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Wilsonian renormalization group versus subtractive renormalization in effective field theories for nucleon–nucleon scattering

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Abstract

We compare the subtractive renormalization and the Wilsonian renormalization group approaches in the context of an effective field theory for the two-nucleon system. Based on an exactly solvable model of contact interactions, we observe that the standard Wilsonian renormalization group approach with a single cutoff parameter does not cover the whole space spanned by the renormalization scale parameters of the subtractive formalism. In particular, renormalization schemes corresponding to Weinberg's power counting in the case of an unnaturally large scattering length are beyond the region covered by the Wilsonian renormalization group approach. In the framework of pionless effective field theory, also extended by the inclusion of a long-range interaction of separable type, we demonstrate that Weinberg's power counting scheme is consistent in the sense that it leads to a systematic order-by-order expansion of the scattering amplitude.

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1. Introduction

More than two decades after the publication of the ground-breaking papers by Weinberg on the chiral effective field theory (EFT) approach to few-nucleon systems [1,2], the problem of renormalization and power counting within this formalism still remains a hotly debated issue. The main difficulty with Weinberg's scheme is related to the fact that the truncated nucleon–nucleon potential within the Lippmann–Schwinger (LS) equation is not renormalizable. Iterations of the integral equation generate ultraviolet (UV) divergences which cannot be absorbed by renormalizing the parameters of the truncated potential. An infinite number of counter terms are needed already at leading order (LO) to cancel UV divergences in iterations of the one-pion exchange (OPE) potential [3]. In certain cases it is possible to obtain finite, cutoff independent results from the LS equation by taking the cutoff Λ to infinity (or, equivalently, much larger than all scales of the considered problem) non-perturbatively while the perturbative iterations remain divergent, see e.g. Refs. [4,5]. However, in EFT, all UV divergences emerging from iterations of the LS equation should be absorbed by counter terms [6].

The above-mentioned renormalization problem can be avoided by treating the pion exchange contributions to the potential perturbatively as proposed by Kaplan, Savage and Wise [7]. Their approach makes use of dimensional regularization supplemented by the power divergence subtraction scheme. It leads to the power counting scheme, which is commonly referred to as the KSW counting (see also Refs. [8,9]). However, it turned out that the perturbative expansion has problems to converge within this approach (at least) in certain spin-triplet channels [10,11].

Plenty of alternative, sometimes contradicting each other, formulations of the chiral EFT in the few-nucleon sector have been suggested and/or are being explored [4–7,12–44], see Refs. [45–50] for review articles. However, it must be said that the original Weinberg approach using a finite momentum- or coordinate-space regulator (or combinations thereof) is extremely successful and remains to be the most widely employed framework. It is the aim of this paper to demonstrate that the often made statement, that the Weinberg scheme is inconsistent and thus its phenomenological success appears unfounded, is misleading.

In the current work we consider exactly renormalizable models of the nucleon–nucleon (NN) potential which lead to well-defined scattering amplitudes. This allows us to compare subtractive renormalization to the Wilsonian RG approach. We show that certain statements made in the literature are not correct. In particular, by exploring the full space of subtractive renormalization, we explicitly demonstrate that both the KSW and Weinberg approaches are consistent schemes and allow for a systematic expansion of the scattering amplitude. This can be understood by performing the subtractions at various scales underlying the corresponding EFTs. In fact, these two approaches simply correspond to two particular choices of the renormalization conditions. Earlier approaches making use of the Wilsonian renormalization group (RG) did not employ this freedom. Thus, while these earlier findings are certainly valid, they do not correspond to the most general situation as considered here. We also argue that the frequently made statement that the Weinberg approach corresponds to the expansion about the trivial fixed point is not correct.

Our paper is organized as follows. We begin in Section 2 with general comments on the unavoidable ambiguities when setting up the power counting schemes for NN scattering with an unnaturally large scattering length. We show qualitatively how the KSW and Weinberg power counting schemes emerge by employing specific counting rules for the building blocks entering the Lippmann–Schwinger equation for the scattering amplitude. Next, in Sec. 3, we introduce an exactly solvable, renormalizable model of contact interactions and give results for the scattering amplitude using the most general subtractive renormalization scheme. In Sec. 4 we perform the

Wilsonian RG analysis of our model following the approach of Ref. [36] (see also Ref. [50] for a recent review and related discussion) and compare the results with both the Weinberg and the KSW schemes. In section 5, we extend our analysis by considering another exactly solvable and renormalizable model of NN scattering which features a long-range interaction of a separable type. The main results of our work are summarized in Sec. 6 while various lengthy expressions can be found in the appendix.

2. On the power counting for the scattering amplitude and the potential

The goal of any EFT is to provide an expansion of observables in powers of ratios of small scales divided by large scales. In this section, we consider an EFT of NN scattering at very low energies. The starting point is a LS equation build upon a two-nucleon potential. We now comment on the relation between the expansion of the scattering amplitude and the expansion of the potential by taking the expansion of the amplitude as an input, fixed by the underlying theory in terms of the low-energy scattering parameters (scattering length, effective range, ...).

To be specific, consider the $^1S_0$ partial wave of NN scattering in non-relativistic EFT with nucleons alone as dynamical degrees of freedom. The inverse scattering length and the three-momenta of the incoming and outgoing nucleons in center-of-mass frame are the only small scales of the considered problem, collectively denoted by Q_S. At low energies, the $^1S_0$ partial wave NN scattering amplitude can be written as a perturbative series corresponding to the effective range expansion (ERE)

$$\begin{aligned} T &= T_{-1} + T_0 + T_1 + \cdots , \\ T_{-1} &= -\frac{4\pi}{m_N\,(-1/a - i\,k)}, \\ T_0 &= \frac{2\pi\, r_e k^2}{m_N\,(-1/a - i\,k)^2}, \\ T_1 &= -\frac{\pi\, r_e^2 k^4}{m_N (-1/a - i\,k)^3}, \\ &\cdots , \end{aligned} \tag{1}$$

where the subscripts indicate the orders in the small parameter Q_S, while a and r_e refer to the scattering length and the effective range, respectively. We take this sequence of approximations to the amplitude and demand that it is reproduced order-by-order by the low-energy EFT. Note that while the LO potential has to be iterated to reproduce the LO amplitude, higher order corrections can be included perturbatively. For the case of contact interactions the amplitude can be calculated analytically and the renormalization can be carried out explicitly. Therefore, it can be shown explicitly that the non-perturbative and perturbative treatments of higher order corrections to the potential differ by higher order contributions which are beyond the accuracy of the calculation (see e.g. Refs. [14,40]). These formally higher order corrections are indeed small provided that the renormalization conditions are appropriately chosen as will be discussed in detail below.

We now make the connection to an underlying NN potential. To assign orders of the small parameter to the various terms in the effective potential corresponding to Eq. (1) we write it as a perturbative series

$$V = V_{\rm LO} + V_{\rm NLO} + V_{\rm NNLO} + \cdots , \tag{2}$$

where the orders corresponding to the different terms need to be obtained by analyzing the integral equation for the scattering amplitude. Each next term in the sequence $V_{\rm LO}$, $V_{\rm NLO}$, $V_{\rm NNLO}, \cdots$ is suppressed by some power of the small parameter compared to the previous term.

To obtain the leading order amplitude T_{-1} we need to solve the LS equation:

$$T_{-1} = V_{\rm LO} + V_{\rm LO}\, G\, T_{-1}, \tag{3}$$

where G is the resolvent-operator of the two-nucleon propagator. The solution to this equation has the form:

$$T_{-1} = (1 - V_{\rm LO}\, G)^{-1} V_{\rm LO}. \tag{4}$$

Eq. (4) can be satisfied by assigning the orders as follows:

$$\begin{aligned} V_{LO} &\sim \epsilon^{x}, \\ 1 - V_{LO}\, G &\sim \epsilon^{1+x}, \\ G &\sim \epsilon^{-x} (\text{or } \epsilon^{1}, \text{ if } x \leq -1), \end{aligned} \tag{5}$$

where $\epsilon \sim Q_S/\Lambda_H$, with Λ_H denoting the hard (breakdown) scale. In the framework of EFT, the order of the potential $V_{\rm LO}$ depends on the choice of the renormalization condition. The choice $x = 0$ corresponds to Weinberg's power counting [1] for the NN potential in EFT. The KSW counting [7] for the LO potential is the realization of the choice $x = -1$, the same counting for the LO potential is advocated in Refs. [8,9] and by the renormalization group approach of Ref. [36]. Below we refer to this case as to "KSW-vK" counting.

Next, let us investigate the power counting for the higher order contributions in the potential for both choices $x = 0$ and $x = -1$. For that we write the amplitude T_0 as

$$T_0 = V_{\rm NLO} + V_{\rm NLO}\, G\, T_{-1} + T_{-1}\, G\, V_{\rm NLO} + T_{-1}\, G\, V_{\rm NLO}\, G\, T_{-1}\,. \tag{6}$$

It follows from Eq. (6) that $V_{\rm NLO} \sim \epsilon^0$ for $x = -1$, and $V_{NLO} \sim \epsilon^2$ for $x = 0$. Higher order terms in the potential can be analyzed analogously.

Thus, we observe that the proper scaling of the physical amplitude in terms of the small parameter can be realized by a potential, whose various contributions behave as

$$V_{\rm LO} \sim \epsilon^0, \qquad V_{\rm NLO} \sim \epsilon^2, \qquad V_{\rm NNLO} \sim \epsilon^4, \qquad \cdots, \tag{7}$$

or

$$V_{\rm LO} \sim \epsilon^{-1}, \qquad V_{\rm NLO} \sim \epsilon^0, \qquad V_{\rm NNLO} \sim \epsilon^2, \qquad \cdots \tag{8}$$

depending on the employed choice of the renormalization conditions. Notice here that under the power counting of Eq. (7), even though V is of order ϵ^0, T is of order ϵ^{-1}. This can be achieved if the potential is chosen such that the kernel of the integral equation has eigenvalues close to 1 (cf. the second equation of Eq. (5)).

To compare the counting rules for the various terms with the actual power counting of the potential obtained in pionless EFT consider the EFT potential for the $^1S_0$ partial wave

$$V = c + c_2 \left(p^2 + p'^2\right) + c_4 (p^4 + p'^4) + c_{22}\, p^2 p'^2, \tag{9}$$

where, for simplicity, we set $c_{22} = 0$. The solution to the LS equation using dimensional regularization has the form

$$T(k) = \frac{c + 2c_2 k^2 \left(c_2 + c_4 k^2\right)}{1 - I\left(k^2\right)\left[c + 2c_2 k^2 \left(c_2 + c_4 k^2\right)\right]}. \tag{10}$$

Subtracting the loop integral $I(k^2) = m\sqrt{-k^2 - i\epsilon}/(4\pi)$ at $k^2 = -\mu^2$ and matching to the ERE one obtains:

$$\frac{1}{T(k)} = \frac{m}{4\pi}\left[\frac{4(1-a\mu)^3}{a^2k^2r_e\left(ak^2r_e - 2a\mu + 2\right) - 4a^2k^4v_2(a\mu - 1) + 4a(a\mu-1)^2} + \mu + i\,k\right],$$
$$c = \frac{4\pi}{m(1/a - \mu)},$$
$$c_2 = \frac{\pi r_e}{m(1/a-\mu)^2},$$
$$c_4 = \frac{\pi r_e^2}{2m(1/a-\mu)^3} + \frac{2\pi v_2}{m(1/a-\mu)^2}. \tag{11}$$

By choosing μ of the order of the hard scale Λ_H and taking into account that $a \sim \epsilon^{-1}$, $r_e \sim v_2 \sim \epsilon^0$, we see from Eq. (11) that the coupling constants are of a natural size, $c_i \sim \epsilon^0$, leading to the scaling of various terms in the potential according to Eq. (7). That is, for this choice of renormalization conditions Eq. (7) corresponds to Weinberg's power counting. On the other hand, if we take μ of the order of the small scale Q_S, the couplings c, c_2 and c_4 comply with the KSW-vK counting.

