Quark masses and Dashen's theorem

Stephan Dürr

University of Wuppertal
Jülich Supercomputing Center

Lattice meets Continuum – Siegen – 20. September 2017

Talk Overview

Including isospin breaking effects in lattice calculations

- Reminder of why QCD with $N_f = 2$ or $2{+}1$ or $2{+}1{+}1$ is so practical
- Reminder of what Dashen says and its implication on schemes
- Options for treating QCD+QED on the lattice
  - Mature calculations: spectroscopy and quark masses
  - Conceptual challenges: decay constants and matrix elements
- Summary from FLAG: $\epsilon = 0.7(3)$ leads to $m_u/m_d = 0.50(3)$ [in $\overline{\mathrm{MS}}$ at 2 GeV]

Setup (1): overall pathway

- QCD with $N_f = 2$ or $2{+}1$ or $2{+}1{+}1$ straightforward to simulate via HMC/RHMC.

- Input quantities $(M_N; M_\pi, M_K, \ldots)$ must be *corrected* for isospin breaking, and correction must account for both QCD ($m_u \neq m_d$) and QED ($\alpha > 0$) sources.

- Systematic treatment initiated by Dashen's theorem (1969) which says:

$$\Delta^\gamma_{\pi^\pm} = \Delta^\gamma_{K^\pm} \text{ and } \Delta^\gamma_{\pi^0} = \Delta^\gamma_{K^0} = 0$$

- FLAG_10/13/16 discuss corrections to Dashen, suggest to use input values

$$\bar{M_\pi}^{\mathrm{phys}} = 134.8(3)\,\mathrm{MeV} \quad \text{and} \quad \bar{M_K}^{\mathrm{phys}} = 494.2(4)\,\mathrm{MeV}$$

for iso-symmetric lattice calculations (error must be propagated).

- Frontier: Treat QCD+QED on the lattice, even though this implies subtle field-theoretic issues (f_π not well defined; m_d/m_s is RGI, m_u/m_s is not). This is relevant for attaining per-mille level accuracy (cf. muon $g{-}2$, ...).

Setup (2): light quark mass ratios in ChPT

$\mathcal{L}^{(2)}$ and higher order Lagrangians contain only Bm_{ud}, Bm_s, which are scheme- and scale-invariant quantities [RGI]; with $m_{ud}=(m_u{+}m_d)/2$ [isospin limit] the LO says:

$$M_{\pi^\pm}^2 = B2m_{ud} \quad , \qquad M_{\pi^0}^2 = B2m_{ud} + O\Big(\frac{[m_u-m_d]^2}{m_s-m_{ud}}\Big)$$

$$M_{K^\pm}^2 = B(m_u+m_s) \quad , \qquad M_{K^0}^2 = B(m_d+m_s)$$

$$M_\eta^2 = B(\frac{2}{3}m_{ud}+\frac{4}{3}m_s) + O\Big(\frac{[m_u-m_d]^2}{m_s-m_{ud}}\Big)$$

This implies $M_{\pi^\pm}^2 - M_{\pi^0}^2 = O([m_u{-}m_d]^2)$ and $M_{K^\pm}^2 - M_{K^0}^2 = B(m_u{-}m_d) < 0$ and

- $M_\pi^2 F_\pi^2 = \Sigma 2m_{ud}$ with $\Sigma \equiv -\langle\bar{u}u\rangle = -\langle\bar{d}d\rangle = -\langle\bar{s}s\rangle$ $(\lim_{m\to 0})$ [GOR]
- $B = \frac{M_\pi^2}{2m_{ud}} = \frac{M_K^2}{m_s+m_u} = \frac{M_\eta^2}{m_s+m_d}$ [Weinberg]
- $3M_\eta^2 = 4M_K^2 - M_\pi^2$ [GellMann Okubo]

Quark mass *ratios* as determined from phenomenology (no handle at quark masses):

	m_u/m_d	m_s/m_d	m_s/m_{ud}
$O(p^2)$	0.55	20.1	25.9
$O(p^4)$	0.55 ± 0.04	18.9 ± 0.8	24.4 ± 1.5

Setup (3): FLAG notation details

Notation: . means observed/PDG mass, $\hat{.}$ means QCD part, $.^\gamma$ means QED part, and $M_P \equiv \hat{M}_P + M_P^\gamma$ (splitting not unique) implies $\Delta_P^\gamma \equiv M_P^2 - \hat{M}_P^2 = 2\hat{M}_P M_P^\gamma + O(\alpha^2)$

- Physical/observed pion mass splitting: $\Delta_\pi \equiv M_{\pi^\pm}^2 - M_{\pi^0}^2 = 1261.2\,\mathrm{MeV}^2$

- Express self-energies and mass differences through it:

$$\Delta_{\pi^0}^\gamma = \epsilon_{\pi^0}\Delta_\pi, \quad \Delta_{K^0}^\gamma = \epsilon_{K^0}\Delta_\pi, \quad \hat{M}_{\pi^\pm}^2 - \hat{M}_{\pi^0}^2 = \epsilon_m \Delta_\pi$$

- In this notation, the self-energies of the charged particles are given by

$$\Delta_{\pi^\pm}^\gamma = (1 + \epsilon_{\pi^0} - \epsilon_m)\Delta_\pi, \quad \Delta_{K^\pm}^\gamma = (1 + \epsilon_{K^0} - \epsilon_m + \epsilon)\Delta_\pi$$

  where ϵ parameterizes violation of Dashen's theorem ("$\epsilon = 0$")

$$\epsilon \equiv \Big(\Delta_{K^\pm}^\gamma - \Delta_{K^0}^\gamma - \Delta_{\pi^\pm}^\gamma + \Delta_{\pi^0}^\gamma\Big)/\Delta_\pi$$

- Determination of $m_{d,u,s}$ based on the masses of $\pi^\pm, \pi^0, K^\pm$ in QCD is equivalent to a determination of $\epsilon_{\pi^0}, \epsilon_{K^0}, \epsilon_m$ and hence ϵ. FLAG scans large body of literature (both lattice and non-lattice) and recommends specific values for $\epsilon_{\pi^0}, \epsilon_{K^0}, \epsilon_m, \epsilon$.

Setup (4): FLAG suggestions for $\epsilon_{\pi^0}, \epsilon_{K^0}, \epsilon_m$ and ϵ

FLAG scans large body of literature (both lattice and non-lattice) and recommends

$$\epsilon_{\pi^0} = 0.07(7)\,,\;\; \epsilon_{K^0} = 0.3(3)\,,\;\; \epsilon_m = 0.04(2)\,,\;\; \epsilon = 0.7(3)$$

which, with naive error propagation, amount to the self-energies

$$\begin{aligned} M^\gamma_{\pi^\pm} &= 4.7(3)\,\mathrm{MeV}\,,\;\; M^\gamma_{\pi^0} = 0.3(3)\,\mathrm{MeV} \quad , \quad M^\gamma_{\pi^\pm} - M^\gamma_{\pi^0} = 4.4(1)\,\mathrm{MeV} \\ M^\gamma_{K^\pm} &= 2.5(5)\,\mathrm{MeV}\,,\;\; M^\gamma_{K^0} = 0.4(4)\,\mathrm{MeV} \quad , \quad M^\gamma_{K^\pm} - M^\gamma_{K^0} = 2.1(1)\,\mathrm{MeV} \end{aligned}$$

and to the pure QCD meson masses

$$\begin{aligned} \hat{M}_{\pi^\pm} &= 134.8(3)\,\mathrm{MeV}\,,\;\; \hat{M}_{\pi^0} = 134.6(3)\,\mathrm{MeV} \quad , \quad \hat{M}_{\pi^\pm} - \hat{M}_{\pi^0} = +0.2(1)\,\mathrm{MeV} \\ \hat{M}_{K^\pm} &= 491.2(5)\,\mathrm{MeV}\,,\;\; \hat{M}_{K^0} = 497.2(4)\,\mathrm{MeV} \quad , \quad \hat{M}_{K^\pm} - \hat{M}_{K^0} = -6.1(4)\,\mathrm{MeV} \end{aligned}$$

from which the aforementioned *corrected* (iso-symmetric) input masses follow as

$$\bar{M}_\pi^{\mathrm{phys}} = \hat{M}_{\pi^\pm} = 134.8(3)\,\mathrm{MeV}\,,\;\; \bar{M}_K^{\mathrm{phys}} = \sqrt{\frac{1}{2}(\hat{M}^2_{K^\pm} + \hat{M}^2_{K^0})} = 494.2(4)\,\mathrm{MeV}$$

