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Mathematical and Numerical Methods for the Nonlinear Hyperbolic Propagation Problem

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A Preliminary Treatment of Collective Space Charge Effects in Accelerator Physics

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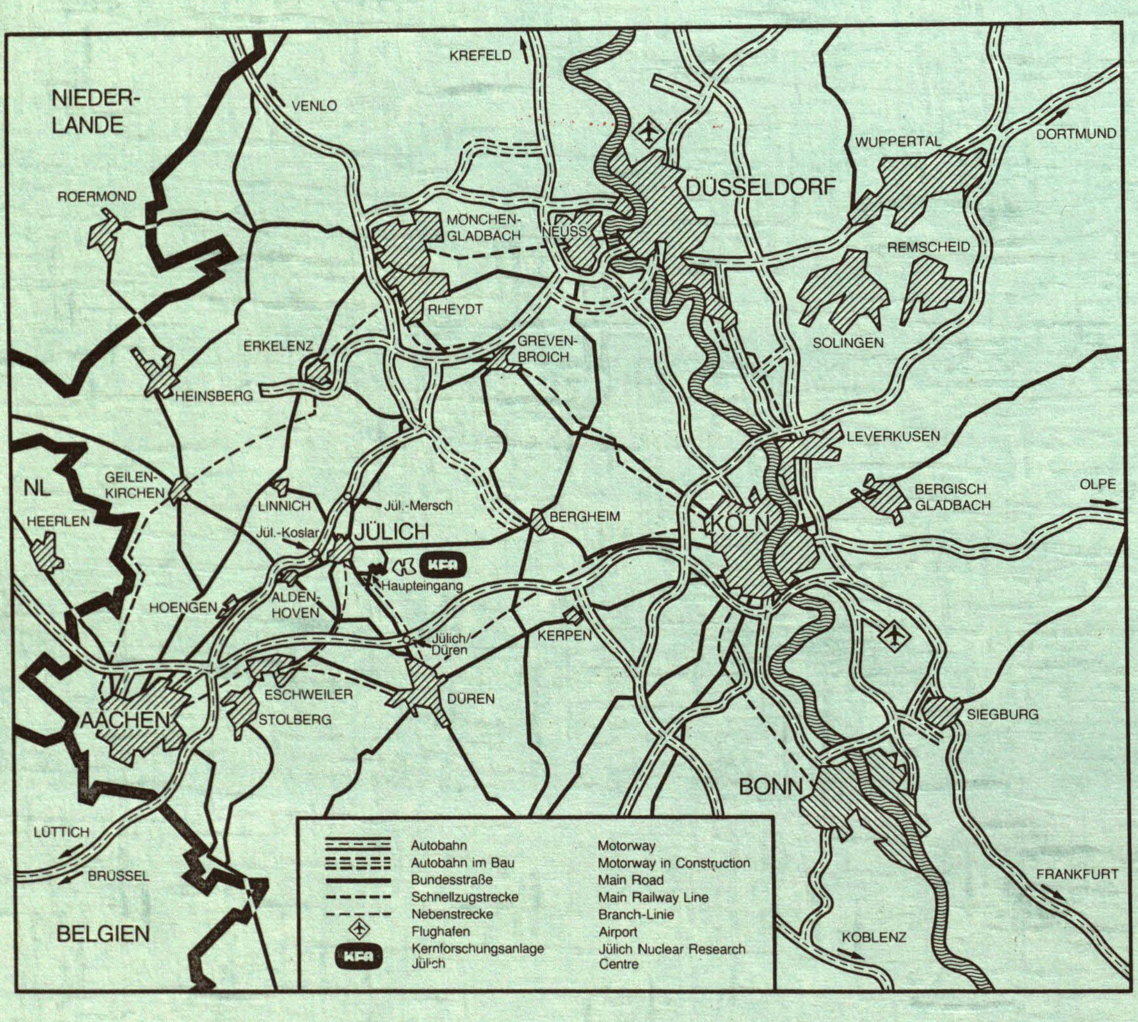

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Abstract

In the first part of this report a physical model is presented, which describes the deforming of a bunch in a storage ring influenced only by its own space charge field. A system of two differential equations for the density and the momentum of the particles is set up, which is independent of any special machine parameter. Due to the sign of the inductance of the chamber walls and the sign of the dispersion of the revolution frequency, we distinguish between a de-bunching and a self-bunching situation. The de-bunching corresponds to a nonlinear hyperbolic propagation problem well-known in gas dynamics, and the self-bunching to a nonlinear elliptic initial value problem.