It is important to emphasize that a certain amount of fine tuning in the scattering amplitude beyond naive dimensional analysis is unavoidable in the case of an unnaturally large scattering length both for the Weinberg and KSW-vK power countings. In the Weinberg case, the fine tuning manifests itself in the second condition in Eq. (5). For the KSW-vK counting, one observes that the constant c_4 actually violates the scaling suggested by Eq. (8). That is, in the KSW-vK counting, the scaling of the KSW-vK amplitude comes out as a result of cancelation of large contributions between the c_2^2 and c_4 contributions, which goes beyond naive dimensional analysis.

To demonstrate in more details the above observations and to compare to the Wilsonian RG approach we now consider solvable toy models of the NN interaction. These models are exactly renormalizable and hence demonstrate an essential feature of consistent EFTs that a perturbative expansion of renormalized non-perturbative expressions, if expanded, reproduce the standard renormalized perturbative series.

3. An exactly solvable model of contact interactions

Consider an exactly solvable model of the fully off-shell LS equation

$$T(p', p, k) = V(p', p, k) + 2m\int \frac{d^3l}{(2\pi)^3}\, V(p', l, k)\, \frac{1}{k^2 - l^2 + i\,\eta}\, T(l, p, k) \tag{12}$$

with the potential

$$V(p', p, k) = \left(1, {p'}^2\right)\, \lambda(k)\, \begin{pmatrix} 1 \\ p^2 \end{pmatrix}, \tag{13}$$

where λ is a 2×2 matrix given by

$$\lambda(k) = \begin{pmatrix} C + k^2 C_{\rm k}(k^2) & C_2 + k^2\, C_{\rm 2E} + k^4 C_{\rm 2k}(k^2) \\ C_2 + k^2\, C_{\rm 2E} + k^4 C_{\rm 2k}(k^2) & C_4 + k^2\, C_{\rm 4E} + k^4\, C_{\rm 4EE} + k^6\, C_{\rm 4k}(k^2) \end{pmatrix}^{-1}. \tag{14}$$

Here, $C_{\rm k}(k^2)$, $C_{\rm 2k}(k^2)$ and $C_{\rm 4k}(k^2)$ are analytic functions of k^2 at $k^2 = 0$, i.e. they can be expanded in a Taylor series in k^2.

By writing

$$T(p',p,k)=\left(1,p'^2\right)\,\tau(k)\begin{pmatrix}1\\ p^2\end{pmatrix} \tag{15}$$

Eq. (12) is reduced to a matrix equation [51]

$$\tau(k)=\lambda(k)+\lambda(k)\mathcal{G}(k)\tau(k)\,, \tag{16}$$

with

$$\mathcal{G}(k)=\begin{pmatrix} I^\Lambda(k) & I^\Lambda(k)k^2+I_3^\Lambda \\ I^\Lambda(k)k^2+I_3^\Lambda & I^\Lambda(k)k^4+I_3^\Lambda k^2+I_5^\Lambda \end{pmatrix}. \tag{17}$$

The cutoff-regularized loop integrals of Eq. (17) are defined as

$$\begin{aligned} I_n^\Lambda &= -\frac{m}{(2\pi)^3}\int d^3l\,l^{n-3}\,\theta(\Lambda-l)=-\frac{m\,\Lambda^n}{2n\pi^2}\,,\quad \text{with }\; n=1,3,5\,,\\ I^\Lambda(k) &= \frac{m}{(2\pi)^3}\int\frac{d^3l\,\theta(\Lambda-l)}{k^2-l^2+i\eta}=I_1^\Lambda-\frac{i\,m\,k}{4\pi}-\frac{mk}{4\pi^2}\ln\frac{\Lambda-k}{\Lambda+k}\,, \end{aligned} \tag{18}$$

where the last equation is valid for $k<\Lambda$. By writing the matrix equation (16) as

$$\tau(k)^{-1}=\lambda(k)^{-1}-\mathcal{G}(k)\,, \tag{19}$$

one can easily see from Eqs. (14), (17) and (18), that the whole Λ-dependence present in $\mathcal{G}(k)$ can be eliminated by choosing

$$\begin{aligned} C &= \alpha-\frac{m\Lambda}{2\pi^2}\equiv-\frac{m\Lambda}{2\pi^2}+\frac{m\mu}{2\pi^2}+C_{\rm R}(\mu),\\ C_2 &= \beta-\frac{m\Lambda^3}{6\pi^2}\equiv-\frac{m\Lambda^3}{6\pi^2}+\frac{m\mu_1^3}{6\pi^2}+C_{2\rm R}(\mu_1),\\ C_4 &= \gamma-\frac{m\Lambda^5}{10\pi^2}\equiv-\frac{m\Lambda^5}{10\pi^2}+\frac{m\mu_3^5}{10\pi^2}+C_{4\rm R}(\mu_3),\\ C_{2\rm E} &= \delta-\frac{m\Lambda}{2\pi^2}\equiv-\frac{m\Lambda}{2\pi^2}+\frac{m\mu_2}{2\pi^2}+C_{2\rm ER}(\mu_2),\\ C_{4\rm E} &= \lambda-\frac{m\Lambda^3}{6\pi^2}\equiv-\frac{m\Lambda^3}{6\pi^2}+\frac{m\mu_4^3}{6\pi^2}+C_{4\rm ER}(\mu_4),\\ C_{4\rm EE} &= \sigma-\frac{m\Lambda}{2\pi^2}\equiv-\frac{m\Lambda}{2\pi^2}+\frac{m\mu_5}{2\pi^2}+C_{4\rm EER}(\mu_5),\\ C_{\rm k}(k^2) &= C_{\rm P}(k^2)-\frac{m\ln\frac{\Lambda-k}{k+\Lambda}}{4k\pi^2}\,,\\ C_{2\rm k}(k^2) &= C_{2\rm P}(k^2)-\frac{m\ln\frac{\Lambda-k}{k+\Lambda}}{4k\pi^2}\,,\\ C_{4\rm k}(k^2) &= C_{4\rm P}(k^2)-\frac{m\ln\frac{\Lambda-k}{k+\Lambda}}{4k\pi^2}\,, \end{aligned} \tag{20}$$

where α, β, γ, δ, λ and σ are some finite parameters. We have introduced the scale-dependent renormalized coupling constants $C_{\rm R}(\mu)$, $C_{2\rm R}(\mu_1)$, etc., where $\mu,\mu_1,\ldots,\mu_5$ are renormalization scale parameters and $C_{\rm P}(k^2)$, $C_{2\rm P}(k^2)$, $C_{4\rm P}(k^2)$ are finite functions of k^2, analytic at $k^2=0$. We set these functions equal to zero for the sake of simplicity. Substituting Eq. (20) into the solution of Eq. (12) we obtain the expression for the inverse of the on-shell scattering amplitude

$$\frac{1}{T(k)} = \frac{N_R}{D_R} + \frac{ikm}{4\pi} \equiv -\frac{k^4(\delta^2-\alpha\sigma)+k^2(2\beta\delta-\alpha\lambda)+\beta^2-\alpha\gamma}{k^4(\alpha-2\delta+\sigma)+k^2(\lambda-2\beta)+\gamma} + \frac{ikm}{4\pi}, \tag{21}$$

where

$$\begin{aligned} N_R &= 6\pi^2\Big\{C_{\rm R}(\mu)\Big[30\pi^2\left(k^4C_{\rm 4EER}(\mu_5)+k^2C_{\rm 4ER}(\mu_4)+C_{\rm 4R}(\mu_3)\right) \\ &\quad + m\left(5k^2\left(3k^2\mu_5+\mu_4^3\right)+3\mu_3^5\right)\Big] \\ &\quad - 5\Big[-3m\mu\left(k^4C_{\rm 4EER}(\mu_5)+k^2C_{\rm 4ER}(\mu_4)+C_{\rm 4R}(\mu_3)\right) \\ &\quad + 2k^2C_{\rm 2ER}(\mu_2)\left(6\pi^2C_{\rm 2R}(\mu_1)+m\left(3k^2\mu_2+\mu_1^3\right)\right)+6\pi^2k^4C_{\rm 2ER}(\mu_2)^2 \\ &\quad + 2m\left(3k^2\mu_2+\mu_1^3\right)C_{\rm 2R}(\mu_1)+6\pi^2C_{\rm 2R}(\mu_1)^2\Big]\Big\} \\ &\quad + m^2\left[9\mu\mu_3^5-5\left(-3\mu k^2\mu_4^3+9k^4\left(\mu_2^2-\mu\mu_5\right)+6k^2\mu_2\mu_1^3+\mu_1^6\right)\right], \\ D_R &= 6\pi^2\Big\{30\pi^2\Big[k^2\Big(k^2\left(C_{\rm 4EER}(\mu_5)-2C_{\rm 2ER}(\mu_2)+C_{\rm R}(\mu)\right)+C_{\rm 4ER}(\mu_4)-2C_{\rm 2R}(\mu_1)\Big) \\ &\quad + C_{\rm 4R}(\mu_3)\Big]+m\left(5k^2\left(3k^2\left(-2\mu_2+\mu_5+\mu\right)-2\mu_1^3+\mu_4^3\right)+3\mu_3^5\right)\Big\}. \end{aligned} \tag{22}$$

As expected, the full scattering amplitude does not depend on the renormalization scale parameters $\mu, \mu_1, \ldots, \mu_5$ as the explicit and implicit (through the renormalized couplings) scale dependence cancels exactly so that each of these parameters can be chosen arbitrarily. On the other hand, the freedom of this choice can be advantageously exploited to obtain a better convergent perturbative series when the expansion is performed in terms of the renormalized coupling(s).

It is convenient to parameterize the model in terms of the standard EFT expansion of the effective potential. To this end, we expand $V(p', p, k)$ in Taylor series in p, p' and k and match the terms up to fourth order to

$$V\left(p', p, k\right) = c + c_2(p'^2+p^2) + c_{\rm E}k^2 + c_{\rm pp}p'^2p^2 + c_{\rm Ep}k^2(p'^2+p^2) + c_{\rm EE}k^4 + \cdots. \tag{23}$$

Notice that the above Taylor expansion is written only for the purpose of matching, and we do not employ truncations of any kind in the potential of our model. Any truncation of this series at any finite order beyond the leading one would render the potential non-renormalizable, i.e. divergences could not be removed by renormalizing parameters appearing in the truncated series. Solving the resulting system of equations we obtain the following relations between the bare parameters

$$
\begin{aligned}
C &= \frac{c_{\rm pp}}{c c_{\rm pp} - c_2^2}\,,\\
C_2 &= \frac{c_2}{c_2^2 - c c_{\rm pp}}\,,\\
C_4 &= \frac{c}{c c_{\rm pp} - c_2^2}\,,\\
C_{2\rm E} &= \frac{c_{\rm E} c_{\rm pp} - c_2 c_{\rm Ep}}{c_2^3 - c c_2 c_{\rm pp}}\,,\\
C_{4\rm E} &= -\frac{c_{\rm E} c_2^2 - 2 c c_{\rm Ep} c_2 + c c_{\rm E} c_{\rm pp}}{c_2^4 - c c_2^2 c_{\rm pp}}\,,\\
C_{4\rm EE} &= -\frac{c_{\rm EE} c_2^2 - 2 c_{\rm E} c_{\rm Ep} c_2 + c c_{\rm Ep}^2 + c_{\rm E}^2 c_{\rm pp} - c c_{\rm EE} c_{\rm pp}}{c_2^4 - c c_2^2 c_{\rm pp}}\,. \qquad (24)
\end{aligned}
$$

Thus, the Taylor expansion of the potential in powers of momenta and energy in this new parametrization has the form analogous to the one of the EFT potential with contact interactions.