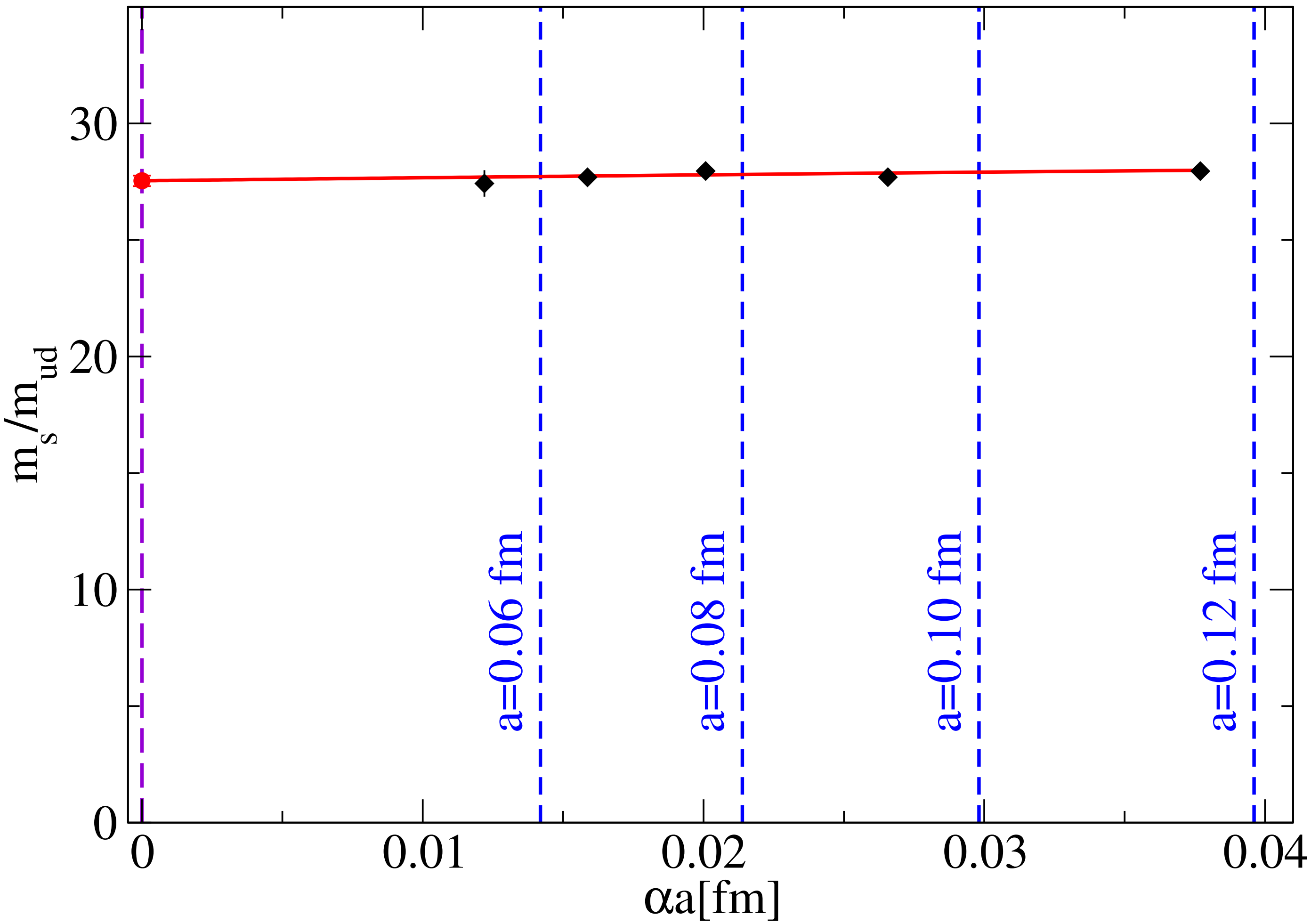
BMW_10 example (1): quark mass computation

1. Choose observables to be "burned", e.g. M_π, M_K, M_Ω in $N_f=2+1$ QCD, and get "polished" experimental values, e.g. $M_\pi=134.8(3)\,\mathrm{MeV}$, $M_K=494.2(4)\,\mathrm{MeV}$ in a world without isospin splitting and without electromagnetism [FLAG_10/13/16].
2. For a given bare coupling β (yields a) tune bare masses $1/\kappa_{ud,s}$ such that the ratios M_π/M_Ω, M_K/M_Ω assume their physical values (in practice: inter-/extrapolation).
3. Read off $1/\kappa_{ud,s}$ or determine bare $am_{ud,s}$ via AWI and convert them (perturbatively or non-perturbatively) to the scheme of your choice (e.g. $\overline{\mathrm{MS}}$ at $\mu=3\,\mathrm{GeV}$).
4. Repeat steps 2 and 3 for at least 3 different lattice spacings and extrapolate the (finite-volume corrected) result to the continuum via Symanzik scaling.

Depending on details, step 3 can be rather demanding [RI/MOM, SF renormalization]. Below, guided tour using plots from BMW-collaboration [arXiv:1011.2403,1011.2711].

BMW_10 example (2): final result for ratio m_s/m_{ud}

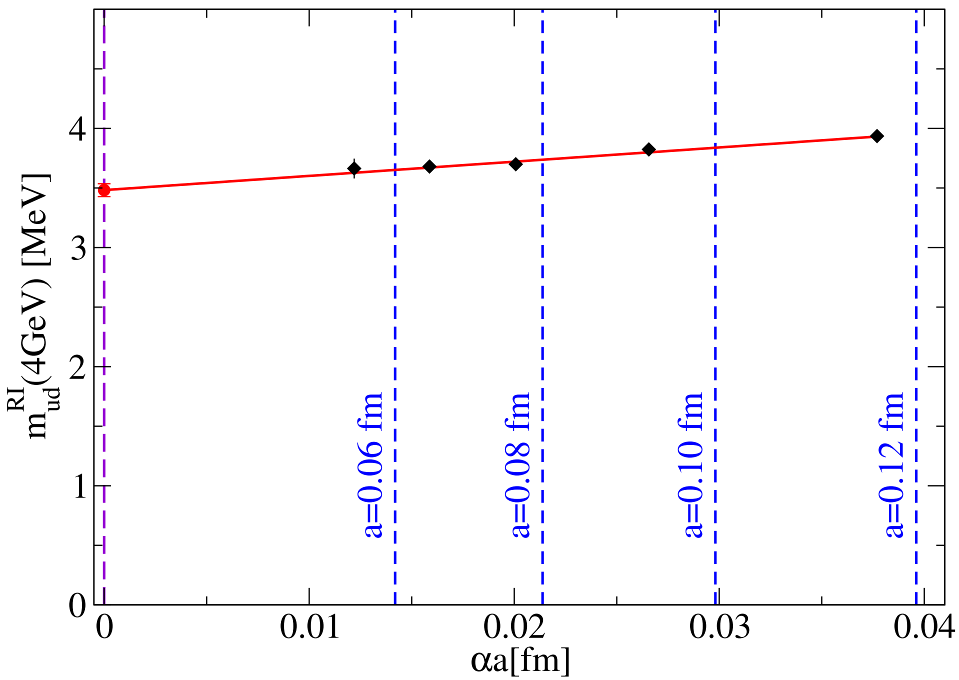
In QCD ratios like m_s/m_{ud} are renormalization group invariant (RGI), hence step 3 in this list is skipped (detail: we invoke αa and a^2 scaling).

