2\cdot W^{eh})\cdot\langle hea\rangle + \\ &+\frac{i}{\hbar}\cdot(S^*-S)\cdot(\langle a\rangle - \langle e^+ea\rangle - \langle h^+ha\rangle)\ ,\end{aligned} \tag{112}$$

where

$$\frac{\partial}{\partial t}\langle\hat{P}^2\rangle = 0 \tag{113}$$

is true. Making use of definitions (105)-(107), Eqs. (111) and (112) read as follows:

$$\frac{\partial}{\partial t}\langle a^+a\rangle = PA + (PA)^* + \frac{i}{\hbar}\cdot M\cdot\langle hea\rangle - \frac{i}{\hbar}\cdot M^*\cdot\langle e^+h^+a^+\rangle\ , \tag{114}$$

$$\begin{aligned}\frac{\partial}{\partial t}PA + \frac{i}{\hbar}\cdot M\cdot\frac{\partial}{\partial t}\langle hea\rangle &= i\cdot(\Omega_E-\Omega)\cdot PA + \frac{|M|^2}{\hbar^2}\cdot\langle\hat{P}^2\rangle + \\ &+\frac{i}{\hbar}\cdot(S^*-S)\cdot(A-EA-HA) + \frac{M}{\hbar}\cdot(\Omega_E+\Omega)\cdot\langle hea\rangle\ .\end{aligned} \tag{115}$$

To obtain $\langle e^+h^+a\rangle, \langle hea\rangle, \langle a\rangle, \langle e^+ea\rangle$ and $\langle h^+ha\rangle$, we have the equations

$$\begin{aligned}\frac{\partial}{\partial t}\langle e^+h^+a\rangle &= -\frac{i}{\hbar}\cdot M^*\cdot\langle e^+h^+he\rangle + \\ &+\frac{i}{\hbar}\cdot(\varepsilon_0+\varphi_0-2\cdot W^{eh}-\hbar\Omega)\cdot\langle e^+h^+a\rangle + \frac{i}{\hbar}\cdot S^*\cdot\Big(\langle a\rangle - \langle e^+ea\rangle - \langle h^+ha\rangle\Big) + \\ &+\frac{i}{\hbar}\cdot M\cdot\Big(\langle a^2\rangle - \langle e^+ea^2\rangle - \langle h^+ha^2\rangle\Big) + \frac{i}{\hbar}\cdot M^*\cdot\Big(\langle a^+a\rangle - \langle e^+ea^+a\rangle - \langle h^+ha^+a\rangle\Big)\ ,\end{aligned} \tag{116}$$

$$\frac{\partial}{\partial t}\langle hea\rangle = -\frac{i}{\hbar}\cdot M^* - \frac{i}{\hbar}\cdot M^*\cdot\Big(\langle e^+h^+he\rangle - \langle e^+e\rangle - \langle h^+h\rangle\Big) + \tag{117}$$
$$+\frac{i}{\hbar}\cdot(-\varepsilon_0 - \varphi_0 + 2\cdot W^{eh} - \hbar\Omega)\cdot\langle hea\rangle - \frac{i}{\hbar}\cdot S\cdot\Big(\langle a\rangle - \langle e^+ea\rangle - \langle h^+ha\rangle\Big) +$$
$$-\frac{i}{\hbar}\cdot M\cdot\Big(\langle a^2\rangle - \langle e^+ea^2\rangle - \langle h^+ha^2\rangle\Big) - \frac{i}{\hbar}\cdot M^*\cdot\Big(\langle a^+a\rangle - \langle e^+ea^+a\rangle - \langle h^+ha^+a\rangle\Big)\ ,$$

$$\frac{\partial}{\partial t}\langle a\rangle = -\frac{i}{\hbar}\cdot\hbar\Omega\cdot\langle a\rangle - \frac{i}{\hbar}\cdot M^*\cdot\langle\hat{P}\rangle\ , \tag{118}$$

$$\frac{\partial}{\partial t}\langle e^+ea\rangle = -\frac{i}{\hbar}\cdot M^*\cdot\langle e^+h^+\rangle - \frac{i}{\hbar}\cdot\hbar\Omega\cdot\langle e^+ea\rangle + \tag{119}$$
$$-\frac{i}{\hbar}\cdot S\cdot\langle e^+h^+a\rangle + \frac{i}{\hbar}\cdot S^*\cdot\langle hea\rangle +$$
$$+\frac{i}{\hbar}\cdot M\cdot\Big(-\langle e^+h^+a^2\rangle + \langle hea^2\rangle\Big) + \frac{i}{\hbar}\cdot M^*\cdot\Big(-\langle e^+h^+a^+a\rangle + \langle hea^+a\rangle\Big)$$

and

$$\frac{\partial}{\partial t}\langle h^+ha\rangle = -\frac{i}{\hbar}\cdot M^*\cdot\langle e^+h^+\rangle - \frac{i}{\hbar}\cdot\hbar\Omega\cdot\langle h^+ha\rangle + \tag{120}$$
$$-\frac{i}{\hbar}\cdot S\cdot\langle e^+h^+a\rangle + \frac{i}{\hbar}\cdot S^*\cdot\langle hea\rangle +$$
$$+\frac{i}{\hbar}\cdot M\cdot\Big(-\langle e^+h^+a^2\rangle + \langle hea^2\rangle\Big) + \frac{i}{\hbar}\cdot M^*\cdot\Big(-\langle e^+h^+a^+a\rangle + \langle hea^+a\rangle\Big)\ .$$

Eqs. (116)-(120) require the knowledge of the expectation values $\langle e^+h^+\rangle, \langle e^+e\rangle, \langle h^+h\rangle$ and $\langle e^+h^+he\rangle$. The corresponding equations of motion read as follows:

$$\frac{\partial}{\partial t}\langle e^+h^+\rangle = \frac{i}{\hbar}\cdot(\varepsilon_0 + \varphi_0 - 2\cdot W^{eh})\cdot\langle e^+h^+\rangle + \frac{i}{\hbar}\cdot S^*\cdot(1 - \langle e^+e\rangle - \langle h^+h\rangle) + \tag{121}$$
$$+\frac{i}{\hbar}\cdot M\cdot(\langle a\rangle - \langle e^+ea\rangle - \langle h^+ha\rangle) + \frac{i}{\hbar}\cdot M^*\cdot(\langle a^+\rangle - \langle e^+ea^+\rangle - \langle h^+ha^+\rangle)\ ,$$

$$\frac{\partial}{\partial t}\langle e^+e\rangle = -\frac{i}{\hbar}\cdot S\cdot\langle e^+h^+\rangle + \frac{i}{\hbar}\cdot S^*\cdot\langle he\rangle + \tag{122}$$
$$+\frac{i}{\hbar}\cdot M\cdot(-\langle e^+h^+a\rangle + \langle hea\rangle) + \frac{i}{\hbar}\cdot M^*\cdot(-\langle e^+h^+a^+\rangle + \langle hea^+\rangle)\ ,$$

$$\frac{\partial}{\partial t}\langle h^+h\rangle = -\frac{i}{\hbar}\cdot S\cdot\langle e^+h^+\rangle + \frac{i}{\hbar}\cdot S^*\cdot\langle he\rangle + \tag{123}$$
$$+\frac{i}{\hbar}\cdot M\cdot(-\langle e^+h^+a\rangle + \langle hea\rangle) + \frac{i}{\hbar}\cdot M^*\cdot(-\langle e^+h^+a^+\rangle + \langle hea^+\rangle)$$

and

$$\frac{\partial}{\partial t}\langle e^+h^+he\rangle = -\frac{i}{\hbar}\cdot S\cdot\langle e^+h^+\rangle + \frac{i}{\hbar}\cdot S^*\cdot\langle he\rangle + \\ +\frac{i}{\hbar}\cdot M\cdot(-\langle e^+h^+a\rangle + \langle hea\rangle) + \frac{i}{\hbar}\cdot M^*\cdot(-\langle e^+h^+a^+\rangle + \langle hea^+\rangle)\ . \tag{124}$$

Eq. (117) is of particular interest, because it contains the source term $(-i/\hbar)\cdot M^*$ (i.e., a constant). As we have already mentioned above, such a source term leads to dynamical vacuum fluctuations of the quantized electromagnetic field. *From a physical point of view, however, this is impossible!* In sections IV-VI, we shall discuss this problem in more detail.

A further remark is concerned with the considerations at the end of section II. From Eqs. (122)-(124), we first conclude that the relations

$$\langle e^+h^+he\rangle - \langle e^+e\rangle = const. = C_1\ ,\ \langle e^+h^+he\rangle - \langle h^+h\rangle = const. = C_2 \tag{125}$$

and

$$\langle e^+e\rangle - \langle h^+h\rangle = const. = C_3 \tag{126}$$

are valid. Consequently, we obtain, in combination with Eq. (98):

$$\langle\hat{P}^2\rangle = 1 + C_1 + C_2\ . \tag{127}$$

If, for example, the initial states $|\Psi(t=0)\rangle = |0\rangle$ or $|\Psi(t=0)\rangle = e^+h^+|0\rangle$ are chosen, we obtain $C_1 = C_2 = 0$ and therefore $\langle\hat{P}^2\rangle(t) = 1$.

We are now going to generalize our results obtained at the end of section II. We choose the initial state

$$|\Psi(t=0)\rangle = \alpha_1\cdot|0\rangle + \alpha_2\cdot e^+|0\rangle + \alpha_3\cdot h^+|0\rangle + \alpha_4\cdot e^+h^+|0\rangle\ , \tag{128}$$

where

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1\ . \tag{129}$$

Under these circumstances, we obtain by a straightforward calculation:

$$\langle e^+h^+he\rangle(t=0) = |\alpha_4|^2\ , \tag{130}$$

$$\langle e^+e\rangle(t=0) = |\alpha_2|^2 + |\alpha_4|^2 \ , \ \langle h^+h\rangle(t=0) = |\alpha_3|^2 + |\alpha_4|^2 \tag{131}$$

and therefore

$$C_1 = -|\alpha_2|^2 \ , \ C_2 = -|\alpha_3|^2 \ . \tag{132}$$

Additionally, we have $PA(t=0) = 0$ and $\langle a^+a\rangle(t=0) = 0$. From Eqs. (127)-(132), we thus conclude that

$$\langle \hat{P}^2\rangle(t) = |\alpha_1|^2 + |\alpha_4|^2 = 1 - |\alpha_2|^2 - |\alpha_3|^2 \tag{133}$$

is true for any $t \geq 0$.

At this point, we introduce the approximation

$$\langle hea\rangle = 0 \ . \tag{134}$$

From Eqs. (114) and (115), we conclude that

$$\frac{\partial}{\partial t}\langle a^+a\rangle = PA + (PA)^* \tag{135}$$

and

$$\frac{\partial}{\partial t}PA = i\cdot(\Omega_E - \Omega)\cdot PA + \frac{|M|^2}{\hbar^2}\cdot\langle\hat{P}^2\rangle + \frac{i}{\hbar}\cdot(S^* - S)\cdot(A - EA - HA) \tag{136}$$

is true under these circumstances.

Now, we choose $\Omega = \Omega_E$ (i.e., the photon energy is equal to the (unrenormalized) excitonic energy) and assume $S = 0$ (i.e., there is no external light field). If these conditions are fulfilled, we obtain from Eqs. (133), (135) and (136) *the exact solution*

$$\langle a^+a\rangle = \frac{|M|^2}{\hbar^2}\cdot(|\alpha_1|^2 + |\alpha_4|^2)\cdot t^2 \ . \tag{137}$$

We thus conclude that in the presence of *electron-hole correlations*, i.e., for $\alpha_4 \neq 0$, approximation (134) is obviously not appropriate to calculate the photon number $\langle a^+a\rangle$ from the Heisenberg equations of motion. It is only for $\alpha_1 = \alpha_4 = 0$, i.e., *for a pure electron-hole gas*, that relation (134) may provide a reasonable approximation. We summarize our results:

Theorem 1:

Consider a purely electronic two-level quantum dot model, as defined by Eqs. (90), (91) and (99) with $\Omega = \Omega_E$. Choose the initial state

$$|\Psi(t=0)\rangle = \alpha_1 \cdot |0\rangle + \alpha_2 \cdot e^+|0\rangle + \alpha_3 \cdot h^+|0\rangle + \alpha_4 \cdot e^+h^+|0\rangle \tag{138}$$

and assume that the exciting light field vanishes. Then, the relation

$$\langle a^+a\rangle = \frac{|M|^2}{\hbar^2} \cdot (|\alpha_1|^2 + |\alpha_4|^2) \cdot t^2 \tag{139}$$

is true, if the approximation $\langle hea\rangle = 0$ is introduced (see text for details). The constant M is the particle-photon coupling constant.

IV. DYNAMICAL VACUUM FLUCTUATIONS OF THE QUANTIZED ELECTROMAGNETIC FIELD IN THE STANDARD MODEL: THE ANALYSIS OF A PURELY ELECTRONIC TWO-LEVEL QUANTUM DOT COUPLED TO A SINGLE PHOTON MODE

In this section, we are going to analyze the dynamical vacuum fluctuations of the quantized electromagnetic field due to the *standard particle-photon interaction* in more detail. To do so, we consider a purely electronic two-level quantum dot model coupled to a single photon mode. If the exciting light field vanishes, the corresponding Hamiltonian H reads as follows:

$$H = H_1 + H_2 \,, \tag{140}$$

where

$$H_1 = \varepsilon_0 \cdot e^+e + \varphi_0 \cdot h^+h - 2 \cdot W^{eh} \cdot e^+h^+he \tag{141}$$

and

$$H_2 = \hbar\Omega \cdot a^+a + \Big(M \cdot a + M^* \cdot a^+\Big) \cdot \Big(e^+h^+ + he\Big) \ . \tag{142}$$

Direct inspection of Eq. (142) shows that the particle-photon interaction operator is in fact of standard type, i.e., the photons couple to the polarization. It is therefore obvious that the vacuum state $|0\rangle$ is not an eigenstate of the complete Hamiltonian H. Consequently, if the initial state

$$|\Psi(t=0)\rangle = |0\rangle \tag{143}$$

is chosen, the solution $|\Psi(t)\rangle$ of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{144}$$

has a non-trivial time evolution. The same statement is true, for example, for the expectation values $\langle\Psi(t)|a^+a|\Psi(t)\rangle, \langle\Psi(t)|e^+e|\Psi(t)\rangle, \langle\Psi(t)|h^+h|\Psi(t)\rangle$ and $\langle\Psi(t)|e^+h^+he|\Psi(t)\rangle$. The expectation value $\langle\Psi(t)|e^+h^+|\Psi(t)\rangle$ is an exception; as we shall see below, the polarization function is equal to zero for any t. In the following part of our considerations, we are going to determine the time evolution of the state $|\Psi(t)\rangle$ under the initial condition (143). We proceed as follows:

First of all, we remark that the electronic part of the Hilbert space under consideration is defined by the basic states

$$|1\rangle = |0_{eh}\rangle \ , \ |2\rangle = e^+|0_{eh}\rangle \ , \ |3\rangle = h^+|0_{eh}\rangle \ , \ |4\rangle = e^+h^+|0_{eh}\rangle \ . \tag{145}$$

From Eqs. (141) and (145), we obtain

$$H_1|i\rangle|\Phi_P\rangle = \lambda_i \cdot |i\rangle|\Phi_P\rangle \ , \ i = 1 \ , \ 2 \ , \ 3 \ , \ 4 \ , \tag{146}$$

where

$$\lambda_1 = 0 \ , \ \lambda_2 = \varepsilon_0 \ , \ \lambda_3 = \varphi_0 \ , \ \lambda_4 = \varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} \ . \tag{147}$$

$|\Phi_P\rangle$ denotes an element of the photonic Fock space. For the following part of our considerations, we need the basic states

$$|e_1\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle + |4\rangle) \ , \ |e_2\rangle = |2\rangle \ , \ |e_3\rangle = |3\rangle \ , \ |e_4\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle - |4\rangle) \ . \tag{148}$$

Making use of definition (148), we obtain

$$H_2|e_i\rangle|\Phi_P\rangle = \Big(\hbar\Omega \cdot a^+ a + P_i \cdot \Big(M \cdot a + M^* \cdot a^+\Big)\Big)\,|e_i\rangle|\Phi_P\rangle \ , \tag{149}$$

where

$$P_1 = 1 \ , \ P_2 = 0 \ , \ P_3 = 0 \ , \ P_4 = -1 \ . \tag{150}$$

At this point, we emphasize again that we are interested in the calculation of the vacuum fluctuations which appear in the model under consideration. We already know from our general results in section II that the existence and the strength of the constant source terms in the Heisenberg equations of motion do not depend on the electronic part of the model Hamiltonian, but only on the particle-photon interaction operator. In order to analyze a non-trivial example, it is therefore reasonable to introduce the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{151}$$

for the Coulomb matrix element W^{eh}. Under these circumstances, we obtain from Eqs. (146) and (148) the relation

$$H_1|e_i\rangle|\Phi_P\rangle = \tilde{E}_i \cdot |e_i\rangle|\Phi_P\rangle \ , \tag{152}$$

where

$$\tilde{E}_1 = 0 \ , \ \tilde{E}_2 = \varepsilon_0 \ , \ \tilde{E}_3 = \varphi_0 \ , \ \tilde{E}_4 = 0 \ . \tag{153}$$

A. Eigenstates and energy spectrum of H

In view of Eqs. (149) and (152), Hamiltonian (140) can immediately be diagonalized. The eigenstates $|\Psi_{in}\rangle$ are given by

$$|\Psi_{in}\rangle = |e_i\rangle|P_{in}\rangle \ , \tag{154}$$

where

$$|P_{in}\rangle = \frac{1}{\sqrt{n!}} \cdot \left(a^+ + \gamma_i\right)^n \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_i|^2\right) \cdot \exp\left(-\gamma_i^* \cdot a^+\right) |0_{Phot.}\rangle \tag{155}$$

and

$$\gamma_i = \frac{M \cdot P_i}{\hbar\Omega} \ . \tag{156}$$

The energy spectrum of H is obtained from the equation

$$E_{in} = \tilde{E}_i + \left(n - |\gamma_i|^2\right) \cdot \hbar\Omega \ . \tag{157}$$

Direct inspection of Eq. (157) shows that our model Hamiltonian H is bounded from below; in particular, the inequality

$$E_{in} \geq E_{i0} \ , \ i = 1 \ , \ 2 \ , \ 3 \ , \ 4 \tag{158}$$

is immediately derived. Because of

$$E_{10} = -|\gamma_1|^2 \cdot \hbar\Omega \ , \ E_{20} = \varepsilon_0 \ , \ E_{30} = \varphi_0 \ , \ E_{40} = -|\gamma_4|^2 \cdot \hbar\Omega \ , \tag{159}$$

the ground-state energy E_0 is found to be

$$E_0 = -\frac{|M|^2}{\hbar\Omega} \ . \tag{160}$$

From Eqs. (150), (156) and (159), we conclude that there exist two eigenstates which belong to the ground-state energy E_0; these are

$$|\Psi_{10}\rangle = |e_1\rangle|P_{10}\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle + |4\rangle) \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_1|^2\right) \cdot \exp\left(-\gamma_1^* \cdot a^+\right) |0_{Phot.}\rangle \tag{161}$$

and

$$|\Psi_{40}\rangle = |e_4\rangle|P_{40}\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle - |4\rangle) \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_4|^2\right) \cdot \exp\left(-\gamma_4^* \cdot a^+\right) |0_{Phot.}\rangle \ . \tag{162}$$

Consequently, Hamiltonian (140) in combination with the additional condition (151) for the Coulomb matrix element W^{eh} does provide an example for *a degenerate ground-state energy.* Obviously, this property is in marked contrast to the basic features of polaronic systems. As has been demonstrated in Refs.[12–15], the ground-state energy of a large class of generalized *Fröhlich models* is in fact a simple eigenvalue.

According to Eqs. (161) and (162), the eigenstates $|\Psi_{10}\rangle$ and $|\Psi_{40}\rangle$ contain electrons, holes and photons. The same statement is true for the excited states

$$|\Psi_{1n}\rangle = |e_1\rangle|P_{1n}\rangle \ , \ |\Psi_{4n}\rangle = |e_4\rangle|P_{4n}\rangle \ , \ n \geq 1 \ ; \tag{163}$$

these eigenstates belong to the energies

$$E_{1n} = E_{4n} = n \cdot \hbar\Omega + E_0 \ , \ n \geq 1 \ . \tag{164}$$

Apart from the solutions (161), (162) and (163), there exist further eigenstates of our model Hamiltonian H which can be obtained from Eq. (155) by taking $i = 2$ or $i = 3$:

$$|\Psi_{2n}\rangle = |e_2\rangle|P_{2n}\rangle = |2\rangle\frac{1}{\sqrt{n!}} \cdot (a^+)^n|0_{Phot.}\rangle \ ; \tag{165}$$

$$|\Psi_{3n}\rangle = |e_3\rangle|P_{3n}\rangle = |3\rangle\frac{1}{\sqrt{n!}} \cdot (a^+)^n|0_{Phot.}\rangle \ . \tag{166}$$

The corresponding energy eigenvalues are found to be

$$E_{2n} = \varepsilon_0 + n \cdot \hbar\Omega \ , \ E_{3n} = \varphi_0 + n \cdot \hbar\Omega \ , \ n \geq 0 \ . \tag{167}$$

From Eq. (155), we obtain the useful relations

$$\left(a^+ + \gamma_i\right)|P_{in}\rangle = \sqrt{n+1} \cdot |P_{in+1}\rangle \tag{168}$$

and

$$(a + \gamma_i^*)\,|P_{in}\rangle = \sqrt{n} \cdot |P_{in-1}\rangle \ ; \tag{169}$$

consequently, a purely photonic operator sequence of type $(a^+)^m a^n$ always leads to *finite* superpositions of eigenstates $|\Psi_{in}\rangle$. In particular,

$$a^+a|P_{in}\rangle = \left(n + |\gamma_i|^2\right) \cdot |P_{in}\rangle - \gamma_i \cdot \sqrt{n} \cdot |P_{in-1}\rangle - \gamma_i^* \cdot \sqrt{n+1} \cdot |P_{in+1}\rangle \tag{170}$$

is true.

The generalization of Eqs. (168) and (169) is straightforward: For $k \geq 0$ and $l \geq 0$, we conclude that

$$(a^+)^k |P_{in}\rangle = \sum_{\nu=0}^{k} \binom{k}{\nu} \cdot (-\gamma_i)^{k-\nu} \cdot \sqrt{\frac{(n+\nu)!}{n!}} \cdot |P_{in+\nu}\rangle \qquad (171)$$

and

$$a^l |P_{in}\rangle = \sum_{\mu=0}^{l} \binom{l}{\mu} \cdot (-\gamma_i^*)^{l-\mu} \cdot \sqrt{\mu! \binom{n}{\mu}} \cdot |P_{in-\mu}\rangle \qquad (172)$$

is true. As far as Eq. (172) is concerned, the additional condition

$$\binom{n}{\mu} = 0 \; , \; if \;\; n < \mu \qquad (173)$$

has to be taken into account. A combination of Eqs. (171) and (172) gives immediately

$$(a^+)^k a^l |P_{in}\rangle = \sum_{m=-l}^{k} \Theta_{klm}^{(i)}(n) \cdot |P_{in+m}\rangle \; , \qquad (174)$$

where

$$\Theta_{klm}^{(i)}(n) := \sum_{-\mu+\nu=m; 0\leq\mu\leq l; 0\leq\nu\leq k} \sqrt{\mu! \binom{n}{\mu}} \cdot \sqrt{\frac{(n+m)!}{(n-\mu)!}} \cdot \binom{l}{\mu} \cdot \binom{k}{\nu} \cdot (-\gamma_i^*)^{l-\mu} \cdot (-\gamma_i)^{k-\nu} \; . \qquad (175)$$

Eq. (175) will be used at the end of this section in connection with expectation values of type $\langle \hat{E} (a^+)^k a^l \rangle (t)$, where $\hat{E}$ denotes an electron-hole operator sequence.

B. Time evolution of the state $|\Psi(t)\rangle$

We are now going to determine the time-evolution of the state $|\Psi(t)\rangle$ under the initial condition (143). First of all, we remark that the vacuum state $|0\rangle$ can be represented by a superposition of the eigenstates $|\Psi_{in}\rangle$:

$$|0\rangle = \sum_{i=1}^{4} \sum_{n=0}^{\infty} c_{in} \cdot |\Psi_{in}\rangle \; , \qquad (176)$$

where

$$c_{in} = \langle 0|\Psi_{in}\rangle^* = \frac{1}{\sqrt{n!}} \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_i|^2\right) \cdot (\gamma_i^*)^n \cdot \langle 0_{eh}|e_i\rangle^* \tag{177}$$

is immediately obtained from Eq. (155). The solution $|\Psi(t)\rangle$ is thus given by the equation

$$|\Psi(t)\rangle = \sum_{i=1}^{4} \sum_{n=0}^{\infty} c_{in} \cdot \exp\left(-\frac{i}{\hbar} \cdot E_{in} \cdot t\right) \cdot |\Psi_{in}\rangle\ . \tag{178}$$

C. Calculation of the expectation value $\langle a^+a\rangle(t)$

It is now a simple task to calculate the time-dependent expectation value $\langle\Psi(t)|a^+a|\Psi(t)\rangle$ under the initial condition (143). Making use of Eqs. (170), (177) and (178), we obtain the result

$$\langle\Psi(t)|a^+a|\Psi(t)\rangle = 2 \cdot |\gamma|^2 \cdot (1 - \cos\Omega \cdot t)\ ,\ \gamma := \frac{M}{\hbar\Omega}\ . \tag{179}$$

Direct inspection of Eq. (179) shows that the photon number $\langle a^+a\rangle(t)$ is proportional to the absolute square of the particle-photon coupling constant M. Equation (179) can also be obtained from Eqs. (111) and (112) in section III.

D. Calculation of the expectation values $\langle e^+e\rangle(t)$, $\langle h^+h\rangle(t)$, $\langle e^+h^+\rangle(t)$ and $\langle e^+h^+he\rangle(t)$

The calculation of the expectation values $\langle e^+e\rangle(t)$, $\langle h^+h\rangle(t)$, $\langle e^+h^+\rangle(t)$ and $\langle e^+h^+he\rangle(t)$ is more difficult. Our starting point is an electron-hole operator sequence $\hat{E}$ which is defined by the equation

$$\hat{E}|e_i\rangle = \sum_{l=1}^{4} \lambda_{il} \cdot |e_l\rangle\ . \tag{180}$$

The calculation of the expectation value $\langle\Psi(t)|\hat{E}|\Psi(t)\rangle$ requires the knowledge of the expressions $\langle P_{jm}|P_{in}\rangle$ for different indices i and j, where $|P_{in}\rangle$ is defined by Eq. (155); a detailed derivation of these expressions will be given now.