Substituting the bare couplings of the new parametrization expressed in terms of renormalized ones as specified in the appendix into the solution to the LS equation we obtain the following renormalized expression for the amplitude:

$$
\begin{aligned}
T(k) &= \frac{D}{N + \frac{i k m}{4\pi} D}\,,\\
N &= 90\pi^2 k^2 \Big[k^2 \left(c_{\rm EpR}^2 - c_{\rm EER} c_{\rm ppR} \right) \left(m\mu c_{\rm R} + 2\pi^2 \right) + k^2 m \mu c_{\rm ER}^2 c_{\rm ppR} \\
&\quad + c_{\rm ER} c_{\rm ppR} \left(m\mu c_{\rm R} + 2\pi^2 \right) \Big] + c_{2\rm R}^2 \Big\{ 6\pi^2 m \Big[5k^2 \Big(3k^2 \left(\mu c_{\rm EER} - 2\mu_2 c_{\rm EpR} + \mu_5 c_{\rm ppR} \right) \\
&\quad - 2\mu_1^3 c_{\rm EpR} + 3\mu c_{\rm ER} + \mu_4^3 c_{\rm ppR} \Big) + 3\mu_3^5 c_{\rm ppR} + 15\mu c_{\rm R} \Big] \\
&\quad + m^2 c_{\rm ppR} c_{\rm R} \left(9\mu\mu_3^5 - 5\Big(9k^4 \left(\mu_2^2 - \mu\mu_5 \right) - 3k^2 \mu\mu_4^3 + 6k^2 \mu_1^3 \mu_2 + \mu_1^6 \Big) \right) + 180\pi^4 \Big\} \\
&\quad - 60\pi^2 k^2 c_{2\rm R} \Big[3 c_{\rm EpR} \left(k^2 m \mu c_{\rm ER} + m\mu c_{\rm R} + 2\pi^2 \right) - m c_{\rm ER} c_{\rm ppR} \left(3k^2 \mu_2 + \mu_1^3 \right) \Big] \\
&\quad + 60\pi^2 m c_{2\rm R}^3 \left(3k^2 \mu_2 + \mu_1^3 \right) + m^2 c_{2\rm R}^4 \Big[5\Big(9k^4 \left(\mu_2^2 - \mu\mu_5 \right) - 3k^2 \mu\mu_4^3 \\
&\quad + 6k^2 \mu_1^3 \mu_2 + \mu_1^6 \Big) - 9\mu\mu_3^5 \Big]\,,\\
D &= 6\pi^2 \Big\{ c_{2\rm R}^2 \Big[30\pi^2 k^2 \left(k^2 \left(c_{\rm EER} - 2c_{\rm EpR} + c_{\rm ppR} \right) + c_{\rm ER} \right) \\
&\quad + c_{\rm R} \left(m c_{\rm ppR} \left(5k^2 \left(3k^2 (\mu - 2\mu_2 + \mu_5) - 2\mu_1^3 + \mu_4^3 \right) + 3\mu_3^5 \right) + 30\pi^2 \right) \Big] \\
&\quad + 30\pi^2 k^2 \Big[k^2 \left(c_{\rm R} \left(c_{\rm EpR}^2 - c_{\rm EER} c_{\rm ppR} \right) + c_{\rm ER}^2 c_{\rm ppR} \right) + c_{\rm ER} c_{\rm ppR} c_{\rm R} \Big]
\end{aligned}
$$

$$- 60\pi^2 k^2 c_{2R}\left[k^2 c_{ER}\left(c_{EpR} - c_{ppR}\right) + c_{EpR} c_R\right]$$
$$- m c_{2R}^4\left[5k^2\left(3k^2(\mu - 2\mu_2 + \mu_5) - 2\mu_1^3 + \mu_4^3\right) + 3\mu_3^5\right] + 60\pi^2 k^2 c_{2R}^3\bigg\}. \tag{25}$$

For our purposes it is sufficient to consider a particular case by taking

$$\lambda = 2\beta, \qquad \sigma = 2\delta - \alpha, \tag{26}$$

for which the inverse amplitude reduces to

$$\frac{1}{T(k)} = \frac{-(\alpha-\delta)^2 k^4 + 2\beta(\alpha-\delta)k^2 + \alpha\gamma - \beta^2}{\gamma} + \frac{ikm}{4\pi}$$
$$\equiv \frac{-((\alpha-\delta)/\beta)^2 k^4 + 2((\alpha-\delta)/\beta)k^2 + \alpha(\gamma/\beta^2) - 1}{(\gamma/\beta^2)} + \frac{ikm}{4\pi}. \tag{27}$$

As can be seen from Eq. (27), the inverse amplitude depends only on three independent parameters which can be conveniently expressed in terms of the scattering length a, effective range r_e and the first shape parameter v_2 via

$$\alpha = \frac{m\left(a r_e^2 + 16 v_2\right)}{64\pi a v_2},$$
$$\frac{\gamma}{\beta^2} = \frac{64\pi v_2}{m r_e^2},$$
$$\frac{\alpha-\delta}{\beta} = -\frac{4v_2}{r_e}. \tag{28}$$

The scattering amplitude then takes the familiar form

$$T(k) = -\frac{4\pi}{m}\left[-\frac{1}{a} + \frac{r_e k^2}{2} + v_2 k^4 - ik\right]^{-1}. \tag{29}$$

Renormalized couplings corresponding to this particular choice are given in Eq. (A.5) of the appendix. As mentioned above, the scattering amplitude does not depend on the redundant constant β, which can be viewed as an off-shell parameter in our model. A generalization to include higher-order shape parameters is straightforward. Notice further that while it is perfectly fine to constrain the finite pieces of the coupling constants as explained above in order to simplify the analysis, all bare parameters of the underlying model have to be taken into account to maintain its explicit renormalizability.

To summarize, we have introduced a solvable, renormalizable model of contact interactions specified by Eqs. (13) and (14). The bare parameters of the model are expressible in terms of the renormalized ones as given in Eq. (20) in such a way that the iterative solution to the LS equation (12) remains finite in the limit of $\Lambda \to \infty$. Notice that renormalization of the amplitude unavoidably introduces the dependence of the renormalized couplings on the subtraction points $\mu, \mu_1, \ldots \mu_5$, whose choice reflects the freedom in the choice of renormalization conditions. Utilizing the standard convention of pionless EFT, the parameters of our model can be expressed in terms of the coupling constants c, c_2, c_E, c_{pp}, c_{Ep} and c_{EE} accompanying zero-range contact interactions. The relations between the bare and renormalized constants in this notation are given in the appendix. Finally, to keep our analysis simple, we restrict ourselves to a particular choice of the parameter space of our model by setting the finite, energy-dependent functions $C_P(k^2)$,

$C_{2\mathrm{P}}(k^2)$ and $C_{4\mathrm{P}}(k^2)$ equal to zero and by constraining the parameters as specified in Eq. (26). For this choice, the real part of the inverse scattering amplitude is given by the first three terms of the effective range expansion. The resulting model provides a simple framework to explore different choices of the renormalization conditions as will be discussed in the next section.

4. Wilsonian renormalization group analysis

We now perform a Wilsonian RG analysis of the model introduced in the previous section along the lines of Ref. [20]. For that, we introduce a scale Λ, which acts as a cut-off on the virtual momenta, and we demand that physics does not depend on it. It is straightforward to check that the potential $V(p', p, k) \equiv V(p', p, k, \Lambda)$ specified by Eqs. (13), (14) and (20) satisfies the RG equation [36]:

$$\frac{\partial V(p', p, k, \Lambda)}{\partial \Lambda} = \frac{m}{2\pi^2}\, V(p', \Lambda, k, \Lambda)\, \frac{\Lambda^2}{\Lambda^2 - k^2}\, V(\Lambda, p, k, \Lambda)\,. \tag{30}$$

Further, the corresponding re-scaled potential

$$\hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda) := \frac{m\Lambda}{2\pi^2}\, V(\hat{p}'\Lambda, \hat{p}\Lambda, \hat{k}\Lambda, \Lambda) \tag{31}$$

satisfies the equation

$$\begin{aligned}\Lambda\, \frac{\partial \hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda)}{\partial \Lambda} &= \hat{p}'\, \frac{\partial \hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda)}{\partial \Lambda} + \hat{p}\, \frac{\partial \hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda)}{\partial \Lambda} + \hat{k}\, \frac{\partial \hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda)}{\partial \Lambda} \\ &\quad + \hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda) + \hat{V}(\hat{p}', 1, \hat{k}, \Lambda)\, \frac{1}{1 - \hat{k}^2}\, \hat{V}(1, \hat{p}, \hat{k}, \Lambda)\,. \end{aligned} \tag{32}$$

For the choice of parameters of Eq. (28), the re-scaled potential has the form

$$\begin{aligned}
\hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda) &= \frac{\hat{P}_1}{\hat{P}_2}\,, \\
\hat{P}_1 &= 3\Lambda \Big\{ 15\pi a m^2 \Lambda^4 r_e^4 \left(\hat{p}'^2 - \hat{k}^2\right)\left(\hat{k}^2 - \hat{p}^2\right) \\
&\quad + 3840\pi^2 a m \hat{k}^2 v_2^2 \beta \Lambda^4 r_e \left(\hat{p}'^2 + \hat{p}^2 - 2\hat{k}^2\right) \\
&\quad + 16 m v_2 \Lambda^2 r_e^2 \Big[15 a m \hat{k} \Lambda^3 \left(\hat{k}^2 - \hat{p}'^2\right)\left(\hat{k}^2 - \hat{p}^2\right) \ln \frac{1 - \hat{k}}{1 + \hat{k}} \\
&\quad + 2a \Big(5\hat{p}'^2 \left(m\Lambda^3 \left(3\hat{p}^2 - 3\hat{k}^2 - 1\right) + 6\pi^2 \beta \right) \\
&\quad + 5\hat{p}^2 \left(6\pi^2\beta - m\left(3\hat{k}^2 + 1\right)\Lambda^3 \right) + 15 m \hat{k}^4 \Lambda^3 + 5 m \hat{k}^2 \Lambda^3 + 3 m \Lambda^3 \\
&\quad - 60\pi^2 \hat{k}^2 \beta \Big) + 15\pi m \Lambda^2 \left(\hat{p}'^2 - \hat{k}^2\right)\left(\hat{k}^2 - \hat{p}^2\right) \Big] - 61440\pi^3 a v_2^2 \beta^2 \Big\}\,, \\
\hat{P}_2 &= 3\pi a m^2 \left(3 - 5\hat{k}^2\right) \Lambda^5 r_e^4 + 7680\pi^2 a \hat{k}^2 v_2^2 \beta \Lambda^2 r_e \left(6\pi^2\beta - m\Lambda^3\right) \\
&\quad + 16 m v_2 \Lambda^3 r_e^2 \Big[-2a \left(m \left(4 - 15\hat{k}^2\right) \Lambda^3 + 60\pi^2 \beta \right)
\end{aligned}$$

$$+ 3am\hat{k}\left(5\hat{k}^2-3\right)\Lambda^3\ln\frac{1-\hat{k}}{1+\hat{k}} + 3\pi m\left(3-5\hat{k}^2\right)\Lambda^2\Bigg]$$
$$+ 92160\pi^3 v_2^2\beta^2\left(\pi a\hat{k}^4 v_2\Lambda^4 + a\hat{k}\Lambda\ln\frac{1-\hat{k}}{1+\hat{k}} + 2a\Lambda - \pi\right). \tag{33}$$