The second part deals with a mathematical and numerical treatment of an approximate equation for the hyperbolic case. For this nonlinear second order partial differential equation we first present three particular integrals: the solution by separating the variables, the similarity solution, and the solution for a parabolic initial distribution of the density. For a more realistic initial condition, we must resort to other methods: Results are obtained in three different ways, first from a highly accurate Taylor series expansion, second from a common finite difference method, and thirdly from the numerical method of characteristics. The appearance of a shock discontinuity is furthermore established in each of these cases.

Contents

Introduction

This report consists of two parts, the first giving a theoretical derivation of two equations describing space charge effects in storage rings, and the second presenting mathematical as well as numerical methods for solving one of these equations, when a certain approximation is introduced.

In Part I we propose a physical model which accounts for the propagation in space and time of a cloud of charged particles (the bunch) in a storage ring when external fields are not present. Therefore only self and mirror fields act on the particles. First the equations of motion which determine the single particle dynamics are derived. Here, as a quite common basic assumption, we use a linear dependence of the potential as a function of the density. Also for the relative particle velocity in the bunch we mainly assume a linear momentum dependence.

The particle bunch dynamics will then be described by proceeding from the Vlasov equation. Conditions are established which allow to set up a continuity equation for the line density. Combined with the equations for the velocity and the momentum change, a system of two nonlinear first order partial differential equations for the density and momentum is deduced. This system depends on two machine parameters: the scaling factor ϕ_o of the potential, which takes into account the kind of interaction of the particles and the surrounding medium, and the constant η_o which measures the energy as deviation from the transition energy.

The signs of ϕ_o and η_o decide whether a de-bunching or a self-bunching will occur. We thus have to distinguish between two scaled pairs of equations, the first being given by

$$\frac{\partial m}{\partial t} = - \frac{\partial \rho}{\partial z} - m \frac{\partial m}{\partial z} ,$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial z} (\rho m) \quad ,$$

where m and ρ are the momentum and the density of the particles.

This system is hyperbolic and describes the de-bunching, whereas the second system,

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial z} - m \frac{\partial m}{\partial z} ,$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial z} (\rho m) ,$$

is elliptic and treats the self-bunching case. Otherwise both systems are free of any machine parameter.

The hyperbolic system is well-known in gas dynamics where the one dimensional isentropic flow of a polytropic gas with an adiabatic exponent $\gamma=2$ obeys the same set of equations [13]. Also the so-called "shallow water" theory uses these equations [14].
The elliptic system is new and very unusual, as elliptic initial value problems are in general not considered. At present we have only been concerned with the hyperbolic case, and the results given in this report even deal only with an approximation to it.

Introducing a scaled action integral $\bar{S}$, a nonlinear hyperbolic partial differential equation of second order can be set up from the first system, which has the form

$$\frac{\partial}{\partial t} \left\{\frac{\partial \bar{S}}{\partial t} + \frac{1}{2} \left(\frac{\partial \bar{S}}{\partial z}\right)^2\right\} = - \frac{\partial}{\partial z} \left\{\frac{\partial \bar{S}}{\partial t} \frac{\partial \bar{S}}{\partial z} + \frac{1}{2} \left(\frac{\partial \bar{S}}{\partial z}\right)^3\right\} .$$

This equation is the starting-point for an equation which approximates the scaled action integral and which will be discussed in Part II.

If in the equation for $\bar{S}$ we neglect the second terms on both sides, we obtain the nonlinear second order differential equation

$$\frac{\partial^2 \Gamma}{\partial t^2} = \frac{\partial}{\partial z} \left\{\frac{\partial \Gamma}{\partial t} \frac{\partial \Gamma}{\partial z}\right\} ,$$

where $-\Gamma$ is the approximation for the scaled action integral $\bar{S}$. This equation will be the subject of a detailed investigation in Part II. First a few remarks seem suitable in order to justify the extensive treatment of a rather restrictive approximation.