Introducing the unitary operator

$$U_i := \exp\left(-\gamma_i^* \cdot a^+ + \gamma_i \cdot a\right) \; , \; \gamma_i := \frac{M \cdot P_i}{\hbar\Omega} \; , \tag{181}$$

we first obtain the result

$$|P_{in}\rangle = U_i \frac{1}{\sqrt{n!}} \cdot \left(a^+\right)^n |0_{Phot.}\rangle \; ; \tag{182}$$

consequently, the desired expression $\langle P_{jm}|P_{in}\rangle$ is given by

$$\langle P_{jm}|P_{in}\rangle = \frac{1}{\sqrt{m!n!}} \cdot \langle 0_{Phot.}|a^m \; U_j^+ \; U_i \; (a^+)^n |0_{Phot.}\rangle \; . \tag{183}$$

In the following part of our considerations, the operator $U_j^+ U_i$ is of particular interest. Making use of the commutation relation

$$\left[\gamma_j^* \cdot a^+ - \gamma_j \cdot a \; , \; -\gamma_i^* \cdot a^+ + \gamma_i \cdot a\right] = \gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i \; , \tag{184}$$

this operator can be represented by an operator of exponential type:

$$U_j^+ \cdot U_i = \exp\left(\frac{1}{2} \cdot \left(\gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i\right)\right) \cdot \exp\left(\left(\gamma_j^* - \gamma_i^*\right) \cdot a^+ - \left(\gamma_j - \gamma_i\right) \cdot a\right) \; . \tag{185}$$

In a next step, we decompose the operator on the right-hand side of Eq. (185). To do so, we introduce the commutation relation

$$\left[-\left(\gamma_j - \gamma_i\right) \cdot a \; , \; \left(\gamma_j^* - \gamma_i^*\right) \cdot a^+\right] = -|\gamma_j - \gamma_i|^2 \; ; \tag{186}$$

the latter equation demonstrates in combination with relation (185) that

$$U_j^+ \cdot U_i = \exp\left(\frac{1}{2} \cdot \left(\gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i + |\gamma_j - \gamma_i|^2\right)\right) \cdot \\ \cdot \exp\left(-\left(\gamma_j - \gamma_i\right) \cdot a\right) \cdot \exp\left(\left(\gamma_j^* - \gamma_i^*\right) \cdot a^+\right) \tag{187}$$

is true.

Let us now assume $m \geq n$. Under these circumstances, the relation

$$\langle P_{jm}|P_{in}\rangle = \frac{1}{\sqrt{m!n!}} \cdot \exp\left(\frac{1}{2} \cdot \left(\gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i + |\gamma_j - \gamma_i|^2\right)\right) \cdot \left(\gamma_j^* - \gamma_i^*\right)^{m-n} \cdot \\ \cdot \sum_{k=0}^{\infty} \frac{(m+k)!}{k!(m-n+k)!} \cdot \left(-|\gamma_j - \gamma_i|^2\right)^k \; , \; m \geq n \tag{188}$$

is immediately obtained from Eqs. (183) and (187). In order to determine the sum

$$\Sigma_{mn}(x) := \sum_{k=0}^{\infty} \frac{(m+k)!}{k!(m-n+k)!} \cdot x^k \ , \ m \geq n \ , \tag{189}$$

we introduce the function

$$f_m(x) := \sum_{k=0}^{\infty} \frac{x^{k+m}}{k!} = x^m \cdot e^x \ . \tag{190}$$

A straightforward calculation shows that

$$\Sigma_{mn}(x) = \frac{f_m^{(n)}(x)}{x^{m-n}} \ , \ m \geq n \tag{191}$$

is true; consequently, the desired expression $\Sigma_{mn}(x)$ is given by

$$\Sigma_{mn}(x) = n! \cdot e^x \cdot \sum_{k=0}^{n} \binom{m}{n-k} \cdot \frac{x^k}{k!} \ . \tag{192}$$

Introducing the *Laguerre polynomials*

$$L_m^{\alpha}(x) = \sum_{k=0}^{m} (-1)^k \cdot \binom{m+\alpha}{m-k} \cdot \frac{x^k}{k!} \ , \tag{193}$$

we thus obtain from Eqs. (188) and (192) the result

$$\begin{aligned} \langle P_{jm} | P_{in} \rangle = \sqrt{\frac{n!}{m!}} \cdot \exp\left(\frac{1}{2} \cdot \left(\gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i - |\gamma_j - \gamma_i|^2\right)\right) \cdot \left(\gamma_j^* - \gamma_i^*\right)^{m-n} \cdot \\ \cdot L_n^{m-n}\left(|\gamma_j - \gamma_i|^2\right) \ , \ m \geq n \ . \end{aligned} \tag{194}$$

The corresponding equation for $m \leq n$ reads as follows:

$$\begin{aligned} \langle P_{jm} | P_{in} \rangle = \sqrt{\frac{m!}{n!}} \cdot \exp\left(\frac{1}{2} \cdot \left(\gamma_j \cdot \gamma_i^* - \gamma_j^* \cdot \gamma_i - |\gamma_j - \gamma_i|^2\right)\right) \cdot (\gamma_i - \gamma_j)^{n-m} \cdot \\ \cdot L_m^{n-m}\left(|\gamma_j - \gamma_i|^2\right) \ , \ m \leq n \ . \end{aligned} \tag{195}$$

At this point, we remark again that the expressions γ_i are defined by the equation $\gamma_i := M \cdot P_i / \hbar\Omega$, where P_i is a real number; making use of this definition, we obtain from Eqs. (194) and (195) the useful relation

$$\langle P_{im} | P_{jn} \rangle = (-1)^{|m-n|} \cdot \langle P_{jm} | P_{in} \rangle \ . \tag{196}$$

From Eqs. (157), (177), (180) and (196), we conclude that

$$\langle\Psi(t)|\hat{E}|\Psi(t)\rangle = \frac{1}{2}\cdot(\lambda_{11}+\lambda_{44}) + (\lambda_{14}+\lambda_{41})\cdot \\ \cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\cdot\exp\left(i(m-n)\Omega\cdot t\right)\cdot c_{1m}^*\cdot c_{1n}\cdot\langle P_{1m}|P_{4n}\rangle \tag{197}$$

is true. Obviously, the remaining task is the determination of the expression

$$D(t) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\cdot\exp\left(i(m-n)\Omega\cdot t\right)\cdot c_{1m}^*\cdot c_{1n}\cdot\langle P_{1m}|P_{4n}\rangle\ . \tag{198}$$

Making use of the general formula

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn} = \sum_{m\leq n}A_{mn} + \sum_{m\geq n}A_{mn} - \sum_{m}A_{mm}\ , \tag{199}$$

we obtain, in combination with Eqs. (196) and (194):

$$D(t) = \exp\left(-3\cdot|\gamma|^2\right)\cdot\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-|\gamma|^2)^m}{(m+k)!}\cdot L_m^k(4|\gamma|^2)\cdot(2|\gamma|^2)^k\cdot\cos k\Omega\cdot t + \\ -\frac{1}{2}\cdot\exp\left(-3\cdot|\gamma|^2\right)\cdot\sum_{m=0}^{\infty}\frac{(-|\gamma|^2)^m}{m!}\cdot L_m^0(4|\gamma|^2)\ ; \tag{200}$$

here, we have introduced the abbreviation

$$\gamma := \frac{M}{\hbar\Omega}\ . \tag{201}$$

From Ref.[16], we already know the general relation

$$\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+\alpha+1)}\cdot L_n^\alpha(x) = J_\alpha(2\sqrt{xz})\cdot e^z\cdot(xz)^{-\frac{1}{2}\alpha}\ ,\ \alpha > -1\ , \tag{202}$$

i.e., the sums of the Laguerre polynomials in Eq. (200) can be performed:

$$D(t) = \exp\left(-4\cdot|\gamma|^2\right)\cdot\sum_{k=0}^{\infty}(-i)^k\cdot J_k\left(4i|\gamma|^2\right)\cdot\cos k\Omega\cdot t + \\ -\frac{1}{2}\cdot\exp\left(-4\cdot|\gamma|^2\right)\cdot J_0\left(4i|\gamma|^2\right)\ . \tag{203}$$

In Eq. (203), we have introduced the *Bessel functions*

$$J_n(z) = \frac{1}{\pi}\cdot\int_0^\pi\cos\left(z\sin t - nt\right)dt\ ,\ n = 0\ ,\ 1\ ,\ 2\ ,\ \ldots\ . \tag{204}$$

The remaining sum of the expressions $(-i)^k \cdot J_k(4i|\gamma|^2) \cdot \cos k\Omega \cdot t$ is related to the Fourier expansion of the function $e^{iz\cos\varphi}$; according to Ref.[16], we have:

$$\exp(iz\cos\varphi) = J_0(z) + 2 \cdot \sum_{k=1}^{\infty} i^k \cdot J_k(z) \cdot \cos k\varphi \ . \tag{205}$$

The *formula of Bessel and Hansen* (Eq. (204)) gives immediately $[J_n(z)]^* = J_n(z^*)$; consequently, the relation

$$\sum_{k=1}^{\infty} (-i)^k \cdot J_k(z) \cdot \cos k\varphi = \frac{1}{2} \cdot \left(e^{-iz\cos\varphi} - J_0(z)\right) \tag{206}$$

is valid, too.

From Eqs. (206) and (203), we obtain:

$$D(t) = \frac{1}{2} \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \ . \tag{207}$$

So we conclude that the expectation value $\langle\Psi(t)|\hat{E}|\Psi(t)\rangle$ is given by

$$\langle\Psi(t)|\hat{E}|\Psi(t)\rangle = \frac{1}{2} \cdot (\lambda_{11} + \lambda_{44}) + (\lambda_{14} + \lambda_{41}) \cdot \frac{1}{2} \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \ . \tag{208}$$

According to Eq. (208), it is now a simple task to calculate the expectation values $\langle e^+e\rangle(t), \langle h^+h\rangle(t)$ and $\langle e^+h^+he\rangle(t)$; here, one obtains immediately

$$\lambda_{11} = \frac{1}{2} \ , \ \lambda_{14} = -\frac{1}{2} \ , \ \lambda_{41} = -\frac{1}{2} \ , \ \lambda_{44} = \frac{1}{2} \tag{209}$$

and therefore

$$\begin{aligned} &\langle e^+e\rangle(t) = \langle h^+h\rangle(t) = \langle e^+h^+he\rangle(t) \equiv \hat{E}1(t) = \\ &= \frac{1}{2} \cdot \left(1 - \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right)\right) \ . \end{aligned} \tag{210}$$

Direct inspection of Eq. (210) shows that $0 \leq \hat{E}1(t) < 1/2$ is true, i.e., the expectation values $\hat{E}1(t)$ are bounded from above by 1/2.

What about the polarization function $\langle e^+h^+\rangle(t)$? Because of the fact that the particle numbers exhibit a non-trivial time evolution, one might expect a similar behaviour for the expectation value $\langle e^+h^+\rangle(t)$, too.

From Eq. (148), however, we obtain

$$\lambda_{11} = \frac{1}{2} \; , \; \lambda_{14} = -\frac{1}{2} \; , \; \lambda_{41} = \frac{1}{2} \; , \; \lambda_{44} = -\frac{1}{2} \tag{211}$$

and therefore

$$\langle e^+ h^+ \rangle (t) = 0 \; . \tag{212}$$

We emphasize that we have checked the validity of our results (210) and (212) in the framework of the *hierarchy equations* in section III. Especially the non-trivial time evolution of the particle numbers $\langle e^+ e \rangle (t) = \langle h^+ h \rangle (t)$ is remarkable, because these expectation values are bounded from above by 1/2. In the strong-coupling limit, the maximum value of the particle numbers is $\approx 1/2$.

E. Expectation values of type $\langle \hat{E} (a^+)^k a^l \rangle (t)$

In our last subsection, we are concerned with combinations of electronic and photonic operators, i.e., we consider expectation values of type $\langle \hat{E} (a^+)^k a^l \rangle (t)$. Making use of the relation

$$\Theta^{(4)}_{kl\sigma}(n) = (-1)^{k+l+\sigma} \cdot \Theta^{(1)}_{kl\sigma}(n) \tag{213}$$

and proceeding along lines similar to the above, we obtain at first

$$\begin{aligned} &\langle \Psi(t) | \hat{E} (a^+)^k a^l | \Psi(t) \rangle = \\ &= \sum_{n=0}^{\infty} \sum_{\sigma=-l}^{k} c^*_{1n+\sigma}(t) \cdot c_{1n}(t) \cdot \Theta^{(1)}_{kl\sigma}(n) \cdot \Big(\lambda_{11} + (-1)^{k+l} \cdot \lambda_{44} \Big) + \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\sigma=-l}^{k} (-1)^{n+\sigma} \cdot \langle P_{1m} | P_{4n+\sigma} \rangle \cdot c^*_{1m}(t) \cdot c_{1n}(t) \cdot \Theta^{(1)}_{kl\sigma}(n) \cdot \Big(\lambda_{14} + (-1)^{k+l} \cdot \lambda_{41} \Big) \; . \end{aligned} \tag{214}$$

As far as our further calculations are concerned, the formula

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} = \sum_{m \geq 0; n \geq 0; m \leq n+\sigma} A_{mn} + \sum_{m \geq 0; n \geq 0; m \geq n+\sigma} A_{mn} - \sum_{m \geq 0; n \geq 0; m = n+\sigma} A_{mn} \tag{215}$$

is extremely useful; a combination of Eqs. (214) and (215) leads to the result

$$\langle \Psi(t)|\hat{E}(a^+)^k a^l|\Psi(t)\rangle = \Sigma_{kl}^{(1)}(t) + \Sigma_{kl}^{(2)}(t) \ , \tag{216}$$

where

$$\Sigma_{kl}^{(1)}(t) = \frac{1}{2}\cdot(\gamma^*)^l\cdot\gamma^k\cdot\Big((-1)^{k+l}\cdot\lambda_{11}+\lambda_{44}\Big)\cdot \tag{217}$$
$$\cdot\sum_{\sigma=-l}^{k}(-1)^{\sigma}\cdot\exp\left(i\sigma\cdot\Omega\cdot t\right)\cdot\sum_{-\mu+\nu=\sigma;0\le\mu\le l;0\le\nu\le k}\binom{l}{\mu}\cdot\binom{k}{\nu}$$

and

$$\Sigma_{kl}^{(2)}(t) = \Upsilon_{kl}^{(1)}(t)+\Upsilon_{kl}^{(2)}(t)+\Upsilon_{kl}^{(3)}(t) \ ; \tag{218}$$

$$\Upsilon_{kl}^{(1)}(t) := \frac{1}{2}\cdot\exp\Big(-3\cdot|\gamma|^2\Big)\cdot\sum_{\sigma=-l}^{k}\exp\left(i\sigma\cdot\Omega\cdot t\right)\cdot\sum_{m\ge0;n\ge0;m+n-\sigma\ge0} \tag{219}$$
$$\Gamma_{kl\sigma}^{(1)}(m,n)\cdot(-1)^m\cdot2^n\cdot\gamma^{m+n}\cdot(\gamma^*)^{m+n-\sigma}\cdot L_m^n(4\cdot|\gamma|^2)\cdot\exp\left(-in\cdot\Omega\cdot t\right) \ ,$$

$$\Upsilon_{kl}^{(2)}(t) := \frac{1}{2}\cdot\exp\Big(-3\cdot|\gamma|^2\Big)\cdot\sum_{\sigma=-l}^{k}\exp\left(i\sigma\cdot\Omega\cdot t\right)\cdot\sum_{m\ge0;n\ge0;m+\sigma\ge0} \tag{220}$$
$$\Gamma_{kl\sigma}^{(2)}(m,n)\cdot(-1)^{m+\sigma}\cdot2^n\cdot\gamma^{m+n+\sigma}\cdot(\gamma^*)^{m+n}\cdot L_{m+\sigma}^n(4\cdot|\gamma|^2)\cdot\exp\left(in\cdot\Omega\cdot t\right) \ ,$$

$$\Upsilon_{kl}^{(3)}(t) := -\frac{1}{2}\cdot\exp\Big(-3\cdot|\gamma|^2\Big)\cdot\sum_{\sigma=-l}^{k}\exp\left(i\sigma\cdot\Omega\cdot t\right)\cdot\sum_{m\ge0;m-\sigma\ge0} \tag{221}$$
$$\Gamma_{kl\sigma}^{(1)}(m,0)\cdot(-1)^m\cdot\gamma^m\cdot(\gamma^*)^{m-\sigma}\cdot L_m^0(4\cdot|\gamma|^2) \ .$$

In Eqs. (219)-(221), we have introduced the abbreviations

$$\Gamma_{kl\sigma}^{(1)}(m,n) := (-1)^{\sigma}\cdot\Big((-1)^{k+l}\cdot\lambda_{14}+\lambda_{41}\Big)\cdot\sum_{-\mu+\nu=\sigma;0\le\mu\le l;0\le\nu\le k} \tag{222}$$
$$\frac{1}{(m+n-\sigma-\mu)!}\cdot\binom{l}{\mu}\cdot\binom{k}{\nu}\cdot(\gamma^*)^{l-\mu}\cdot\gamma^{k-\nu}$$

and

$$\Gamma_{kl\sigma}^{(2)}(m,n) := (-1)^{\sigma}\cdot\Big((-1)^{k+l}\cdot\lambda_{14}+\lambda_{41}\Big)\cdot\sum_{-\mu+\nu=\sigma;0\le\mu\le l;0\le\nu\le k} \tag{223}$$
$$\frac{(m+\sigma)!}{(m-\mu)!(m+n+\sigma)!}\cdot\binom{l}{\mu}\cdot\binom{k}{\nu}\cdot(\gamma^*)^{l-\mu}\cdot\gamma^{k-\nu} \ .$$

Remark: According to Eq. (175), we have to keep in mind that for the simplified expression (222) the additional condition $m+n-\sigma-\mu \geq 0$ and for the simplified expression (223) the additional condition $m-\mu \geq 0$ has to be taken into account. Under these circumstances, the inequalities $m+\sigma \geq 0$ and $m+n+\sigma \geq 0$ (see Eq. (223)) are automatically fulfilled.

In the following part of our considerations, we are going to discuss a large class of expectation values in more detail. By taking $\hat{E} = 1$, for example, we obtain $\lambda_{ij} = \delta_{ij}$ and therefore

$$\langle \Psi(t)|(a^+)^k a^l|\Psi(t)\rangle = \frac{1}{2} \cdot (\gamma^*)^l \cdot \gamma^k \cdot \Big((-1)^{k+l} + 1\Big) \cdot \cdot \sum_{\sigma=-l}^{k} (-1)^\sigma \cdot \exp\left(i\sigma \cdot \Omega \cdot t\right) \cdot \sum_{-\mu+\nu=\sigma;0\leq\mu\leq l;0\leq\nu\leq k} \binom{l}{\mu} \cdot \binom{k}{\nu} . \tag{224}$$

Our next example are expectation values of type $\langle \hat{E} a^l \rangle(t)$. Because of the fact that Eqs. (218)-(221) come into play, extra care has to be taken.

From Eqs. (219), (220) and (221), we first conclude that $\sigma \leq 0$ is true; consequently, we obtain:

$$\sum_{m\geq 0;n\geq 0;m+n-\sigma\geq 0} \rightarrow \sum_{m\geq 0;n\geq 0} , \sum_{m\geq 0;n\geq 0;m+\sigma\geq 0} \rightarrow \sum_{n\geq 0;m+\sigma\geq 0} , \sum_{m\geq 0;m-\sigma\geq 0} \rightarrow \sum_{m\geq 0} . \tag{225}$$

In view of the second relation in Eq. (225), we should emphasize that the minimum value of $m+\sigma$ is in fact zero. As far as the coefficients $\Gamma^{(1)}_{0l\sigma}(m,n)$ and $\Gamma^{(2)}_{0l\sigma}(m,n)$ are concerned, the relations

$$\Gamma^{(1)}_{0l\sigma}(m,n) = (-1)^\sigma \cdot \Big((-1)^l \cdot \lambda_{14} + \lambda_{41}\Big) \cdot \frac{1}{(m+n)!} \cdot \binom{l}{-\sigma} \cdot (\gamma^*)^{l+\sigma} \tag{226}$$

and

$$\Gamma^{(2)}_{0l\sigma}(m,n) = (-1)^\sigma \cdot \Big((-1)^l \cdot \lambda_{14} + \lambda_{41}\Big) \cdot \frac{1}{(m+n+\sigma)!} \cdot \binom{l}{-\sigma} \cdot (\gamma^*)^{l+\sigma} \tag{227}$$

are immediately derived from Eqs. (222) and (223). After these preparations, the desired expression $\Sigma^{(2)}_{0l}(t)$ can now be calculated; making use of *Bessel functions* again, we obtain:

$$\Sigma^{(2)}_{0l}(t) = \Big((-1)^l \cdot \lambda_{14} + \lambda_{41}\Big) \cdot (\gamma^*)^l \cdot \cdot \left(1 - \exp\left(-i \cdot \Omega \cdot t\right)\right)^l \cdot \frac{1}{2} \cdot \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big) . \tag{228}$$

According to Eqs. (217) and (228), the expectation value $\langle \hat{E} a^l \rangle(t)$ is thus given by

$$\langle \Psi(t)|\hat{E} a^l|\Psi(t)\rangle = \frac{1}{2} \cdot (1 - \exp(-i \cdot \Omega \cdot t))^l \cdot (\gamma^*)^l \cdot \tag{229}$$
$$\cdot \Big((-1)^l \cdot \lambda_{11} + \lambda_{44} + \Big[(-1)^l \cdot \lambda_{14} + \lambda_{41} \Big] \cdot \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1) \Big) \Big) \ .$$

Because of the fact that the electron-hole operator $\hat{E}$ can be chosen arbitrarily, the expectation values of type $\langle \hat{E}(a^+)^k \rangle(t)$ are also contained in our general result (229). By taking $\hat{E} = e^+ h^+$ and $l = 1$, for example, we obtain the *photon-assisted polarization function* $\langle e^+ h^+ a \rangle(t)$:

$$\langle e^+ h^+ a \rangle(t) = -\frac{1}{2} \cdot (1 - \exp(-i \cdot \Omega \cdot t)) \cdot \gamma^* \cdot \Big(1 - \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1) \Big) \Big) \ . \tag{230}$$

Finally, we should remember that the expectation value $\langle hea \rangle(t)$ was decisive in our previous investigations and calculations; by taking $\hat{E} = he$ and $l = 1$, we obtain our most important result:

$$\langle hea \rangle(t) = -\frac{1}{2} \cdot (1 - \exp(-i \cdot \Omega \cdot t)) \cdot \gamma^* \cdot \Big(1 + \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1) \Big) \Big) \ . \tag{231}$$

It is now obvious that the approximation $\langle hea \rangle(t) = 0$ does not provide an appropriate approach to the Heisenberg equations of motion. In fact, we conclude from Eqs. (230) and (231) that

$$|\langle e^+ h^+ a \rangle(t)| = |\langle hea \rangle(t)| \cdot \frac{1 - \exp(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1))}{1 + \exp(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1))} \tag{232}$$

is true; consequently, the absolute value of $\langle e^+ h^+ a \rangle(t)$ can never exceed the absolute value of $\langle hea \rangle(t)$. At this point, we refer additionally to our general results in sections II and III. As far as the expectation value $\langle hea \rangle(t)$ is concerned, the physical interpretation that an electron-hole pair *and* a photon are removed from the system under consideration is often used in the literature. If this interpretation is correct, the absolute value of $\langle hea \rangle(t)$ is expected to be comparatively small[8]. In the framework of the standard model, however, Eq. (232) demonstrates explicitly that such a picture is obviously *not* correct.

We summarize our basic results:

Theorem 2:

Consider a purely electronic two-level quantum dot model, as defined by Eqs. (140), (141) and (142) (i.e., the exciting light field vanishes). Assume (i) that the Coulomb matrix element W^{eh} *fulfills the condition* $\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0$ *and (ii) that the initial state is the vacuum state. Then, analytical solutions are available and the following relations are valid for any* t*:*

$$\langle a^+ a\rangle(t) = 2 \cdot |\gamma|^2 \cdot (1 - \cos\Omega \cdot t) \ , \tag{233}$$

$$\langle e^+ e\rangle(t) = \langle h^+ h\rangle(t) = \langle e^+ h^+ h e\rangle(t) = \frac{1}{2} \cdot \Big(1 - \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big)\Big) \ , \tag{234}$$

$$\langle e^+ h^+\rangle(t) = \langle a^+\rangle(t) = \langle a\rangle(t) = 0 \ , \tag{235}$$

$$\begin{aligned} &\langle e^+ h^+ a\rangle(t) = \\ &= -\frac{1}{2} \cdot (1 - \cos\Omega \cdot t + i \cdot \sin\Omega \cdot t) \cdot \gamma^* \cdot \Big(1 - \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big)\Big) \ , \end{aligned} \tag{236}$$

$$\begin{aligned} &\langle h e a\rangle(t) = \\ &= -\frac{1}{2} \cdot (1 - \cos\Omega \cdot t + i \cdot \sin\Omega \cdot t) \cdot \gamma^* \cdot \Big(1 + \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big)\Big) \ , \end{aligned} \tag{237}$$

$$\begin{aligned} &\langle h e a^+\rangle(t) = \\ &= -\frac{1}{2} \cdot (1 - \cos\Omega \cdot t - i \cdot \sin\Omega \cdot t) \cdot \gamma \cdot \Big(1 - \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big)\Big) \ , \end{aligned} \tag{238}$$

$$\begin{aligned} &\langle e^+ h^+ a^+\rangle(t) = \\ &= -\frac{1}{2} \cdot (1 - \cos\Omega \cdot t - i \cdot \sin\Omega \cdot t) \cdot \gamma \cdot \Big(1 + \exp\Big(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\Big)\Big) \ . \end{aligned} \tag{239}$$

The expression γ *is related to the particle-photon coupling constant* M*:* $\gamma := M/\hbar\Omega$*.*

V. FURTHER PROPERTIES OF THE TWO-LEVEL SYSTEM: TIME EVOLUTION OF THE EXPECTATION VALUES IN THE PRESENCE OF AN EXTERNAL LIGHT FIELD

In section IV, we have analyzed the dynamical vacuum fluctuations of a simplified electron-hole system coupled to a single photon mode. Because of the fact that analytical solutions are available at least for a specific choice of the Coulomb matrix element, it is an interesting task to analyze the time evolution of the expectation values *in the presence of an external light field* and to calculate, for example, the *optical absorption spectrum* $A(\omega)$.