Following the Wilsonian RG approach of Ref. [36] for an unnaturally large scattering length, we expand the potential $\hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda)$ of Eq. (33) in powers of ϵ by counting $a\Lambda \sim \epsilon^0$, $\Lambda \sim \epsilon$ and obtain:

$$\hat{V}(\hat{p}', \hat{p}, \hat{k}, \Lambda) = \hat{V}_0(\hat{p}', \hat{p}, \hat{k}, \Lambda) + \hat{V}_1(\hat{p}', \hat{p}, \hat{k}, \Lambda) + \cdots, \tag{34}$$

where

$$\hat{V}_0(\hat{p}', \hat{p}, \hat{k}, \Lambda) = \frac{2}{-\hat{k}\ln\frac{1-\hat{k}}{1+\hat{k}} - 2 + \pi/(a\Lambda)} \sim \epsilon^0,$$
$$\hat{V}_1(\hat{p}', \hat{p}, \hat{k}, \Lambda) = \frac{\pi\hat{k}^2\Lambda r_e}{\left(-\hat{k}\ln\frac{1-\hat{k}}{1+\hat{k}} - 2 + \pi/(a\Lambda)\right)^2} \sim \epsilon^1,$$
$$\cdots. \tag{35}$$

The leading order term $\hat{V}_0$ corresponds to a non-trivial fixed point with an unstable perturbation summed up to all orders [36]. The next term $\hat{V}_1$ is of order ϵ^1. The corresponding un-scaled potentials have the orders

$$V_0(p', p, k, \Lambda) = \frac{4\pi^2 a}{m\left(\pi - 2a\Lambda - ak\ln\frac{1-k/\Lambda}{1+k/\Lambda}\right)} \sim \frac{1}{\epsilon},$$
$$V_1(p', p, k, \Lambda) = \frac{2\pi^3 a^2 k^2 r_e}{m\left(\pi - 2a\Lambda - ak\ln\frac{1-k/\Lambda}{1+k/\Lambda}\right)^2} \sim \epsilon^0,$$
$$\cdots. \tag{36}$$

This scaling behavior for the various contributions to the potential based on the Wilsonian RG analysis can now be compared with the one corresponding to different choices of renormalization conditions in the subtractive approach. Notice, however, that the potential cannot be fully determined by reproducing the on-shell scattering amplitude. Thus, we also need to make a choice for the off-shell parameter β. Taking $\beta \sim \epsilon^0$ and choosing the subtraction points as

$$\mu \sim \mu_{1,\ldots,5} \sim \epsilon\,, \tag{37}$$

the renormalized potential takes the form

$$V_R(p', p'k) = \frac{4\pi^2 a}{\epsilon\, m(\pi - 2a\mu)} + \frac{2\pi^3 a^2 r_e k^2}{m(\pi - 2a\mu)^2}$$
$$+ \epsilon\left[\frac{\pi^4 a^3 r_e^2 k^4}{m(\pi - 2a\mu)^3} + \frac{\pi a r_e^2\left(3(\pi - 2a\mu)k^2 - 2a\mu^3\right)}{24\beta v_2(\pi - 2a\mu)^2} - \frac{\pi a r_e^2\left(p^2 + p'^2\right)}{16\beta v_2(\pi - 2a\mu)}\right] + \cdots, \tag{38}$$

where we have symbolically included here the factors of ϵ to account for the orders of various terms. It fulfills the standard KSW-vK counting with $c_R \sim \epsilon^{-1}$, $c_{ER} \sim \epsilon^{-2}$, $c_{EER} \sim \epsilon^{-3}$, except for the coupling $c_{2R} \sim \epsilon^{-1}$ which is suppressed by one power of the soft scale relative to the KSW-vK scaling. Exact KSW-vK power counting with $c_{2R} \sim \epsilon^{-2}$ is restored for the choice of $\beta \sim \epsilon$.

On the other hand, renormalized couplings of natural size $\sim \epsilon^0$ are obtained for another choice of subtraction points. In fact, taking

$$\mu \sim \Lambda_H, \quad \mu_{1,\ldots,5} \sim \epsilon\,, \tag{39}$$

and setting $\beta \sim \epsilon^0$, we obtain for the renormalized potential:

$$V_R(p', p, k) = \frac{4\pi^2 a}{m(\pi - 2a\mu)} + \epsilon^2 \left[\frac{\pi^2 a r_e k^2 (16\pi a\beta v_2 + m r_e)}{8\beta m v_2 (\pi - 2a\mu)^2} - \frac{\pi a r_e^2 \left(p^2 + p'^2\right)}{16\beta v_2 (\pi - 2a\mu)} \right] + \cdots . \tag{40}$$

Equation (40) corresponds to renormalized coupling constants of natural size as assumed in Weinberg's power counting. The scaling of the coupling constants differs from the one suggested by the standard Wilsonian RG analysis since the latter uses a single cutoff scale and does not cover that region of the multi-dimensional space of renormalization scale parameters which corresponds to Weinberg's power counting. Moreover, for the choice of the renormalization conditions corresponding to Weinberg's power counting, the scaling of renormalized coupling constants is the same regardless of whether the scattering length is natural or unnaturally large (i.e. the potential provides a smooth interpolation between the two regimes). Notice further that in case of an unnaturally large scattering length, Weinberg's power counting does *not* correspond to the expansion of the amplitude around the trivial fixed point as it is sometimes erroneously claimed.

As specified below, it is not difficult to introduce two cutoff parameters in the considered exactly solvable model such that the scattering amplitude is cutoff-independent and, unlike the standard RG case, various terms in the expansion of the cutoff-dependent potential scale analogously to those of the potential with two subtraction points, considered above.

Consider the Lippmann–Schwinger equation (12) for the potential given as

$$\begin{aligned} V(p, q, k, \Lambda_i) &= V_0(\Lambda_i, k)\,\theta(\Lambda_1 - p)\theta(\Lambda_1 - q) \\ &\quad + V_1(\Lambda_i, k)\left[p^2\,\theta(\Lambda_2 - p)\theta(\Lambda_1 - q) + q^2\,\theta(\Lambda_1 - p)\theta(\Lambda_2 - q)\right] \\ &\quad + V_2(\Lambda_i, k)\, p^2 q^2 \theta(\Lambda_2 - p)\theta(\Lambda_2 - q), \end{aligned} \tag{41}$$

where Λ_1 and Λ_2 ($\Lambda_1 > \Lambda_2$) are two cutoffs and $V_j(\Lambda_i, k)$ for $j = 0, 1, 2$ are such functions of the cutoffs that the off-shell amplitude is cutoff independent.

Let us write the potential as

$$V(p, q, k, \Lambda_i) = \left(\theta(\Lambda_1 - p),\ p^2\theta(\Lambda_2 - p)\right) \begin{pmatrix} V_0(\Lambda_i, k), & V_1(\Lambda_i, k) \\ V_1(\Lambda_i, k), & V_2(\Lambda_i, k) \end{pmatrix} \begin{pmatrix} \theta(\Lambda_1 - q) \\ q^2\theta(\Lambda_2 - q) \end{pmatrix}, \tag{42}$$

and take

$$\begin{pmatrix} V_0(\Lambda_i, k), & V_1(\Lambda_i, k) \\ V_1(\Lambda_i, k), & V_2(\Lambda_i, k) \end{pmatrix} = \lambda^{-1}, \tag{43}$$

with

$$
\begin{aligned}
\lambda &= \begin{pmatrix} \lambda_{11}, & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}, \\
\lambda_{11} &= \frac{m}{4\pi a} + \frac{m r_e^2}{64\pi v_2} - \frac{k\, m \ln \frac{\Lambda_1 - k}{\Lambda_1 + k}}{4\pi^2} - \frac{\Lambda_1 m}{2\pi^2}, \\
\lambda_{12} &= \frac{k^2 m}{4\pi a} + \beta + \frac{k^2 m r_e^2}{64\pi v_2} + \frac{4\beta k^2 v_2}{r_e} - \frac{k^3 m \ln \frac{\Lambda_2 - k}{\Lambda_2 + k}}{4\pi^2} - \frac{k^2 \Lambda_2 m}{2\pi^2} - \frac{\Lambda_2^3 m}{6\pi^2}, \\
\lambda_{22} &= \frac{k^4 m}{4\pi a} + \frac{k^4 m r_e^2}{64\pi v_2} + \frac{8\beta k^4 v_2}{r_e} + \frac{64\pi \beta^2 v_2}{m r_e^2} - \frac{k^5 m \ln \frac{\Lambda_2 - k}{\Lambda_2 + k}}{4\pi^2} \\
&\quad - \frac{k^4 \Lambda_2 m}{2\pi^2} + 2\beta k^2 - \frac{k^2 \Lambda_2^3 m}{6\pi^2} - \frac{\Lambda_2^5 m}{10\pi^2},
\end{aligned}
\tag{44}
$$

where β is an off-shell parameter and the on-shell amplitude does not depend on it. The corresponding on shell T-matrix has the form

$$
T(k,k,k) = -\frac{4\pi}{m} \frac{1}{-\frac{1}{a} + \frac{k^2 r_e}{2} + k^4 v_2 - ik}. \tag{45}
$$

Next let us define the rescaled potential

$$
\hat{V}(\hat{p}, \hat{q}, \hat{k}, \Lambda_1, \Lambda_2) = \frac{\Lambda_1 m}{2\pi^2}\, V\left(\Lambda_2 \hat{p}, \Lambda_2 \hat{q}, \Lambda_2 \hat{k}, \Lambda_1, \Lambda_2\right), \tag{46}
$$

and by counting $\Lambda_2 (<< \Lambda_1)$ as a small parameter and Λ_1 as a large one expand it as follows:

$$
\begin{aligned}
\hat{V} &= \frac{2a\Lambda_1}{\pi - 2a\Lambda_1} \\
&+ \frac{\Lambda_2^2 a \left[\Lambda_1 m r_e^2 \left(2a\Lambda_1 \left(\hat{p}^2 + \hat{q}^2\right) + \pi \left(2\hat{k}^2 - \hat{p}^2 - \hat{q}^2\right)\right) + 32\pi a \beta \hat{k}^2 v_2 \left(\pi \Lambda_1 r_e - 4\right)\right]}{32\pi \beta v_2 \left(\pi - 2a\Lambda_1\right)^2} \\
&- \frac{\Lambda_2^3 a^2 \Lambda_1 m r_e^2 \left(3\hat{k}^3 \ln \frac{1-\hat{k}}{1+\hat{k}} + 6\hat{k}^2 + 2\right)}{48\pi \beta v_2 \left(\pi - 2a\Lambda_1\right)^2} \\
&+ \cdots .
\end{aligned}
\tag{47}
$$