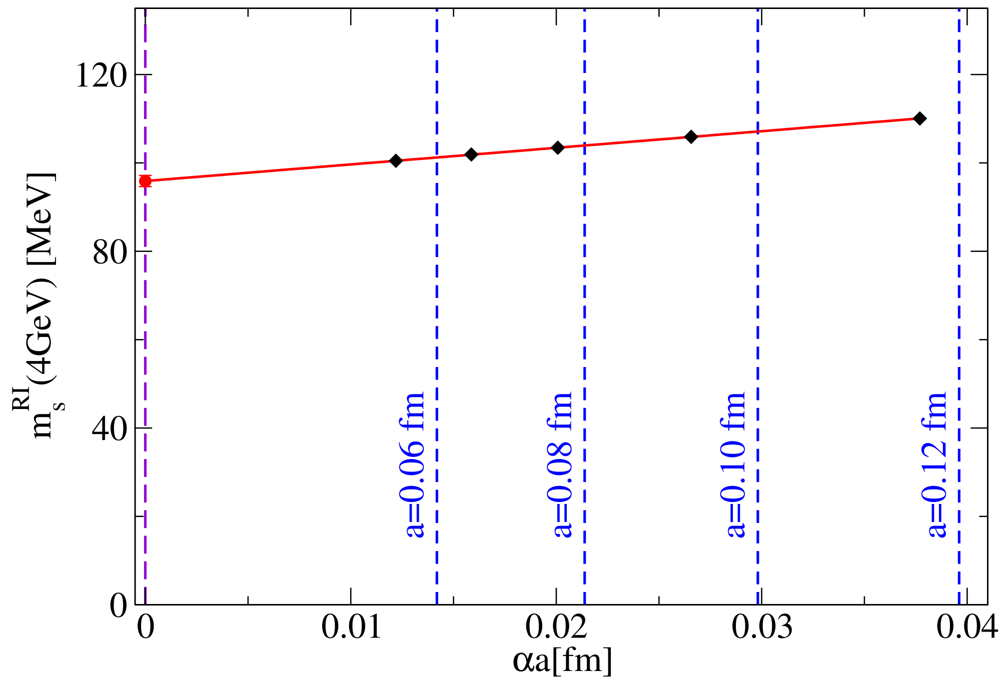


Final result $m_s/m_{ud} = 27.53(20)(08)$ amounts to 0.78% precision.

BMW_10 example (3): final results for m_s and m_{ud}

Good scaling of $m_{ud,s}^{\mathrm{RI}}(4\,\mathrm{GeV})$ out to the coarsest lattice ($a\sim 0.116\,\mathrm{fm}$):





Conversion with analytical 4-loop formula at $4\,\mathrm{GeV}$ and downwards running in $\overline{\mathrm{MS}}$:

	m_{ud}	m_s
RI$(4\,\mathrm{GeV})$	3.503(48)(49)	96.4(1.1)(1.5)
RGI	4.624(63)(64)	127.3(1.5)(1.9)
$\overline{\mathrm{MS}}(2\,\mathrm{GeV})$	3.469(47)(48)	95.5(1.1)(1.5)

RGI/$\overline{\mathrm{MS}}$ results (table 1.9% prec.) need to be augmented by a $\sim 1\%$ conversion error.

BMW_10 example (4): splitting m_{ud} with input from $\eta \to 3\pi$

The process $\eta \to 3\pi$ is highly sensitive to QCD isospin breaking (from $m_u \neq m_d$) but rather insensitive to QED isospin breaking (from $q_u \neq q_d$), and this is captured in Q.

Rewrite the Leutwyler ellipse in the form

$$\frac{1}{Q^2} = 4\left(\frac{m_{ud}}{m_s}\right)^2 \frac{m_d - m_u}{m_d + m_u}$$

and use the conservative estimate $Q = 22.3(8)$ of [Leutwyler, Chiral Dynamics 09] together with our result $m_s/m_{ud} = 27.53(20)(08)$ to get the asymmetry parameter

$$\frac{m_d - m_u}{m_d + m_u} = 0.381(05)(27) \quad \longleftrightarrow \quad m_u/m_d = 0.448(06)(29)$$

from which we then obtain these individual m_u, m_d values:

	m_u	m_d	m_s
RI$(4\,\mathrm{GeV})$	2.17(04)(10)	4.84(07)(12)	96.4(1.1)(1.5)
RGI	2.86(05)(13)	6.39(09)(15)	127.3(1.5)(1.9)
$\overline{\mathrm{MS}}(2\,\mathrm{GeV})$	2.15(03)(10)	4.79(07)(12)	95.5(1.1)(1.5)

Dashen theorem (1): EM-effects in ChPT

R. Urech and R. Baur extended the chiral Lagrangian to include electromagnetic effects; usual *power counting rule* is extended to read $p^2 \sim m \sim e^2$.

$$\begin{aligned}
\mathcal{L}_2 &= \frac{F_0^2}{4}\langle d^\mu U^\dagger d_\mu U + \chi U^\dagger + \chi^\dagger U\rangle \\
d_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \\
v_\mu &= QA_\mu + \cdots \\
Q &= \frac{e}{3}\,\mathrm{diag}\,(2,-1,-1) \\
\chi &= 2B_0(s+ip) \\
s &= \mathrm{diag}\,(m_u, m_d, m_s) \\
F_\pi &= F_0\,[1+O(m_q)] \\
B_0 &= -\frac{1}{F_0^2}\langle 0|\bar{u}u|0\rangle\,[1+O(m_q)] \quad .
\end{aligned}$$

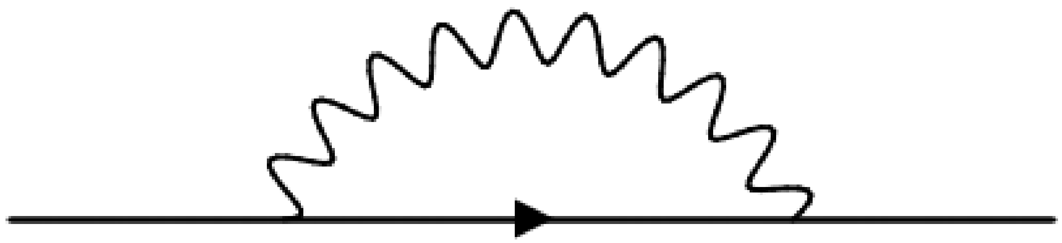
$$\Phi = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

$$V_{\mu\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0_{\mu\nu} + \frac{1}{\sqrt{6}}\omega_{8\,\mu\nu} & \rho^+_{\mu\nu} & K^{*+}_{\mu\nu} \\ \rho^-_{\mu\nu} & -\frac{1}{\sqrt{2}}\rho^0_{\mu\nu} + \frac{1}{\sqrt{6}}\omega_{8\,\mu\nu} & K^{*0}_{\mu\nu} \\ K^{*-}_{\mu\nu} & \overline{K^{*0}}_{\mu\nu} & -\frac{2}{\sqrt{6}}\omega_{8\,\mu\nu} \end{pmatrix}$$

$$\begin{aligned}
\mathcal{L}_2^V &= \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{iG_V}{2\sqrt{2}}\langle V_{\mu\nu}[u^\mu, u^\nu]\rangle \\
\mathcal{L}_2^A &= \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu}\rangle \\
f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu} u \\
F_{R,L}^{\mu\nu} &= \partial^\mu(v^\nu \pm a^\nu) - \partial^\nu(v^\mu \pm a^\mu) - i[v^\mu \pm a^\mu, v^\nu \pm a^\nu] \\
u^\mu &= iu^\dagger d^\mu U u^\dagger = u^{\dagger\,\mu} \\
U &= u^2 \quad .
\end{aligned}$$

Dashen theorem (2): leading order in ChPT

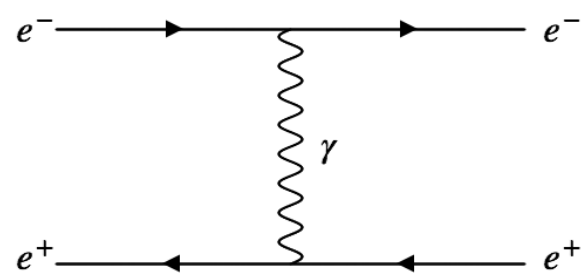
Dashen theorem follows from LO graph in ChPT (full line is $\pi^\pm$ or $K^\pm$):



Dashen theorem is in chiral limit, it receives corrections from finite quark masses. Studies with finite m_q suggest that corrections *are* large or *may be* large.