In solid state physics, the calculation of optical absorption spectra is often based on a *semiclassical approach*. Here, the electron-hole system is coupled to a *classical exciting light field*, and the absorption coefficient $A(\omega)$ is obtained from the well-known equation

$$A(\omega) = 2 \cdot \frac{\omega}{c_0} \cdot n_b \cdot Im\sqrt{1 + \frac{\chi(\omega)}{\varepsilon_b}} \ . \tag{240}$$

Basic definitions as well as various illustrations of Eq. (240) can be found in many textbooks (see, e.g., Ref[9]). For this reason, we are not going to explain further details here (as these are not decisive for our further considerations). In the limit $\chi(\omega)/\varepsilon_b \ll 1$, $A(\omega)$ is - apart from unimportant factors - given by the formula[9]

$$A(\omega) = Im\frac{P(\omega)}{E(\omega)} \ . \tag{241}$$

Eqs. (240) and (241) are well established in the literature. In many cases, the desired optical absorption spectrum can only be obtained in the framework of a numerical approach. In order to perform a numerical calculation explicitly, the introduction of phenomenological damping constants is indispensable. From a strictly formal point of view, however, there is no doubt that such a procedure has to be regarded mainly as a technical trick: *Any relaxation process - if it exists at all - has its origin in the Hamiltonian.* For this reason, Eqs. (240) and (241) should also be valid in the limit of vanishing (phenomenological) damping constants - provided that $A(\omega)$ is introduced as a *distribution.*

Surprisingly enough, it turns out that the two-level system from section IV *in the presence of an external light field* does provide a counter-example: For the same choice of the Coulomb matrix element and for a specific class of initial states, the complete infinite hierarchy of coupled equations for the expectation values can be solved analytically in closed form *for arbitrary electric field amplitudes.* Making use of the *exact formula* for the polarization function $\langle e^+h^+\rangle(t)$, we shall demonstrate that neither Eq. (241) nor Eq. (240) can be applied to calculate the optical absorption spectrum $A(\omega)$ in the limit of vanishing (phenomenological) damping constants. Our statements are valid *even in the case that the particle-photon coupling constant vanishes*, i.e., the coupling of the electron-hole system to the quantized electromagnetic field is not decisive in this context.

A. Introduction of the model Hamiltonian

In the following part of our considerations, we are concerned with an extended version of the model Hamiltonian from section IV:

$$H = H_1 + H_2 + V_{ehS}(t) \ , \tag{242}$$

$$H_1 = \varepsilon_0 \cdot e^+e + \varphi_0 \cdot h^+h - 2 \cdot W^{eh} \cdot e^+h^+he \ , \tag{243}$$

$$H_2 = \hbar\Omega \cdot a^+a + \Big(M \cdot a + M^* \cdot a^+\Big) \cdot \Big(e^+h^+ + he\Big) \ , \tag{244}$$

$$V_{ehS}(t) = S(t) \cdot \Big(e^+h^+ + he\Big) \ . \tag{245}$$

Here, $S(t)$ contains the amplitude $E(t)$ of the exciting light field as well as the coupling matrix element. We require in advance that $E(t)$ fulfills the following conditions:

$$E(t) = 0 \ , \ if \ t \leq 0 \ ; \ E(t) \to 0 \ , \ if \ t \to \infty \ ; \ \int_0^{+\infty} |E(t)| dt < +\infty \ . \tag{246}$$

B. Solution of the time-dependent Schrödinger equation

As before, we introduce the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{247}$$

for the Coulomb matrix element W^{eh}. If we split $H(t)$ into

$$H(t) = H_0 + S(t) \cdot \left(e^+ h^+ + he\right) \ , \tag{248}$$

we conclude from our results in section IV that

$$H_0 |\Psi_{in}\rangle = E_{in} \cdot |\Psi_{in}\rangle \tag{249}$$

is true. The solution $|\Psi(t)\rangle$ of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle \tag{250}$$

can be represented by a superpositon of the eigenstates $|\Psi_{in}\rangle$:

$$|\Psi(t)\rangle = \sum_{i=1}^{4} \sum_{n=0}^{\infty} c_{in}(t) \cdot |\Psi_{in}\rangle \ . \tag{251}$$

According to our assumption from above, the coefficients $c_{in}(t)$ are explicitly given by the equations

$$c_{1n}(t) = c_{1n}(0) \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\left(n - |\gamma|^2\right) \cdot \hbar\Omega \cdot t + \int_0^t S(\tau) d\tau\right]\right) \ , \tag{252}$$

$$c_{2n}(t) = c_{2n}(0) \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\varepsilon_0 + n \cdot \hbar\Omega\right] \cdot t\right) \ , \tag{253}$$

$$c_{3n}(t) = c_{3n}(0) \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\varphi_0 + n \cdot \hbar\Omega\right] \cdot t\right) \ , \tag{254}$$

$$c_{4n}(t) = c_{4n}(0) \cdot \exp\left(-\frac{i}{\hbar} \cdot \left[\left(n - |\gamma|^2\right) \cdot \hbar\Omega \cdot t - \int_0^t S(\tau) d\tau\right]\right) \ . \tag{255}$$

Here, the coefficients $c_{in}(0)$ correspond to the initial state $|\Psi(t = 0)\rangle$.

C. Optical absorption spectrum: Part I

Our results from above exhibit an interesting feature of the model Hamiltonian under consideration: Because of the fact that $S(t)$ is a real number for any t, the relation

$$|c_{in}(t)| = |c_{in}(0)| \tag{256}$$

is immediately derived from Eqs. (252)-(255). The latter equation demonstrates in combination with relation (249) that

$$\langle\Psi(t)|H_0|\Psi(t)\rangle = \langle\Psi(0)|H_0|\Psi(0)\rangle \tag{257}$$

is valid for any $t \geq 0$: The total energy of the particle-photon system is just a constant. In combination with condition (246), we obtain additionally

$$\langle H(t \to -\infty)\rangle = \langle H(t \to +\infty)\rangle \ , \tag{258}$$

i.e., the optical absorption spectrum $A(\omega)$ of the particle-photon system is equal to zero:

$$A(\omega) = 0 \ . \tag{259}$$

Later on, we shall see whether the formula $A(\omega) = Im[P(\omega)/E(\omega)]$ leads to the same result - or not.

D. Time evolution of the expectation values: General part

After these preparations, we are now going to determine the time evolution of an expectation value of type $\langle\hat{E}(a^+)^k a^l\rangle(t)$, where $\hat{E}$ denotes an electron-hole operator sequence. Making use of the relations

$$\hat{E}|e_i\rangle = \sum_{j=1}^{4} \lambda_{ij} \cdot |e_j\rangle \ , \ (a^+)^k a^l |P_{in}\rangle = \sum_{\sigma=-l}^{+k} \Theta_{kl\sigma}^{(i)}(n) \cdot |P_{in+\sigma}\rangle \tag{260}$$

and proceeding along lines similar to those in section IV, we obtain by a lengthy but straightforward calculation the general result

$$\langle\Psi(t)|\hat{E}(a^+)^k a^l|\Psi(t)\rangle = \sum_{i\neq j}\lambda_{ij}\cdot\exp\left(-\frac{1}{2}\cdot|\gamma_j-\gamma_i|^2\right)\cdot \tag{261}$$
$$\cdot\sum_{\sigma=-l}^{k}\left(A_{kl\sigma}(i,j)+B_{kl\sigma}(i,j)-C_{kl\sigma}(i,j)\right)+$$
$$+\sum_{i=1}^{4}\lambda_{ii}\cdot\sum_{\sigma=-l}^{k}\sum_{n=0}^{\infty}\sqrt{n!(n+\sigma)!}\cdot c^*_{in+\sigma}(t)\cdot c_{in}(t)\cdot X^{(i)}_{kl\sigma}(\sigma,n)\ ,$$

where

$$A_{kl\sigma}(i,j) := \sum_{m\geq 0;n\geq 0;m+n-\sigma\geq 0}\sqrt{m!(m+n-\sigma)!}\cdot c^*_{jm}(t)\cdot c_{im+n-\sigma}(t)\cdot \tag{262}$$
$$\cdot(\gamma_j-\gamma_i)^n\cdot(-1)^n\cdot L^n_m\left(|\gamma_j-\gamma_i|^2\right)\cdot X^{(i)}_{kl\sigma}(m,n)\ ,$$

$$B_{kl\sigma}(i,j) := \sum_{m\geq 0;n\geq 0;m+n+\sigma\geq 0}\sqrt{m!(m+n+\sigma)!}\cdot c^*_{jm+n+\sigma}(t)\cdot c_{im}(t)\cdot \tag{263}$$
$$\cdot\left(\gamma^*_j-\gamma^*_i\right)^n\cdot L^n_{m+\sigma}\left(|\gamma_j-\gamma_i|^2\right)\cdot Y^{(i)}_{kl\sigma}(m,n)$$

and

$$C_{kl\sigma}(i,j) := \sum_{m\geq 0;m-\sigma\geq 0}\sqrt{m!(m-\sigma)!}\cdot c^*_{jm}(t)\cdot c_{im-\sigma}(t)\cdot \tag{264}$$
$$\cdot L^0_m\left(|\gamma_j-\gamma_i|^2\right)\cdot X^{(i)}_{kl\sigma}(m,0)\ .$$

In Eqs. (261)-(264), we have introduced the abbreviations

$$X^{(i)}_{kl\sigma}(m,K) := \sum_{-\mu+\nu=\sigma;0\leq\mu\leq l;0\leq\nu\leq k} \tag{265}$$
$$\frac{1}{(m+K-\sigma-\mu)!}\cdot\binom{l}{\mu}\cdot\binom{k}{\nu}\cdot(-\gamma^*_i)^{l-\mu}\cdot(-\gamma_i)^{k-\nu}$$

and

$$Y^{(i)}_{kl\sigma}(m,K) := \sum_{-\mu+\nu=\sigma;0\leq\mu\leq l;0\leq\nu\leq k} \tag{266}$$
$$\frac{(m+\sigma)!}{(m+K+\sigma)!(m-\mu)!}\cdot\binom{l}{\mu}\cdot\binom{k}{\nu}\cdot(-\gamma^*_i)^{l-\mu}\cdot(-\gamma_i)^{k-\nu}\ .$$

As far as Eqs. (265) and (266) are concerned, we refer additionally to the detailed remarks in section IV.

E. Time evolution of the expectation values: Special part

We already know that there exist two eigenstates of H_0 which belong to the ground-state energy E_0; these are

$$|\Psi_{10}\rangle = |e_1\rangle|P_{10}\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle + |4\rangle) \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_1|^2\right) \cdot \exp\left(-\gamma_1^* \cdot a^+\right) |0_{Phot.}\rangle \tag{267}$$

and

$$|\Psi_{40}\rangle = |e_4\rangle|P_{40}\rangle = \frac{1}{\sqrt{2}} \cdot (|1\rangle - |4\rangle) \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_4|^2\right) \cdot \exp\left(-\gamma_4^* \cdot a^+\right) |0_{Phot.}\rangle \ . \tag{268}$$

In the following part of our considerations, we are going to analyze the time evolution of the expectation values $\langle \hat{E}(a^+)^k a^l\rangle(t)$ under the initial condition

$$|\Psi(t=0)\rangle\rangle = c_{10} \cdot |\Psi_{10}\rangle + c_{40} \cdot |\Psi_{40}\rangle \ , \ |c_{10}|^2 + |c_{40}|^2 = 1 \ , \tag{269}$$

where $c_{10} \equiv c_{10}(0)$ and $c_{40} \equiv c_{40}(0)$. From Eq. (261), we obtain:

$$\begin{aligned} \langle\Psi(t)|\hat{E}(a^+)^k a^l|\Psi(t)\rangle = \gamma^k \cdot (\gamma^*)^l \cdot \left((-1)^{k+l} \cdot \lambda_{11} \cdot |c_{10}|^2 + \lambda_{44} \cdot |c_{40}|^2\right) + \\ + \exp\left(-2 \cdot |\gamma|^2\right) \cdot \gamma^k \cdot (\gamma^*)^l \cdot \Lambda(t) \ , \end{aligned} \tag{270}$$

where

$$\begin{aligned} \Lambda(t) := (-1)^l \cdot \lambda_{14} \cdot c_{10} \cdot c_{40}^* \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) + \\ +(-1)^k \cdot \lambda_{41} \cdot c_{10}^* \cdot c_{40} \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \ . \end{aligned} \tag{271}$$

Eq. (270) is remarkable because it contains the solution for the complete infinite system of coupled equations for the expectation values under the initial condition (269). We have checked the validity of relation (270) for various equations of motion at the beginning of the infinite hierarchy (see section III again).

By taking $\hat{E} = 1$ and $k = l = 1$, for example, we obtain the photon number $\langle a^+ a\rangle(t)$:

$$\langle a^+ a\rangle(t) = |\gamma|^2 \ . \tag{272}$$

We thus obtain the surprising result that the photon number is just a constant, i.e., it does not depend on the strength of the external light field.

By taking $k = l = 0$, we obtain the occupation numbers of the electron-hole system:

$$\langle e^+e\rangle(t) = \langle h^+h\rangle(t) = \frac{1}{2} - \frac{1}{2} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot \Lambda_{occ.}(t) \ , \tag{273}$$

where

$$\Lambda_{occ.}(t) := c_{10} \cdot c_{40}^* \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) + \\ +c_{10}^* \cdot c_{40} \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \tag{274}$$

determines the time evolution of the expectation values under consideration. Because of $|c_{10}|^2 + |c_{40}|^2 = 1$, we conclude from Eq. (274) that

$$-1 \leq \Lambda_{occ.}(t) \leq 1 \tag{275}$$

is true, i.e., we may derive upper and lower bounds for the expectation values $\langle e^+e\rangle(t)$ and $\langle h^+h\rangle(t)$:

$$\frac{1}{2} - \frac{1}{2} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \leq \langle e^+e\rangle(t) = \langle h^+h\rangle(t) \leq \frac{1}{2} + \frac{1}{2} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \ . \tag{276}$$

Obviously, both expectation values are not negative and bounded from above by 1 - as it must be. In the strong-coupling limit, we obtain $\langle e^+e\rangle(t) = \langle h^+h\rangle(t) \approx 1/2$.

A further important example is the *polarization function*[17] $\langle e^+h^+\rangle(t)$. By taking $\hat{E} = e^+h^+$ and $k = l = 0$, we obtain from Eq. (270):

$$\langle e^+h^+\rangle(t) = \frac{1}{2} \cdot \left(|c_{10}|^2 - |c_{40}|^2\right) - \frac{1}{2} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot \Lambda_{pol.}(t) \ ; \tag{277}$$

here, we have introduced the abbreviation

$$\Lambda_{pol.}(t) := c_{10} \cdot c_{40}^* \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) + \\ -c_{10}^* \cdot c_{40} \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \ . \tag{278}$$

According to Eqs. (245), (277) and (278), the semiclassical interaction energy $\langle V_{ehS}(t)\rangle$ is thus given by

$$\langle V_{ehS}(t)\rangle = S(t) \cdot \left(|c_{10}|^2 - |c_{40}|^2\right) \ . \tag{279}$$

Remark: In the following part of our considerations, the coefficients c_{10} *and* c_{40} *are chosen to be real numbers.*

From Eqs. (277) and (278), we conclude that

$$\begin{aligned} P(t) := \langle e^+h^+\rangle(t) = \frac{1}{2} \cdot \left(|c_{10}|^2 - |c_{40}|^2\right) + \\ +i \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot c_{10} \cdot c_{40} \cdot \sin\left(\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \end{aligned} \tag{280}$$

is true under these circumstances.

F. Optical absorption spectrum: Part II

We have already seen at the very beginning of this section that the optical absorption spectrum $A(\omega)$ of the particle-photon system under consideration is equal to zero: $A(\omega) = 0$. This statement is true because of the fact that the total energy $\langle H_0\rangle$ of the particle-photon system is just a constant and condition (246) is additionally assumed to be valid. What do we obtain from the well-known formula $A(\omega) = Im[P(\omega)/E(\omega)]$?

As far as $P(\omega)$ is concerned, we first remark that

$$\begin{aligned} P(\omega) = \int_{-\infty}^{+\infty} P(t) \cdot e^{i\omega t} dt = \\ = \int_{-\infty}^{0} P(0) \cdot e^{i\omega t} dt + \int_0^{+\infty} (P(t) - P(\infty)) \cdot e^{i\omega t} dt + \int_0^{+\infty} P(\infty) \cdot e^{i\omega t} dt \end{aligned} \tag{281}$$

is true. Introducing the distributions

$$\int_{-\infty}^{+\infty} e^{i\omega t} dt = 2\pi \cdot \delta(\omega) \ ; \ \int_0^{+\infty} e^{i\omega t} dt = \pi \cdot \delta(\omega) + i \cdot P\left(\frac{1}{\omega}\right) \ , \tag{282}$$

we obtain from Eqs. (280) and (281):

$$P(\omega) = \pi \cdot Q \cdot \delta(\omega) + \exp\left(-2 \cdot |\gamma|^2\right) \cdot c_{10} \cdot c_{40} \cdot R(\omega) \ , \tag{283}$$

where

$$Q := |c_{10}|^2 - |c_{40}|^2 + i \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot c_{10} \cdot c_{40} \cdot \sin\left(\frac{2}{\hbar} \cdot \int_0^{+\infty} S(\tau) d\tau\right) \tag{284}$$

and

$$\begin{aligned} R(\omega) := & -\sin\left(\frac{2}{\hbar} \cdot \int_0^{+\infty} S(\tau) d\tau\right) \cdot \left(P\left(\frac{1}{\omega}\right) - \frac{1}{\omega}\right) + \\ & -\frac{2}{\hbar\omega} \cdot \int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \cdot S(t) \cdot \cos\omega t \, dt + \\ & -\frac{2i}{\hbar\omega} \cdot \int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \cdot S(t) \cdot \sin\omega t \, dt \ . \end{aligned} \tag{285}$$

In a next step, we split the Fourier transforms of $P(t)$ and $E(t)$ into

$$E(\omega) = E_R(\omega) + i \cdot E_I(\omega) \tag{286}$$

and

$$P(\omega) = P_R(\omega) + i \cdot P_I(\omega) \ ; \tag{287}$$

consequently, the imaginary part of $P(\omega)/E(\omega)$ is given by

$$Im \frac{P(\omega)}{E(\omega)} = \frac{P_I(\omega) \cdot E_R(\omega) - P_R(\omega) \cdot E_I(\omega)}{E_R^2(\omega) + E_I^2(\omega)} \ . \tag{288}$$

Remark: In order to avoid confusions, we omit the expression $\sim (P(1/\omega) - 1/\omega)$ *(see Eq. (285)) in the following part of our considerations. It is needless to emphasize that our further conclusions are not affected.*

From Eq. (283), we thus obtain

$$\begin{aligned} P_I(\omega) \cdot E_R(\omega) - P_R(\omega) \cdot E_I(\omega) = D(\omega) \cdot \pi \cdot \delta(\omega) + \\ -\frac{2}{\hbar\omega} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot c_{10} \cdot c_{40} \cdot \left(\Sigma_I(\omega) \cdot E_R(\omega) - \Sigma_R(\omega) \cdot E_I(\omega)\right) \end{aligned} \tag{289}$$

under these circumstances. Here, we have introduced the abbreviation

$$\begin{aligned} D(\omega) := & \left(|c_{40}|^2 - |c_{10}|^2\right) \cdot E_I(\omega) + \\ & + \exp\left(-2 \cdot |\gamma|^2\right) \cdot c_{10} \cdot c_{40} \cdot \sin\left(\frac{2}{\hbar} \cdot \int_0^{+\infty} S(\tau) d\tau\right) \cdot E_R(\omega) \end{aligned} \tag{290}$$

and the *non-linear integral transforms*

$$\Sigma_R(\omega) := \int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \cdot S(t) \cdot \cos \omega t \; dt \tag{291}$$

and

$$\Sigma_I(\omega) := \int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \cdot S(t) \cdot \sin \omega t \; dt \; . \tag{292}$$

Now, it is obvious that the second term on the right-hand side of Eq. (289) is - in general - different from zero. The proof proceeds by *reductio ad absurdum:* We assume that the equation $\Sigma_R(\omega)/\Sigma_I(\omega) = E_R(\omega)/E_I(\omega)$ is valid for arbitrary amplitudes $E(t)$ and replace $E(t) \to \lambda \cdot E(t)$; under these circumstances, we obtain the relation

$$\frac{\int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \lambda \cdot \int_0^t S(\tau)d\tau\right) \cdot S(t) \cdot \cos \omega t \; dt}{\int_0^{+\infty} \cos\left(\frac{2}{\hbar} \cdot \lambda \cdot \int_0^t S(\tau)d\tau\right) \cdot S(t) \cdot \sin \omega t \; dt} = \frac{E_R(\omega)}{E_I(\omega)} \; . \tag{293}$$

Introducing the power series expansion of the cosine-function, Eq. (293) reads as follows:

$$\frac{\sum_{k=0}^{\infty} a_k(\omega) \cdot \lambda^{2k}}{\sum_{k=0}^{\infty} b_k(\omega) \cdot \lambda^{2k}} = \frac{E_R(\omega)}{E_I(\omega)} \; , \tag{294}$$

where

$$a_k(\omega) := \frac{(-1)^k}{(2k)!} \cdot \left(\frac{2}{\hbar}\right)^{2k} \cdot \int_0^{+\infty} \left[\int_0^t S(\tau)d\tau\right]^{2k} \cdot S(t) \cdot \cos \omega t \; dt \tag{295}$$

and

$$b_k(\omega) := \frac{(-1)^k}{(2k)!} \cdot \left(\frac{2}{\hbar}\right)^{2k} \cdot \int_0^{+\infty} \left[\int_0^t S(\tau)d\tau\right]^{2k} \cdot S(t) \cdot \sin \omega t \; dt \; . \tag{296}$$

According to our assumptions from above (see Eq. (246)), we obtain the inequalities

$$|a_k(\omega)| \leq G \cdot \frac{1}{(2k)!} \cdot \left(\frac{2 \cdot G}{\hbar}\right)^{2k} \; , \; |b_k(\omega)| \leq G \cdot \frac{1}{(2k)!} \cdot \left(\frac{2 \cdot G}{\hbar}\right)^{2k} \; , \tag{297}$$

where

$$G := \int_0^{+\infty} |S(\tau)| d\tau < +\infty \; . \tag{298}$$

These inequalities ensure that the power series on the left-hand side of Eq. (294) are in fact *absolutely convergent*, i.e., we obtain well-defined expressions for arbitrary values of λ and ω.

Now, direct inspection of Eqs. (295) and (296) shows that the coefficients $a_k(\omega)$ and $b_k(\omega)$ are - in general - different from each other; consequently, Eq. (294) exhibits a contradiction, because the left-hand side does - in general - explicitly depend on λ. This finishes our proof.

In the last part of our considerations, we choose $c_{10} = \pm 1/\sqrt{2}$ and $c_{40} = \pm 1/\sqrt{2}$; under these circumstances, *we obtain* $P(t \leq 0) = 0$, $\langle V_{ehS}(t) \rangle = 0$ *and*

$$P(\omega) = \pi \cdot i \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot \left(\pm \frac{1}{2}\right) \cdot \sin\left(\frac{2}{\hbar} \cdot \int_0^{+\infty} S(\tau) d\tau\right) \cdot \delta(\omega) + \\ + \exp\left(-2 \cdot |\gamma|^2\right) \cdot \left(\pm \frac{1}{2}\right) \cdot R(\omega) \ , \tag{299}$$

$$P_I(\omega) \cdot E_R(\omega) - P_R(\omega) \cdot E_I(\omega) = \\ = \exp\left(-2 \cdot |\gamma|^2\right) \cdot \left(\pm \frac{1}{2}\right) \cdot \sin\left(\frac{2}{\hbar} \cdot \int_0^{+\infty} S(\tau) d\tau\right) \cdot E_R(\omega) \cdot \pi \cdot \delta(\omega) + \\ - \frac{2}{\hbar\omega} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot \left(\pm \frac{1}{2}\right) \cdot \left(\Sigma_I(\omega) \cdot E_R(\omega) - \Sigma_R(\omega) \cdot E_I(\omega)\right) \ . \tag{300}$$

We thus conclude that *the optical absorption spectrum is - in general - different from zero, too; moreover,* $A(\omega)$ *may become negative.* We emphasize that these results are valid *even in the case that the particle-photon coupling constant vanishes*, i.e., the standard coupling of the electron-hole system to the quantized electromagnetic field is not decisive in this context. Instead, our findings clearly demonstrate that the formula $A(\omega) = Im[P(\omega)/E(\omega)]$ does - in general - *not* provide an appropriate approach to the optical properties of an electron-hole system. Last but not least, it is important to realize that our conclusions are only based on the mathematical structure of Eq. (280) - and not on specific examples for the exciting light field.

According to our results from above, it is necessary to demonstrate for any specific model that the expression $Im[P(\omega)/E(\omega)]$ is in fact equal to the optical absorption spectrum. If such a proof is missing, a numerical calculation of $A(\omega)$ (which is based on the formula $A(\omega) = Im[P(\omega)/E(\omega)]$ in combination with phenomenological damping constants) is at least highly questionable. If we use the improved formula $A(\omega) = 2 \cdot (\omega/c_0) \cdot n_b \cdot Im\sqrt{1 + \chi(\omega)/\varepsilon_b}$, Eq. (283) leads to a serious mathematical problem, the reason being that the square root contains a distribution. Here, the optical absorption spectrum $A(\omega)$ does in fact not exist.

We summarize our basic results:

Theorem 3:

Consider a particle-photon system, as defined by Eqs. (242), (243) and (244). Assume (i) that the Coulomb matrix element W^{eh} *fulfills the condition* $\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0$ *and (ii) that the initial state is given by*

$$|\Psi(t=0)\rangle = c_{10} \cdot |\Psi_{10}\rangle + c_{40} \cdot |\Psi_{40}\rangle \ , \ |c_{10}|^2 + |c_{40}|^2 = 1 \ , \tag{301}$$

where $|\Psi_{10}\rangle$ *and* $|\Psi_{40}\rangle$ *are defined by Eqs. (267) and (268), respectively. Then, the following relations are valid for any* $t \geq 0$*:*

$$\begin{aligned} \langle \Psi(t)|\hat{E}(a^+)^k a^l|\Psi(t)\rangle &= \gamma^k \cdot (\gamma^*)^l \cdot \left((-1)^{k+l} \cdot \lambda_{11} \cdot |c_{10}|^2 + \lambda_{44} \cdot |c_{40}|^2\right) + \\ &+ \exp\left(-2 \cdot |\gamma|^2\right) \cdot \gamma^k \cdot (\gamma^*)^l \cdot \Lambda(t) \ , \end{aligned} \tag{302}$$

$$\begin{aligned} \Lambda(t) &:= (-1)^l \cdot \lambda_{14} \cdot c_{10} \cdot c_{40}^* \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) + \\ &+(-1)^k \cdot \lambda_{41} \cdot c_{10}^* \cdot c_{40} \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \ ; \end{aligned} \tag{303}$$

$$\langle e^+h^+\rangle(t) = \frac{1}{2} \cdot \left(|c_{10}|^2 - |c_{40}|^2\right) - \frac{1}{2} \cdot \exp\left(-2 \cdot |\gamma|^2\right) \cdot \Lambda_{pol.}(t) \ , \tag{304}$$

$$\begin{aligned} \Lambda_{pol.}(t) &:= c_{10} \cdot c_{40}^* \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) + \\ &-c_{10}^* \cdot c_{40} \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \ . \end{aligned} \tag{305}$$

Here, the expressions λ_{ij} *are defined by Eq. (260);* γ *is related to the particle-photon coupling constant* M*:* $\gamma := M/\hbar\Omega$*. As far as the optical absorption spectrum is concerned, the following statements are true: (i) The total energy of the particle-photon system is a constant. (ii) The expression* $Im[P(\omega)/E(\omega)]$ *is - in general - different from zero and may become negative.*

VI. DYNAMICAL VACUUM FLUCTUATIONS OF THE QUANTIZED ELECTROMAGNETIC FIELD IN THE STANDARD MODEL: THE ANALYSIS OF A PURELY ELECTRONIC TWO-LEVEL QUANTUM DOT COUPLED TO AN UNLIMITED NUMBER OF PHOTON MODES

According to our general results in section II, *a particle-photon interaction operator of standard type* always leads to dynamical vacuum fluctuations of the quantized electromagnetic field. Here, the existence and the strength of the constant source terms in the Heisenberg equations of motion do not depend on the electronic part of the model Hamiltonian, but only on the particle-photon interaction operator itself. The detailed analysis of the time evolution of various expectation values in sections IV and V is based on a purely electronic two-level quantum dot model coupled to a single photon mode. It is important to realize that analytical solutions are available at least for a specific choice of the Coulomb matrix element. In view of our basic Hamiltonian (2), however, a more realistic model is highly desirable, the reason being that the standard interaction operator $V_1 + V_2$ (as defined by Eqs. (9) and (11)) describes a coupling of the electron-hole system to an *unlimited* number of photon modes. In combination with the additional property that the photon dispersion $\hbar\Omega_k$ depends explicitly on the $k-$vector, *relaxation phenomena* are therefore expected.