Obviously the equation for Γ has a much simpler form than the equation for $\bar{S}$, and thus facilitates considerably the discussion. Yet many qualitative features are retained, as can be observed from a number of results obtained from the equation for $\bar{S}$, which is under investigation at present. In the following we shall point out the analogies or similarities, but also the differences, for both equations, whenever this seems to be appropriate.

Irrespective of the particular type, both equations may serve to test different numerical methods and algorithms designed to solve hyperbolic initial value problems. This will be done to a large extent in Chapters 3 and 4 using the equation for Γ .

Quite generally, however, the detailed investigation of a simple, but by no means trivial, nonlinear second order partial differential equation such as the one for Γ deserves, from a purely theoretical point of view, its own right.

Only very few nonlinear partial differential equations of second order possess a general solution. Since an equation very similar to the one for Γ does have a general solution, it was tempting to look for the general solution of the equation for Γ (see Appendix A2). A standard technique known from gas dynamics can be applied to set up an equation for the time as a function of the Riemann invariants, the so-called hyperbolic normal form. This equation reveals such a degree of difficulty that to discover the general solution (if this exists at all) seems to be impossible.

This is the only occasion where a totally different situation is encountered when the same procedure is applied to the equation for $\bar{S}$. The hyperbolic normal form has a surprisingly simple structure given by the Poisson-Euler-Darboux equation [17]

$$t_{\xi\eta} + \frac{\mu}{\xi+\eta} (t_\xi + t_\eta) = o \ ,$$

where ξ and η are the Riemann invariants. For integer values of μ the general solution is known. In our case we have $\mu = 3/2$, which is not soluble but still allows to construct the Riemann function so that the Riemann integration formula can be applied [24].

Since the general solution is not available, any particular integral of the equation for Γ will be of great value since it leads to a deeper insight into the variety of solutions admitted by this equation. In Chapter 1 three such integrals are presented which satisfy the initial conditions

$$\Gamma(z,o) = o \ ,$$

$$\Gamma_t(z,o) = f(z) \ ,$$

where $f(z)$ is even, and otherwise depends on the actual integral.

In particular, the solutions can be characterized by one of the following forms:

a) separation of variables,
b) the similarity solution,
c) parabolic initial distribution $f(z) = a - bz^2$.

When these three types of solution are applied to the equation for $\bar{S}$, it follows that

a) separating the variables now leads to the restricted z dependence $\bar{S}(z,t) = \varphi(t)z^2$, where the equation for $\varphi(t)$ can no longer be solved explicitly,
b) the similarity solution possesses the same similarity behaviour $\bar{S}(z,t) = t^2\psi(x)$ with $x = tz^{-2/3}$,
c) the parabolic case can also be solved analytically, except that now the solution is given in implicit form.

On the whole we observe that the differences are of quantitative nature only. Of special interest will be the singular behaviour of the three types of solution.

If one prescribes more realistic initial distributions $f(z)$, one can no longer expect to find any analytic results. For this study, the case where $f(z)$ represents a Gaussian distribution, will have great relevance and will be considered exclusively in all other chapters.

In Chapter 2 a powerful method is derived which allows to obtain results via a Taylor series expansion. This series which is based on the Gaussian initial distribution yields highly accurate results whenever it is convergent - in fact the full accuracy in fourfold precision (about 31 decimal places for IBM) could be achieved in a large region of the (t,z) plane - but it breaks down along a curve $\zeta(t)$ due to a singularity of the solution at a certain point $\zeta(t_s)$, where t_s is the time for the onset of a shock discontinuity.

In order to obtain results in that range of the (t,z) plane, where the Taylor series no longer converges, we have applied a simple discretization method as explained in Chapter 3. Although this method is known to be unstable, we have obtained very accurate results, which coincide with those of the Taylor series whenever available. Around t_s this solution begins to oscillate and eventually will break down. On the other hand, a stable method, which introduces numerical diffusion, yields smooth results also beyond t_s, but the accuracy is very poor for all times.

Since the differential equation for Γ is hyperbolic, the most natural access to solve the initial value problem numerically consists in the method of characteristics, as e.g. realized in the Massau construction. In Chapter 4 the power inherent in the availability of characteristics will be demonstrated for the present equation. As the outstanding result we only mention the location of the shock discontinuity. This singularity also occurs when the equation for $\bar{S}$ is solved, except that the time for break-down is at about half the value for t_s.