Dashen theorem ignores (replace $e^- \to q$ and $e^+ \to \bar{q}$):

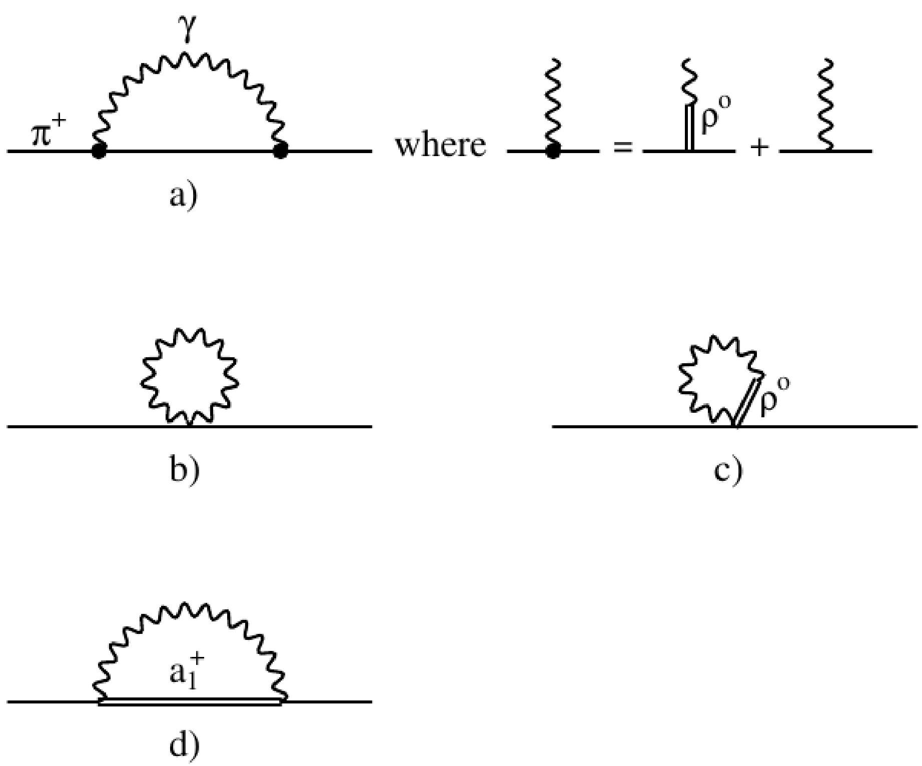
- photons connecting $u \leftrightarrow \bar{d}$ or $d \leftrightarrow \bar{u}$ within $\pi^\pm$
- photons connecting $u \leftrightarrow \bar{u}$ or $d \leftrightarrow \bar{d}$ within π^0
- photons connecting $u \leftrightarrow \bar{s}$ or $s \leftrightarrow \bar{u}$ within $K^\pm$
- photons connecting $d \leftrightarrow \bar{s}$ or $s \leftrightarrow \bar{d}$ within π^0



Such effects are captured – at least in part – by higher orders in ChPT (with photons), built into finite parts of higher-order chiral counterterms !

Dashen theorem (3): corrections in ChPT

R.Baur and R.Urech calculate effect at order $O(e^2 m_q)$, based on resonance saturation estimates for higher-order low-energy coefficients.



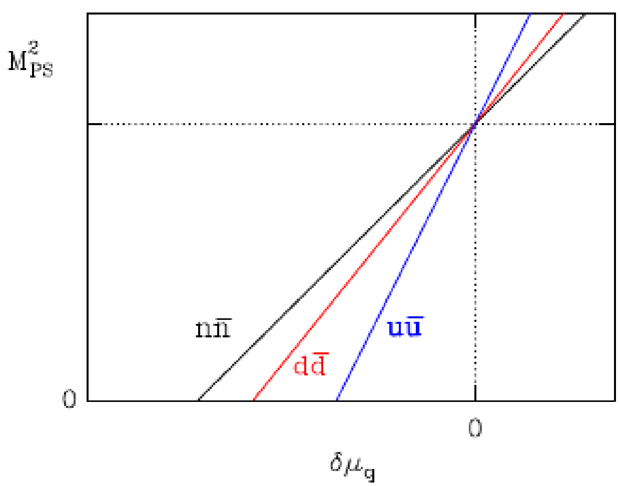
Within this framework they reach the conclusion that corrections to $\epsilon = 0$ are *small*.

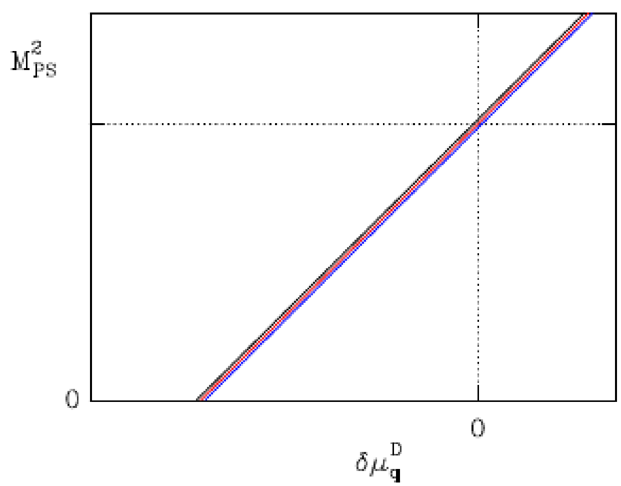
Dashen theorem (4): Dashen scheme by QCDSF

QCD+QED renormalization intertwined (1PI quark self-energy with 1 gluon-loop and 1 photon-loop); mass anomalous dimension depends on electric charge.

$$
\begin{aligned}
M^2(a\bar{b}) \;=\; & M^2 + \alpha(\delta\mu_a + \delta\mu_b) + c(\delta m_u + \delta m_d + \delta m_s) \\
& + \beta_0 \tfrac{1}{6}(\delta m_u^2 + \delta m_d^2 + \delta m_s^2) + \beta_1(\delta\mu_a^2 + \delta\mu_b^2) + \beta_2(\delta\mu_a - \delta\mu_b)^2 \\
& + \beta_0^{EM}(e_u^2 + e_d^2 + e_s^2) + \beta_1^{EM}(e_a^2 + e_b^2) + \beta_2^{EM}(e_a - e_b)^2 \\
& + \gamma_0^{EM}(e_u^2 \delta m_u + e_d^2 \delta m_d + e_s^2 \delta m_s) + \gamma_1^{EM}(e_a^2 \delta\mu_a + e_b^2 \delta\mu_b) \\
& + \gamma_2^{EM}(e_a - e_b)^2(\delta\mu_a + \delta\mu_b) + \gamma_3^{EM}(e_a^2 - e_b^2)(\delta\mu_a - \delta\mu_b) \\
& + \gamma_4^{EM}(e_u^2 + e_d^2 + e_s^2)(\delta\mu_a + \delta\mu_b) \\
& + \gamma_5^{EM}(e_a + e_b)(e_u \delta m_u + e_d \delta m_d + e_s \delta m_s)\;.
\end{aligned}
$$

QCDSF specialty: Dashen scheme [arXiv:1508.06401, 1509.00799] where, starting from a flavor breaking expansion, quark masses of charged quarks are rescaled (renormalized) to give equal slope as for (fictitious) neutral $q\bar{q}$ state.