Intuitively, one might argue that such an intrinsic relaxation leads to a *damping* of the dynamical vacuum fluctuations. But is this really true? At this point, the equations of motion for expectation values of type $\langle \ldots a \rangle$, $\langle \ldots a^+ \rangle$, $\langle \ldots a^2 \rangle$ and $\langle \ldots (a^+)^2 \rangle$ become important again. Direct inspection of these equations demonstrates that there must exist at least one expectation value or at least one derivative which remains finite even in the limit $t \to +\infty$ (otherwise, Eqs. (86) and (88) would exhibit an intrinsic contradiction). Consequently, *a smooth damping* of the dynamical vacuum fluctuations (i.e., a long-time behaviour with the properties $\lim_{t\to\infty}\langle \ldots \rangle = 0$ and $\lim_{t\to\infty} \partial/\partial t \langle \ldots \rangle = 0$ for *any* expectation value) can be excluded.

In order to analyze the long-time behaviour of the expectation values further, we now introduce a purely electronic two-level quantum dot model coupled to an unlimited number of photon modes; if the exciting light field vanishes, the corresponding Hamiltonian H reads as follows:

$$H = H_1 + H_2 \ , \tag{306}$$

where

$$H_1 = \varepsilon_0 \cdot e^+ e + \varphi_0 \cdot h^+ h - 2 \cdot W^{eh} \cdot e^+ h^+ h e \tag{307}$$

and

$$H_2 = \int d^3k \sum_\lambda \hbar\Omega_k \cdot a^+_{k\lambda} a_{k\lambda} + \Big(e^+ h^+ + he\Big) \cdot \int d^3k \sum_\lambda \Big(M_{k\lambda} \cdot a_{k\lambda} + M^*_{k\lambda} \cdot a^+_{k\lambda}\Big) \ . \tag{308}$$

As before, we introduce the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{309}$$

for the Coulomb matrix element W^{eh}; moreover, we require that the particle-photon coupling function $M_{k\lambda}$ fulfills the inequalities

$$\int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} < +\infty \ , \ \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{\hbar\Omega_k} < +\infty \ . \tag{310}$$

We shall see that Hamiltonian (306) does provide a basic model for a particle-boson system, which is well suited to study the phenomenon of *relaxation* in the framework of quantum field theory. In particular, condition (310) ensures that H is well-defined and bounded from below: *No renormalizations are necessary in order to get finite results.* Last but not least, we should emphasize once more that the particle-photon interaction operator is in fact of standard type, i.e., the photons couple to the polarization.

A. Eigenstates and energy spectrum of H

Making use of condition (309) and proceeding along lines similar to those in section IV, we obtain the eigenstates $|\Psi_{ik_1\lambda_1 \ldots k_n\lambda_n}\rangle$ and the energy spectrum $E_{ik_1\lambda_1 \ldots k_n\lambda_n}$ of Hamiltonian (306):

$$|\Psi_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = |e_i\rangle|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle \ , \tag{311}$$

$$|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = \exp\left(-\frac{1}{2}\cdot\int d^3k\sum_\lambda|\gamma_{ik\lambda}|^2\right)\cdot\left(a^+_{k_1\lambda_1}+\gamma_{ik_1\lambda_1}\right)\cdot\ldots \tag{312}$$
$$\ldots\cdot\left(a^+_{k_n\lambda_n}+\gamma_{ik_n\lambda_n}\right)\cdot\exp\left(-\int d^3k\sum_\lambda\gamma^*_{ik\lambda}\cdot a^+_{k\lambda}\right)|0\rangle \ ;$$

$$E_{ik_1\lambda_1\ldots k_n\lambda_n} = \tilde{E}_i + \hbar\Omega_{k_1} + \ldots + \hbar\Omega_{k_n} - \int d^3k\sum_\lambda\hbar\Omega_k\cdot|\gamma_{ik\lambda}|^2 \ ; \tag{313}$$

$$\gamma_{ik\lambda} := \frac{M_{k\lambda}\cdot P_i}{\hbar\Omega_k} \ . \tag{314}$$

For mathematical convenience, we have introduced the indices $(ik_1\lambda_1\ldots k_n\lambda_n)$ instead of the quantum numbers β to number the eigenstates and the corresponding energy eigenvalues of Hamiltonian (306). Note that there exists a map

$$(ik_1\lambda_1\ldots k_n\lambda_n)\to\beta \ ; \tag{315}$$

in particular,

$$|\Psi_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = |\Psi_{iP(k_1\lambda_1\ldots k_n\lambda_n)}\rangle \ , \ E_{ik_1\lambda_1\ldots k_n\lambda_n} = E_{iP(k_1\lambda_1\ldots k_n\lambda_n)} \tag{316}$$

is true for an arbitrary permutation $P(k_1\lambda_1\ldots k_n\lambda_n)$ of the indices $(k_1\lambda_1)\ldots(k_n\lambda_n)$.

From Eq. (312), we conclude that

$$a^+_{k_0\lambda_0}|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = -\gamma_{ik_0\lambda_0}\cdot|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle + |P_{ik_0\lambda_0k_1\lambda_1\ldots k_n\lambda_n}\rangle \ , \tag{317}$$

$$\begin{aligned} a_{k_0\lambda_0}|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle &= -\gamma^*_{ik_0\lambda_0}\cdot|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle + \\ &+\sum_{j=1}^n\delta\left(k_j-k_0\right)\cdot\delta_{\lambda_j\lambda_0}\cdot|P_{ik_1\lambda_1\ldots k_{j-1}\lambda_{j-1}k_{j+1}\lambda_{j+1}\ldots k_n\lambda_n}\rangle \end{aligned} \tag{318}$$

and

$$\begin{aligned} &a^+_{k_0\lambda_0}a_{k_0\lambda_0}|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = \\ &= -\gamma_{ik_0\lambda_0}\sum_{j=1}^n\delta\left(k_j-k_0\right)\cdot\delta_{\lambda_j\lambda_0}\cdot|P_{ik_1\lambda_1\ldots k_{j-1}\lambda_{j-1}k_{j+1}\lambda_{j+1}\ldots k_n\lambda_n}\rangle + \\ &+|\gamma_{ik_0\lambda_0}|^2\cdot|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle + \sum_{j=1}^n\delta\left(k_j-k_0\right)\cdot\delta_{\lambda_j\lambda_0}\cdot|P_{ik_1\lambda_1\ldots k_{j-1}\lambda_{j-1}k_0\lambda_0k_{j+1}\lambda_{j+1}\ldots k_n\lambda_n}\rangle + \\ &-\gamma^*_{ik_0\lambda_0}\cdot|P_{ik_0\lambda_0k_1\lambda_1\ldots k_n\lambda_n}\rangle \end{aligned} \tag{319}$$

is true.

B. Time evolution of the state $|\Psi(t)\rangle$

The time evolution of the state $|\Psi(t)\rangle$ under the initial condition

$$|\Psi(t=0)\rangle = |0\rangle \tag{320}$$

is determined by the equation

$$|\Psi(t)\rangle = \sum_{i=1}^{4}\sum_{n=0}^{\infty}\int d^3k_1\ldots d^3k_n \sum_{\lambda_1\ldots\lambda_n}\frac{1}{n!}\cdot c_{ik_1\lambda_1\ldots k_n\lambda_n}(t)\cdot|\Psi_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle\ , \tag{321}$$

where

$$c_{ik_1\lambda_1\ldots k_n\lambda_n}(t) = \exp\left(-\frac{i}{\hbar}\cdot E_{ik_1\lambda_1\ldots k_n\lambda_n}\cdot t\right)\cdot c_{ik_1\lambda_1\ldots k_n\lambda_n}(0) \tag{322}$$

and

$$c_{ik_1\lambda_1\ldots k_n\lambda_n}(0) = \exp\left(-\frac{1}{2}\cdot\int d^3k\sum_{\lambda}|\gamma_{ik\lambda}|^2\right)\cdot[\langle 0_{eh}|e_i\rangle]^*\cdot\gamma^*_{ik_1\lambda_1}\cdot\ldots\cdot\gamma^*_{ik_n\lambda_n}\ . \tag{323}$$

The coefficient $1/n!$ in Eq. (321) has been introduced for mathematical convenience.

C. Time evolution of the expectation values: General part

We are now going to determine the time evolution of an expectation value of general type $\langle\Psi(t)|\hat{E}\hat{A}|\Psi(t)\rangle$, where $\hat{E}$ denotes an electron-hole operator sequence and $\hat{A}$ describes the photonic part. Making use of Eqs. (321)-(323), we obtain by a straightforward calculation the basic formula

$$\begin{aligned}\langle\Psi(t)|\hat{E}\hat{A}|\Psi(t)\rangle &= \frac{1}{2}\cdot\exp\left(-\int d^3k\sum_{\lambda}|\gamma_{1k\lambda}|^2\right)\cdot \\ &\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!}\cdot\frac{1}{n!}\cdot\int d^3q_1\ldots d^3q_m\sum_{\mu_1\ldots\mu_m}\int d^3k_1\ldots d^3k_n\sum_{\lambda_1\ldots\lambda_n} \\ &\gamma_{1q_1\mu_1}\cdot\ldots\cdot\gamma_{1q_m\mu_m}\cdot\gamma^*_{1k_1\lambda_1}\cdot\ldots\cdot\gamma^*_{1k_n\lambda_n}\cdot \\ &\cdot\exp\left(i\left(\Omega_{q_1}+\ldots+\Omega_{q_m}-\Omega_{k_1}\ldots-\Omega_{k_n}\right)\cdot t\right)\cdot\mho_{q_1\mu_1\ldots q_m\mu_m k_1\lambda_1\ldots k_n\lambda_n}\ ;\end{aligned} \tag{324}$$

here, we have introduced the abbreviation

$$\mho_{q_1\mu_1\ldots q_m\mu_m k_1\lambda_1\ldots k_n\lambda_n} := \langle e_1|\hat{E}|e_1\rangle \cdot \langle P_{1q_1\mu_1\ldots q_m\mu_m}|\hat{A}|P_{1k_1\lambda_1\ldots k_n\lambda_n}\rangle + \tag{325}$$
$$+(-1)^n \cdot \langle e_1|\hat{E}|e_4\rangle \cdot \langle P_{1q_1\mu_1\ldots q_m\mu_m}|\hat{A}|P_{4k_1\lambda_1\ldots k_n\lambda_n}\rangle +$$
$$+(-1)^m \cdot \langle e_4|\hat{E}|e_1\rangle \cdot \langle P_{4q_1\mu_1\ldots q_m\mu_m}|\hat{A}|P_{1k_1\lambda_1\ldots k_n\lambda_n}\rangle +$$
$$+(-1)^{m+n} \cdot \langle e_4|\hat{E}|e_4\rangle \cdot \langle P_{4q_1\mu_1\ldots q_m\mu_m}|\hat{A}|P_{4k_1\lambda_1\ldots k_n\lambda_n}\rangle \ .$$

D. Calculation of the photon number

By taking $\hat{E} = 1$ and $\hat{A} = \int d^3k \sum_\lambda a^+_{k\lambda} a_{k\lambda}$, we obtain from Eqs. (319) and (324) the time evolution of the photon number:

$$N_{Ph.}(t) := \langle\Psi(t)| \int d^3k \sum_\lambda a^+_{k\lambda} a_{k\lambda} |\Psi(t)\rangle = 2 \cdot \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} \cdot (1 - \cos\Omega_k \cdot t) \ . \tag{326}$$

The coupling matrix element $M_{k\lambda}$ has the structure

$$M_{k\lambda} = g(k) \cdot (p \cdot \varepsilon_{k\lambda}) \ , \ g(k) = g(|k|) \ , \tag{327}$$

where p denotes a fixed vector; consequently, we have:

$$\int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} \cdot \cos\Omega_k \cdot t = \tag{328}$$
$$= \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \sum_\lambda \sin\vartheta \cdot (p \cdot \varepsilon_{\vartheta\varphi\lambda})^2 \cdot \int_0^{+\infty} dq \ q^2 \cdot \frac{(g(q))^2}{(\hbar\Omega_q)^2} \cdot \cos\Omega_q \cdot t \ .$$

According to the *lemma of Riemann-Lebesgue*, we thus conclude from Eqs. (326) and (328) that

$$\lim_{t\to\infty} N_{Ph.}(t) = 2 \cdot \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} \tag{329}$$

is true: *For sufficiently large t, the photon number approaches a positive value.*

Eq. (329) is in accordance with our previous statement that there must exist at least one expectation value or at least one derivative which remains finite in the limit $t \to \infty$. From a physical point of view, the relaxation of the photon number illustrates the coupling of the electron-hole system to an *unlimited number of photon modes.* It is obvious that our nonperturbative results exhibit once more a serious problem of the present theory of particle-photon interaction:

(i) In the absence of an external light field, Hamiltonian (2) does not prefer a specific point of time; consequently, if the initial state is the *vacuum state*, the time evolution of the expectation values has to be *homogeneous* (because there is nothing which can disturb the homogeneity of the dynamics).

(ii) The existence of dynamical vacuum fluctuations demonstrates explicitly that the time evolution of the expectation values is *not homogeneous*.

Statements (i) and (ii) taken together constitute an *intrinsic contradiction on a physical level*. It is needless to emphasize that our conclusion is correct even in the weak-coupling limit, i.e., a particle-photon interaction operator of standard type is possible from a mathematical point of view, but can not be accepted for *physical* reasons.

E. Expectation values of type $\langle \hat{E} \rangle(t)$

In our next subsection, we are going to analyze the time evolution of expectation values of type $\langle \hat{E} \rangle(t)$, where $\hat{E}$ is an electron-hole operator sequence which is defined by the equation

$$\hat{E}|e_i\rangle = \sum_{l=1}^{4} \lambda_{il} \cdot |e_l\rangle \ . \tag{330}$$

By taking $\hat{A} = 1$, we obtain from Eqs. (324) and (330) the formula

$$\begin{aligned} \langle \Psi(t)|\hat{E}|\Psi(t)\rangle = \frac{1}{2} \cdot (\lambda_{11} + \lambda_{44}) + \frac{1}{2} \cdot \exp\left(- \int d^3k \sum_{\lambda} |\gamma_{k\lambda}|^2 \right) \cdot \\ \cdot \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!} \cdot \frac{1}{n!} \cdot \int d^3q_1 \ldots d^3q_m \sum_{\mu_1 \ldots \mu_m} \int d^3k_1 \ldots d^3k_n \sum_{\lambda_1 \ldots \lambda_n} \\ \gamma_{q_1\mu_1} \cdot \ldots \cdot \gamma_{q_m\mu_m} \cdot \gamma^*_{k_1\lambda_1} \cdot \ldots \cdot \gamma^*_{k_n\lambda_n} \cdot \\ \cdot \exp\left(i\left(\Omega_{q_1} + \ldots + \Omega_{q_m} - \Omega_{k_1} \ldots - \Omega_{k_n}\right) \cdot t\right) \cdot D_{q_1\mu_1 \ldots q_m\mu_m k_1\lambda_1 \ldots k_n\lambda_n} \ , \end{aligned} \tag{331}$$

where

$$\begin{aligned} D_{q_1\mu_1 \ldots q_m\mu_m k_1\lambda_1 \ldots k_n\lambda_n} := (-1)^n \cdot \lambda_{41} \cdot \langle P_{1q_1\mu_1 \ldots q_m\mu_m} | P_{4k_1\lambda_1 \ldots k_n\lambda_n} \rangle + \\ +(-1)^m \cdot \lambda_{14} \cdot \langle P_{4q_1\mu_1 \ldots q_m\mu_m} | P_{1k_1\lambda_1 \ldots k_n\lambda_n} \rangle \end{aligned} \tag{332}$$

and

$$\gamma_{k\lambda} := \frac{M_{k\lambda}}{\hbar\Omega_k} \ . \tag{333}$$

Now, we introduce the representation

$$|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle = U_i \cdot a^+_{k_1\lambda_1} \ldots a^+_{k_n\lambda_n}|0_{Phot.}\rangle \ , \tag{334}$$

$$U_i := \exp\left(-\int d^3k \sum_\lambda \gamma^*_{ik\lambda} \cdot a^+_{k\lambda} + \int d^3k \sum_\lambda \gamma_{ik\lambda} \cdot a_{k\lambda}\right) \tag{335}$$

for the eigenstates $|P_{ik_1\lambda_1\ldots k_n\lambda_n}\rangle$. In view of Eq. (332), the relation

$$\begin{aligned} U_1^+ \cdot U_4 = \exp&\left(2 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right) \cdot \\ \cdot \exp&\left(-2 \cdot \int d^3k \sum_\lambda \gamma_{k\lambda} \cdot a_{k\lambda}\right) \cdot \exp\left(2 \cdot \int d^3k \sum_\lambda \gamma^*_{k\lambda} \cdot a^+_{k\lambda}\right) \end{aligned} \tag{336}$$

and the corresponding formula for the operator sequence $U_4^+ \cdot U_1$ prove to be extremely useful. Making use of Eqs. (334) and (336), Eq. (331) reads as follows:

$$\begin{aligned} \langle\Psi(t)|\hat{E}|\Psi(t)\rangle &= \frac{1}{2} \cdot (\lambda_{11} + \lambda_{44}) + \frac{1}{2} \cdot \exp\left(\int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right) \cdot \\ &\cdot \sum_{m+j=n+i} \frac{2^{j+i}}{m!j!n!i!} \cdot \left[(-1)^{n+j} \cdot \lambda_{41} + (-1)^{m+i} \cdot \lambda_{14}\right] \cdot \\ &\cdot \int d^3k_1 \ldots d^3k_{n+i} \sum_{\lambda_1\ldots\lambda_{n+i}} |\gamma_{k_1\lambda_1}|^2 \cdot \ldots \cdot |\gamma_{k_{n+i}\lambda_{n+i}}|^2 \cdot \\ &\cdot \sum_{u_1\ldots u_{m+j}=P(k_1\ldots k_{n+i})} \exp\left(i\left(\Omega_{u_1} + \ldots + \Omega_{u_m} - \Omega_{k_1} \ldots - \Omega_{k_n}\right) \cdot t\right) \ . \end{aligned} \tag{337}$$

Here, $P(k_1 \ldots k_{n+i})$ denotes a permutation of the wave vectors $k_1 \ldots k_{n+i}$. Note that the condition $m + j = n + i$ in Eq. (337) leads to the relation

$$(-1)^{m+i} = (-1)^{m+i+j-j} = (-1)^{n+2i-j} = (-1)^{n+j} \ ; \tag{338}$$

consequently, we have:

$$\begin{aligned} \langle\Psi(t)|\hat{E}|\Psi(t)\rangle &= \frac{1}{2} \cdot (\lambda_{11} + \lambda_{44}) + \frac{1}{2} \cdot \exp\left(\int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right) \cdot (\lambda_{41} + \lambda_{14}) \cdot \\ &\cdot \sum_{m+j=n+i} \frac{1}{m!} \cdot \frac{(-1)^n}{n!} \cdot \frac{(-2)^j}{j!} \cdot \frac{2^i}{i!} \cdot \int d^3k_1 \ldots d^3k_{n+i} \sum_{\lambda_1\ldots\lambda_{n+i}} |\gamma_{k_1\lambda_1}|^2 \cdot \ldots \cdot |\gamma_{k_{n+i}\lambda_{n+i}}|^2 \cdot \\ &\cdot \sum_{u_1\ldots u_{m+j}=P(k_1\ldots k_{n+i})} \exp\left(i\left(\Omega_{u_1} + \ldots + \Omega_{u_m} - \Omega_{k_1} \ldots - \Omega_{k_n}\right) \cdot t\right) \ . \end{aligned} \tag{339}$$

Making use of the *lemma of Riemann-Lebesgue*, the long-time limit in Eq. (339) is immediately obtained: Direct inspection of this equation shows that only those contributions have to be taken into account, which fulfill the conditions

$$m = n \ , \ j = i \ , \ u_1 \ldots u_n = P(k_1 \ldots k_n) \ , \ u_{n+1} \ldots u_{n+i} = P(k_{n+1} \ldots k_{n+i}) \ . \qquad (340)$$

We thus conclude that

$$\langle \hat{E} \rangle (t \to \infty) = \frac{1}{2} \cdot (\lambda_{11} + \lambda_{44}) + \frac{1}{2} \cdot (\lambda_{41} + \lambda_{14}) \cdot \exp \left(-4 \cdot \int d^3k \sum_{\lambda} |\gamma_{k\lambda}|^2 \right) \qquad (341)$$

is true. As important examples, we mention the expectation values $\langle e^+ e \rangle$, $\langle h^+ h \rangle$ and $\langle e^+ h^+ h e \rangle$:

$$\begin{aligned} &\langle e^+ e \rangle (t \to \infty) = \langle h^+ h \rangle (t \to \infty) = \langle e^+ h^+ h e \rangle (t \to \infty) = \\ &= \frac{1}{2} \cdot \left(1 - \exp \left(-4 \cdot \int d^3k \sum_{\lambda} |\gamma_{k\lambda}|^2 \right) \right) \ . \end{aligned} \qquad (342)$$

Again, any expectation value approaches a finite value for sufficiently large t.

The polarization function $\langle e^+ h^+ \rangle (t)$, however, proves to be an exception: From Eq. (339), we obtain additionally

$$\langle e^+ h^+ \rangle (t) = 0 \ . \qquad (343)$$

F. Expectation values of type $\langle \hat{E} \int d^3k \sum_{\lambda} f_{k\lambda} \cdot a^+_{k\lambda} \rangle$

In our last subsection, we are concerned with expectation values of type $\langle \hat{E} \int d^3k \sum_{\lambda} f_{k\lambda} \cdot a^+_{k\lambda} \rangle$. From Eq. (324), we obtain by a lengthy but straightforward calculation the formula

$$\langle \hat{E} \int d^3k \sum_{\lambda} f_{k\lambda} \cdot a^+_{k\lambda} \rangle (t) = (\lambda_{11} - \lambda_{44}) \cdot (\Sigma_1 + \Sigma_2) + (\lambda_{41} - \lambda_{14}) \cdot (\Pi_1 + \Pi_2) \ , \qquad (344)$$

where

$$\Sigma_1 = -\frac{1}{2} \cdot \int d^3k \sum_{\lambda} f_{k\lambda} \cdot \gamma_{k\lambda} \ , \ \Sigma_2 = \frac{1}{2} \cdot \int d^3k \sum_{\lambda} \gamma_{k\lambda} \cdot f_{k\lambda} \cdot \exp (i \Omega_k \cdot t) \ , \qquad (345)$$

$$\Pi_1 = \frac{1}{2} \cdot \exp\left(\int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right) \cdot \int d^3k \sum_\lambda f_{k\lambda} \cdot \gamma_{k\lambda} \cdot \tag{346}$$
$$\cdot \sum_{m+j=n+i} \frac{1}{m!} \cdot \frac{(-1)^n}{n!} \cdot \frac{(-2)^j}{j!} \cdot \frac{2^i}{i!} \cdot \int d^3k_1 \ldots d^3k_{n+i} \sum_{\lambda_1 \ldots \lambda_{n+i}} |\gamma_{k_1\lambda_1}|^2 \cdot \ldots \cdot |\gamma_{k_{n+i}\lambda_{n+i}}|^2 \cdot$$
$$\cdot \sum_{q_1 \ldots q_{m+j} = P(k_1 \ldots k_{n+i})} \exp\left(i\left(\Omega_{q_1} + \ldots + \Omega_{q_m} - \Omega_{k_1} \ldots - \Omega_{k_n}\right) \cdot t\right)$$

and

$$\Pi_2 = \frac{1}{2} \cdot \exp\left(\int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right) \cdot \tag{347}$$
$$\cdot \sum_{m+j=n+1+i} \frac{1}{m!} \cdot \frac{(-1)^n}{n!} \cdot \frac{(-2)^j}{j!} \cdot \frac{2^i}{i!} \cdot \int d^3k_1 \ldots d^3k_{n+1+i} \sum_{\lambda_1 \ldots \lambda_{n+1+i}}$$
$$|\gamma_{k_1\lambda_1}|^2 \cdot \ldots \cdot |\gamma_{k_{n+i}\lambda_{n+i}}|^2 \cdot \gamma_{k_{n+1+i}\lambda_{n+1+i}} \cdot f_{k_{n+1+i}\lambda_{n+1+i}} \cdot$$
$$\cdot \sum_{q_1 \ldots q_{m+j} = P(k_1 \ldots k_{n+1+i})} \exp\left(i\left(\Omega_{q_1} + \ldots + \Omega_{q_m} - \Omega_{k_1} \ldots - \Omega_{k_n}\right) \cdot t\right) \ .$$

In the following part of our considerations, we assume that the function $f_{k\lambda}$ is chosen such that the *lemma of Riemann-Lebesgue* can be applied to perform the long-time limit in Eq. (344); in particular, $f_{k\lambda}$ should not contain a distribution. Under these circumstances, we obtain from Eq. (344) the relation

$$\langle \hat{E} \int d^3k \sum_\lambda f_{k\lambda} \cdot a^+_{k\lambda}\rangle(t \to \infty) = \left(-\frac{1}{2}\right) \cdot \int d^3k \sum_\lambda f_{k\lambda} \cdot \gamma_{k\lambda} \cdot \tag{348}$$
$$\cdot \left(\lambda_{11} - \lambda_{44} + [\lambda_{41} - \lambda_{14}] \cdot \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)\right) \ .$$

Note that the electron-hole operator $\hat{E}$ can be chosen arbitrarily; consequently, the expectation values of type $\langle \hat{E} \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle$ are also contained in our general results. In particular, the long-time limit is given by

$$\langle \hat{E} \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle(t \to \infty) = \left(-\frac{1}{2}\right) \cdot \int d^3k \sum_\lambda f_{k\lambda} \cdot \gamma^*_{k\lambda} \cdot \tag{349}$$
$$\cdot \left(\lambda_{11} - \lambda_{44} + [\lambda_{14} - \lambda_{41}] \cdot \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)\right) \ .$$

From Eq. (349), we obtain, for example

$$\langle e^+h^+ \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle(t \to \infty) = \tag{350}$$
$$= \left(-\frac{1}{2}\right) \cdot \int d^3k \sum_\lambda f_{k\lambda} \cdot \gamma^*_{k\lambda} \cdot \left(1 - \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)\right) \ ,$$

$$\langle he \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle(t \to \infty) = \tag{351}$$
$$= \left(-\frac{1}{2}\right) \cdot \int d^3k \sum_\lambda f_{k\lambda} \cdot \gamma^*_{k\lambda} \cdot \left(1 + \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)\right)$$

and therefore

$$|\langle e^+h^+ \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle(t \to \infty)| = \tag{352}$$
$$= |\langle he \int d^3k \sum_\lambda f_{k\lambda} \cdot a_{k\lambda}\rangle(t \to \infty)| \cdot \frac{1 - \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)}{1 + \exp\left(-4 \cdot \int d^3k \sum_\lambda |\gamma_{k\lambda}|^2\right)} \ .$$

Eq. (352) demonstrates explicitly that the expectation values $\langle hea_{k\lambda}\rangle$ are - in general - *not* comparatively small. Consequently, the physical interpretation that an electron-hole pair *and* a photon are removed from the system under consideration is incorrect. In fact, we only know that the expectation values are determined by the hierarchy equations and by the initial state: Any further assumption (which is often motivated by a so called "physical interpretation") may cause *intrinsic contradictions or contradictions to physical laws.*

We summarize our results:

Theorem 4:

Consider a purely electronic two-level quantum dot coupled to an unlimited number of photon modes, as defined by Eqs. (306), (307) and (308) (i.e., the exciting light field vanishes). Assume (i) that the Coulomb matrix element W^{eh} *fulfills the condition* $\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0$ *and (ii) that the initial state is the vacuum state. Then, the following relations are valid for any* $t \geq 0$*:*

$$N_{Ph.}(t) := \langle\Psi(t)| \int d^3k \sum_\lambda a^+_{k\lambda} a_{k\lambda} |\Psi(t)\rangle = 2 \cdot \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} \cdot (1 - \cos\Omega_k \cdot t) \ , \tag{353}$$

$$N_{Ph.}(t \to \infty) = 2 \cdot \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2} \ , \tag{354}$$

$$\langle e^+e\rangle(t \to \infty) = \langle h^+h\rangle(t \to \infty) = \langle e^+h^+he\rangle(t \to \infty) = \tag{355}$$
$$= \frac{1}{2} \cdot \left(1 - \exp\left(-4 \cdot \int d^3k \sum_\lambda \frac{|M_{k\lambda}|^2}{(\hbar\Omega_k)^2}\right)\right) \ ,$$

$$\langle e^+h^+\rangle(t) = 0 \ . \tag{356}$$

$M_{k\lambda}$ *denotes the particle-photon coupling function.*

VII. GENERAL RELATIONSHIP BETWEEN THE FULLY QUANTIZED AND THE SEMICLASSICAL LIGHT-MATTER INTERACTION OPERATOR. FURTHER CONCLUSIONS

We have already emphasized that the standard particle-photon interaction operator does not fulfill the basic condition of solid state physics that the vacuum state has to be an eigenstate of the complete Hamiltonian, if the exciting light-field vanishes. Apart from trivial cases, this observation leads to the conclusion that the ground state of the standard Hamiltonian contains, in general, electrons, holes and photons. Moreover, dynamical vacuum fluctuations of the quantized electromagnetic field do appear, if the initial state $|\Psi(t=0)\rangle = |0\rangle$ is chosen. *According to a basic physical experience, however, dynamical vacuum fluctuations do not exist!* It is therefore necessary to replace the standard interaction operator by another one.