This expansion translates in the following expansion for the unscaled potential

$$
\begin{aligned}
V &= \frac{4\pi^2 a}{\pi m - 2a\Lambda_1 m} \\
&+ \frac{\pi a \left[\Lambda_1 m r_e^2 \left(2a\Lambda_1 \left(p^2 + q^2\right) + \pi \left(2k^2 - p^2 - q^2\right)\right) + 32\pi a \beta k^2 v_2 \left(\pi \Lambda_1 r_e - 4\right)\right]}{16\beta \Lambda_1 m v_2 \left(\pi - 2a\Lambda_1\right)^2} \\
&- \frac{\pi a^2 r_e^2 \left(3k^3 \ln \frac{\Lambda_2 - k}{\Lambda_2 + k} + 6k^2 \Lambda_2 + 2\Lambda_2^3\right)}{24\beta v_2 \left(\pi - 2a\Lambda_1\right)^2} \\
&+ \cdots .
\end{aligned}
\tag{48}
$$

The above example demonstrates that the freedom of the choice of different renormalization scales can be also advantageously exploited in the cutoff EFT. Generalization of the Wilsonian

RG approach to the NN scattering for arbitrary potentials by introducing a multitude of cutoff parameters is a subject of a work in progress.

It is instructive to look at the perturbative expansion of the amplitude T of Eq. (25) within Weinberg's approach. Taking $\mu \sim \epsilon^0$, $\mu_i \sim \epsilon$ we obtain for $k \sim \epsilon$

$$T(k) = \frac{1}{1/c_{\rm R} + m(2\mu + i\pi k)/(4\pi^2)} + \frac{16\pi^2\left[3\pi^2k^2c_{\rm ER} + c_{2\rm R}\left(mc_{\rm R}\left(3k^2(\mu-\mu_2) - \mu_1^3\right) + 6\pi^2k^2\right)\right]}{3\left[4\pi^2 + mc_{\rm R}(2\mu + i\pi k)\right]^2} + \cdots . \tag{49}$$

Naively one might be tempted to conclude that for μ of the order of the hard scale, the LO amplitude

$$T_{\rm LO} = \frac{1}{1/c_{\rm R} + m(2\mu + i\pi k)/(4\pi^2)} \tag{50}$$

is of order ϵ^0. However, due to the cancelation between μ and $1/c_{\rm R}$, it is actually of order ϵ^{-1} while the next term in the expansion of the amplitude is of the order ϵ^0, etc. Notice that as already pointed out in section 2, the appearance of cancelations in the amplitude beyond naive dimensional analysis is unavoidable in the case of an unnaturally large scattering length.

In the next section, we extend our analysis to another exactly solvable toy model with a long-range interaction and show that all our conclusions remain valid in this case too.

5. A toy model with a long-range interaction

Consider a spin-singlet S-wave interaction of two nucleons specified by the following exactly solvable toy-model potential (for more details, including the form of corresponding LS equation, see Ref. [6])

$$V(p,p') = v_l F_l(p) F_l(p') + v_s F_s(p) F_s(p'), \quad F_l(p) \equiv \frac{\sqrt{p^2+m_s^2}}{p^2+m_l^2}, \quad F_s(p) \equiv \frac{1}{\sqrt{p^2+m_s^2}}, \tag{51}$$

where m_l and m_s are the small and large mass scales corresponding to the long- and the short-range interaction. We choose the strength of the long-range interaction $v_l = \alpha\, m_l^4$ such that the LO long-range potential is of order zero when m_l and the momenta are counted as soft quantities. To generate the phase shifts similar to those of the $^1S_0$ partial wave of np scattering, we use $m_l = 135$ MeV, $m_s = 750$ MeV and tune the coupling constants v_s and α such that the scattering length and the effective range are $a = -1/(8\text{ MeV})$ and $r_e = 1/(100\text{ MeV})$, respectively. For the switched-off long-range interaction the ERE parameters turn out to be $a = -1/(68.71\text{ MeV})$ and $r_e = 1/(343.53\text{ MeV})$. The resulting phase shifts are shown in Fig. 1.

To reproduce the phase shifts of the "underlying theory" in the EFT approach we count m_l and the three-momenta as small quantities and consider the following NLO effective potential

$$V_{\rm EFT}(p,p') = c + \frac{\alpha_0\, m_l^4}{\left(m_l^2+p^2\right)\left(m_l^2+p'^2\right)} + c_2(p^2+p'^2) + d\, m_l^2 + \frac{\alpha_0' m_l^4\left(p^2+p'^2\right) + \alpha_0'' m_l^6}{2m_s^2\left(m_l^2+p^2\right)\left(m_l^2+p'^2\right)}, \tag{52}$$

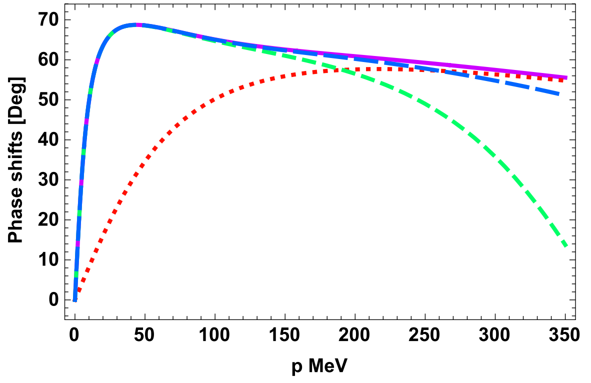

Fig. 1. S-wave phase shift of the toy model with a long-range interaction as a function of the momentum in the center-of-mass frame. The solid (magenta) and the dotted (red) lines correspond to the exact phase shifts of the toy model and to the switched-off long-range potential, respectively. The dashed (green) and long-dashed (blue) lines represent phase shifts of the NLO EFT for the choices of the renormalization scale $\mu = 135$ MeV and $\mu = 750$ MeV, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where α_0, α_0' and α_0'' are bare parameters of the long-range part and c, c_2 and d are the bare couplings of the contact interaction (short-range) terms. All divergences appearing in the NLO amplitude can be absorbed in the renormalization of the parameters of the potential of Eq. (52) provided that the NLO terms of the potential are treated perturbatively. Notice that due to the separability of the long-range part of the considered toy model potential, the parameters of the long-range part also need to be renormalized in order to remove divergences of loop diagrams obtained by iterations of the potential. We treat the full NLO potential non-perturbatively by substituting it into the integral equation and applying the subtractive renormalization. To match the results for the scattering amplitude with the ones based on the underlying model, we take the renormalized couplings of the long-range EFT potential as $\alpha_R = \alpha_R' = \alpha_R'' = \alpha$. We implement here the proposal from section 4 and use different subtraction points for the LO interaction and for subleading interactions by discriminating between the renormalization scale μ, corresponding to the LO interaction, and all other renormalization scales which we put equal to zero. Tuning the renormalized couplings of the contact interaction terms c_R and $c_{2\mathrm{R}}$ to reproduce the scattering length and the effective range (we take $d_R = 0$ as it cannot be disentangled from c_R) we obtain the values

$$c_\mathrm{R} = -3.770\,\frac{4\pi}{m\,m_s} = -0.679\,\frac{4\pi}{m\,\mu},\;\; c_{2\mathrm{R}} = 7.727\,\frac{4\pi}{m\,m_s^3} = 0.022\,\frac{4\pi}{m\,\mu^3}\quad \text{for } \mu = 135\text{ MeV},$$

$$c_\mathrm{R} = -0.916\,\frac{4\pi}{m\,m_s}\,,\qquad c_{2\mathrm{R}} = 0.447\,\frac{4\pi}{m\,m_s^3}\qquad \text{for}\quad \mu = 750\text{ MeV}, \tag{53}$$

where the factors of $4\pi/m$ emerge from the employed normalization of the potential and T-matrix, see Eq. (29). Thus, by choosing the renormalization scale of the order of the soft scale m_l of the problem, the renormalized couplings of the contact interactions are enhanced, as suggested by KSW-vK-like counting (and by the standard Wilsonian RG analysis with a single cutoff scale), and they are natural (i.e. dimensionless couplings are ~ 1 when the dimension is taken away by the large scale of the problem) for the renormalization scale of the order of the hard scale.

We also plot in Fig. 1 the resulting phase shifts at LO and NLO for $\mu = m_l = 135$ MeV and $\mu = m_s = 750$ MeV. Notice that the dependence of the phase shifts on the renormalization scale

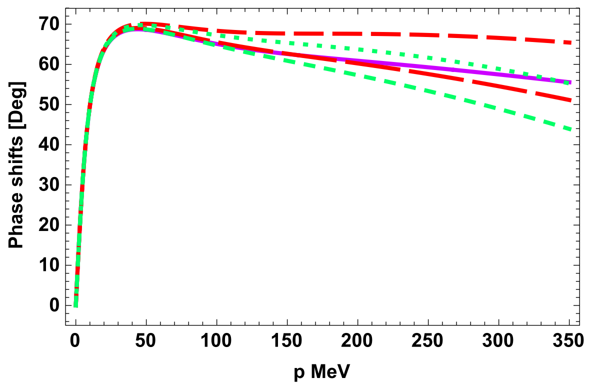

Fig. 2. S-wave phase shift of the toy model with a long-range interaction as a function of the momentum in the center-of-mass frame. The solid (magenta) line corresponds to the exact phase shifts. The dotted and short-dashed (green) lines represent the LO and NLO phase shifts for the choice of the cutoff $\Lambda = 500$ MeV, respectively, and the middle- and long-dashed (red) lines represent analogous results for $\Lambda = 800$ MeV. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

μ emerges as a consequence of the non-perturbative inclusion of the subleading contributions to the potential. One observes that the choice of the renormalization scale of the order of the hard scale in the problem leads to a better reproduction of the phase shift at higher energy, fully in line with the findings of Ref. [53]. It is shown in that paper (in the framework of pionless EFT), that the choice of the subtraction scale μ of the order of the soft scale results in enhanced scheme- and μ-dependent contributions to the phase shifts for the case of an unnaturally large scattering length.

While the considered model with a long-range interaction of a separable type leads to simple, analytical expressions for the scattering amplitude, which can be explicitly renormalized by replacing the bare coupling constants in terms of the renormalized ones, the situation is much more complicated for a realistic case of the two-nucleon force, whose long-range tail is governed by the OPE. It is not feasible to perform subtractive renormalization of all iterations of the LS equation when the OPE is treated non-perturbatively. Instead, one usually introduces a finite cutoff chosen of the order of the hard scale in the problem and performs *implicit* renormalization by (effectively) expressing the bare coupling constants in terms of observable quantities such as NN phase shifts. We have followed the same cutoff EFT approach for our toy model and calculated the LO and NLO phase shifts. Our results are displayed in Fig. 2 for the cutoffs $\Lambda = 500$ MeV and $\Lambda = 800$ MeV along with the phase shifts of the underlying model. The quality of the description of the phase shifts at NLO is comparable to the ones obtained after explicit renormalization of the amplitude as visualized in Fig. 1.