PART I

Deforming of a Bunch Due to Space Charge Effects

A Collective Phenomenon in Accelerator Physics

I.1 Motivation

A main part of recent accelerator developments is directed to increase the current of the beam. High intense accelerators are of special interest for example in material irradiation facilities, for medical irradiation projects, to study heavy ion fusion, for spallation neutron sources and military applications.

By increasing the current of a particle beam, the collective phenomena of space charge effects can no longer be ignored. In addition to the conventional accelerator problems, the difficulties occurring from space charge fields must be understood in order to construct high intense machines.

As a contribution to the proposal for the German spallation neutron source (SNQ), we considered the time development of the position and momentum distribution of a bunch in a compressor ring (IKOR) [1-3]. In this rather new kind of study, only the influence of the longitudinal electric field due to self and mirror fields is taken into account. As proposed in the design study for the compressor ring, we exclude any additional external longitudinal electric field. Using the general tools of fluid dynamics, we study a nonlinear hyperbolic differential equation.

However, space charge fields are not only a disturbance in conventional accelerator physics, these fields can also be exploited for a completely new kind of machines: the collective ion accelerator. In conventional accelerators the accelerating fields are built up in the structures external to the beam path and are limited by electric break-down at the accelerating gap. Collective accelerators promise the possibility of a compact, very high gradient, strongly focused, high energy ion acceleration. A short overview "Recent Advances in Collective Ion Acceleration" with further references was given recently by M. Reiser [4].

I.2 General aspects

Particles with equal charge, mass and energy can be kept on an orbit by the use of a storage ring [5,6]. For example, we may discuss the situation of a particle bunch of about $5 \cdot 10^9$ protons with an energy of 1 GeV circulating on an orbit with a radius of 3o m . A typical transverse dimension of such a beam is about three orders of magnitude smaller than the radius of the orbit. The particles are initially concentrated in a bunch of about o.1 m length.

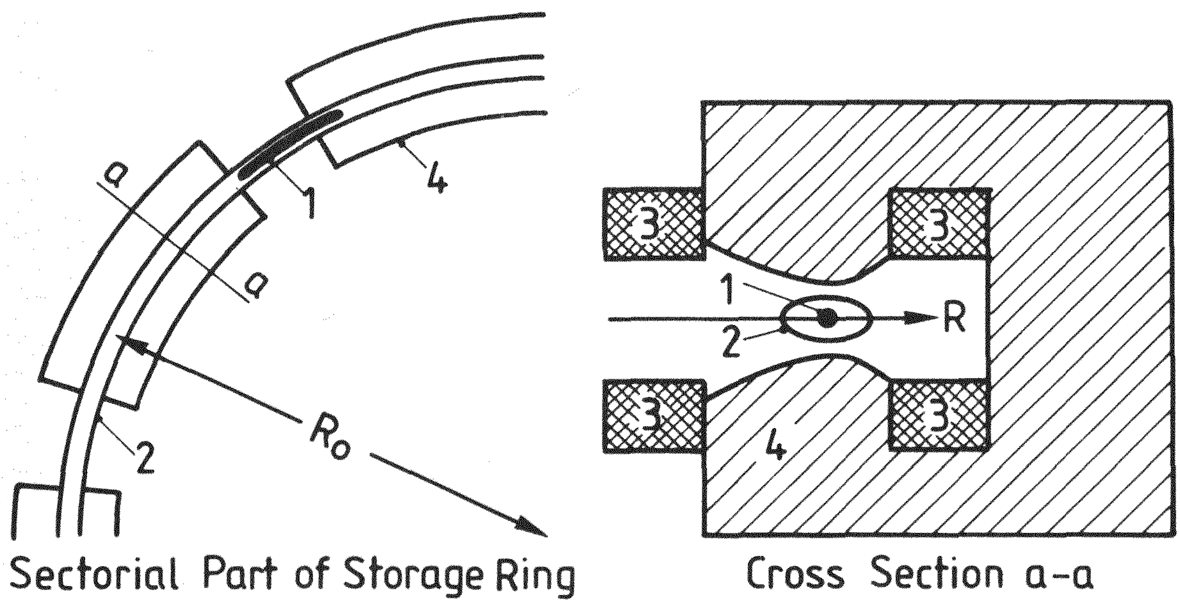

Fig. I : Sketch of a storage ring with a particle bunch (1: particle bunch, 2: vacuum chamber, 3: magnetic coils, 4: magnets for guide field).