In this section, we shall not discuss possible alternative approaches in detail. Instead, it proves useful to analyze *the general relationship between the fully quantized and the semiclassical light-matter interaction operator* in advance. To do so, we introduce at first the general particle-photon interaction operator

$$V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n \ , \tag{357}$$

where $\hat{A}_n$ are purely photonic operator sequences in normal order. We assume that any electron-hole operator sequence $\hat{E}\hat{H}_n$ fulfills the condition

$$\langle 0|\hat{E}\hat{H}_n|0\rangle = 0 \ . \tag{358}$$

We proceed as follows:

(i) The coupling of the electron-hole system to the classical exciting light field can be described by an interaction operator of general type

$$V_{ehS}(t) = \sum_m S_m(t) \cdot \hat{F}\hat{G}_m \ ; \ \langle 0|\hat{F}\hat{G}_m|0\rangle = 0 \ , \tag{359}$$

where $\hat{F}\hat{G}_m$ are electron-hole operator sequences in normal order. In order to demonstrate this, we start from the general expression $V_{ehS}(t) = \sum_m S_m(t) \cdot \hat{\Phi}\hat{\Gamma}_m$ and split $\hat{\Phi}\hat{\Gamma}_m$ into $\hat{\Phi}\hat{\Gamma}_m = \mu_m \cdot \underline{1} + \hat{F}\hat{G}_m$, $\langle 0|\hat{F}\hat{G}_m|0\rangle = 0$. *For physical reasons*, we require $\langle 0|V_{ehS}(t)|0\rangle = 0$. This finishes our proof.

(ii) Now, let $\hat{I}$ be an arbitrary operator of the electron-hole system. Under these circumstances, the Heisenberg equation of motion leads to the expectation values

$$\left\langle \left[V_{ehp}, \hat{I}\right]\right\rangle = \sum_n \left\langle \hat{A}_n \left[\hat{E}\hat{H}_n, \hat{I}\right]\right\rangle \tag{360}$$

and

$$\left\langle \left[V_{ehS}, \hat{I}\right]\right\rangle = \sum_m S_m(t) \cdot \left\langle \left[\hat{F}\hat{G}_m, \hat{I}\right]\right\rangle \ . \tag{361}$$

Performing the semiclassical limit in Eq. (360), we obtain

$$\left\langle \left[V_{ehp}, \hat{I}\right]\right\rangle \to \sum_n \left\langle \hat{A}_n\right\rangle \cdot \left\langle \left[\hat{E}\hat{H}_n, \hat{I}\right]\right\rangle \ . \tag{362}$$

Because of the fact that *the expressions (361) and (362) must contain the same basic physics*, we conclude (without loss of generality) that

$$\left[\hat{E}\hat{H}_n, \hat{I}\right] = \left[\hat{F}\hat{G}_n, \hat{I}\right] \tag{363}$$

and

$$\left[\hat{E}\hat{H}_n - \hat{F}\hat{G}_n, \hat{I}\right] = 0 \tag{364}$$

must be true. We remark that the electron-hole operator $\hat{I}$ in Eq. (364) can be chosen arbitrarily; consequently, we have

$$\hat{E}\hat{H}_n - \hat{F}\hat{G}_n = \lambda_n \cdot \underline{1} \ . \tag{365}$$

By taking vacuum expectation values, we obtain, in combination with Eqs. (358) and (359), $\lambda_n = 0$ and

$$V_{ehp} = \sum_n \hat{A}_n\hat{E}\hat{H}_n \ , \ V_{ehS}(t) = \sum_n S_n(t) \cdot \hat{E}\hat{H}_n \ , \tag{366}$$

i.e., the electronic parts of the interaction operators V_{ehp} and $V_{ehS}(t)$ have the same mathematical structure.

We summarize our results:

Theorem 5:

Consider an electron-hole system, interacting with phonons and photons. Let $V_{ehS}(t)$ be the semiclassical light-matter interaction operator. Assume that the electronic constituents of the particle-photon interaction operator

$$V_{ehp} = \sum_n \hat{A}_n \hat{E} \hat{H}_n \tag{367}$$

fulfill the condition

$$\langle 0|\hat{E}\hat{H}_n|0\rangle = 0 \; . \tag{368}$$

Then, the electronic parts of the interaction operators V_{ehp} and $V_{ehS}(t)$ have the same mathematical structure:

$$V_{ehS}(t) = \sum_n S_n(t) \cdot \hat{E}\hat{H}_n \; . \tag{369}$$

After these preparations, we now consider a particle-photon interaction operator V_{ehp} which is represented by a functional form of the vector potential $\hat{A}(r)$ and the electronic field operators $\hat{\Psi}^+(r), \hat{\Psi}(r)$:

$$V_{ehp} = \sum_{m=1}^{\infty} \hat{F}_m \; , \tag{370}$$

$$\begin{aligned} &\hat{F}_m = \qquad (371)\\ &\bigcup_{m1} \ldots \hat{A}_{\nu_m(1,1)}(r_{\mu_m(1,1)}) \ldots \hat{A}_{\nu_m(1,j)}(r_{\mu_m(1,j)}) \ldots \hat{A}_{\nu_m(1,m)}(r_{\mu_m(1,m)}) \ldots + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\bigcup_{ml} \ldots \hat{A}_{\nu_m(l,1)}(r_{\mu_m(l,1)}) \ldots \; \hat{A}_{\nu_m(l,j)}(r_{\mu_m(l,j)}) \ldots \; \hat{A}_{\nu_m(l,m)}(r_{\mu_m(l,m)}) \ldots \; + \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$\bigcup_{ml}$ denotes the (linear) "operator environment" of the $l-$th constituent of $\hat{F}_m$. *Our basic assumption is that photonic operators do only appear in $\hat{A}(r)$.* The electronic field operators are contained in $\bigcup_{ml}$. The functions $\nu_m(l,j)$ and $\mu_m(l,j)$ decribe the distribution of the indices ν and μ for a fixed functional form $\hat{F}_m$. Any constituent of $\hat{F}_m$ contains exactly m expressions of type $\hat{A}_\nu(r_\mu)$.

According to Eqs. (7) and (8), the vector potential $\hat{A}(r) = (\hat{A}_1(r), \hat{A}_2(r), \hat{A}_3(r))$ reads as follows:

$$\hat{A}_\nu(r) = \int d^3k \sum_\lambda B_{k\lambda\nu}(r)\; a_{k\lambda} + \int d^3k \sum_\lambda B^*_{k\lambda\nu}(r)\; a^+_{k\lambda}\;, \tag{372}$$

$$B_{k\lambda\nu}(r) = G_k\; \varepsilon^{(\nu)}_{k\lambda}\; e^{ik\cdot r}\;,\; G_k = \sqrt{\frac{\hbar}{2\cdot(2\pi)^3\cdot\varepsilon_0\cdot\Omega_k}}\;. \tag{373}$$

For the following part of our considerations, it proves useful to introduce the abbreviations

$$B^{(1)}_{k\lambda\nu}(r) := B_{k\lambda\nu}(r) = G_k\; \varepsilon^{(\nu)}_{k\lambda}\; e^{ik\cdot r}\;, \tag{374}$$

$$B^{(2)}_{k\lambda\nu}(r) := B^*_{k\lambda\nu}(r) = G_k\; \varepsilon^{(\nu)}_{k\lambda}\; e^{-ik\cdot r} \tag{375}$$

and

$$a^{(1)}_{k\lambda} := a_{k\lambda}\;,\; a^{(2)}_{k\lambda} := a^+_{k\lambda}\;. \tag{376}$$

Making use of these definitions, we obtain

$$\hat{A}_\nu(r) = \sum_{i=1}^{2} \int d^3k \sum_\lambda B^{(i)}_{k\lambda\nu}(r)\; a^{(i)}_{k\lambda}\;. \tag{377}$$

In view of Eq. (371), we now introduce the expansions

$$\hat{A}_{\nu_m(l,j)}(r_{\mu_m(l,j)}) = \sum_{i_j=1}^{2} \int d^3k_j \sum_{\lambda_j} B^{(i_j)}_{k_j\lambda_j\nu_m(l,j)}(r_{\mu_m(l,j)})\; a^{(i_j)}_{k_j\lambda_j}\;. \tag{378}$$

We emphasize that the indices i,k,λ are labelled only by the column index j. A combination of Eqs. (371) and (378) leads to the result

$$\hat{F}_m = \int d^3k_1 \sum_{\lambda_1} \ldots \int d^3k_m \sum_{\lambda_m} \sum_{i_1=1}^{2} \ldots \sum_{i_m=1}^{2} \hat{E}\hat{H}^{(i_1 \ldots i_m)}_{k_1\lambda_1 \ldots k_m\lambda_m} : a^{(i_1)}_{k_1\lambda_1} \ldots a^{(i_m)}_{k_m\lambda_m} :\;, \tag{379}$$

where

$$\hat{E}\hat{H}^{(i_1\dots i_j\dots i_m)}_{k_1\lambda_1\dots k_j\lambda_j\dots k_m\lambda_m} := \tag{380}$$
$$\bigcup_{m1}\dots B^{(i_1)}_{k_1\lambda_1\nu_m(1,1)}(r_{\mu_m(1,1)})\dots B^{(i_j)}_{k_j\lambda_j\nu_m(1,j)}(r_{\mu_m(1,j)})\dots B^{(i_m)}_{k_m\lambda_m\nu_m(1,m)}(r_{\mu_m(1,m)})\dots +$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\bigcup_{ml}\dots B^{(i_1)}_{k_1\lambda_1\nu_m(l,1)}(r_{\mu_m(l,1)})\dots\ B^{(i_j)}_{k_j\lambda_j\nu_m(l,j)}(r_{\mu_m(l,j)})\dots\ B^{(i_m)}_{k_m\lambda_m\nu_m(l,m)}(r_{\mu_m(l,m)})\dots\ +$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Note that $\hat{E}\hat{H}^{(i_1\dots i_m)}_{k_1\lambda_1\dots k_m\lambda_m}$ does not contain photonic operators. The symbol $:\dots:$ in Eq. (379) denotes *normal order*. We remark that $\bigcup_{ml}$ does neither act on (i,k,λ) nor on $a^{(i)}_{k\lambda}$.

So far the representation of V_{ehp} is quite general. However, if we are concerned with a *physical* theory, we have to require that at least the following *symmetry conditions* are fulfilled:

$$\hat{E}\hat{H}^{(1\dots 1)}_{k_1\lambda_1\dots k_m\lambda_m} = \hat{E}\hat{H}^{(1\dots 1)}_{P(k_1\lambda_1\dots k_m\lambda_m)}\ ,\ \hat{E}\hat{H}^{(2\dots 2)}_{k_1\lambda_1\dots k_m\lambda_m} = \hat{E}\hat{H}^{(2\dots 2)}_{P(k_1\lambda_1\dots k_m\lambda_m)}\ . \tag{381}$$

Here, $P(k_1\lambda_1\dots k_m\lambda_m)$ denotes an arbitrary permutation of the indices $(k_1\lambda_1)\dots(k_m\lambda_m)$. It is needless to emphasize that condition (381) is due to the Bose character of the photons.

In the absence of an external light field, the vacuum state $|0\rangle$ has to be an eigenstate of the Hamiltonian under consideration. As usual, it is understood that any eigenvalue of $|0\rangle$ is equal to zero; consequently,

$$0 = V_{ehp}|0\rangle = \tag{382}$$
$$= \sum_{m=1}^{\infty}\int d^3k_1\sum_{\lambda_1}\dots\int d^3k_m\sum_{\lambda_m}\sum_{i_1=1}^{2}\dots\sum_{i_m=1}^{2}\hat{E}\hat{H}^{(i_1\dots i_m)}_{k_1\lambda_1\dots k_m\lambda_m} : a^{(i_1)}_{k_1\lambda_1}\dots a^{(i_m)}_{k_m\lambda_m} : |0\rangle =$$
$$= \sum_{m=1}^{\infty}\int d^3k_1\sum_{\lambda_1}\dots\int d^3k_m\sum_{\lambda_m}\ \hat{E}\hat{H}^{(2\dots 2)}_{k_1\lambda_1\dots k_m\lambda_m}\ a^{+}_{k_1\lambda_1}\dots a^{+}_{k_m\lambda_m}\ |0\rangle$$

is true.

Now, let $(q_1\mu_1\dots q_n\mu_n)$ be a fixed configuration of $q-$vectors and $\mu-$indices. From Eq. (382), we obtain immediately

$$
\begin{aligned}
0 = \int d^3k_1 \sum_{\lambda_1} \ldots \int d^3k_n \sum_{\lambda_n} \\
\langle 0_{Phot.}| a_{q_1\mu_1} \ldots a_{q_n\mu_n}\; a^+_{k_1\lambda_1} \ldots a^+_{k_n\lambda_n}\; |0_{Phot.}\rangle \cdot \hat{E}\hat{H}^{(2\ldots2)}_{k_1\lambda_1\ldots k_n\lambda_n}\; |0_{eh}\rangle = \\
= \sum_{k_1\lambda_1\ldots k_n\lambda_n = P(q_1\mu_1\ldots q_n\mu_n)} \hat{E}\hat{H}^{(2\ldots2)}_{k_1\lambda_1\ldots k_n\lambda_n}\; |0_{eh}\rangle \; . \qquad (383)
\end{aligned}
$$

Making use of the symmetry condition (381), we conclude from Eq. (383) that

$$
\hat{E}\hat{H}^{(2\ldots2)}_{q_1\mu_1\ldots q_n\mu_n} |0_{eh}\rangle = 0 \qquad (384)
$$

is valid, too.

In a next step, we consider an expression of type $\hat{E}\hat{H}^{(i_1\ldots2\ldots i_m)}_{k_1\lambda_1\ldots k_j\lambda_j\ldots k_m\lambda_m}$ and replace $k_j \to -k_j$ for a fixed index j $(1 \le j \le m)$. We remark that k_j does only appear in the expressions $B^{(2)}_{k_j\lambda_j\nu_m(l,j)}(r_{\mu_m(l,j)}), l \ge 1$ (see Eq. (380) again). From Eqs. (374) and (375), we obtain, in combination with the formula

$$
\varepsilon_{-k\lambda} = (-1)^{\lambda} \cdot \varepsilon_{k\lambda} \; , \qquad (385)
$$

the relations

$$
B^{(2)}_{-k_j\lambda_j\nu_m(l,j)}(r_{\mu_m(l,j)}) = (-1)^{\lambda_j} \cdot B^{(1)}_{k_j\lambda_j\nu_m(l,j)}(r_{\mu_m(l,j)}) \quad for\ any\ l \ge 1 \qquad (386)
$$

and

$$
\hat{E}\hat{H}^{(i_1\ldots2\ldots i_m)}_{k_1\lambda_1\ldots -k_j\lambda_j\ldots k_m\lambda_m} = (-1)^{\lambda_j} \cdot \hat{E}\hat{H}^{(i_1\ldots1\ldots i_m)}_{k_1\lambda_1\ldots k_j\lambda_j\ldots k_m\lambda_m} \; , \qquad (387)
$$

i.e., $(\ldots 2 \ldots)$ *is replaced by* $(\ldots 1 \ldots)$ *and* $-k_j$ *is replaced by* k_j. From Eq. (384), we thus conclude that

$$
\hat{E}\hat{H}^{(i_1\ldots i_n)}_{q_1\mu_1\ldots q_n\mu_n} |0_{eh}\rangle = 0 \qquad (388)
$$

is valid for an *arbitrary* configuration $(i_1 \ldots i_n)$. Consequently, the complete interaction operator V_{ehp} reads as follows:

$$
V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n \quad with \quad \hat{E}\hat{H}_n|0\rangle = 0 \; . \qquad (389)
$$

Roughly speaking, Eq. (389) tells us that the particle operators $\hat{E}\hat{H}_n$ are of *density type*.

We summarize our last result:

Theorem 6:

Consider an electron-hole system, interacting with phonons and photons. Assume (i) that the particle-photon interaction operator V_{ehp} *is a functional form of the vector potential* $\hat{A}(r)$ *and the electronic field operators* $\hat{\Psi}^+(r), \hat{\Psi}(r)$ *(see Eq. (370)) and (ii) that the vacuum state is an eigenstate of the Hamiltonian under consideration, if the exciting light field vanishes (as usual, it is understood that any eigenvalue of* $|0\rangle$ *is equal to zero). Then,* V_{ehp} *has the structure*

$$V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n \quad \textit{with} \quad \hat{E}\hat{H}_n|0\rangle = 0 \ . \tag{390}$$

In view of *Theorems 5* and *6* from above, it is now a simple task to derive one of our most important results. To do so, we assume that conditions (i) and (ii) of *Theorem 6* are fulfilled. Under these circumstances, the particle-photon interaction operator V_{ehp} has the structure

$$V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n \quad \textit{with} \quad \hat{E}\hat{H}_n|0\rangle = 0 \ . \tag{391}$$

From Eq. (391), we conclude that *Theorem 5* can be applied; consequently, the electronic parts of the interaction operators V_{ehp} and $V_{ehS}(t)$ have the same mathematical structure:

$$V_{ehS}(t) = \sum_n S_n(t) \cdot \hat{E}\hat{H}_n \ . \tag{392}$$

In a last step, we assume that the initial state is the vacuum state:

$$|\Psi(t=0)\rangle = |0\rangle \ . \tag{393}$$

In view of Eqs. (391), (392) and (393), we thus conclude that

$$|\Psi(t)\rangle = |0\rangle \tag{394}$$

is the solution of the time-dependent Schrödinger equation. Consequently, neither a change of the particle numbers nor an emission of photons is possible under these circumstances.

Theorem 7:

Consider an electron-hole system, interacting with phonons and photons. Assume (i) that the particle-photon interaction operator V_{ehp} *is a functional form of the vector potential* $\hat{A}(r)$ *and the electronic field operators* $\hat{\Psi}^+(r), \hat{\Psi}(r)$ *(see Eq. (370)) and (ii) that the vacuum state is an eigenstate of the Hamiltonian under consideration, if the exciting light field vanishes (as usual, it is understood that any eigenvalue of* $|0\rangle$ *is equal to zero). Choose the initial state* $|\Psi(t=0)\rangle = |0\rangle$*. Then, the relation*

$$|\Psi(t)\rangle = |0\rangle \tag{395}$$

is true for any $t \geq 0$*.*

We add four comments:

(i) Apart from the symmetry conditions (381), the details of the mathematical structure of the functional form $\hat{F}_m$ in Eq. (371) are unimportant. Consequently, it is essentially the Bose character of the photons which leads to our basic result in *Theorem 7.* However, we have to emphasize again that the following condition has to be fulfilled: *Photonic operators do only appear in* $\hat{A}(r)$*.* This condition is violated, if further bosonic field operators due to the quantized electromagnetic field are contained in $\hat{F}_m$ (as an important example, we mention the so-called "time derivative" of $\hat{A}(r)$). In *Theorem 7*, the existence of further Bose fields is excluded by the requirement that the particle-photon interaction operator V_{ehp} does only contain the vector potential $\hat{A}(r)$ and the electronic field operators $\hat{\Psi}^+(r), \hat{\Psi}(r)$.

(ii) The essential physical statement of *Theorem 7* can be summarized as follows: If we start from the initial state $|\Psi(t=0)\rangle = |0\rangle$, a *non-trivial* time evolution is possible only if the quantum-field theoretical Hamiltonian under consideration leads to *dynamical vacuum fluctuations.*

(iii) The reader will verify by a careful inspection that the proofs for *Theorem 5* and *Theorem 6* remain valid, if the light-matter interaction depends on further *non-photonic* field operators. In view of condition (i) of *Theorem 7*, it is therefore sufficient to require that - apart from the vector potential $\hat{A}(r)$ - only *non-photonic* field operators do appear in the matter-photon interaction operator V_{ehp}.

(iv) A final remark is concerned with the so-called *rotating wave approximation.* Here, the corresponding Hamiltonian H can not be represented by a functional form of the vector potential $\hat{A}(r)$ and the electronic field operators $\hat{\Psi}^+(r), \hat{\Psi}(r)$, i.e., H is not a quantum-field theoretical Hamiltonian; consequently, *Theorem 7* can not be applied (condition (i) is violated).

A. Introduction of photonic densities

Our previous investigations are based on a particle-photon interaction operator V_{ehp} which is represented by the functional form (370) of the vector potential $\hat{A}(r)$ and the electronic field operators $\hat{\Psi}^+(r), \hat{\Psi}(r)$. As has been demonstrated above, condition (ii) of *Theorem 6* leads to the conclusion that the electronic parts of V_{ehp} have to be of *density type.*

Although the interaction operator V_{ehp} from above is quite general, it does not contain, for example, the Bose field

$$\hat{B}(r) = \int d^3k \sum_{\lambda} B_{k\lambda}(r) \cdot (-i\Omega_k) \cdot a_{k\lambda} + \int d^3k \sum_{\lambda} B^*_{k\lambda}(r) \cdot i\Omega_k \cdot a^+_{k\lambda} \tag{396}$$

which is the so-called "time derivative" of $\hat{A}(r)$. By a suitable combination of $\hat{A}(r)$, $\hat{B}(r)$ and further Bose fields, it may be possible to create an interaction operator of type

$$V_{ehp} = \sum_n \hat{A}_n \hat{E} \hat{H}_n \quad \textit{with} \quad \hat{A}_n|0\rangle = 0 \; . \tag{397}$$

In the following part of our considerations, we take Eq. (397) for granted and discuss the question whether V_{ehp} leads to a realistic model or not. Obviously, the complete Hamiltonian

$$H(t) = H_0 + V_{ehS}(t) =: H_M + \int d^3k \sum_{\lambda} \hbar\Omega_k a^+_{k\lambda} a_{k\lambda} + V_{ehp} + V_{ehS}(t) \tag{398}$$

fulfills the condition $H_0|0\rangle = 0$, provided that the vacuum state is an eigenstate of H_M (H_M corresponds to the particle-phonon system). Consequently, the statement $\hat{E}\hat{H}_n|0\rangle = 0$ is - in general - no longer valid.

We proceed as follows:

First of all, it proves useful to introduce the time-dependent operator

$$\hat{O}(t) := H_M + V_{ehS}(t) \ . \tag{399}$$

We are going to analyze the solution $|\Psi(t)\rangle$ of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{O}(t)|\Psi(t)\rangle \tag{400}$$

under the initial conditon

$$|\Psi(t=0)\rangle = \hat{E}\hat{H}\hat{P}|0\rangle \ , \tag{401}$$

where $\hat{E}\hat{H}\hat{P}$ is a particle-phonon operator sequence. We remark that both the initial state $|\Psi(t=0)\rangle$ as well as the operator $\hat{O}(t)$ itself are restricted to the particle-phonon subspace; consequently, the same statement is true for the solution $|\Psi(t)\rangle$ of Eqs. (400) and (401). We thus obtain

$$a_{k\lambda}|\Psi(t)\rangle = 0 \ , \ \hat{A}_n|\Psi(t)\rangle = 0 \tag{402}$$

and therefore

$$\hat{O}(t)|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \ , \tag{403}$$

i.e., $|\Psi(t)\rangle$ is the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \ , \tag{404}$$

too.