QCD+QED (1): compact versus non-compact QED

- Compact QED:

  Fundamental object is gauge link $U_\mu(x) \equiv \mathcal{P}\{\exp(-\mathrm{ie}\int_x^{x+\hat{\mu}} A(s)ds)\}$ and everything expressed in terms of gauge-invariant objects, e.g. Wilson gauge action

$$S(x) \equiv \beta \sum_{x,\mu<\nu} \{1 - \cos(\theta_{\mu\nu}(x))\} = \beta \sum_{x,\mu<\nu} \{1 - \mathrm{Re}(U_{\mu\nu}(x))\}$$

  in which rescaled field $\theta_\mu(x) = eA_\mu(x)$ takes values in $]-\pi, \pi[$.

- Non-compact QED:

  Fundamental object is gauge field $A_\mu(x + \frac{1}{2}\hat{\mu})$ which takes values in $]-\infty, \infty[$.
  Electromagnetic backgrounds must be fixed to some gauge (e.g. Lorenz gauge).

Technical issue: Compact QED has *bulk* phase transition at $\beta \simeq 1.01$

Conceptual issue: QED is likely trivial (as signaled by Landau pole at 1-loop), though running over practical mass scales on the lattice is ridiculously small.

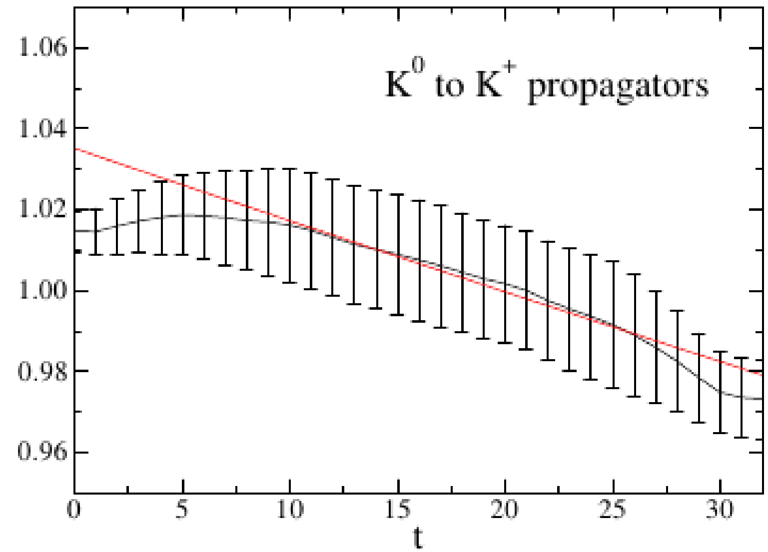
Phenomenological efforts for QCD+QED tend to use non-compact QED formulation.

QCD+QED (2): pioneering works

In standard $N_f = 2+1$ lattice studies two sources of isospin breaking are ignored (up-down mass difference, electromagnetic). Since they are both small, it would appear reasonable to include both of them a posteriori, by reweighting the configurations.

PACS-CS has long experience with reweighting in the quark mass; they used reweighting in m_{ud} to shift M_π from $156\,\mathrm{MeV}$ to $135\,\mathrm{MeV}$.

In arXiv:1205.2961 they extend this approach to account for QED effects and the up-down quark mass difference; they find $M_{K^0} > M_K^{\pm}$.



Pioneering publication for QCD+QED on the lattice is Duncan et al, Phys. Rev. Lett. 76 (1996) 3894-3897 [hep-lat/9602005].
Continuation by RBC/UKQCD Phys.Rev. D76 (2007), Phys.Rev. D82 (2010) 094508.

Still, there remain issues relating to finite-volume corrections, see e.g. Hayakawa Uno, Prog.Theor.Phys. 120 (2008) 413 and Portelli et al, PoS LATTICE2011 (2011) 136.

QCD+QED (3): survey of options – part 1

Gauss law forbids states with non-zero total electric charge in 3D box with periodic boundary conditions, since surface-integral is just zero:

$$Q = \int_{T^3} j_4(\vec{x},t)\,\mathrm{d}^3x = \int_{T^3} \partial_k E_k(\vec{x},t)\,\mathrm{d}^3x = 0$$

Something must be done, and for a summary of the various options tried in the lattice community I draw on A.Patella, PoS LATTICE2016 (2017) 020 [arXiv:1702.03857].

1. If $\tilde{A}_\mu(p)$ is photon field in Fourier space, the constraint $\tilde{A}_\mu(0) = 0$ defines QED$_{TL}$. QED$_{TL}$ does not have a transfer matrix with a regular perturbative expansion. QED$_{TL}$ has been used in the electro-quenched study BMW_16 (see later).

2. The QED$_{\mathrm{SF}}$ prescription restricts $e\tilde{A}_\mu(0)/V$ to the interval $]-\pi/L_\mu, \pi/L_\mu[$. In QED$_{\mathrm{SF}}$ charged two-point functions are non-local in time, the existence of a transfer matrix seems unlikely. QED$_{\mathrm{SF}}$ has been used by QCDSF in many instances.

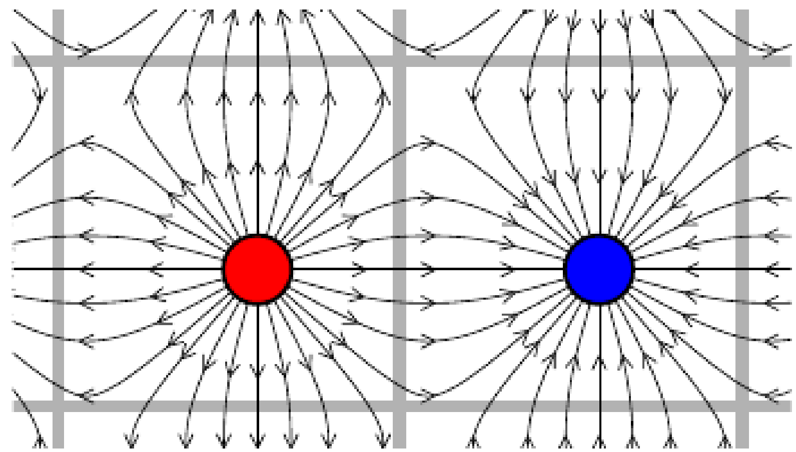
3. The constraint $\tilde{A}_\mu(\vec{0}, p_4) = 0$ for any p_4 (equivalently for each timeslice) defines QED$_L$, originally advocated in Hayakawa Uno, Prog. Theor. Phys. 120, 413 (2008).

QCD+QED (4): survey of options – part 2

QED$_L$ has a transfer matrix, but non-local terms spoil renormalization of composite operators. QED$_L$ was used by BMW_14 (baryon mass splittings) and Roma-Soton.

4. QED with a massive photon is a consistent QFT, albeit gauge-invariance is broken. This scheme is referred to as QED$_m$, it has two IR regulators (m and L). Care must be exercised to make sure that the two limits are taken in the right order.

5. QED with C-parity violating boundary conditions in the spatial directions is a perfectly consistent QFT, too. Flavor is broken, albeit in a purely local fashion. The respective flavor mixing is exponentially suppressed as $L \to \infty$, and absent in the renormalization of composite operators. QED$_C$ allows a gauge-invariant description of charged interpolating operators.



All these approaches are equivalent if the $L \to \infty$ limit is take before any other limit (large-t limit in 2-point functions, continuum limit, massless photon limit). In general the infinite-volume limit does not commute with the other limits.