Finally, we have analyzed the exact Wilsonian RG trajectory of the cutoff regularized potential corresponding to the exact potential of Eq. (51). By applying the sharp cutoff to the LS equation we obtained the corresponding cutoff dependent potential leading to the cutoff independent off-shell scattering amplitude. This energy-dependent potential satisfies the RG equation of Eq. (30). In exact analogy to the case of the contact interactions alone, the Wilsonian RG analysis leads to the KSW-vK-like power counting for couplings of the contact interactions also in the presence of a long-range interaction. These results are in line with the ones found earlier by Birse and collaborators [36]. However, we stress again that the Wilsonian RG analysis of Ref. [36] does not cover the full range in the space of renormalization scale parameters, in particular, the range corresponding to Weinberg's power counting. This means that both the KSW-vK and the Weinberg approach are consistent with the exact RG but correspond to different choices of the subtraction scales.

6. Summary and conclusions

Using two examples of exactly renormalizable toy model nucleon–nucleon potentials we have compared the most general subtractive renormalization and the Wilsonian renormalization group (RG) approach of Ref. [36]. We find that the scaling of coupling constants uncovered by the Wilsonian RG analysis corresponds to the choice of the renormalization scheme when all subtraction points are chosen of the order of the soft scale of the problem. This scaling is also shared by the KSW-vK power counting of Ref. [7]. On the other hand, by choosing the renormalization point corresponding to the coupling constant of the momentum- and energy-independent contact interaction of the order of the hard scale of the problem while taking all other renormalization points of the order of the soft scale, one recovers Weinberg's power counting [1,2] with renormalized coupling constants being of natural size both for natural as well as unnaturally large scattering lengths.

In the KSW approach of Ref. [7], dimensional regularization along with PDS subtraction scheme has been used. Therefore, within this approach, all renormalization points are taken either zero or of the order of the scale of dimensional regularization. In standard Wilsonian RG approach one also uses a single scale, the cutoff parameter, and hence in both these cases one studies the behavior of couplings in a one-parameter subspace of the multi-dimensional space of the renormalization group of the corresponding EFT. As a result of this restriction, in both the KSW and the standard Wilsonian RG approaches one does not cover that area in the space of renormalization parameters which is appropriate for the Weinberg approach to nucleon–nucleon scattering problem for the case of an unnaturally large scattering length. We also emphasize that in the case of an unnaturally large scattering length, Weinberg's power counting does *not* correspond to the expansion around the trivial fixed point as it is sometimes claimed. Further conceptual issues of the Weinberg's power counting for potentials with long-range interactions, including the ones raised in Ref. [54], will be addressed in a forthcoming publication.

When performing realistic chiral EFT calculations of NN scattering, the Lippmann–Schwinger equation is usually regularized with a finite cutoff, chosen of the order of the hard scale in the problem as done e.g. in Refs. [37,38]. It is not known to us how to practically implement a subtractive renormalization with the pion-exchange potentials being treated non-perturbatively. In such calculations, renormalization is carried out *implicitly* by adjusting the bare low-energy constants to experimental data or phase shifts, see Ref. [52] for a discussion. We conjecture that this approach is equivalent to the choice of renormalization conditions specified above, i.e. with the subtraction scale corresponding to the LO (higher-order) contact interactions chosen of the order of the hard (soft) scales of the problem, provided the determined bare LECs are of a natural size. This conjecture can be easily verified for the case of the pionless EFT approach, where the analytical expressions for the scattering amplitude are available.

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Appendix A. Various expressions

The potential of section 3 in the new parametrization:

$$
\begin{aligned}
V\left(p', p, k\right) &= \frac{P_1}{P_2}\,,\\
P_1 &= -c_2^2 k m \left(c_2^2 - c c_{\mathrm{pp}}\right)\left(k^2 - p^2\right)\left(k^2 - p'^2\right)\ln\frac{\Lambda - k}{k+\Lambda}\\
&\quad - 4\pi^2\Big[c_2^2\left(k^2 c_{\mathrm{E}} + k^4 c_{\mathrm{EE}} - k^2 c_{\mathrm{Ep}} p'^2 - k^2 p^2 c_{\mathrm{Ep}} + p^2 c_{\mathrm{pp}} p'^2 + c\right)\\
&\quad + k^2 c_{\mathrm{pp}}\left(k^2 c_{\mathrm{E}}^2 + c c_{\mathrm{E}} - c k^2 c_{\mathrm{EE}}\right)\\
&\quad + c_2 k^2\left(c_{\mathrm{E}} c_{\mathrm{pp}}\left(p'^2 + p^2\right) - 2 c_{\mathrm{Ep}}\left(k^2 c_{\mathrm{E}} + c\right)\right) + c k^4 c_{\mathrm{Ep}}^2 + c_2^3\left(p'^2 + p^2\right)\Big]\,,\\
P_2 &= k m \ln\frac{\Lambda - k}{k+\Lambda}\Big\{k^2\Big[c_2^2\left(c_{\mathrm{E}} + k^2\left(c_{\mathrm{EE}} - 2c_{\mathrm{Ep}} + c_{\mathrm{pp}}\right)\right) + 2 c_2 k^2 c_{\mathrm{E}}\left(c_{\mathrm{pp}} - c_{\mathrm{Ep}}\right)\\
&\quad + k^2 c_{\mathrm{E}}^2 c_{\mathrm{pp}} + 2 c_2^3\Big] + c\left(k^2 c_{\mathrm{pp}}\left(c_{\mathrm{E}} - k^2 c_{\mathrm{EE}}\right) + k^4 c_{\mathrm{Ep}}^2 - 2 c_2 k^2 c_{\mathrm{Ep}} + c_2^2\right)\Big\}\\
&\quad - 4\pi^2\left(k^2 c_{\mathrm{pp}}\left(c_{\mathrm{E}} - k^2 c_{\mathrm{EE}}\right) + k^4 c_{\mathrm{Ep}}^2 - 2 c_2 k^2 c_{\mathrm{Ep}} + c_2^2\right)\,.
\end{aligned}
\tag{A.1}
$$

The on-shell amplitude in the new parametrization:

$$
\begin{aligned}
\frac{1}{T} &= \frac{N_0}{D_0} + \frac{i k m}{4\pi}\,,\\
N_0 &= 90\pi^2 m \Lambda\Big\{k^4\Big[c_2\left(c_2 c_{\mathrm{EE}} - 2\left(c_{\mathrm{E}} + c_2\right) c_{\mathrm{Ep}}\right) + \left(c_{\mathrm{E}} + c_2\right)^2 c_{\mathrm{pp}}\Big]\\
&\quad + c\Big[k^2 c_{\mathrm{E}} c_{\mathrm{pp}} + k^4\left(c_{\mathrm{Ep}}^2 - c_{\mathrm{EE}} c_{\mathrm{pp}}\right) - 2 c_2 k^2 c_{\mathrm{Ep}} + c_2^2\Big] + c_2^2 k^2\left(c_{\mathrm{E}} + 2c_2\right)\Big\}\\
&\quad + 180\pi^4\Big[k^2 c_{\mathrm{E}} c_{\mathrm{pp}} + k^4\left(c_{\mathrm{Ep}}^2 - c_{\mathrm{EE}} c_{\mathrm{pp}}\right) - 2 c_2 k^2 c_{\mathrm{Ep}} + c_2^2\Big]\\
&\quad + 30\pi^2 c_2 m \Lambda^3\Big[k^2\left(2 c_{\mathrm{E}} c_{\mathrm{pp}} + c_2\left(c_{\mathrm{pp}} - 2c_{\mathrm{Ep}}\right)\right) + 2c_2^2\Big]\\
&\quad + 15 c_2^2 k^2 m^2 \Lambda^4\left(c_2^2 - c c_{\mathrm{pp}}\right) - 4 c_2^2 m^2 \Lambda^6\left(c_2^2 - c c_{\mathrm{pp}}\right) + 18\pi^2 c_2^2 m \Lambda^5 c_{\mathrm{pp}}\,,\\
D_0 &= 6\pi^2\Big\{30\pi^2\Big[k^4\left(c_2\left(c_2 c_{\mathrm{EE}} - 2\left(c_{\mathrm{E}} + c_2\right) c_{\mathrm{Ep}}\right) + \left(c_{\mathrm{E}} + c_2\right)^2 c_{\mathrm{pp}}\right)
\end{aligned}
$$

$$+ c\left(k^2 c_{\rm E} c_{\rm pp} + k^4\left(c_{\rm Ep}^2 - c_{\rm EE} c_{\rm pp}\right) - 2c_2 k^2 c_{\rm Ep} + c_2^2\right) + c_2^2 k^2 \left(c_{\rm E} + 2c_2\right)\Big]$$

$$+ 5c_2^2 k^2 m\Lambda^3\left(c_2^2 - c c_{\rm pp}\right) - 3c_2^2 m\Lambda^5\left(c_2^2 - c c_{\rm pp}\right)\Big\}. \tag{A.2}$$

The bare couplings of the new parametrization expressed in terms of the renormalized ones:

$$x := \frac{2\pi^2}{c_{\rm 2R}^2 - c_{\rm ppR} c_{\rm R}},$$

$$c = \frac{18\pi^2\left[m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right]}{9\left[m(\mu - \Lambda) - c_{\rm ppR} x\right]\left[m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right] - 5\left[3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right]^2},$$

$$c_2 = \frac{30\pi^2\left[3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right]}{5\left(3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right)^2 - 9\left(m(\mu - \Lambda) - c_{\rm ppR} x\right)\left(m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right)},$$

$$c_{\rm pp} = \frac{90\pi^2\left(m(\mu - \Lambda) - c_{\rm ppR} x\right)}{9\left(m(\mu - \Lambda) - c_{\rm ppR} x\right)\left(m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right) - 5\left(3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right)^2},$$

$$\begin{aligned}
c_E = {} & \frac{-1}{\left[5\left(3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right)^2 - 9\left(m(\mu - \Lambda) - c_{\rm ppR} x\right)\left(m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right)\right]^2} \\
& \times \Bigg\{30\pi^2\left(3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right)\Bigg[5\left(3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right) \\
& \times \left(m\left(\mu_4^3 - \Lambda^3\right) - \frac{3\left(c_{ER} c_{\rm 2R}^2 - 2c_{EpR} c_{\rm R} c_{\rm 2R} + c_{ER} c_{\rm ppR} c_{\rm R}\right) x}{c_{\rm 2R}^2}\right) \\
& - 18\left(m(\mu_2 - \Lambda) - \frac{(c_{\rm 2R} c_{EpR} - c_{ER} c_{\rm ppR}) x}{c_{\rm 2R}}\right)\left(m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right)\Bigg]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
c_{\rm EE} = {} & \frac{90\pi^2}{\left\{5\left[3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right]^2 - 9\left[m(\mu - \Lambda) - c_{\rm ppR} x\right]\left[m\left(\mu_3^5 - \Lambda^5\right) - 5c_{\rm R} x\right]\right\}^3} \\
& \times \Bigg\{25\Bigg[\frac{\left(c_{EER} c_{\rm 2R}^2 - 2c_{EpR} c_{ER} c_{\rm 2R} + c_{ER}^2 c_{\rm ppR} + \left(c_{EpR}^2 - c_{EER} c_{\rm ppR}\right) c_{\rm R}\right) x}{c_{\rm 2R}^2} \\
& + m(\Lambda - \mu_5)\Bigg] \\
& \times \left[3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right]^4 + 50\left[m(\mu_2 - \Lambda) - \frac{(c_{\rm 2R} c_{EpR} - c_{ER} c_{\rm ppR}) x}{c_{\rm 2R}}\right] \\
& \times \left[m\left(\mu_4^3 - \Lambda^3\right) - \frac{3\left(c_{ER} c_{\rm 2R}^2 - 2c_{EpR} c_{\rm R} c_{\rm 2R} + c_{ER} c_{\rm ppR} c_{\rm R}\right) x}{c_{\rm 2R}^2}\right] \\
& \times \left[3c_{\rm 2R} x + m\left(\mu_1^3 - \Lambda^3\right)\right]^3
\end{aligned}$$