So we conclude that a non-trivial time evolution of $|\Psi(t)\rangle$ is in fact possible, if we use a particle-photon interaction operator of type (397) which violates condition (i) of *Theorem 7*. Surprisingly enough, however, it turns out that the photon number is just a constant: From Eq. (402), we obtain immediately

$$\langle\Psi(t)|\int d^3k \sum_\lambda a^+_{k\lambda}a_{k\lambda}|\Psi(t)\rangle = 0 \ . \tag{405}$$

We should emphasize that the validity of Eq. (405) has only been demonstrated for a *subset* of initial states (see Eq. (401) again). In other words: If condition (397) is fulfilled, the photon number is - in general - not a constant of motion.

Eq. (405) can be extended to an arbitrary photonic operator sequence $\hat{A}$ which is in normal order and does not contain a constituent $\sim \underline{1}_{Phot.}$:

$$\langle \Psi(t)|\hat{A}|\Psi(t)\rangle = \langle 0_{Phot.}|\langle \Psi_{e\,h\,phon.}(t)|\hat{A}|\Psi_{e\,h\,phon.}(t)\rangle|0_{Phot.}\rangle = 0 \; . \tag{406}$$

We summarize our last result:

Theorem 8:

Consider an electron-hole system, interacting with phonons and photons. Assume that any photonic constituent $\hat{A}_n$ of the particle-photon interaction operator $V_{ehp} = \sum_n \hat{A}_n \hat{E}\hat{H}_n$ fulfills the condition $\hat{A}_n|0\rangle = 0$. Choose the initial state $|\Psi(t=0)\rangle = \hat{E}\hat{H}\hat{P}|0\rangle$, where $\hat{E}\hat{H}\hat{P}|0\rangle$ is an element of the particle-phonon subspace. Then, the relation

$$\langle \Psi(t)|\hat{A}|\Psi(t)\rangle = 0 \tag{407}$$

is true for any $t \geq 0$, if $\hat{A}$ is a purely photonic operator sequence in normal order which does not contain a constituent $\sim \underline{1}_{Phot.}$.

VIII. GENERAL PARTICLE-PHOTON INTERACTION OPERATOR OF PRODUCT TYPE

In our next section, we are concerned again with an electron-hole system, interacting with phonons and photons. The particle-photon interaction operator V_{ehp} is chosen to be of product type:

$$V_{ehp} = \hat{A} \cdot \hat{E}\hat{H} \; ; \tag{408}$$

here, $\hat{A}$ denotes the photonic part and $\hat{E}\hat{H}$ is an electron-hole operator sequence in normal order.

Our complete Hamiltonian H reads as follows:

$$H(t) = H_M + \int d^3k \sum_\lambda \hbar\Omega_k a^+_{k\lambda} a_{k\lambda} + V_{ehp} + V_{ehS}(t) \; . \tag{409}$$

H_M corresponds to the particle-phonon system and fulfills the condition

$$H_M|0\rangle = 0 \; . \tag{410}$$

We assume that the initial state is the vacuum state:

$$|\Psi(t = 0)\rangle = |0\rangle \; . \tag{411}$$

In order to proceed, we are now going to analyze two different cases:

A. First case: $\hat{A}|0\rangle = 0$

Here, we introduce the operator

$$\hat{O}(t) := H_M + V_{ehS}(t) \; . \tag{412}$$

We remark that the solution $|\Psi(t)\rangle$ of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{O}(t)|\Psi(t)\rangle \tag{413}$$

under the initial condition (411) is restricted to the particle-phonon subspace. According to our assumption from above ($\hat{A}|0\rangle = 0$), we thus obtain

$$\hat{O}(t)|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \; , \tag{414}$$

i.e., $|\Psi(t)\rangle$ is the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \; , \tag{415}$$

too. Moreover,

$$\langle\Psi(t)| \int d^3k \sum_\lambda a^+_{k\lambda} a_{k\lambda} |\Psi(t)\rangle = 0 \tag{416}$$

is true for any $t \geq 0$. *In view of Eq. (416), we should emphasize that the photon number $\langle \int d^3k \sum_\lambda a^+_{k\lambda} a_{k\lambda}\rangle(t)$ is, in general, not a constant of motion.*

B. Second case: $\hat{A}|0\rangle \neq 0$

If the vacuum state $|0\rangle$ is not an eigenstate of $\hat{A}$, any constituent of the operator sequence $\hat{E}\hat{H}$ must contain at least one annihilation operator of the electron-hole system, the reason being that the vacuum state has to be an eigenstate of the complete Hamiltonian, if the exciting light field vanishes. Consequently,

$$\hat{E}\hat{H}|0\rangle = 0 \tag{417}$$

is true. According to *Theorem 5* in section VII, the electronic constituents of the interaction operators V_{ehp} and $V_{ehS}(t)$ must have the same mathematical structure under these circumstances:

$$V_{ehp} = \hat{A} \cdot \hat{E}\hat{H} \ , \ V_{ehS}(t) = S(t) \cdot \hat{E}\hat{H} \ . \tag{418}$$

In view of Eqs. (417) and (418), we thus conclude that

$$|\Psi(t)\rangle = |0\rangle \tag{419}$$

is the solution of the time-dependent Schrödinger equation, i.e., Eq. (416) remains valid.

We summarize our results:

Theorem 9:

Consider an electron-hole system, interacting with phonons and photons. Assume (i) that the vacuum state is an eigenstate of the complete Hamiltonian, if the exciting light field vanishes (as usual, it is understood that any eigenvalue of $|0\rangle$ is equal to zero) and (ii) that the particle-photon interaction operator is of product type. Choose the initial state $|\Psi(t=0)\rangle = |0\rangle$. Then, the relation

$$\langle\Psi(t)| \int d^3k \sum_{\lambda} a^{+}_{k\lambda} a_{k\lambda} |\Psi(t)\rangle = 0 \tag{420}$$

is true for any $t \geq 0$.

IX. ANALYSIS OF A PURELY ELECTRONIC TWO-LEVEL QUANTUM DOT COUPLED TO THE QUANTIZED ELECTROMAGNETIC FIELD IN THE FRAMEWORK OF AN EXTENDED MODEL HAMILTONIAN

Our previous investigations are essentially based on the particle-photon interaction operators V_1 and V_2 (defined by Eqs. (9) and (11)), where only *interband transitions* have been taken into account. Here, the coupling of the electron-hole system to the quantized electromagnetic field is characterized by operator products of *polarization type*:

$$V_1 + V_2 = \sum_{nm} \hat{A}_{nm} \cdot e_n^+ h_m^+ + \sum_{nm} \hat{A}_{nm}^+ \cdot h_m e_n \ , \tag{421}$$

where

$$\begin{aligned} \hat{A}_{nm} = & \int d^3k \sum_{\lambda} U_{nmk\lambda} \cdot a_{k\lambda} + \int d^3k \sum_{\lambda} V_{nmk\lambda} \cdot a_{k\lambda}^+ + \\ & + \int d^3k \int d^3k' \sum_{\lambda\lambda'} P_{nmkk'\lambda\lambda'} \cdot a_{k\lambda} a_{k'\lambda'} + \int d^3k \int d^3k' \sum_{\lambda\lambda'} Q_{nmkk'\lambda\lambda'} \cdot a_{k\lambda}^+ a_{k'\lambda'}^+ + \\ & + \int d^3k \int d^3k' \sum_{\lambda\lambda'} R_{nmkk'\lambda\lambda'} \cdot a_{k\lambda}^+ a_{k'\lambda'} \ . \end{aligned} \tag{422}$$

Strictly speaking, however, the complete interaction operator

$$V = \int d^3r \hat{\Psi}^+(r) \left(-\frac{e}{m} \cdot \hat{A}(r) \cdot p + \frac{e^2 \hat{A}^2(r)}{2m} \right) \hat{\Psi}(r) \tag{423}$$

leads to further contributions which are not contained in Eq. (421); these contributions are characterized by operator products of *density type* and may change some of our analytical results obtained in sections III, IV, V and VI. In the following part of our considerations, we are going to analyze this problem in more detail.

A. Two-level quantum dot coupled to a single photon mode

In a first step, we consider an extended version of Hamiltonian (140) in section IV:

$$H = H_1 + H_2 \ , \tag{424}$$

where

$$H_1 = \varepsilon_0 \cdot e^+ e + \varphi_0 \cdot h^+ h - 2 \cdot W^{eh} \cdot e^+ h^+ h e \ , \tag{425}$$

$$H_2 = \hbar\Omega \cdot a^+ a + V_{ehp} \tag{426}$$

and

$$V_{ehp} = \Big(e^+ + h\Big)\Big(M \cdot a + M^* \cdot a^+\Big)\Big(e + h^+\Big) \ . \tag{427}$$

Making use of *the basic rule of quantum field theory that any operator sequence has to be in normal order*, the particle-photon interaction operator V_{ehp} reads as follows:

$$V_{ehp} = \Big(M \cdot a + M^* \cdot a^+\Big)\Big(e^+ h^+ + he + e^+ e - h^+ h\Big) \ . \tag{428}$$

In view of the additional contribution $e^+e - h^+h$ in Eq. (428), we first remark that

$$\Big(e^+ e - h^+ h\Big)|e_i\rangle = Q_i \cdot |e_i\rangle \tag{429}$$

is true; here, the eigenvalues Q_i are defined as follows:

$$Q_1 = 0 \ , \ Q_2 = 1 \ , \ Q_3 = -1 \ , \ Q_4 = 0 \ . \tag{430}$$

From Eqs. (429) and (430), we obtain, in combination with our previous result (149) and (150),

$$H_2|e_i\rangle|\Phi_P\rangle = \Big(\hbar\Omega \cdot a^+ a + R_i \cdot \Big(M \cdot a + M^* \cdot a^+\Big)\Big)|e_i\rangle|\Phi_P\rangle \ , \tag{431}$$

where now

$$R_1 = 1 \ , \ R_2 = 1 \ , \ R_3 = -1 \ , \ R_4 = -1 \ . \tag{432}$$

As before, we introduce the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{433}$$

for the Coulomb matrix element W^{eh}; consequently, the eigenvalue equation

$$H_1|e_i\rangle|\Phi_P\rangle = \tilde{E}_i \cdot |e_i\rangle|\Phi_P\rangle \tag{434}$$

remains valid, where

$$\tilde{E}_1 = 0 \ , \ \tilde{E}_2 = \varepsilon_0 \ , \ \tilde{E}_3 = \varphi_0 \ , \ \tilde{E}_4 = 0 \ . \tag{435}$$

A comparison of Eqs. (431) and (434) with Eqs. (149) and (152) in section IV demonstrates immediately that the eigenstates and the energy spectrum of Hamiltonian (424) can be obtained from Eqs. (154) and (157) by the replacement

$$P_2 \to 1 \ , \ P_3 \to -1 \ . \tag{436}$$

As an example, we mention the energy spectrum E_{in}:

$$E_{in} = \tilde{E}_i + \left(n - |\gamma_i|^2\right) \cdot \hbar\Omega \ , \ \gamma_i := \frac{M \cdot P_i}{\hbar\Omega} \ . \tag{437}$$

Because of $|P_i| = 1$, we obtain from Eq. (437):

$$E_{in} = \tilde{E}_i + n \cdot \hbar\Omega - \frac{|M|^2}{\hbar\Omega} \ , \tag{438}$$

i.e., the energy eigenvalues E_{2n} and E_{3n} now have an additional *photon shift*. It should be emphasized, however, that this additional shift has no influence on the ground-state energy E_0:

$$E_0 = E_{10} = E_{40} = -\frac{|M|^2}{\hbar\Omega} \ . \tag{439}$$

Turning to the time evolution of the system under consideration, we now choose the initial state

$$|\Psi(t=0)\rangle = |0\rangle \ . \tag{440}$$

From Eq. (177), we conclude that $c_{2n} = c_{3n} = 0$ is true, i.e., only the expressions for $i = 1, 4$ do appear in Eq. (178). Here, it is important to realize again that the constants P_1 and P_4 do *not* change their values. Consequently, we obtain the same solution $|\Psi(t)\rangle$ as before, and *Theorem 2* remains valid.

B. Two-level quantum dot coupled to a single photon mode in the presence of an external light field

The extended version of Hamiltonian (242) in section V reads as follows:

$$H = H_0 + V_{ehS}(t) \ , \tag{441}$$

where

$$H_0 = H_1 + H_2 \ , \tag{442}$$

$$H_1 = \varepsilon_0 \cdot e^+ e + \varphi_0 \cdot h^+ h - 2 \cdot W^{eh} \cdot e^+ h^+ h e \ , \tag{443}$$

$$H_2 = \hbar\Omega \cdot a^+ a + \left(M \cdot a + M^* \cdot a^+\right)\left(e^+ h^+ + h e + e^+ e - h^+ h\right) \ . \tag{444}$$

According to *Theorem 5* in section VII, the mathematical structure of the semiclassical light-matter interaction operator $V_{ehS}(t)$ is determined by the particle-photon interaction operator V_{ehp} in Eq. (444):

$$V_{ehS}(t) = S(t) \cdot \left(e^+ h^+ + h e + e^+ e - h^+ h\right) \ . \tag{445}$$

As before, analytical results are available at least for a specific choice of the Coulomb matrix element W^{eh}:

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \ . \tag{446}$$

If condition (446) is fulfilled, the eigenvalue equation

$$H_0 |\Psi_{in}\rangle = E_{in} \cdot |\Psi_{in}\rangle \tag{447}$$

is true, provided that the constants P_2 and P_3 in Eqs. (154) and (157) are replaced by 1 and -1, respectively.

Turning to the solution

$$|\Psi(t)\rangle = \sum_{i=1}^{4}\sum_{n=0}^{\infty} c_{in}(t)\cdot|\Psi_{in}\rangle \tag{448}$$

of the time-dependent Schrödinger equation, we obtain by a straightforward calculation the relation

$$c_{in}(t) = c_{in}(0)\cdot\exp\left(-\frac{i}{\hbar}\cdot\left(E_{in}\cdot t + P_i\cdot\int_0^t S(\tau)d\tau\right)\right) , \tag{449}$$

where now

$$P_1 = 1\ ,\ P_2 = 1\ ,\ P_3 = -1\ ,\ P_4 = -1\ . \tag{450}$$

Again, the constants P_1 and P_4 do not change their values. Because of the fact that $S(t)$ is a real number for any t, we obtain additionally

$$|c_{in}(t)| = |c_{in}(0)| \tag{451}$$

and therefore

$$\langle\Psi(t)|H_0|\Psi(t)\rangle = \langle\Psi(0)|H_0|\Psi(0)\rangle , \tag{452}$$

i.e., our previous result (257) remains valid. It is needless to emphasize that we can proceed along the same lines as in section V.C to demonstrate that the optical absorption spectrum $A(\omega)$ of the particle-photon system is equal to zero: $A(\omega) = 0$.

Now, we choose again the initial state

$$|\Psi(t=0)\rangle = c_{10}\cdot|\Psi_{10}\rangle + c_{40}\cdot|\Psi_{40}\rangle\ ,\ |c_{10}|^2 + |c_{40}|^2 = 1\ , \tag{453}$$

i.e., a superposition of those eigenstates which belong to the ground-state energy E_0. From Eqs. (448) and (449), we conclude that $c_{2n}(t) = c_{3n}(t) = 0$ is true under these circumstances, i.e., only the expressions for $i = 1, 4$ do appear on the right-hand side of Eq. (448). Consequently, we obtain the same solution $|\Psi(t)\rangle$ as before, and *Theorem 3* remains valid, too.

C. Two-level quantum dot coupled to an unlimited number of photon modes

Here, the model Hamiltonian (306) from section VI should be replaced by the Hamiltonian

$$H = H_1 + H_2 \ , \tag{454}$$

where

$$H_1 = \varepsilon_0 \cdot e^+ e + \varphi_0 \cdot h^+ h - 2 \cdot W^{eh} \cdot e^+ h^+ h e \ , \tag{455}$$

$$H_2 = \int d^3k \sum_\lambda \hbar\Omega_k \cdot a^+_{k\lambda} a_{k\lambda} + V_{ehp} \tag{456}$$

and

$$\begin{aligned} V_{ehp} &= \left(e^+ + h\right) \int d^3k \sum_\lambda \left(M_{k\lambda} \cdot a_{k\lambda} + M^*_{k\lambda} \cdot a^+_{k\lambda}\right) \left(e + h^+\right) \to \\ &\to \int d^3k \sum_\lambda \left(M_{k\lambda} \cdot a_{k\lambda} + M^*_{k\lambda} \cdot a^+_{k\lambda}\right) \cdot \left(e^+ h^+ + he + e^+ e - h^+ h\right) \ . \end{aligned} \tag{457}$$

Under the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{458}$$

for the Coulomb matrix element W^{eh}, the eigenstates and the energy spectrum of Hamiltonian (454) can be obtained from Eqs. (311) and (313) in section VI by the replacement

$$P_2 \to 1 \ , \ P_3 \to -1 \ , \tag{459}$$

i.e., *the constants* P_1 *and* P_4 *do not change their values.*

Turning to the dynamical vacuum fluctuations, we conclude from Eq. (323) that only the expressions for $i = 1, 4$ do appear in Eq. (321). Consequently, the time evolution of the system under consideration is determined by the same solution $|\Psi(t)\rangle$ as before, and *Theorem 4* remains valid.

D. Analysis of the standard approximation $\langle hea\rangle = 0$

Last but not least, we are going to analyze the standard approximation $\langle hea\rangle = 0$ in the framework of the extended model Hamiltonian (424):

$$H = H_1 + \hbar\Omega \cdot a^+ a + \left(M \cdot a + M^* \cdot a^+\right) \hat{R} \;, \tag{460}$$

where

$$\hat{R} := e^+ h^+ + he + e^+ e - h^+ h \;. \tag{461}$$

In a first step, we obtain by a straightforward calculation the basic commutation relation

$$\begin{aligned}&[H, \hat{E}(a^+)^i a^j] = [H_1, \hat{E}](a^+)^i a^j + (i-j) \cdot \hbar\Omega \cdot \hat{E}(a^+)^i a^j + \\ &+ i \cdot M \cdot \hat{R}\hat{E}(a^+)^{i-1} a^j - j \cdot M^* \cdot \hat{E}\hat{R}(a^+)^i a^{j-1} + \\ &+ [\hat{R}, \hat{E}](M \cdot (a^+)^i a^{j+1} + M^* \cdot (a^+)^{i+1} a^j) \;.\end{aligned} \tag{462}$$

As before,

$$\hat{E} = \hat{E}\left(e, e^+, h, h^+\right) \tag{463}$$

is a functional form of the electron and hole operators e, e^+ and h, h^+, respectively.

Introducing the further abbreviation

$$\hat{P} := e^+ h^+ + he \;, \tag{464}$$

the useful relations

$$\left(e^+ e - h^+ h\right) \hat{P} = 0 \;, \; \hat{P}\left(e^+ e - h^+ h\right) = 0 \;, \tag{465}$$

$$\hat{R}\hat{P} = \hat{P}^2 \;, \; \hat{P}\hat{R} = \hat{P}^2 \tag{466}$$

and

$$\hat{R}\hat{P}^n = \hat{P}^{n+1} \;, \; \hat{P}^n \hat{R} = \hat{P}^{n+1} \;, \; n \geq 1 \tag{467}$$

are immediately derived. From Eqs. (98) and (465), we obtain additionally

$$\hat{R}^2 = 1 \ . \tag{468}$$

A combination of Eqs. (462) and (466)-(468) leads to the commutation relations

$$\left[H, a^+ a\right] = M \cdot \hat{R}a - M^* \cdot \hat{R}a^+ \ , \tag{469}$$

$$\left[H, \hat{R}a\right] = \left[H_1, \hat{R}\right] a - \hbar\Omega \cdot \hat{R}a - M^* \ , \tag{470}$$

$$\left[H, \hat{P}a\right] = \left[H_1, \hat{P}\right] a - \hbar\Omega \cdot \hat{P}a - M^* \cdot \hat{P}^2 \tag{471}$$

and

$$\left[H, \hat{P}^2\right] = \left[H_1, \hat{P}^2\right] = 0 \ . \tag{472}$$

Because of $[H_1, e^+e - h^+h] = 0$, we may replace $[H_1, \hat{R}] \rightarrow [H_1, \hat{P}]$ in Eq. (470). From Eqs. (469)-(472), we thus obtain the following equations of motion:

$$\frac{\partial}{\partial t}\langle a^+ a\rangle = \frac{i}{\hbar} \cdot M \cdot \langle \hat{R}a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{R}a^+\rangle \ , \tag{473}$$

$$\begin{aligned} \frac{\partial}{\partial t}\langle \hat{R}a\rangle = &-\frac{i}{\hbar} \cdot \hbar\Omega \cdot \langle \hat{R}a\rangle - \frac{i}{\hbar} \cdot M^* + \\ &+\frac{i}{\hbar} \cdot (\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh}) \cdot \langle e^+ h^+ a\rangle + \frac{i}{\hbar} \cdot (-\varepsilon_0 - \varphi_0 + 2 \cdot W^{eh}) \cdot \langle hea\rangle \ , \end{aligned} \tag{474}$$

$$\begin{aligned} \frac{\partial}{\partial t}\langle \hat{P}a\rangle = &-\frac{i}{\hbar} \cdot \hbar\Omega \cdot \langle \hat{P}a\rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{P}^2\rangle + \\ &+\frac{i}{\hbar} \cdot (\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh}) \cdot \langle e^+ h^+ a\rangle + \frac{i}{\hbar} \cdot (-\varepsilon_0 - \varphi_0 + 2 \cdot W^{eh}) \cdot \langle hea\rangle \end{aligned} \tag{475}$$

and

$$\frac{\partial}{\partial t}\langle \hat{P}^2\rangle = 0 \ . \tag{476}$$

Again, we introduce the initial state

$$|\Psi(t=0)\rangle = \alpha_1 \cdot |0\rangle + \alpha_2 \cdot e^+|0\rangle + \alpha_3 \cdot h^+|0\rangle + \alpha_4 \cdot e^+ h^+|0\rangle \ , \tag{477}$$

where

$$|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1 \ ; \tag{478}$$

under these circumstances, the solution of Eq. (476) reads as follows:

$$\langle \hat{P}^2 \rangle(t) = |\alpha_1|^2 + |\alpha_4|^2 = 1 - |\alpha_2|^2 - |\alpha_3|^2 =: 1 - K \ . \tag{479}$$

Making use of the abbreviation

$$y := \langle \hat{R}a \rangle - \langle \hat{P}a \rangle \ , \tag{480}$$

we conclude from Eqs. (474), (475) and (479) that

$$\frac{dy}{dt} = -i\Omega \cdot y - \frac{i}{\hbar} \cdot M^* \cdot K \tag{481}$$

is true. Because of $y(t=0) = 0$, we thus obtain

$$y(t) = \langle \hat{R}a \rangle - \langle \hat{P}a \rangle = \frac{M^*}{\hbar\Omega} \cdot K \cdot \exp\left(-i\Omega \cdot t\right) - \frac{M^*}{\hbar\Omega} \cdot K \ . \tag{482}$$

Obviously, we can use Eq. (482) to eliminate the expectation values $\langle \hat{R}a \rangle$ and $\langle \hat{R}a^+ \rangle$ from Eq. (473):

$$\frac{\partial}{\partial t}\langle a^+ a \rangle = \frac{i}{\hbar} \cdot M \cdot \langle \hat{P}a \rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{P}a^+ \rangle + 2 \cdot \frac{|M|^2}{\hbar^2\Omega} \cdot K \cdot \sin \Omega t \ . \tag{483}$$

So far everything is exact. Now, we introduce the approximation

$$\langle hea \rangle = 0 \ . \tag{484}$$

From Eq. (475), we conclude that the expectation value $\langle \hat{P}a \rangle$ fulfills the equation

$$\frac{\partial}{\partial t}\langle \hat{P}a \rangle = \frac{i}{\hbar} \cdot \left(\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} - \hbar\Omega\right) \cdot \langle \hat{P}a \rangle - \frac{i}{\hbar} \cdot M^* \cdot \langle \hat{P}^2 \rangle \tag{485}$$

under these circumstances.

In a last step, we choose

$$\Omega = \Omega_E := \frac{1}{\hbar} \cdot \left(\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh}\right) \ , \tag{486}$$

i.e., the photon energy is equal to the (unrenormalized) excitonic energy. A combination of Eqs. (479), (483), (485) and (486) leads to the final result

$$\langle a^+ a\rangle(t) = \frac{|M|^2}{\hbar^2} \cdot \Big(|\alpha_1|^2 + |\alpha_4|^2\Big) \cdot t^2 + 2 \cdot \frac{|M|^2}{\hbar^2 \Omega^2} \cdot \Big(|\alpha_2|^2 + |\alpha_3|^2\Big) \cdot (1 - \cos \Omega t) \ . \qquad (487)$$

If, for example, α_4 is different from zero, the photon number $\langle a^+ a\rangle(t)$ is dominated again by an expression $\sim t^2$.

We summarize our last result:

Theorem 10:

Consider a purely electronic two-level quantum dot model, as defined by Eqs. (425), (460) and (461) with $\Omega = \Omega_E$. Choose the initial state

$$|\Psi(t=0)\rangle = \alpha_1 \cdot |0\rangle + \alpha_2 \cdot e^+|0\rangle + \alpha_3 \cdot h^+|0\rangle + \alpha_4 \cdot e^+ h^+|0\rangle \ . \qquad (488)$$

Then, the relation

$$\langle a^+ a\rangle(t) = \frac{|M|^2}{\hbar^2} \cdot (|\alpha_1|^2 + |\alpha_4|^2) \cdot t^2 + 2 \cdot \frac{|M|^2}{\hbar^2 \Omega^2} \cdot (|\alpha_2|^2 + |\alpha_3|^2) \cdot (1 - \cos \Omega t) \qquad (489)$$

is true, if the approximation $\langle hea\rangle = 0$ is introduced (see text for details). The constant M is the particle-photon coupling constant.

X. DYNAMICAL VACUUM FLUCTUATIONS OF A PURE ELECTRON-HOLE SYSTEM DUE TO THE COULOMB INTERACTION

A. General remarks

We have already seen that the standard coupling of the electron-hole system to the quantized electromagnetic field exhibits a serious and fundamental difficulty, the reason being that *constant source terms* do appear in the equations of motion for a specific class of expectation values. In order to get an idea of the origin of these constant source terms, it is important to realize the basic fact that the problem of *dynamical vacuum fluctuations* is by no means restricted to the standard particle-photon interaction operator. If - in the absence of any external perturbation - the vacuum state $|0\rangle$ is *not* an eigenstate of the underlying Hamiltonian H, the existence of constant source terms is obviously *a necessary feature* of the quantum-field theoretical hierarchy equations.

As a further and quiet important example, we mention the Coulomb interaction between electrons and holes in a purely electronic system. Here, the standard interaction operator $V_{Coul.}$ reads as follows:

$$V_{Coul.} = :\ \int d^3r \int d^3r' \ \hat{\Psi}^+(r)\ \hat{\Psi}^+(r')\ V(r-r')\ \hat{\Psi}(r')\ \hat{\Psi}(r)\ : \tag{490}$$

As usual, the symbol $:\ \ldots\ :$ in Eq. (490) denotes *normal order* (here, we refer additionally to Ref.[18]); moreover, $V(r-r')$ is the two-particle potential.