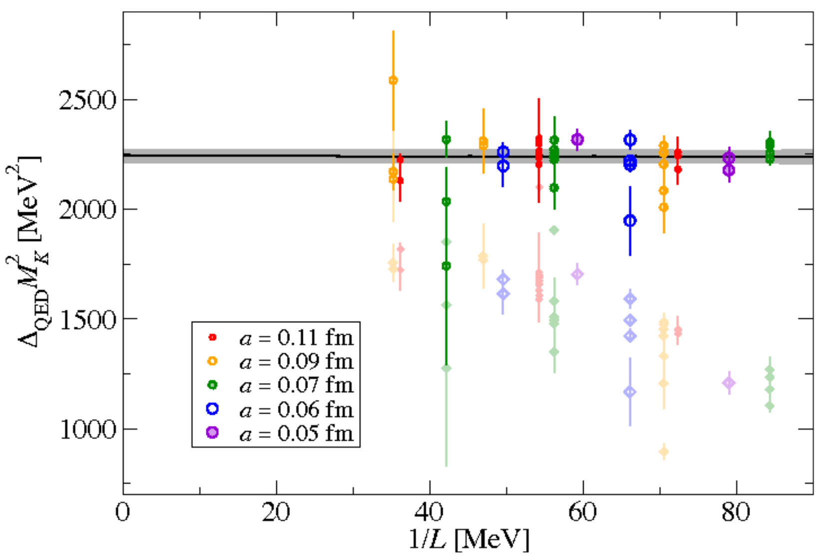
BMW_16 example (1): electroquenched setup

BMW_16 [arXiv:1604.07112, 1702.00309] uses quenched QED$_{TL}$ ($k_\mu = 0$ mode removed) on top of dynamical $N_f = 2+1$ QCD configurations (Symanzik gauge action, $c_{\mathrm{SW}} = 1$, 2 HEX, 5 lattice spacings, M_π^2 down below physical value).

Physics: Tune 5 *parameters* to their physical values, sc. $m_u, m_d, m_s, \alpha_{\mathrm{em}}, \alpha_{\mathrm{st}}$.

Lattice: Tune 5 *observables* to their physical values, e.g. $M_\chi^2 \equiv \frac{1}{2}(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2)$, $M_{\pi^\pm}^2$, $\Delta M_K^2 \equiv M_{K^\pm}^2 - M_{K^0}^2$, α_{em}, scale-quantity (M_Ω or M_Ξ).

Kaon splitting is interpolated to physical values of $M_\chi^2, M_{\pi^\pm}^2, \alpha_{\mathrm{em}}$ by means of the separating ansatz $\Delta M_K^2 = C_K(M_{\pi^\pm}^2, M_\chi^2, a, L)\alpha_{\mathrm{em}} + E_K(M_{\pi^\pm}^2, M_\chi^2, a)\delta m$.

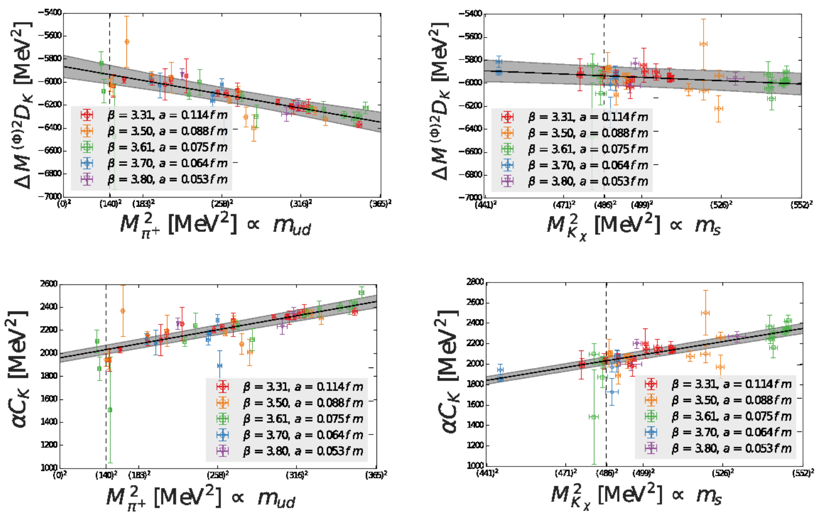


Subtle: In QED$_{TL}$ finite-volume effects are polynomial (no gap), for charged scalar particles first two terms are structure-independent and read

$$\frac{M(L)}{M(\infty)} = -\frac{2.837...}{M(\infty)L}\left[1 + \frac{2}{M(\infty)L}\left(1 - \frac{\pi}{5.674...}\frac{T}{L}\right)\right]$$

BMW_16 example (2): anlysis and results

To avoid dealing with a renormalized δm, the LO relation $\Delta M^2 \equiv M^2_{\bar{u}u} - M^2_{\bar{d}d} = 2B_2\delta m + O(m_{ud}\alpha_{\rm em}, m_{ud}\delta m, \alpha^2_{\rm em}, \alpha_{\rm em}\delta m, \delta m^2)$ is used, where $M^2_{\bar{u}u}, M^2_{\bar{d}d}, B_2$ have been determined before; this gives the more practical relation $\Delta M_K^2 = C_K(M^2_{\pi^\pm}, M^2_\chi, a, L)\alpha_{\rm em} + D_K(M^2_{\pi^\pm}, M^2_\chi, a)\Delta M^2$.



Combining this relation with lattice data and using the physical ΔM_K^2 gives ΔM^2 at the physical point. Since B_2 is known, the latter can be converted to δm. With the separation $\Delta M_K^2 = C_K\alpha_{\rm em} + D_K 2B_2\delta m$ in hand, one can determine the correction to Dashen's theorem.

$$\epsilon = \frac{\Delta_{\rm QED}M_K^2 - \Delta_{\rm QED}M_\pi^2}{\Delta M_\pi^2} \overset{G}{\simeq} \frac{\Delta_{\rm QED}M_K^2 - \Delta_{\rm exp}M_\pi^2}{\Delta_{\rm exp}M_\pi^2} = 0.73(2)_{\rm stat}(5)_{\rm latt}(17)_{\rm qQED}$$

$$\delta m = -2.41(6)(4)(9)\,\text{MeV},\ m_u = 2.27(6)(5)(4)\,\text{MeV},\ m_d = 4.67(6)(5)(4)\,\text{MeV}$$

$$\frac{m_u}{m_d} = 0.485(11)(8)(14),\ R = \frac{m_s - m_{ud}}{m_d - m_u} = 33(1)(1)(1),\ Q = \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2} = 23.4(4)(3)(4)$$

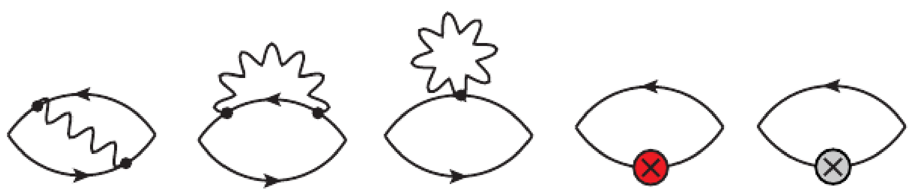
Conceptual challenges (1): Rome-Southampton approach

- The decay amplitude $\pi \to \ell\bar{\nu}$ is *infinite* at $O(\alpha_{\rm em})$ due to IR divergences.

- The physical quantity is the decay rate of $\pi \to \ell\bar{\nu}$ plus an arbitrary number of undetected soft photons (energy below detector resolution ΔE) in the final state, with only one photon at $O(\alpha_{\rm em})$. [Bloch and Nordsieck, Phys.Rev. 52, 54 (1937)]

- The RM123-SOTON collaboration proposes a smart method to split the decay amplitude into a universal perturbative part and a non-universal structure-dependent part. [Carrasco et al., Phys.Rev. D91 (2015) 074506]

Convenient starting point is insertion of both δm and EM-operators in iso-symmetric QCD correlation functions, tantamount to a perturbative expansion in $O(\delta m, \alpha_{\rm em})$, originally proposed by the same people. [Divitiis et al., Phys.Rev. D87 (2013) 114505]



(a) (b) (c) (d) (e)

FIG. 1: *Fermionic connected diagrams contributing at $\mathcal{O}(e^2)$ and $\mathcal{O}(m_d - m_u)$ to the IB corrections to meson masses: exchange (a), self energy (b), tadpole (c), pseudoscalar insertion (d) and scalar insertion (e).*

Conceptual challenges (2): Rome-Southampton approach

Crucial ingredients for RM123-SOTON calculation of $\Gamma(\Delta E)$ with $\Delta E \sim 30\,\mathrm{MeV}$:

- ◦ Finite volume regulates IR divergences.
- ◦ Calculation of the finite structure-dependent part of $\pi \to \ell\bar{\nu} + 0_\gamma$.
- ◦ Structure dependent part of $\pi \to \ell\bar{\nu} + 1_\gamma$ is shown to be negligible.
- ◦ Universal part of $\pi \to \ell\bar{\nu}[\gamma]$ calculated analytically in $1/L$ and $\log(L)$.
- ◦ Finite-volume corrections to structure dependent part vanish like $1/L^2$.
- ◦ First numerical data available [J.Phys.Conf.Ser. 800 (2017) 012005].