$$
\begin{aligned}
&- 5\Bigg[27\left(m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right)\left(m(\mu_2-\Lambda)-\frac{(c_{2\rm R}c_{EpR}-c_{ER}c_{\rm ppR})x}{c_{2\rm R}}\right)^2\\
&+5\left(m(\mu-\Lambda)-c_{\rm ppR}x\right)\\
&\times\left(m\left(\mu_4^3-\Lambda^3\right)-\frac{6\left(c_{ER}c_{2\rm R}^2-2c_{EpR}c_{\rm R}c_{2\rm R}+c_{ER}c_{\rm ppR}c_{\rm R}\right)\pi^2}{c_{2\rm R}^4-c_{2\rm R}^2c_{\rm ppR}c_{\rm R}}\right)^2\\
&-9\left[m(\mu-\Lambda)-c_{\rm ppR}x\right]\left[m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right]\\
&\times\left(m(\mu_5-\Lambda)-\frac{\left(c_{\rm EER}c_{2\rm R}^2-2c_{\rm EpR}c_{\rm ER}c_{2\rm R}+c_{\rm ER}^2c_{\rm ppR}+\left(c_{\rm EpR}^2-c_{\rm EER}c_{\rm ppR}\right)c_{\rm R}\right)x}{c_{2\rm R}^2}\right)\Bigg]\\
&\times\left(3c_{2\rm R}x+m\left(\mu_1^3-\Lambda^3\right)\right)^2+90\left(m(\mu-\Lambda)-c_{\rm ppR}x\right)\\
&\times\left(m(\mu_2-\Lambda)-\frac{(c_{2\rm R}c_{\rm EpR}-c_{\rm ER}c_{\rm ppR})x}{c_{2\rm R}}\right)\left(m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right)\\
&\times\left[m\left(\mu_4^3-\Lambda^3\right)-\frac{3\left(c_{\rm ER}c_{2\rm R}^2-2c_{\rm EpR}c_{\rm R}c_{2\rm R}+c_{ER}c_{\rm ppR}c_{\rm R}\right)x}{c_{2\rm R}^2}\right]\\
&\times\left[3c_{2\rm R}x+m\left(\mu_1^3-\Lambda^3\right)\right]\\
&+81\left[c_{\rm ppR}x+m(\Lambda-\mu)\right]\left[\frac{(c_{2\rm R}c_{\rm EpR}-c_{\rm ER}c_{\rm ppR})x}{c_{2\rm R}}+m(\Lambda-\mu_2)\right]^2\\
&\times\left[m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right]^2\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
c_{Ep}=&\;\frac{90\pi^2}{\left\{5\left[3c_{2\rm R}x+m\left(\mu_1^3-\Lambda^3\right)\right]^2-9\left[m(\mu-\Lambda)-c_{\rm ppR}x\right]\left[m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right]\right\}^2}\\
&\times\Bigg\{5\left[\frac{(c_{2\rm R}c_{\rm EpR}-c_{\rm ER}c_{\rm ppR})x}{c_{2\rm R}}+m(\Lambda-\mu_2)\right]\left[m\left(\Lambda^3-\mu_1^3\right)-3c_{2\rm R}x\right]^2\\
&-9\left[m(\mu-\Lambda)-c_{\rm ppR}x\right]\left[m(\mu_2-\Lambda)-\frac{(c_{2\rm R}c_{\rm EpR}-c_{\rm ER}c_{\rm ppR})x}{c_{2\rm R}}\right]\\
&\times\left[m\left(\mu_3^5-\Lambda^5\right)-5c_{\rm R}x\right]+5\left[m(\mu-\Lambda)-c_{\rm ppR}x\right]\\
&\times\left[3c_{2\rm R}x+m\left(\mu_1^3-\Lambda^3\right)\right]\\
&\times\left[m\left(\mu_4^3-\Lambda^3\right)-\frac{3\left(c_{\rm ER}c_{2\rm R}^2-2c_{\rm EpR}c_{\rm R}c_{2\rm R}+c_{\rm ER}c_{\rm ppR}c_{\rm R}\right)x}{c_{2\rm R}^2}\right]\Bigg\}.
\end{aligned}
\tag{A.3}
$$

The renormalization scheme dependence of the renormalized couplings has the form

$$
\begin{aligned}
c_{\rm R}&=\frac{C_{4\rm R}(\mu_3)}{C_{4\rm R}(\mu_3)C_{\rm R}(\mu)-C_{2\rm R}(\mu_1)^2},\\
c_{2\rm R}&=\frac{C_{2\rm R}(\mu_1)}{C_{2\rm R}(\mu_1)^2-C_{4\rm R}(\mu_3)C_{\rm R}(\mu)},
\end{aligned}
$$

$$
\begin{aligned}
c_{\text{ppR}} &= \frac{C_{\text{R}}(\mu)}{C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu) - C_{2\text{R}}(\mu_1)^2}\,,\\
c_{\text{ER}} &= \frac{C_{2\text{R}}(\mu_1)\left[2C_{2\text{ER}}(\mu_2)C_{4\text{R}}(\mu_3) - C_{2\text{R}}(\mu_1)C_{4\text{ER}}(\mu_4)\right]}{\left[C_{2\text{R}}(\mu_1)^2 - C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu)\right]^2}\,,\\
c_{\text{EER}} &= -\frac{1}{\left[c_{2\text{R}}(\mu_1)^2 - C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu)\right]^3}\Bigg\{C_{4\text{EER}}(\mu_5)C_{2\text{R}}(\mu_1)^4\\
&\quad - 2C_{2\text{ER}}(\mu_2)C_{4\text{ER}}(\mu_4)c_{2\text{R}}(\mu_1)^3 + \Big[3C_{4\text{R}}(\mu_3)C_{2\text{ER}}(\mu_2)^2\\
&\quad + \Big(C_{4\text{ER}}(\mu_4)^2 - C_{4\text{EER}}(\mu_5)C_{4\text{R}}(\mu_3)\Big)C_{\text{R}}(\mu)\Big]c_{2\text{R}}(\mu_1)^2\\
&\quad - 2C_{2\text{ER}}(\mu_2)C_{4\text{ER}}(\mu_4)C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu)C_{2\text{R}}(\mu_1) + C_{2\text{ER}}(\mu_2)^2C_{4\text{R}}(\mu_3)^2C_{\text{R}}(\mu)\Bigg\}\,,\\
c_{\text{EpR}} &= \frac{C_{2\text{R}}(\mu_1)C_{4\text{ER}}(\mu_4)C_{\text{R}}(\mu) - C_{2\text{ER}}(\mu_2)\left[C_{2\text{R}}(\mu_1)^2 + C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu)\right]}{\left[C_{2\text{R}}(\mu_1)^2 - C_{4\text{R}}(\mu_3)C_{\text{R}}(\mu)\right]^2}\,.
\end{aligned}
\tag{A.4}
$$

The renormalized couplings corresponding to the parameters of Eq. (28)

$$
\begin{aligned}
c_{\text{R}} &= \frac{n_1}{d_1},\\
n_1 &= 576\pi^2 a v_2\left(m^2\mu_3^5 r_e^2 - 640\pi^3 v_2\beta^2\right),\\
d_1 &= 9\pi a m^3\mu_3^5 r_e^4 + 16 m v_2\Big[m r_e^2\left(10 a m\mu_1^6 + 9m\mu_3^5(\pi - 2a\mu) - 120\pi^2 a\beta\mu_1^3\right)\\
&\quad - 5760\pi^3 v_2\beta^2(\pi - 2a\mu)\Big],\\
c_{2\text{R}} &= \frac{n_2}{d_2},\\
n_2 &= 960\pi^2 a v_2 r_e^2\left(6\pi^2\beta - m\mu_1^3\right),\\
d_2 &= 9\pi a m^2\mu_3^5 r_e^4 + 16 m v_2 r_e^2\left[10 a m\mu_1^6 + 9m\mu_3^5(\pi - 2a\mu) - 120\pi^2 a\beta\mu_1^3\right]\\
&\quad -92160\pi^3 v_2^2\beta^2(\pi - 2a\mu),\\
c_{ER} &= \frac{n_3}{d_3},\\
n_3 &= 1920\pi^2 a v_2 r_e\left(m\mu_1^3 - 6\pi^2\beta\right)\Big\{-960\pi^2 v_2\beta\Big[a m^2\mu_1^3 r_e^3\\
&\quad + 96\pi m v_2\beta\,(\pi - 2a\mu_2)\,r_e + 1536\pi^3 a v_2^2\beta^2\Big] + 9m^2\mu_3^5 r_e^2\Big[16 m v_2\,(\pi - 2a\mu_2)\,r_e\\
&\quad + \pi a m r_e^3 + 256\pi^2 a v_2^2\beta\Big] + 80 a m^2 v_2\mu_4^3 r_e^3\left(m\mu_1^3 - 6\pi^2\beta\right)\Big\},
\end{aligned}
$$

$$d_3 = m\Bigg\{9\pi a m^2\mu_3^5 r_e^4 + 16 m v_2 r_e^2\Big[10 a m\mu_1^6 + 9m\mu_3^5(\pi - 2a\mu) - 120\pi^2 a\beta\mu_1^3\Big]$$
$$- 92160\pi^3 v_2^2\beta^2(\pi - 2a\mu)\Bigg\}^2,$$

$$c_{\rm ppR} = \frac{n_4}{d_4},$$
$$n_4 = 90\pi^2 m r_e^2\Big(\pi a r_e^2 + 16 v_2(\pi - 2a\mu)\Big),$$
$$d_4 = -9\pi a m^2\mu_3^5 r_e^4 + 16 m v_2 r_e^2\Big(-10 a m\mu_1^6 - 9m\mu_3^5(\pi - 2a\mu) + 120\pi^2 a\beta\mu_1^3\Big)$$
$$+ 92160\pi^3 v_2^2\beta^2(\pi - 2a\mu),$$