Introducing the *exact band structure expansions*

$$\hat{\Psi}^+(r) = \sum_{n_1} \Psi^*_{n_1}(r) \cdot e^+_{n_1} + \sum_{m_1} \Phi^*_{m_1}(r) \cdot h_{m_1}\ , \tag{491}$$

$$\hat{\Psi}^+(r') = \sum_{n_2} \Psi^*_{n_2}(r') \cdot e^+_{n_2} + \sum_{m_2} \Phi^*_{m_2}(r') \cdot h_{m_2}\ , \tag{492}$$

$$\hat{\Psi}(r') = \sum_{n_3} \Psi_{n_3}(r') \cdot e_{n_3} + \sum_{m_3} \Phi_{m_3}(r') \cdot h^+_{m_3} \tag{493}$$

and

$$\hat{\Psi}(r) = \sum_{n_4} \Psi_{n_4}(r) \cdot e_{n_4} + \sum_{m_4} \Phi_{m_4}(r) \cdot h^+_{m_4} \tag{494}$$

into Eq. (490), it is immediately seen that the existence of a constant source term at least in one equation of motion is due to the constituents of type

$$e^+_n\ e^+_{n'}\ h^+_{m'}\ h^+_m\ , \tag{495}$$

provided that the sum of these expressions is different from zero. According to the anticommutation relations for Fermi operators, an expression of type (495) can never appear in a two-level system. In order to obtain *dynamical vacuum fluctuations* in a pure electron-hole system, it is therefore necessary to choose an *N-level system with $N \geq 3$*.

In the following part of our considerations, we denote the sum of the expressions of type (495) by $\hat{V}$. From Eqs. (490) - (494), we obtain immediately:

$$\hat{V} = \sum_{nn'm'm} W^{eehh}_{nn'm'm} \cdot e^+_n \, e^+_{n'} \, h^+_{m'} \, h^+_m \ , \tag{496}$$

where

$$W^{eehh}_{nn'm'm} := \int d^3r \int d^3r' \, V(r-r') \cdot \Psi^*_n(r) \cdot \Psi^*_{n'}(r') \cdot \Phi_{m'}(r') \cdot \Phi_m(r) \ . \tag{497}$$

If on the right-hand side of Eq. (496) the condition $n = n'$ or $m = m'$ is fulfilled, the corresponding contribution vanishes; consequently, we may replace

$$\sum_{nn'} A_{nn'} \ \rightarrow \ \sum_{n<n'} (A_{nn'} + A_{n'n}) \tag{498}$$

and

$$\sum_{mm'} B_{mm'} \ \rightarrow \ \sum_{m<m'} (B_{mm'} + B_{m'm}) \ . \tag{499}$$

From Eqs. (496) and (497), we thus obtain

$$\hat{V} = \sum_{n<n'} \sum_{m<m'} V^{eh}_{nn'm'm} \cdot e^+_n \, e^+_{n'} \, h^+_{m'} \, h^+_m \ ; \tag{500}$$

here, we have introduced the abbreviation

$$V^{eh}_{nn'm'm} := W^{eehh}_{nn'm'm} - W^{eehh}_{n'nm'm} + W^{eehh}_{n'nmm'} - W^{eehh}_{nn'mm'} \ . \tag{501}$$

According to the Fermi character of electrons and holes, we must now require that the Coulomb matrix elements $W^{eehh}_{nn'm'm}$ satisfy the *anti-symmetry conditions*

$$W^{eehh}_{nn'm'm} = -W^{eehh}_{n'nm'm} \ , \ W^{eehh}_{nn'm'm} = -W^{eehh}_{nn'mm'} \ . \tag{502}$$

These physical conditions ensure that the complete expressions *(Coulomb matrix element times operator)* in Eq. (496) remain unchanged, if (nn') is replaced by $(n'n)$ or $(m'm)$ is replaced by (mm'). [4]

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[4]If we introduce the exact band structure expansion into Eq. (490), the theory describes *physical electrons* and *physical holes.* It is well known that these particles are *identical.*

From a strictly mathematical point of view, condition (502) is not obvious. It is in fact the *exact band structure expansion* (491) - (494) which must deliver these important anti-symmetry conditions. For this reason, Eq. (502) does also provide an important check for the validity of the well-known *enveloppe approximation*.

A combination of Eqs. (501) and (502) gives immediately

$$V^{eh}_{nn'm'm} = 4 \cdot W^{eehh}_{nn'm'm} \; . \tag{503}$$

In the following part of our considerations, we shall not use the complete quantum-field theoretical expression $V_{Coul.}$ (as defined by Eq. (490)); instead, we shall restrict ourselves to a simplified model which is defined by the interaction operator

$$V_C := \hat{W}^{eh} + \hat{W}^{ee} + \hat{W}^{hh} + \hat{V} + \hat{V}^{+} \; , \tag{504}$$

where

$$\hat{W}^{eh} := -2 \cdot \sum_{nmm'n'} W^{eh}_{nmm'n'} \cdot e^{+}_{n} \; h^{+}_{m} \; h_{m'} \; e_{n'} \; , \tag{505}$$

$$\hat{W}^{ee} := \sum_{n_1 n_2 n_3 n_4} W^{ee}_{n_1 n_2 n_3 n_4} \cdot e^{+}_{n_1} \; e^{+}_{n_2} \; e_{n_3} \; e_{n_4} \tag{506}$$

and

$$\hat{W}^{hh} := \sum_{m_1 m_2 m_3 m_4} W^{hh}_{m_1 m_2 m_3 m_4} \cdot h^{+}_{m_1} \; h^{+}_{m_2} \; h_{m_3} \; h_{m_4} \; . \tag{507}$$

Direct inspection of Eqs. (4) and (505)-(507) shows that the interaction operator $\hat{W}^{eh} + \hat{W}^{ee} + \hat{W}^{hh}$ agrees formally with the Coulomb interaction operator introduced in Eq. (4). However, we emphasize again that the Coulomb matrix elements W^{eh}, W^{ee} and W^{hh} are now determined by the *exact band structure expansion* (491) - (494); for this reason, the matrix elements W^{ee} and W^{hh} have to fulfill the *anti-symmetry conditions*

$$W^{ee}_{n_1 n_2 n_3 n_4} = -W^{ee}_{n_2 n_1 n_3 n_4} \; , \; W^{ee}_{n_1 n_2 n_3 n_4} = -W^{ee}_{n_1 n_2 n_4 n_3} \tag{508}$$

and

$$W^{hh}_{m_1m_2m_3m_4} = -W^{hh}_{m_2m_1m_3m_4} \; , \; W^{hh}_{m_1m_2m_3m_4} = -W^{hh}_{m_1m_2m_4m_3} \; . \tag{509}$$

In the framework of our simplified model, the complete Hamiltonian H thus reads as follows:

$$H = \sum_n \varepsilon_n e_n^+ \, e_n + \sum_m \varphi_m h_m^+ \, h_m + V_C \; . \tag{510}$$

Before going on, we should emphasize that the anti-symmetry conditions (502), (508) and (509) are *not essential for the following part of our considerations.* However, it is quiet convenient to use them, the reason being that various mathematical expressions can be simplified considerably.

B. Analysis of a three-level system: Analytical solutions I.

We are now going to analyze a purely electronic three-level system (as defined by Eq. (510)) in more detail. Here, the possible values of the indices n and m are given by

$$n = 1 \; , \; 2 \qquad \textit{and} \qquad m = 1 \; , \; 2 \; . \tag{511}$$

As far as the time evolution of the expectation values of interest is concerned, it is important to gain some insight into possible mathematical structures of *exact solutions.* For this reason, we now introduce two additional conditions:

Condition 1:

$$\left(V^{eh}_{1221}\right)^* = V^{eh}_{1221} =: V \; ; \tag{512}$$

Condition 2:

$$\varepsilon_1 + \varepsilon_2 + \varphi_2 + \varphi_1 + C_{1221} + 4 \cdot W^{ee}_{1221} + 4 \cdot W^{hh}_{1221} = 0 \; , \tag{513}$$

where

$$C_{1221} := -2 \cdot \left(W^{eh}_{1221} + W^{eh}_{1111} + W^{eh}_{2222} + W^{eh}_{2112}\right) \; . \tag{514}$$

Here, it is understood that *the Coulomb matrix elements are independent system parameters.* It is obvious that conditions (512) and (513) do not cover the most general case. However, the corresponding analytical solutions (which will be discussed below) may lead to a critical judgement of, e.g., perturbational approaches to the dynamics of an electron-hole system.

Making use of Eqs. (512)-(514), we obtain immediately

$$H|e_1\rangle = V \cdot |e_1\rangle \ , \ H|e_2\rangle = -V \cdot |e_2\rangle \ ; \tag{515}$$

here, we have introduced the abbreviations

$$|e_1\rangle := \frac{1}{\sqrt{2}} \cdot \Big(|0\rangle + e_1^+ \ e_2^+ \ h_2^+ \ h_1^+|0\rangle\Big) \ , \ |e_2\rangle := \frac{1}{\sqrt{2}} \cdot \Big(|0\rangle - e_1^+ \ e_2^+ \ h_2^+ \ h_1^+|0\rangle\Big) \ . \tag{516}$$

The knowledge of the two eigenstates $|e_1\rangle$ and $|e_2\rangle$ enables one to analyze the dynamical vacuum fluctuations of the system under consideration *analytically*. In fact, if we start from the initial state

$$|\Psi(t=0)\rangle = |0\rangle \ , \tag{517}$$

the time evolution of the expectation values is determined by the state

$$|\Psi(t)\rangle = \cos\left(\frac{V}{\hbar} \cdot t\right) \cdot |0\rangle - i \cdot \sin\left(\frac{V}{\hbar} \cdot t\right) \cdot e_1^+ \ e_2^+ \ h_2^+ \ h_1^+|0\rangle \ . \tag{518}$$

The exact solution (518) exhibits a remarkable feature: It does only depend on the combination (*coupling times t*)! In other words: If the Coulomb matrix element V is different from zero, the coupling of the electron-hole system is always *strong*.

Turning to the expectation values, we obtain from Eq. (518) by a simple calculation the result

$$\langle e_N^+ e_N \rangle = \langle h_M^+ h_M \rangle = \langle e_N^+ h_M^+ h_M e_N \rangle = \sin^2\left(\frac{V}{\hbar} \cdot t\right) \ . \tag{519}$$

Direct inspection of Eq. (519) shows that any occupation number of the electron-hole system is not negative and bounded from above by 1 - as it must be. Furthermore, the maximum value of the particle numbers is actually 1 - *even if V is arbitrarily small!*

As before, the polarization functions $\langle e_N^+ h_M^+ \rangle$ prove to be an exception: From Eq. (518), it follows immediately that

$$\langle e_N^+ h_M^+ \rangle = 0 \tag{520}$$

is true for arbitrary values of N and M. In combination with relation (519), this equation is in complete contradiction to our physical expectation.

Last but not least, we mention the expectation value $\langle e_1^+ e_2^+ h_2^+ h_1^+ \rangle$. Here, the time evolution is determined by the equation

$$\langle e_1^+ e_2^+ h_2^+ h_1^+ \rangle = \frac{i}{2} \cdot \sin\left(2 \cdot \frac{V}{\hbar} \cdot t\right) \ . \tag{521}$$

As far as the latter expectation value is concerned, the physical interpretation that two electrons and two holes are created in the system under consideration is often used in the literature. If this interpretation is correct, the absolute value of $\langle e_1^+ e_2^+ h_2^+ h_1^+ \rangle$ is expected to be comparatively small. In the framework of the standard theory, however, Eq. (521) demonstrates again that such a picture is obviously *not* correct. This statement is further supported by the fact that the maximum value of $|\langle e_1^+ e_2^+ h_2^+ h_1^+ \rangle|$ does not depend on the numerical value of the coupling constant V.

C. Analysis of a three-level system: Analytical solutions II.

Our preceding investigations can be extended considerably. To begin with, we first introduce the abbreviations

$$|n_1 \dots n_k\rangle := e_{n_1}^+ \dots e_{n_k}^+ |0_e\rangle \ , \ |m_1 \dots m_l\rangle := h_{m_1}^+ \dots h_{m_l}^+ |0_h\rangle \tag{522}$$

and

$$A_{n_1 n m_1 m} := -2 \cdot W_{n m m_1 n_1}^{eh} \ , \tag{523}$$

$$B_{n_1 n m_1 m_2}^{(1)} := -2 \cdot \left(W_{n m_1 m_1 n_1}^{eh} + W_{n m_2 m_2 n_1}^{eh}\right) \ , \tag{524}$$

$$B_{n_1 n_2 m_1 m}^{(2)} := -2 \cdot \left(W_{n_1 m m_1 n_1}^{eh} + W_{n_2 m m_1 n_2}^{eh}\right) \ , \tag{525}$$

$$C_{n_1n_2m_1m_2} := -2 \cdot \left(W^{eh}_{n_1m_1m_1n_1} + W^{eh}_{n_1m_2m_2n_1} + W^{eh}_{n_2m_1m_1n_2} + W^{eh}_{n_2m_2m_2n_2}\right) \ . \tag{526}$$

Making use of these abbreviations, we obtain by a straightforward calculation the following equations:

$$\hat{W}^{eh}|n_1 \ldots n_k\rangle|m_1 \ldots m_l\rangle = 0 \ , \ if \ k = 0 \ or \ l = 0 \ , \tag{527}$$

$$\hat{W}^{eh}|n_1\rangle|m_1\rangle = \sum_{nm} A_{n_1nm_1m} \cdot |n\rangle|m\rangle \ , \tag{528}$$

$$\hat{W}^{eh}|n_1\rangle|m_1m_2\rangle = \sum_{n} B^{(1)}_{n_1nm_1m_2} \cdot |n\rangle|m_1m_2\rangle \ , \tag{529}$$

$$\hat{W}^{eh}|n_1n_2\rangle|m_1\rangle = \sum_{m} B^{(2)}_{n_1n_2m_1m} \cdot |n_1n_2\rangle|m\rangle \ , \tag{530}$$

$$\hat{W}^{eh}|n_1n_2\rangle|m_1m_2\rangle = C_{n_1n_2m_1m_2} \cdot |n_1n_2\rangle|m_1m_2\rangle \ ; \tag{531}$$

$$\left(\hat{W}^{ee} + \hat{W}^{hh}\right)|n_1 \ldots n_k\rangle|m_1 \ldots m_l\rangle = 0 \ , \ if \ k \leq 1 \ and \ l \leq 1 \ , \tag{532}$$

$$\left(\hat{W}^{ee} + \hat{W}^{hh}\right)|n_1 \ldots n_k\rangle|21\rangle = 4 \cdot W^{hh}_{1221} \cdot |n_1 \ldots n_k\rangle|21\rangle \ , \ if \ k \leq 1 \ , \tag{533}$$

$$\left(\hat{W}^{ee} + \hat{W}^{hh}\right)|12\rangle|m_1 \ldots m_l\rangle = 4 \cdot W^{ee}_{1221} \cdot |12\rangle|m_1 \ldots m_l\rangle \ , \ if \ l \leq 1 \ , \tag{534}$$

$$\left(\hat{W}^{ee} + \hat{W}^{hh}\right)|12\rangle|21\rangle = \left(4 \cdot W^{ee}_{1221} + 4 \cdot W^{hh}_{1221}\right) \cdot |12\rangle|21\rangle \ . \tag{535}$$

In combination with condition (512) from above, we obtain additionally:

$$\left(\hat{V} + \hat{V}^{+}\right)|e_1\rangle = V \cdot |e_1\rangle \ ; \tag{536}$$

$$\left(\hat{V} + \hat{V}^{+}\right)|n_1 \ldots n_k\rangle|m_1 \ldots m_l\rangle = 0 \ , \tag{537}$$

$$if \quad |n_1 \ldots n_k\rangle|m_1 \ldots m_l\rangle \neq |0\rangle \quad and \quad |n_1 \ldots n_k\rangle|m_1 \ldots m_l\rangle \neq |12\rangle|21\rangle \ ;$$

$$\left(\hat{V} + \hat{V}^{+}\right)|e_2\rangle = -V \cdot |e_2\rangle \ . \tag{538}$$

Now, it turns out that Hamiltonian (510) can immediately be diagonalized, if one uses the preceding condition (513) and if one introduces the following additional condition for the Coulomb matrix elements W^{eh}:

Condition 3:

$$W^{eh}_{nmm'n'} = 0 \ , \quad if \quad n \neq n' \quad or \quad m \neq m' \ . \tag{539}$$

In fact, making use of conditions (513) and (539), it follows immediately that the following eigenvalue equations are true:

$$H|e_1\rangle = V \cdot |e_1\rangle \ , \tag{540}$$

$$H|0_e\rangle|m_1\rangle = \varphi_{m_1} \cdot |0_e\rangle|m_1\rangle \ , \tag{541}$$

$$H|0_e\rangle|21\rangle = \left(\varphi_2 + \varphi_1 + 4 \cdot W^{hh}_{1221}\right) \cdot |0_e\rangle|21\rangle \ , \tag{542}$$

$$H|n_1\rangle|0_h\rangle = \varepsilon_{n_1} \cdot |n_1\rangle|0_h\rangle \ , \tag{543}$$

$$H|n_1\rangle|m_1\rangle = (\varepsilon_{n_1} + \varphi_{m_1} + A_{n_1 n_1 m_1 m_1}) \cdot |n_1\rangle|m_1\rangle \ , \tag{544}$$

$$H|n_1\rangle|21\rangle = \left(\varepsilon_{n_1} + \varphi_2 + \varphi_1 + B^{(1)}_{n_1 n_1 21} + 4 \cdot W^{hh}_{1221}\right) \cdot |n_1\rangle|21\rangle \ , \tag{545}$$

$$H|12\rangle|0_h\rangle = (\varepsilon_1 + \varepsilon_2 + 4 \cdot W^{ee}_{1221}) \cdot |12\rangle|0_h\rangle \ , \tag{546}$$

$$H|12\rangle|m_1\rangle = \left(\varepsilon_1 + \varepsilon_2 + \varphi_{m_1} + B^{(2)}_{12 m_1 m_1} + 4 \cdot W^{ee}_{1221}\right) \cdot |12\rangle|m_1\rangle \ , \tag{547}$$

$$H|e_2\rangle = -V \cdot |e_2\rangle \ . \tag{548}$$

Note that the validity of condition (513) is not affected by condition (539). Furthermore, the Coulomb matrix element V does not appear in conditions (513) and (539).

In the following part of our considerations, V is chosen to be *positive and sufficiently large*; if these conditions are fulfilled, the ground state energy E_0 and the ground state $|\Psi_0\rangle$ are given by

$$E_0 = -V \quad (V > 0) \tag{549}$$

and

$$|\Psi_0\rangle = |e_2\rangle \ . \tag{550}$$

It is obvious that the ground state $|\Psi_0\rangle$ is *unique* under these circumstances. Basic ground-state expectation values are contained in the following equation:

$$\langle\Psi_0|e_N^+ e_N|\Psi_0\rangle = \frac{1}{2} \ , \ \langle\Psi_0|h_M^+ h_M|\Psi_0\rangle = \frac{1}{2} \ , \ \langle\Psi_0|e_N^+ h_M^+|\Psi_0\rangle = 0 \ . \tag{551}$$

D. Analysis of a three-level system: Analytical solutions III.

It is an interesting task to analyze the time evolution of expectation values in the presence of an external light field. For this purpose, we now introduce the time-dependent Hamiltonian

$$H(t) := H + V_{ehS}(t) \ , \tag{552}$$

where

$$V_{ehS}(t) := \sum_{nm} S_{nm}(t) \cdot e_n^+ h_m^+ + \sum_{nm} S_{nm}^*(t) \cdot h_m e_n \tag{553}$$

describes the coupling of the classical exciting light field to the electron-hole system in accordance with the standard model. Introducing the abbreviations

$$\lambda_1(t) := \frac{1}{\sqrt{2}} \cdot (S_{11}(t) + S_{22}^*(t)) \ , \ \lambda_2(t) := \frac{1}{\sqrt{2}} \cdot (S_{12}(t) - S_{21}^*(t)) \ , \tag{554}$$

$$\lambda_3(t) := \frac{1}{\sqrt{2}} \cdot (S_{11}(t) - S_{22}^*(t)) \ , \ \lambda_4(t) := \frac{1}{\sqrt{2}} \cdot (S_{12}(t) + S_{21}^*(t)) \ , \tag{555}$$

we obtain by a straightforward calculation the following equations:

$$V_{ehS}(t)|e_1\rangle = \lambda_1 \cdot |1\rangle|1\rangle + \lambda_2 \cdot |1\rangle|2\rangle - \lambda_2^* \cdot |2\rangle|1\rangle + \lambda_1^* \cdot |2\rangle|2\rangle \ , \tag{556}$$

$$V_{ehS}(t)|e_2\rangle = \lambda_3 \cdot |1\rangle|1\rangle + \lambda_4 \cdot |1\rangle|2\rangle + \lambda_4^* \cdot |2\rangle|1\rangle - \lambda_3^* \cdot |2\rangle|2\rangle \ , \tag{557}$$

$$V_{ehS}(t)|1\rangle|1\rangle = \lambda_1^* \cdot |e_1\rangle + \lambda_3^* \cdot |e_2\rangle \ , \tag{558}$$

$$V_{ehS}(t)|1\rangle|2\rangle = \lambda_2^* \cdot |e_1\rangle + \lambda_4^* \cdot |e_2\rangle \ , \tag{559}$$

$$V_{ehS}(t)|2\rangle|1\rangle = -\lambda_2 \cdot |e_1\rangle + \lambda_4 \cdot |e_2\rangle \ , \tag{560}$$

$$V_{ehS}(t)|2\rangle|2\rangle = \lambda_1 \cdot |e_1\rangle - \lambda_3 \cdot |e_2\rangle \ . \tag{561}$$

Eqs. (556) - (561) demonstrate explicitly that the three-level system under consideration exhibits a remarkable - and unexpected - feature. In order to illustrate this in more detail, we now introduce the additional - and by no means unrealistic - *Condition 4*:

Condition 4 (First alternative):

$$S_{11} = S_{22}^* \ ; \ S_{12} = -S_{21}^* \ . \tag{562}$$

Because of the fact that the amplitude of the exciting light field is a real valued function of t, Eq. (562) contains nothing else but *additional conditions for the corresponding transition matrix elements.*

From Eq. (555), we conclude that $\lambda_3 = 0$ and $\lambda_4 = 0$ is true under these circumstances. Consequently, Eq. (557) yields immediately the simple relation

$$V_{ehS}(t)|e_2\rangle = 0 \ . \tag{563}$$

In a next step, we choose the initial state

$$|\Psi(t = 0)\rangle = |\Psi_0\rangle = |e_2\rangle \ . \tag{564}$$

Turning to the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle = H|\Psi(t)\rangle + V_{ehS}(t)|\Psi(t)\rangle \ , \tag{565}$$

we thus obtain the surprising result

$$|\Psi(t)\rangle = \exp\left(\frac{i}{\hbar}\cdot V\cdot t\right)\cdot|e_2\rangle = \exp\left(\frac{i}{\hbar}\cdot V\cdot t\right)\cdot|\Psi_0\rangle \ , \tag{566}$$

i.e., the time evolution of the three-level system under consideration is in fact *trivial* (provided that the initial condition (564) is fulfilled).

In a second step, we replace the first version of *Condition 4* by the following alternative:

Condition 4 (Second alternative):

(i) S_{ij} *is a real valued function for arbitrary indices* i, j.

(ii) $S_{12} = 0 \ ; \ \ S_{21} = 0$.

(iii) $\varepsilon_1 + \varphi_1 - 2\cdot W^{eh}_{1111} = \varepsilon_2 + \varphi_2 - 2\cdot W^{eh}_{2222} =: E$.

From Eqs. (554) and (555), we first conclude that $\lambda_2 = 0$ and $\lambda_4 = 0$ is true under these circumstances. Consequently, if we choose the initial state

$$|\Psi(t=0)\rangle = |\Psi_0\rangle = |e_2\rangle \tag{567}$$

again, the time evolution of the state $|\Psi(t)\rangle$ has the general form

$$|\Psi(t)\rangle = c_1(t)\cdot|e_1\rangle + c_2(t)\cdot|e_2\rangle + d_1(t)\cdot|1\rangle|1\rangle + d_2(t)\cdot|2\rangle|2\rangle \ . \tag{568}$$

Here, the functions $c_i(t)$ and $d_j(t)$ are determined by the following equations of motion:

$$i\hbar\dot{c}_1 = V\cdot c_1 + \lambda_1\cdot(d_1 + d_2) \ , \tag{569}$$

$$i\hbar\dot{c}_2 = -V\cdot c_2 + \lambda_3\cdot(d_1 - d_2) \ , \tag{570}$$

$$i\hbar\dot{d}_1 = \lambda_1\cdot c_1 + \lambda_3\cdot c_2 + E\cdot d_1 \ , \tag{571}$$

$$i\hbar\dot{d}_2 = \lambda_1\cdot c_1 - \lambda_3\cdot c_2 + E\cdot d_2 \ . \tag{572}$$

One realizes immediately that the complete system (569)-(572) can be replaced by the following one:

$$i\hbar\dot{c}_1 = V \cdot c_1 + \lambda_1 \cdot (d_1 + d_2) \ , \tag{573}$$

$$i\hbar\,(d_1 + d_2)^{\cdot} = 2\lambda_1 c_1 + E \cdot (d_1 + d_2) \ , \tag{574}$$

$$i\hbar\dot{c}_2 = -V \cdot c_2 + \lambda_3 \cdot (d_1 - d_2) \ , \tag{575}$$

$$i\hbar\,(d_1 - d_2)^{\cdot} = 2\lambda_3 c_2 + E \cdot (d_1 - d_2) \ . \tag{576}$$

From Eqs. (573) and (574), we obtain, in combination with the initial condition (567), the solutions

$$c_1(t) = 0 \tag{577}$$

and

$$d_1(t) + d_2(t) = 0 \ . \tag{578}$$

Introducing the abbreviations

$$x := c_2 \ , \ y := 2d_1 \ , \tag{579}$$

the last two equations (575) and (576) now read as follows:

$$i\hbar\dot{x} = -V \cdot x + \lambda_3 \cdot y \ , \tag{580}$$

$$i\hbar\dot{y} = 2\lambda_3 \cdot x + E \cdot y \ . \tag{581}$$

In the following part of our considerations, we restrict ourselves to a static electric field, i.e., we choose $\lambda_3(t) = const$. *Moreover, we require* $E + V \neq 0$.