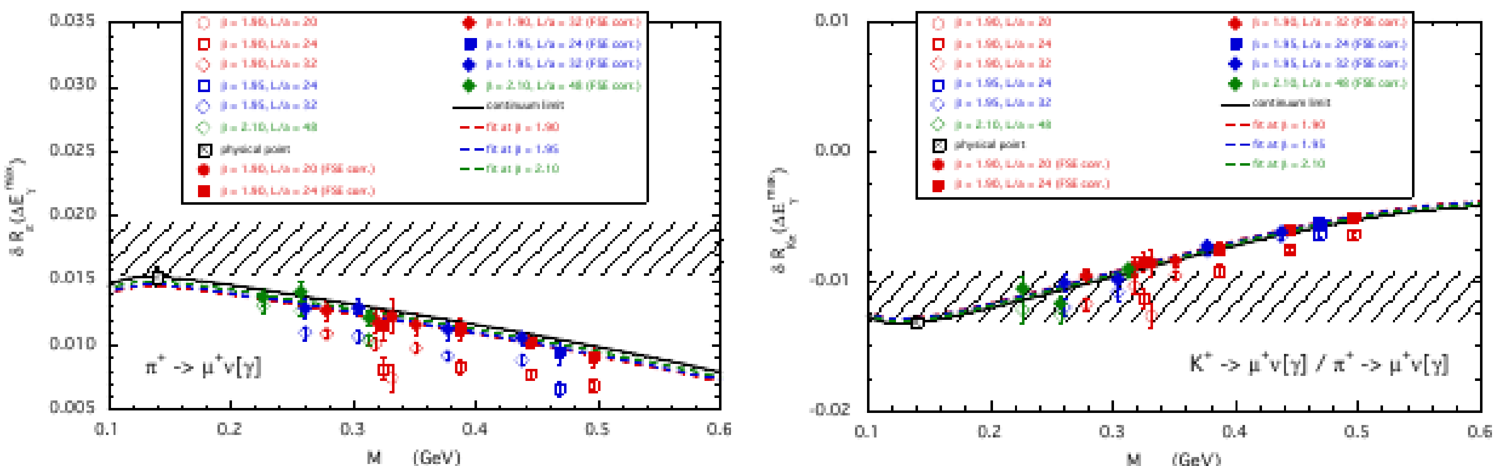


Figure 3: *Results for the corrections δR_π (left panel) and $\delta R_{K\pi} \equiv \delta R_K - \delta R_\pi$ (right panel) obtained after the subtraction of the "universal" FV terms computed in perturbation theory [7]. The full markers correspond to the lattice data corrected by the residual FV corrections obtained using the fitting function [12] including the chiral log. The dashed lines represent the results in the infinite volume limit at each value of the lattice spacing, while the solid lines are the results in the continuum limit. The crosses represent the values δR_π^{phys} and $\delta R_{K\pi}^{phys}$ at the physical point. The shaded areas correspond respectively to the values 0.0176(21) and −0.0112(21) at 1-sigma level, obtained using ChPT (see Refs. [10, 11]).*

Summary from FLAG

For each quantity FLAG provides [see http://itpwiki.unibe.ch/flag]:

- complete list of references
- summary of essential ingredients of each study (N_f, action, ...)
- averages for "mature" quantities (separately for each N_f)
- pressure on reader to *cite original papers*

FLAG_10 [arXiv:1011.4408, Eur.Phys.J. C71 (2011) 1695] covers:

1. light quark masses m_{ud}, m_s
2. V_{us} and V_{ud} via decay constants and form factors
3. chiral low-energy constants (LECs)
4. kaon bag parameter B_K

FLAG_13 [arXiv:1310.8555, Eur.Phys.J. C74 (2014) 2890] in addition:

5. D-meson decay constants and form factors
6. B-meson decay constants, form factors, and mixing parameter
7. strong coupling constant

FLAG_16 [arXiv:1607.00299, Eur.Phys.J. C77 (2017) 112] provides update.

- **FLAG_16 color coding**

Chiral extrapolation:

★ $M_{\pi,\mathrm{min}} < 200$ MeV
○ $200\,\mathrm{MeV} \leq M_{\pi,\mathrm{min}} \leq 400\,\mathrm{MeV}$
■ $M_{\pi,\mathrm{min}} > 400$ MeV

Continuum extrapolation:

★ 3 or more lattice spacings *and* $(a_{\mathrm{max}}/a_{\mathrm{min}})^2 \geq 2.0$
○ 2 or more lattice spacings *and* $(a_{\mathrm{max}}/a_{\mathrm{min}})^2 \geq 1.4$
■ otherwise

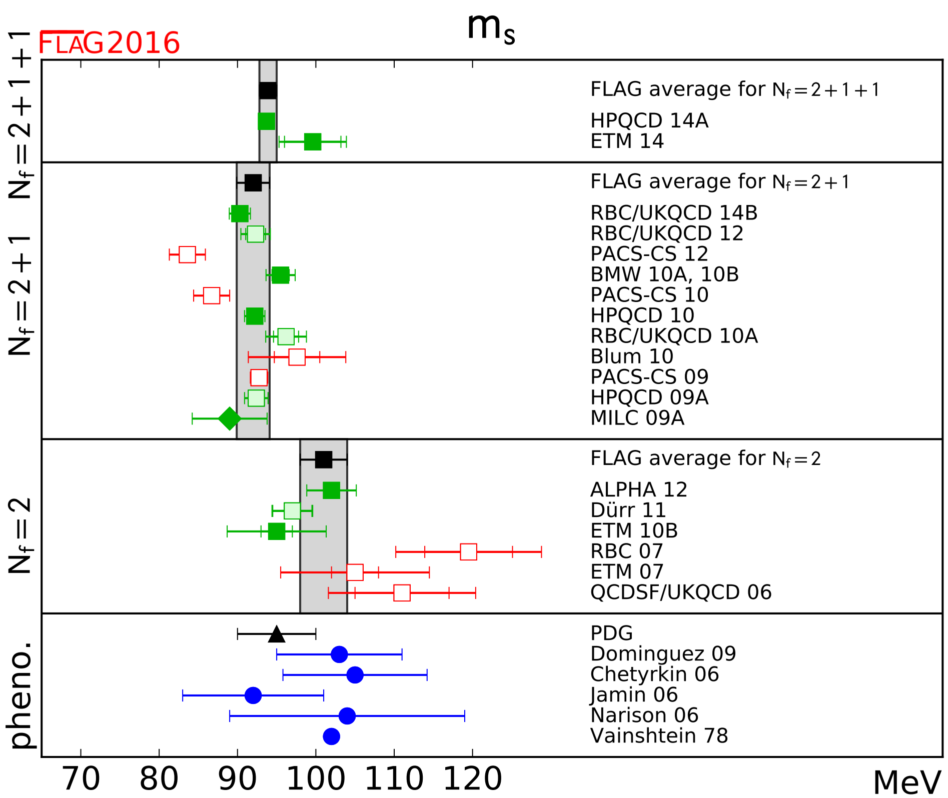
Finite-volume effects:

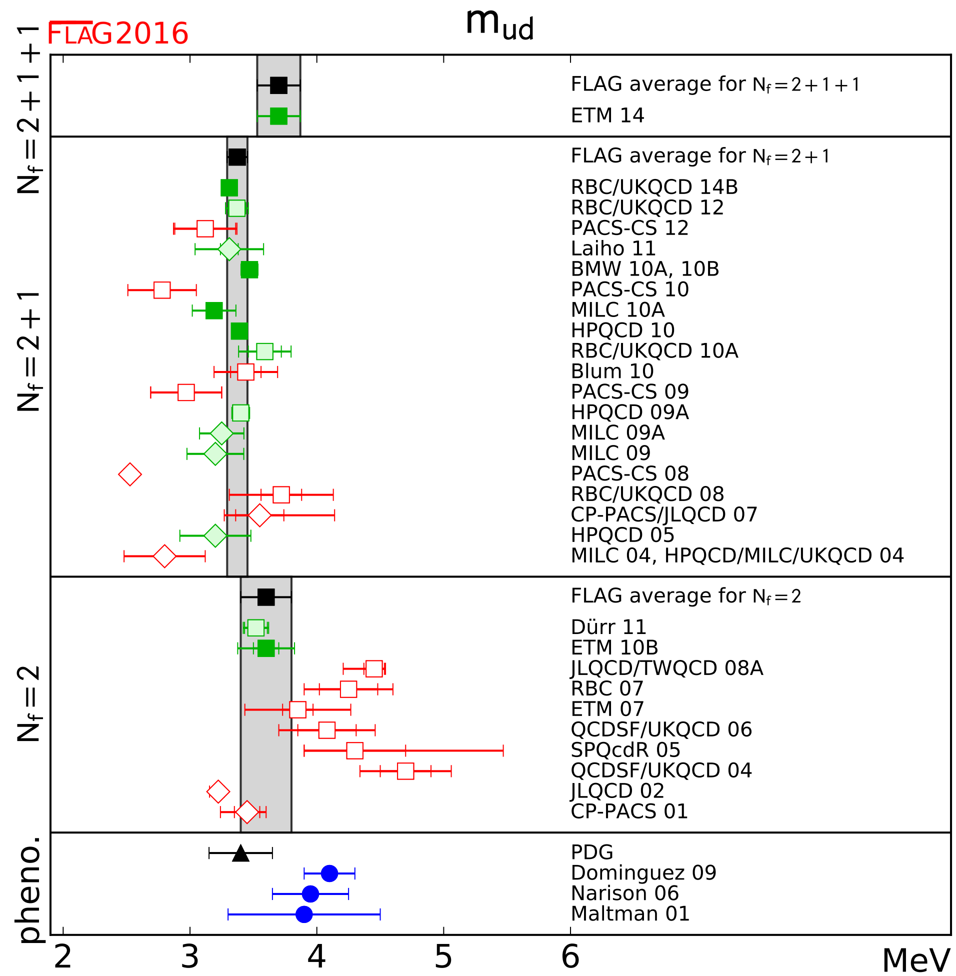
★ $[M_{\pi,\mathrm{min}}/200\,\mathrm{MeV}]^2 \cdot \exp(4 - M_{\pi,\mathrm{min}} L_{\mathrm{max}}(M_{\pi,\mathrm{min}})) \leq 1$
○ $[M_{\pi,\mathrm{min}}/200\,\mathrm{MeV}]^2 \cdot \exp(3 - M_{\pi,\mathrm{min}} L_{\mathrm{max}}(M_{\pi,\mathrm{min}})) \leq 1$
■ otherwise

Renormalization (where applicable):

★ non-perturbatively
○ any-loop perturbation theory with reasonable error estimate
■ otherwise

- **FLAG compilation (iso-symmetric)**





- strong inconsistencies (for given N_f) only among red results
- all green points (for given N_f) reasonably consistent
- m_{ud}^{phys} and m_s^{phys} depend only mildly on N_f; ratio essentially N_f-independent
- reasonable consistency with non-lattice results

- **FLAG compilation (regarding ϵ)**

Some works use parameterizations of mass splittings which constrain $\epsilon_{\pi^0}, \epsilon_{K^0}, \epsilon_m, \epsilon$.

* DuncanEichtenThacker_96, within $N_f = 0$ QCD, use a parameterization which amounts to $\epsilon_m = 0$ ab initio. FLAG converts their findings to $\epsilon = 0.50(8)$.

* RBC_07, within electro-quenched $N_f = 2$ QCD, fixes the quark masses via the physical masses of $\pi^0, K^\pm, K^0$; they find the rather small value $M^\gamma_{K^\pm} - M^\gamma_{K^0} = 1.443(55)\,\mathrm{MeV}$. Blum_10, an update in electro-quenched $N_f = 2+1$ QCD, finds $M^\gamma_{K^\pm} - M^\gamma_{K^0} = 1.87(10)\,\mathrm{MeV}$. The latter value corresponds to $\epsilon = 0.5(1)$.

* MILC sets $\epsilon_{\pi^0} = \epsilon_{K^0} = \epsilon_m = 0$ and $\epsilon = 1(1)$ or $\epsilon = 1.2(5)$ in MILC_04, MILC_09. Subsequently, MILC determines this coefficient, finding $\epsilon = 0.81(5)(18)$.

* BMW determines ϵ in electro-quenched $N_f = 2+1$ QCD, finding $\epsilon = 0.57(6)(6)$.

* RM123, using operator insertions in $N_f = 2$ QCD, finds $\epsilon = 0.78(18)(18)$.

* QCDSF, using dynamical QCD+QED, finds $\epsilon = 0.50(6)$ [one a only, stat. error].

* $\eta \to 3\pi$ decays, along with dispersive techniques, yield $\epsilon = 0.70(28)$.

- **FLAG recommendation (regarding ϵ)**

1. FLAG considers the following values a sound summary of all available information

$$\epsilon_{\pi^0} = 0.07(7)\,,\ \epsilon_{K^0} = 0.3(3)\,,\ \epsilon_m = 0.04(2)\,,\ \underline{\epsilon = 0.7}(3)$$

2. FLAG starts from the LO formula in ChPT

$$\frac{m_u}{m_d} \stackrel{\text{LO}}{=} \frac{\hat{M}^2_{K^\pm} - \hat{M}^2_{K^0} + \hat{M}^2_{\pi^\pm}}{\hat{M}^2_{K^0} - \hat{M}^2_{K^\pm} + \hat{M}^2_{\pi^\pm}}$$

which, upon using the physical masses and linearizing the corrections, turns into

$$\frac{m_u}{m_d} \stackrel{\text{LO}}{=} 0.558 - \underline{0.084\epsilon} - 0.02\epsilon_{\pi^0} + 0.11\epsilon_m$$

and hence amounts to $m_u/m_d = 0.50(3)$ in QCD (or in $\overline{\text{MS}}$ at $2\,\text{GeV}$).
Accordingly $m_{ud} \simeq 3.5\,\text{MeV}$ is split into $m_u \simeq 2.3\,\text{MeV}$ and $m_d \simeq 4.7\,\text{MeV}$.

3. This rules out $m_u = 0$ *within QCD* by 17σ, no matter how elegant it would be ...

Summary

- In ChPT isospin-breaking in pion mass is quadratic, $M_{\pi^\pm}^2 - M_{\pi^0}^2 \propto (m_d - m_u)^2$, while such breakings scale $\propto m_d - m_u$ in general (e.g. in nucleon mass).
- In ChPT electromagnetic isospin-breakings affect only the charged mesons, i.e. $M_{\pi^\pm}^\gamma = M_{K^\pm}^\gamma > 0$ and $M_{\pi^0}^\gamma = M_{K^0}^\gamma = 0$ ["Dashen's theorem" $\leftrightarrow \epsilon = 0$].
- The QCD/QED-isospin-breakup is, in general, scheme and scale-dependent. In joint theory only same-charge mass-ratios are RGI quantities, e.g. m_d/m_s.
- Reasonably mature lattice calculations suggest $\epsilon = 0.7(3)$, i.e. sizable corrections, see e.g. FLAG3, Eur.Phys.J. C77 (2017) 112 [arXiv:1607.00299].
- Conceptual issues with decay constants; many interesting strategies and results, see e.g. A.Patella, PoS LATTICE2016 (2017) 020 [arXiv:1702.03857].

Thanks to the BMW collaboration and the FLAG consortium.