$$c_{EpR} = \frac{n_5}{d_5},$$
$$n_5 = 90\pi^2 r_e\Bigg\{9\pi^2 a^2 m^3\mu_3^5 r_e^7 + 2304\pi^3 a^2 m^2 v_2^2\beta\mu_3^5 r_e^4$$
$$- 256 m v_2^2 r_e^3\Big[10 a m^2\mu_1^6(\pi - 2a\mu_2) - 9m^2\mu_3^5(\pi - 2a\mu)(\pi - 2a\mu_2)$$
$$+ 10 a m\mu_1^3\Big(24\pi^2 a\beta(\mu_2 - \mu) - m\mu_4^3(\pi - 2a\mu)\Big)$$
$$+ 60\pi^2 a m\beta\mu_4^3(\pi - 2a\mu) + 360\pi^4 a\beta^2(2a\mu - 4a\mu_2 + \pi)\Big]$$
$$- 4096\pi^2 a v_2^3\beta r_e^2\Big[10a\Big(m^2\mu_1^6 - 12\pi^2 m\beta\mu_1^3 + 72\pi^4\beta^2\Big) - 9m^2\mu_3^5(\pi - 2a\mu)\Big]$$
$$- 32\pi a m^2 v_2 r_e^5\Big(5a\mu_4^3\Big(6\pi^2\beta - m\mu_1^3\Big) + 5 a m\mu_1^6 + 9m\mu_3^5(a\mu + a\mu_2 - \pi)\Big)$$
$$- 1474560\pi^3 m v_2^3\beta^2(\pi - 2a\mu)(\pi - 2a\mu_2) r_e - 23592960\pi^5 a v_2^4\beta^3(\pi - 2a\mu)\Bigg\},$$
$$d_5 = \Big[-9\pi a m^2\mu_3^5 r_e^4 + 16 m v_2 r_e^2\Big(-10 a m\mu_1^6 - 9m\mu_3^5(\pi - 2a\mu) + 120\pi^2 a\beta\mu_1^3\Big)$$
$$+ 92160\pi^3 v_2^2\beta^2(\pi - 2a\mu)\Big]^2,$$

$$c_{EER} = \frac{n_6}{d_6},$$
$$n_6 = -90\pi^2\Bigg\{81 a^3 m^6\pi^3\mu_3^{10} r_e^{12} - 144 a^2 m^5\pi^2 v_2\mu_3^5\Big[20 a m\mu_1^6$$
$$+ 9m(2a\mu + 4a\mu_2 - 3\pi)\mu_3^5 + 20a\Big(6\pi^2\beta - m\mu_1^3\Big)\mu_4^3\Big] r_e^{10}$$
$$+ 41472 a^3 m^5\pi^4\beta v_2^2\mu_3^{10} r_e^9 + 256 a m^4\pi v_2^2\Big[100 a^2 m^2\mu_1^{12} - 200 a^2 m^2\mu_4^3\mu_1^9$$
$$- 20 a m\Big(9m\mu_3^5(a\mu - 6a\mu_2 + a\mu_5 + 2\pi) - 5a\mu_4^3\Big(m\mu_4^3 + 12\pi^2\beta\Big)\Big)\mu_1^6$$

$$
\begin{aligned}
&- 120am\Big(10a\pi^2\beta\mu_4^6 + 3\mu_3^5\left(m(a\mu-\pi)\mu_4^3 + a\mu_2\left(m\mu_4^3 + 24\pi^2\beta\right)\right.\\
&\left.- 6a\pi^2\beta\left(3\mu+\mu_5\right)\right)\Big)\mu_1^3 + 9\Big(9m^2\left(\pi-2a\mu_2\right)\left(-4a\mu-2a\mu_2+3\pi\right)\mu_3^{10}\\
&+ 240a\pi^2\beta\Big(m(a\mu-\pi)\mu_4^3 + a\mu_2\left(m\mu_4^3+18\pi^2\beta\right)\\
&- 3\pi^2\beta\left(3a\mu+a\mu_5+\pi\right)\Big)\mu_3^5 + 400a^2\pi^4\beta^2\mu_4^6\Big)\Big]r_e^8\\
&- 147456a^2m^3\pi^3\beta v_2^3\mu_3^5\Big[360a\pi^4\beta^2 + m\Big(10am\mu_1^6 - 5a\left(m\mu_4^3+12\pi^2\beta\right)\mu_1^3\\
&+ 9m\left(a\mu+a\mu_2-\pi\right)\mu_3^5 + 30a\pi^2\beta\mu_4^3\Big)\Big]r_e^7\\
&+ 4096m^3v_2^3\Big[100a^2m^3\left(\pi-2a\mu_5\right)\mu_1^{12} + 200a^2m^2\Big(2a\mu_2\left(m\mu_4^3-12\pi^2\beta\right)\\
&+ \pi\left(24a\pi\beta\mu_5 - m\mu_4^3\right)\Big)\mu_1^9 + 20am\Big(-54a^2m^2\mu_2^2\mu_3^5\\
&- 9m^2\left(\pi(a\mu+\pi) + a(\pi-2a\mu)\mu_5\right)\mu_3^5\\
&+ 18a\pi\mu_2\left(3m^2\mu_3^5 + 20a\pi\beta\left(6\pi^2\beta - m\mu_4^3\right)\right)\\
&+ 5a\Big(m^2(\pi-2a\mu)\mu_4^6 + 12m\pi^2\beta(4a\mu+\pi)\mu_4^3\\
&+ 72\pi^4\beta^2\left(-3a\mu-5a\mu_5+\pi\right)\Big)\Big)\mu_1^6 + 60a\Big(216a^2m^2\pi^2\beta\mu_2^2\mu_3^5\\
&- 6a\mu_2\Big(m^2\left(m(\pi-2a\mu)\mu_4^3 + 24\pi^2\beta(a\mu+\pi)\right)\mu_3^5 + 160a\pi^4\beta^2\left(3\pi^2\beta - m\mu_4^3\right)\Big)\\
&+ \pi\Big(3m^2\left(36a\pi^2\beta\mu + (\pi-2a\mu)\left(m\mu_4^3 + 12a\pi\beta\mu_5\right)\right)\mu_3^5\\
&+ 20a\pi\beta\left(-m^2(\pi-2a\mu)\mu_4^6 - 6m\pi^2\beta(6a\mu+\pi)\mu_4^3 + 72a\pi^4\beta^2\left(\mu+\mu_5\right)\right)\Big)\Big)\mu_1^3\\
&+ 9\Big(9\left(16m\pi^5\beta^2v_2a^3 + m^3(\pi-2a\mu)\left(\pi-2a\mu_2\right)^2\right)\mu_3^{10}\\
&- 120am\pi^2\beta\Big(60a^2\pi^2\beta\mu_2^2 + 2a\left(-m(\pi-2a\mu)\mu_4^3 - 30\pi^3\beta\right)\mu_2\\
&+ 6\pi^3\beta(a\mu+2\pi) + \pi(\pi-2a\mu)\left(m\mu_4^3 + 6a\pi\beta\mu_5\right)\Big)\mu_3^5\\
&+ 400a^2\pi^4\beta^2\mu_4^3\left(m(\pi-2a\mu)\mu_4^3 + 12\pi^2\beta\left(2a\mu-4a\mu_2+\pi\right)\right)\Big)\Big]r_e^6\\
&+ 131072am^2\pi^2\beta v_2^4\Big[100a^2m^3\mu_1^{12} - 100a^2m^2\left(m\mu_4^3+12\pi^2\beta\right)\mu_1^9\\
&- 180am\Big(m^2\left(a\mu-3a\mu_2+\pi\right)\mu_3^5 - 10a\pi^2\beta\left(m\mu_4^3+4\pi^2\beta\right)\Big)\mu_1^6
\end{aligned}
$$

$$+ 90am\left(m\mu_3^5\left(m(\pi-2a\mu)\mu_4^3+12\pi^2\beta\left(4a\mu-6a\mu_2+\pi\right)\right)-160a\pi^4\beta^2\mu_4^3\right)\mu_1^3$$

$$+ 81m^3(\pi-2a\mu)\left(\pi-2a\mu_2\right)\mu_3^{10}+43200a^2\pi^6\beta^3\mu_4^3$$

$$- 540am\pi^2\beta\mu_3^5\left(m(\pi-2a\mu)\mu_4^3+12\pi^2\beta\left(a\mu-5a\mu_2+2\pi\right)\right)\Bigg]r_e^5$$

$$- 9437184m^2\pi^3\beta^2v_2^4\Bigg[10am^2\left(a\left(3a\pi v_2\mu_3^5-30a\mu_2^2+30\pi\mu_2-5(\pi-2a\mu)\mu_5\right)\right.$$

$$\left.- 5\pi(a\mu+\pi)\right)\mu_1^6$$

$$+ 10am\Bigg(360a^2\pi^2\beta\mu_2^2-10a\left(m(\pi-2a\mu)\mu_4^3+24\pi^2\beta(a\mu+\pi)\right)\mu_2$$

$$+ \pi\left(-36a^2\pi^2\beta v_2\mu_3^5+180a\pi^2\beta\mu+5(\pi-2a\mu)\left(m\mu_4^3+12a\pi\beta\mu_5\right)\right)\Bigg)\mu_1^3$$

$$+ 3\Bigg(-60a^2\left(80a\pi^4\beta^2-m^2(\pi-2a\mu)\mu_3^5\right)\mu_2^2+20a\pi\Bigg(60a\pi^3(2a\mu+3\pi)\beta^2$$

$$+ m(\pi-2a\mu)\left(10a\pi\beta\mu_4^3-3m\mu_3^5\right)\Bigg)\mu_2$$

$$+ \pi\Bigg(3a^2m^2(2a\mu-\pi)v_2\mu_3^{10}+15\pi\left(40\pi^3\beta^2v_2a^3+m^2(\pi-2a\mu)\right)\mu_3^5$$

$$- 100a\pi^2\beta\left(3\pi^2\beta(6a\mu+\pi)+(\pi-2a\mu)\left(m\mu_4^3+6a\pi\beta\mu_5\right)\right)\Bigg)\Bigg)\Bigg]r_e^4$$

$$+ 1509949440am\pi^5\beta^3v_2^5\Bigg[10am^2\left(a\mu-3a\mu_2+\pi\right)\mu_1^6+5am\Bigg(-m(\pi-2a\mu)\mu_4^3$$

$$- 12\pi^2\beta\left(4a\mu-6a\mu_2+\pi\right)\Bigg)\mu_1^3-9m^2(\pi-2a\mu)\left(\pi-2a\mu_2\right)\mu_3^5$$

$$+ 30am\pi^2\beta(\pi-2a\mu)\mu_4^3+360a\pi^4\beta^2\left(2a\mu-4a\mu_2+\pi\right)\Bigg]r_e^3$$

$$+ 9059696640\pi^6\beta^4v_2^5\Bigg[4\pi v_2\left(240a\pi^4\beta^2+m\left(5am\mu_1^6-60a\pi^2\beta\mu_1^3\right.\right.$$

$$\left.\left.- 3m(\pi-2a\mu)\mu_3^5\right)\right)a^2+15m^2(\pi-2a\mu)\left(\pi-2a\mu_2\right)^2\Bigg]r_e^2$$

$$+ 4348654387200am\pi^8\beta^5(\pi-2a\mu)v_2^6\left(\pi-2a\mu_2\right)r_e$$

$$+ 34789235097600a^2\pi^{10}\beta^6(\pi-2a\mu)v_2^7\Bigg\},$$

$$d_6 = m\Bigg[9am^2\pi r_e^4\mu_3^5-92160\pi^3\beta^2(\pi-2a\mu)v_2^2$$

$$+ 16mr_e^2v_2\left(10am\mu_1^6-120a\pi^2\beta\mu_1^3+9m(\pi-2a\mu)\mu_3^5\right)\Bigg]^3. \qquad \text{(A.5)}$$

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