Making use of the definitions

$$f_1 := \frac{E+V}{2} \ , \ f_2 := \sqrt{\left(\frac{E+V}{2}\right)^2 + 2\lambda_3^2} \ , \tag{582}$$

$$\hbar\omega_1 := \frac{E-V}{2} + \sqrt{\left(\frac{E+V}{2}\right)^2 + 2\lambda_3^2} \ , \ \hbar\omega_2 := \frac{E-V}{2} - \sqrt{\left(\frac{E+V}{2}\right)^2 + 2\lambda_3^2} \tag{583}$$

and

$$Z_1(t) := \frac{1}{2} \cdot \left(e^{-i\omega_1 \cdot t} + e^{-i\omega_2 \cdot t}\right) \ , \ Z_2(t) := \frac{1}{2} \cdot \left(e^{-i\omega_1 \cdot t} - e^{-i\omega_2 \cdot t}\right) \ , \tag{584}$$

one obtains by a straightforward calculation the result

$$c_2(t) = Z_1(t) - \frac{f_1}{f_2} \cdot Z_2(t) \ , \ d_1(t) = \frac{\lambda_3}{f_2} \cdot Z_2(t) \ . \tag{585}$$

The time evolution of the state $|\Psi(t)\rangle$ is thus determined by the equation

$$\begin{aligned} |\Psi(t)\rangle = & \frac{1}{\sqrt{2}} \cdot c_2(t) \cdot |0\rangle + d_1(t) \cdot e_1^+ \ h_1^+ \ |0\rangle - d_1(t) \cdot e_2^+ \ h_2^+ \ |0\rangle + \\ & - \frac{1}{\sqrt{2}} \cdot c_2(t) \cdot e_1^+ \ e_2^+ \ h_2^+ \ h_1^+|0\rangle \ . \end{aligned} \tag{586}$$

Before going on, we should remark that the functions $Z_1(t)$ and $Z_2(t)$ fulfill the equations

$$|Z_1(t)|^2 + |Z_2(t)|^2 = 1 \ , \ Z_1^*(t) \cdot Z_2(t) + Z_1(t) \cdot Z_2^*(t) = 0 \ ; \tag{587}$$

consequently, the norm of the state $|\Psi(t)\rangle$ is given by

$$\begin{aligned} \langle\Psi(t)|\Psi(t)\rangle &= |c_2(t)|^2 + 2 \cdot |d_1(t)|^2 = \\ &= |Z_1(t)|^2 + |Z_2(t)|^2 - \frac{f_1}{f_2} \cdot (Z_1^*(t) \cdot Z_2(t) + Z_1(t) \cdot Z_2^*(t)) = 1 \end{aligned} \tag{588}$$

- as it must be. This basic check confirms that our calculation is in fact correct.

Turning to the occupation numbers of the electron-hole system, one obtains immediately

$$\langle e_1^+ \ e_1\rangle(t) = \langle e_2^+ \ e_2\rangle(t) = \langle h_1^+ \ h_1\rangle(t) = \langle h_2^+ \ h_2\rangle(t) = \frac{1}{2} \ , \tag{589}$$

i.e., a change of the particle numbers can not take place under the conditions from above.

Our next result is somewhat strange. For this reason, we decided to write down some basic steps of the underlying (and by no means difficult) calculation. From Eq. (586), we first obtain:

$$e_1^+ \; h_1^+ \; |\Psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot c_2(t) \cdot e_1^+ \; h_1^+ \; |0\rangle - d_1(t) \cdot e_1^+ \; e_2^+ \; h_2^+ \; h_1^+ \; |0\rangle \tag{590}$$

and

$$\langle \Psi(t)| e_1^+ \; h_1^+ \; |\Psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot (c_2^*(t) \cdot d_1(t) + c_2(t) \cdot d_1^*(t)) \; ; \tag{591}$$

the corresponding results for $e_2^+ \; h_2^+$ read as follows:

$$e_2^+ \; h_2^+ \; |\Psi(t)\rangle = \frac{1}{\sqrt{2}} \cdot c_2(t) \cdot e_2^+ \; h_2^+ \; |0\rangle + d_1(t) \cdot e_1^+ \; e_2^+ \; h_2^+ \; h_1^+ \; |0\rangle \tag{592}$$

and

$$\langle \Psi(t)| e_2^+ \; h_2^+ \; |\Psi(t)\rangle = -\frac{1}{\sqrt{2}} \cdot (c_2^*(t) \cdot d_1(t) + c_2(t) \cdot d_1^*(t)) \; . \tag{593}$$

From a strictly physical point of view, both subsystems (with indices 1 and 2) are equivalent. On the other hand, however, there is *a change of sign* in Eqs. (591) and (593) (if $\lambda_3 \neq 0$). Although the basic features of the three-level system under consideration are already indicated by Eqs. (556)-(561) and (554)-(555), one has to realize that these purely mathematical properties itself can - of course - never provide a physical explanation for the observed behaviour.

A combination of Eqs. (591), (593), (585), (587) and (584) leads to the result

$$\langle e_2^+ \; h_2^+ \rangle = -\langle e_1^+ \; h_1^+ \rangle = \frac{\lambda_3}{\sqrt{2}} \cdot \frac{f_1}{f_2^2} \cdot \left(1 - \cos \frac{2 f_2}{\hbar} \cdot t\right) \; . \tag{594}$$

Under the conditions from above, the time evolution of the semiclassical interaction energy is thus determined by the equation

$$\langle \Psi(t)| V_{ehS}(t) |\Psi(t)\rangle = -2 \cdot \lambda_3^2 \cdot \frac{f_1}{f_2^2} \cdot \left(1 - \cos \frac{2 f_2}{\hbar} \cdot t\right) \; . \tag{595}$$

From a strictly physical point of view again, the conclusion $\lambda_3 = 0$ seems to be the only possibility to avoid the difficulties in connection with the change of sign in Eq. (594). But this would mean that *conditions (i), (ii) and (iii) from above actually imply* $S_{11} = S_{22}$ *and therefore* $|\Psi(t)\rangle = \exp\left(\frac{i}{\hbar} \cdot V \cdot t\right) \cdot |\Psi_0\rangle$ *for arbitrary external fields!*

We summarize our basic results:

Theorem 11:

Consider a purely electronic three-level system, as defined by Eqs. (504) and (510). Assume that the following conditions are fulfilled:

Condition 1:

$$\left(V^{eh}_{1221}\right)^* = V^{eh}_{1221} =: V \; , \tag{596}$$

Condition 2:

$$\begin{aligned} &\varepsilon_1 + \varepsilon_2 + \varphi_1 + \varphi_2 + \\ &-2 \cdot \left(W^{eh}_{1111} + W^{eh}_{1221} + W^{eh}_{2112} + W^{eh}_{2222}\right) + 4 \cdot W^{ee}_{1221} + 4 \cdot W^{hh}_{1221} = 0 \; . \end{aligned} \tag{597}$$

Statement 1:

If the initial state $|\Psi(t=0)\rangle = |0\rangle$ is chosen, the following relations are valid for any $t \geq 0$:

$$\langle e^+_N \, e_N \rangle(t) = \langle h^+_M \, h_M \rangle(t) = \langle e^+_N \, h^+_M \, h_M \, e_N \rangle(t) = \sin^2\left(\frac{V}{\hbar} \cdot t\right) \; , \tag{598}$$

$$\langle e^+_N \, h^+_M \rangle(t) = 0 \quad , \quad \langle e^+_1 \, e^+_2 \, h^+_2 \, h^+_1 \rangle(t) = \frac{i}{2} \cdot \sin\left(2 \cdot \frac{V}{\hbar} \cdot t\right) \; . \tag{599}$$

Statement 2:

Assume additionally that

Condition 3:

$$W^{eh}_{nmm'n'} = 0 \quad , \quad if \quad n \neq n' \quad or \quad m \neq m' \tag{600}$$

is fulfilled and that V is chosen to be positive and sufficiently large. Then, the three-level system under consideration has a unique ground state $|\Psi_0\rangle$ with energy eigenvalue E_0.In particular, the following relations are valid:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} \cdot \left(|0\rangle - e^+_1 \, e^+_2 \, h^+_2 \, h^+_1 |0\rangle\right) \; , \; E_0 = -V \; , \tag{601}$$

$$\langle\Psi_0|e_N^+e_N|\Psi_0\rangle = \langle\Psi_0|h_M^+h_M|\Psi_0\rangle = \frac{1}{2} \quad , \quad \langle\Psi_0|e_N^+h_M^+|\Psi_0\rangle = 0 \; . \tag{602}$$

Statement 3:

Consider now the time-dependent Hamiltonian $H(t) := H + V_{ehS}(t)$, where $V_{ehS}(t)$ describes the coupling of the classical exciting light field to the electron-hole system (see Eq. (553)). Assume additionally that the transition matrix elements are chosen such that

Condition 4 (First alternative):

$$S_{11} = S_{22}^* \; ; \; S_{12} = -S_{21}^* \tag{603}$$

is fulfilled. Choose the initial state $|\Psi(t=0)\rangle = |\Psi_0\rangle$ (ground state). Under these circumstances, the time evolution of the three-level system under consideration is determined by

$$|\Psi(t)\rangle = \exp\left(\frac{i}{\hbar} \cdot V \cdot t\right) \cdot |\Psi_0\rangle \; . \tag{604}$$

Statement 4:

Consider again the time-dependent Hamiltonian $H(t)$ and replace Condition 4 from above by the following alternative:

Condition 4 (Second alternative):

(i) S_{ij} is a real valued function for arbitrary indices i, j.

(ii) $S_{12} = 0 \;\; ; \;\; S_{21} = 0$.

(iii) $\varepsilon_1 + \varphi_1 - 2 \cdot W_{1111}^{eh} = \varepsilon_2 + \varphi_2 - 2 \cdot W_{2222}^{eh} =: E$.

Choose a static electric field and assume $E + V \neq 0$. If the initial state is the ground state again, the following relation is valid for any $t \geq 0$:

$$\langle e_2^+ \; h_2^+\rangle = -\langle e_1^+ \; h_1^+\rangle = \frac{\lambda_3}{\sqrt{2}} \cdot \frac{f_1}{f_2^2} \cdot \left(1 - \cos\frac{2f_2}{\hbar} \cdot t\right) \; . \tag{605}$$

Here, the expressions λ_3, f_1 and f_2 are defined by Eqs. (555) and (582).

XI. THE QUANTUM-FIELD THEORETICAL VACUUM STATE IN THE PRESENCE OF AN EXTERNAL LIGHT FIELD

Our previous analysis in section IV is restricted to the case $S(t) = 0$. Here, one of our most important results was the statement that the particle numbers $\langle e^+e\rangle(t)$ and $\langle h^+h\rangle(t)$ are bounded from above by 1/2. At this point, the following question arises: Is it possible to achieve the maximum value $\langle e^+e\rangle = \langle h^+h\rangle = 1$ by a suitable choice of $E(t)$?

Actually, it is an interesting task to determine the time evolution of the expectation values *in the presence of an external light field* and under the initial condition

$$|\Psi(t=0)\rangle = |0\rangle \ . \tag{606}$$

The underlying Hamiltonian in the extended version of section IX.B reads as follows:

$$H = H_0 + V_{ehS}(t) \ , \tag{607}$$

where

$$H_0 = H_1 + H_2 \ , \tag{608}$$

$$H_1 = \varepsilon_0 \cdot e^+e + \varphi_0 \cdot h^+h - 2 \cdot W^{eh} \cdot e^+h^+he \ , \tag{609}$$

$$H_2 = \hbar\Omega \cdot a^+a + \left(M \cdot a + M^* \cdot a^+\right)\left(e^+h^+ + he + e^+e - h^+h\right) \ , \tag{610}$$

$$V_{ehS}(t) = S(t) \cdot \left(e^+h^+ + he + e^+e - h^+h\right) \ . \tag{611}$$

It turns out that the general formula (261) in section V leads to exact analytical results also for $S(t) \neq 0$ provided that the additional condition

$$\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0 \tag{612}$$

is fulfilled; under these circumstances, no further restrictions on the electric field amplitude $E(t)$ have to be introduced.

In order to apply Eq. (261), we first remark that Eq. (263) can be replaced by the relation

$$B_{kl\sigma}(i,j) = \sum_{m\geq 0; n\geq 0; m+\sigma\geq 0} \sqrt{m!(m+n+\sigma)!} \cdot c^*_{jm+n+\sigma}(t) \cdot c_{im}(t) \cdot \tag{613}$$
$$\cdot \left(\gamma_j^* - \gamma_i^*\right)^n \cdot L^n_{m+\sigma}\left(|\gamma_j - \gamma_i|^2\right) \cdot Y^{(i)}_{kl\sigma}(m,n) \; .$$

In the following part of our considerations, we are going to calculate the time evolution of the expectation values $\langle \hat{E}a^l \rangle(t)$. Because of $k = 0$, we conclude from Eq. (261) that $\sigma \leq 0$ is true; consequently, we may replace

$$\sum_{m\geq 0; n\geq 0; m+n-\sigma\geq 0} \rightarrow \sum_{m\geq 0; n\geq 0} \quad , \quad \sum_{m\geq 0; n\geq 0; m+\sigma\geq 0} \rightarrow \sum_{n\geq 0; m+\sigma\geq 0} \quad , \quad \sum_{m\geq 0; m-\sigma\geq 0} \rightarrow \sum_{m\geq 0} \tag{614}$$

in Eqs. (262), (613) and (264). Note that the minimum value of $m+\sigma$ in the second relation in Eq. (614) is equal to zero. From Eqs. (265) and (266), we obtain additionally

$$X^{(i)}_{0l\sigma}(m,K) = \frac{1}{(m+K)!} \cdot \binom{l}{-\sigma} \cdot (-\gamma_i^*)^{l+\sigma} \tag{615}$$

and

$$Y^{(i)}_{0l\sigma}(m,K) = \frac{1}{(m+K+\sigma)!} \cdot \binom{l}{-\sigma} \cdot (-\gamma_i^*)^{l+\sigma} \; . \tag{616}$$

We already know that the initial condition (606) leads to the explicit expressions

$$c_{in}(0) = \frac{1}{\sqrt{2\cdot n!}} \cdot \exp\left(-\frac{1}{2}\cdot|\gamma_i|^2\right) \cdot (\gamma_i^*)^n \quad , \quad i = 1,4 \quad , \tag{617}$$

where

$$\gamma_1 = \gamma \; , \; \gamma_4 = -\gamma \; ; \; \gamma := \frac{M}{\hbar\Omega} \; . \tag{618}$$

According to our previous investigations, the time evolution of the system under consideration is thus determined by the equations

$$c_{1n}(t) = c_{1n}(0) \cdot \exp\left(-\frac{i}{\hbar}\cdot E_{1n}\cdot t\right) \cdot \exp\left(-\frac{i}{\hbar}\cdot \int_0^t S(\tau)d\tau\right) \tag{619}$$

and

$$c_{4n}(t) = (-1)^n \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \cdot c_{1n}(t) \ . \tag{620}$$

We are now prepared to calculate the expressions

$$A_{0l\sigma}(i,j) + B_{0l\sigma}(i,j) - C_{0l\sigma}(i,j) \tag{621}$$

for $(i,j) = (1,4)$ and $(i,j) = (4,1)$ (see again Eq. (261)). Introducing the abbreviations

$$Z_1(t) := \sum_{n=0}^{\infty} \exp\left(-in\Omega \cdot t\right) \cdot \left(2 \cdot |\gamma|^2\right)^n \cdot \sum_{m=0}^{\infty} \frac{1}{(m+n)!} \cdot \left(-|\gamma|^2\right)^m \cdot L_m^n\left(4 \cdot |\gamma|^2\right) \ , \tag{622}$$

$$\begin{aligned} Z_2(t) : = & \sum_{n=0}^{\infty} \exp\left(in\Omega \cdot t\right) \cdot \left(2 \cdot |\gamma|^2\right)^n \cdot \\ & \cdot \sum_{m+\sigma \geq 0} \frac{1}{(m+\sigma+n)!} \cdot \left(-|\gamma|^2\right)^{m+\sigma} \cdot L_{m+\sigma}^n\left(4 \cdot |\gamma|^2\right) \ , \end{aligned} \tag{623}$$

$$Z_3(t) := -\sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(-|\gamma|^2\right)^m \cdot L_m^0\left(4 \cdot |\gamma|^2\right) \tag{624}$$

and

$$Z(t) := Z_1(t) + Z_2(t) + Z_3(t) \ , \tag{625}$$

one obtains by a lengthy but straightforward calculation the following results:

$$\begin{aligned} & A_{0l\sigma}(1,4) + B_{0l\sigma}(1,4) - C_{0l\sigma}(1,4) = \\ & = \frac{1}{2} \cdot \exp\left(-|\gamma|^2\right) \cdot Z(t) \cdot (\gamma^*)^l \cdot \binom{l}{-\sigma} \cdot (-1)^\sigma \cdot \exp\left(i\sigma\Omega \cdot t\right) \cdot \\ & \cdot (-1)^l \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \ , \end{aligned} \tag{626}$$

$$\begin{aligned} & A_{0l\sigma}(4,1) + B_{0l\sigma}(4,1) - C_{0l\sigma}(4,1) = \\ & = \frac{1}{2} \cdot \exp\left(-|\gamma|^2\right) \cdot Z(t) \cdot (\gamma^*)^l \cdot \binom{l}{-\sigma} \cdot (-1)^\sigma \cdot \exp\left(i\sigma\Omega \cdot t\right) \cdot \\ & \cdot \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \ . \end{aligned} \tag{627}$$

In order to calculate $Z(t)$, we have to remember that the minimum value of $m+\sigma$ in Eq. (623) is in fact zero; consequently,

$$Z(t) = \sum_{n=0}^{\infty} 2 \cdot \cos n\Omega \cdot t \cdot \left(2 \cdot |\gamma|^2\right)^n \cdot \sum_{m=0}^{\infty} \frac{1}{(m+n)!} \cdot \left(-|\gamma|^2\right)^m \cdot L_m^n \left(4 \cdot |\gamma|^2\right) + - \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \left(-|\gamma|^2\right)^m \cdot L_m^0 \left(4 \cdot |\gamma|^2\right) \tag{628}$$

is true. Making use of Eqs. (202) and (206) in section IV, the remaining sums can be performed and our final result for $Z(t)$ reads as follows:

$$Z(t) = \exp\left(-|\gamma|^2\right) \cdot \exp\left(4 \cdot |\gamma|^2 \cdot \cos \Omega \cdot t\right) \ . \tag{629}$$

From Eqs. (626), (627) and (629), we thus conclude that the first expression on the right-hand side of Eq. (261) in section V is given by

$$\begin{aligned} &\sum_{i \neq j} \lambda_{ij} \cdot \exp\left(-\frac{1}{2} \cdot |\gamma_j - \gamma_i|^2\right) \cdot \sum_{\sigma=-l}^{0} \left(A_{0l\sigma}(i,j) + B_{0l\sigma}(i,j) - C_{0l\sigma}(i,j)\right) = \\ &= \frac{1}{2} \cdot (\gamma^*)^l \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos \Omega \cdot t - 1)\right) \cdot \left(1 - \exp\left(-i\Omega \cdot t\right)\right)^l \cdot \\ &\cdot \left((-1)^l \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \cdot \lambda_{14} + \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \cdot \lambda_{41}\right) \ . \end{aligned} \tag{630}$$

In order to calculate the second expression on the right-hand side of Eq. (261), we use the relations

$$(-1)^{-\mu} \cdot (-1)^{-\nu} = (-1)^{\sigma} \quad , \quad \gamma^{\sigma} \cdot \gamma^{k-\nu} = \gamma^{k-\mu} \tag{631}$$

and refer additionally to the detailed remarks in section IV. We thus obtain:

$$\begin{aligned} &\sum_{i=1}^{4} \lambda_{ii} \cdot \sum_{\sigma=-l}^{k} \sum_{n=0}^{\infty} \sqrt{n!(n+\sigma)!} \cdot c_{in+\sigma}^*(t) \cdot c_{in}(t) \cdot X_{kl\sigma}^{(i)}(\sigma, n) = \\ &= \frac{1}{2} \cdot (\gamma^*)^l \cdot \gamma^k \cdot \left[(-1)^{k+l} \cdot \lambda_{11} + \lambda_{44}\right] \cdot \sum_{\sigma=-l}^{k} (-1)^{\sigma} \cdot \exp\left(i\sigma\Omega \cdot t\right) \cdot \\ &\cdot \sum_{-\mu+\nu=\sigma; 0 \leq \mu \leq l; 0 \leq \nu \leq k} \binom{l}{\mu} \cdot \binom{k}{\nu} \ . \end{aligned} \tag{632}$$

Note that the latter expression does not depend on the external light field.

From Eqs. (630) and (632), the desired expectation value $\langle \hat{E}a^l \rangle(t)$ is now immediately obtained:

$$\langle \hat{E} a^l \rangle(t) = \frac{1}{2} \cdot (\gamma^*)^l \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot (1 - \exp(-i\Omega \cdot t))^l \cdot \tag{633}$$
$$\cdot \left((-1)^l \cdot \exp\left(-2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \cdot \lambda_{14} + \exp\left(2 \cdot \frac{i}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \cdot \lambda_{41}\right) +$$
$$+\frac{1}{2} \cdot (\gamma^*)^l \cdot \left((-1)^l \cdot \lambda_{11} + \lambda_{44}\right) \cdot (1 - \exp(-i\Omega \cdot t))^l \ .$$

Making use of the formula[16]

$$\sum_{k=0}^{n-p} \binom{n}{k} \cdot \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!} \ , \tag{634}$$

we derive additionally from Eq. (632):

$$\langle (a^+)^n a^n \rangle(t) = \left(|\gamma|^2\right)^n \cdot \left[\frac{(2n)!}{(n!)^2} + 2 \cdot (2n)! \cdot \sum_{p=1}^{n} (-1)^p \cdot \frac{\cos p\Omega \cdot t}{(n-p)!(n+p)!}\right] \ . \tag{635}$$

Eq. (635) leads to the surprising result that the photon number

$$\langle a^+ a \rangle(t) = 2 \cdot |\gamma|^2 \cdot (1 - \cos\Omega \cdot t) \tag{636}$$

does actually not depend on the external light field. On the other hand, however, there are many other expectation values which depend *explicitly* on $E(t)$. As important examples, we mention the particle numbers of the electron-hole system, the polarization function and the photon-assisted expectation values $\langle e^+ h^+ a \rangle(t)$ and $\langle hea \rangle(t)$:

$$\langle e^+ e \rangle(t) = \langle h^+ h \rangle(t) = \langle e^+ h^+ h e \rangle(t) = \tag{637}$$
$$= \frac{1}{2} \cdot \left(1 - \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right) \ ,$$

$$\langle e^+ h^+ \rangle(t) = \frac{i}{2} \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \sin\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau \ , \tag{638}$$

$$\langle e^+ h^+ a \rangle(t) = -\frac{1}{2} \cdot \gamma^* \cdot (1 - \exp(-i\Omega \cdot t)) \cdot \tag{639}$$
$$\cdot \left[1 - \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right] \ ,$$

$$\langle hea \rangle(t) = -\frac{1}{2} \cdot \gamma^* \cdot (1 - \exp(-i\Omega \cdot t)) \cdot \tag{640}$$
$$\cdot \left[1 + \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau)d\tau\right] \ .$$

From Eqs. (637) and (638), we obtain, in combination with Eq. (611), the remarkable result

$$\langle V_{ehS}(t)\rangle = 0 \quad for\ any\ t \geq 0\ . \tag{641}$$

Because of the fact that $\langle H_0\rangle(t)$ is a constant *under all circumstances* (see again Eq. (452) in section IX.B), we conclude that there is actually no energy transfer from the external field to the rest of the system.

In the following part of our considerations, we are going to discuss some specific cases in more detail:

First case:

$$|\gamma| \ll 1 \quad , \quad \cos\frac{2}{\hbar}\cdot\int_0^{t_0} S(\tau)d\tau = -1\ . \tag{642}$$

Here, our generalized equation (637) demonstrates immediately that the upper bound 1/2 for the particle numbers is in fact no longer valid. Under the conditions from above, we obtain:

$$\langle e^+e\rangle(t_0) = \langle h^+h\rangle(t_0) \cong 1\ . \tag{643}$$

Second case:

$$\frac{2}{\hbar}\cdot\int_0^{+\infty} S(\tau)d\tau = \frac{\pi}{2}\ . \tag{644}$$

Here, we obtain from Eqs. (637)-(640):

$$\langle e^+e\rangle(t) = \langle h^+h\rangle(t) = \langle e^+h^+he\rangle(t) \cong \frac{1}{2} \quad , \quad large\ t\ , \tag{645}$$

$$\langle e^+h^+\rangle(t) \cong \frac{i}{2}\cdot\exp\Big(4\cdot|\gamma|^2\cdot(\cos\Omega\cdot t - 1)\Big) \quad , \quad large\ t\ , \tag{646}$$

$$\langle e^+h^+a\rangle(t) \cong \langle hea\rangle(t) \cong -\frac{1}{2}\cdot\gamma^*\cdot\left(1 - \exp\left(-i\Omega\cdot t\right)\right) \quad , \quad large\ t\ . \tag{647}$$

Eq. (647) demonstrates once more that even in the presence of an external light field the photon-assisted expectation value $\langle hea\rangle(t)$ is - in general - not comparatively small. Our previous conclusions in sections II, III, IV and VI are thus confirmed (see also Ref.[2] for an additional discussion).

We summarize our basic results:

Theorem 12:

Consider a particle-photon system in the presence of an external light field, as defined by Eq. (607). Assume (i) that the Coulomb matrix element W^{eh} *fulfills the condition* $\varepsilon_0 + \varphi_0 - 2 \cdot W^{eh} = 0$ *and (ii) that the initial state is the vacuum state. Then, the following relations are valid for any* $t \geq 0$*:*

$$\langle a^+ a\rangle(t) = 2 \cdot |\gamma|^2 \cdot (1 - \cos\Omega \cdot t) \ , \tag{648}$$

$$\begin{aligned} \langle e^+ e\rangle(t) &= \langle h^+ h\rangle(t) = \langle e^+ h^+ h e\rangle(t) = \\ &= \frac{1}{2} \cdot \left(1 - \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau\right) \ , \end{aligned} \tag{649}$$

$$\langle e^+ h^+\rangle(t) = \frac{i}{2} \cdot \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \sin\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau \ , \tag{650}$$

$$\begin{aligned} \langle e^+ h^+ a\rangle(t) &= -\frac{1}{2} \cdot \gamma^* \cdot (1 - \exp(-i\Omega \cdot t)) \cdot \\ &\cdot \left[1 - \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau\right] \ , \end{aligned} \tag{651}$$

$$\begin{aligned} \langle h e a\rangle(t) &= -\frac{1}{2} \cdot \gamma^* \cdot (1 - \exp(-i\Omega \cdot t)) \cdot \\ &\cdot \left[1 + \exp\left(4 \cdot |\gamma|^2 \cdot (\cos\Omega \cdot t - 1)\right) \cdot \cos\frac{2}{\hbar} \cdot \int_0^t S(\tau) d\tau\right] \ . \end{aligned} \tag{652}$$

The expression γ *is related to the particle-photon coupling constant* M*:* $\gamma := M/\hbar\Omega$*. The semiclassical interaction energy* $\langle V_{ehS}(t)\rangle$ *fulfills the equation*

$$\langle V_{ehS}(t)\rangle = 0 \quad for\ any\ t \geq 0 \ , \tag{653}$$

and there is no energy transfer from the external field to the rest of the system.

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