The particle cloud is subject to internal and external electric and magnetic forces, which are acting on the particles and changing their relative position. In more detail we want to study the influence of the electric forces, due to self fields of the particle cloud and mirror fields of the chamber wall. The superposition of these two electric fields will be called "space charge field"[1]. The goal of this report is to discuss the behaviour of such a particle beam, especially the time dependence of the density and the momentum distribution. As a concept we use a continuously smoothed density distribution to describe the cloud. This means that we ignore a short range particle-particle interaction and consider only a collective Coulomb field (the space charge field), which behaves like an external field.

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[1] Even some authors restrict this label to the former fields.

I.3 Single particle dynamics

It is convenient to describe the particle momentum p by the variable $m=\Delta p/p_o$, where p_o is the nominal longitudinal momentum value and Δp a longitudinal momentum deviation with

$$p = p_o(1+m) \ .$$

As far as possible, all values which depend on p , are related to m .

In a similar way we use the variable $\Delta\Theta/\Theta_o$ to describe the particle position

$$\Theta = \Theta_o(1+\Delta\Theta/\Theta_o) \ .$$

The angle Θ_o is always oriented to the center of the bunch, and the yield of a position deviation from the center is given by $\Delta\Theta$.

Instead of using $\Delta\Theta/\Theta_o$, which should be defined in the laboratory system, we mainly shall use the variable z , defined in the bunch system, as

$$z = \Delta\Theta \ R_o \ \gamma_o \ ,$$

where γ_o is the relativistic parameter and R_o is the mean radial position measured from the center of the machine for a particle with the nominal momentum p_o (see also Fig. I).

I.3.1 The influence of the magnetic field

As a first step we line out the influence of the magnetic guide field which keeps the particle on a closed circular orbit, at least in a first approximation. All these circles have the same center, and the radius of each orbit is, for a given ring structure, only defined by the particle momentum m . Using this concept, we ignore any oscillation around the mean particle path, which is described by the betatron oscillation ([5], pp. 14).

The radial position

$$R = R_o(1+\Delta R/R_o)$$

and the particle momentum m are related via the fundamental equation ([5], p. 14o)

$$\frac{\Delta R}{R_o} = \alpha_1 m ,$$

where α_1 is the "momentum compaction factor" and is defined only by parameters of the storage ring.

Similarly, the orbit length L changes with the momentum as

$$\frac{\Delta L}{L_o} = \alpha_1 m(1+\alpha_2 m) ,$$

where α_2 is an additional parameter [7].

For the revolution frequency $\Delta\omega/\omega_o$, the momentum dependence is given in second order approximation by [7]

$$\frac{\Delta\omega}{\omega_o} = \frac{d}{d\tau}\,\frac{\Delta\Theta}{\omega_o} = \eta_o m - \eta_1 m^2 ,$$

where

$$\eta_o = \frac{1}{\gamma_o^2} - \frac{1}{\gamma_{tro}^2} ,$$

$$\eta_1 = \frac{\alpha_2}{\gamma_{tro}^2} + \frac{\eta_o}{\gamma_{tro}^2} + \frac{3}{2}\,\frac{\beta_o^2}{\gamma_o^2} .$$

Here τ is the time in the laboratory system, and we shall call t the time in the bunch system $(t=\tau/\gamma_o)$.

The relativistic parameters β_o , γ_o are related to the nominal momentum value $m=o$. Between the "transition energy" γ_{tro} and the momentum compaction factor α_1 there exists the relation

$$\gamma_{tro} = \frac{1}{\sqrt{\alpha_1}} .$$

The second order approximation of $\Delta\omega/\omega_o$ becomes important, if the machine works close to the transition energy ($\gamma_o = \gamma_{tro}$, or $\eta_o = o$), as proposed for the compressor ring [1].

Depending on the sign of $\Delta\omega/\omega_o$ (and due to the sign of the space charge field), it is possible to have a self-bunching or a de-bunching of the particle cloud.
From the $\Delta\omega/\omega_o$ relation we can derive the relative particle speed inside the bunch,

$$v = \frac{dz}{dt} = \beta_o c \gamma_o^2 \frac{\Delta\omega}{\omega_o} = \beta_o c \gamma_o^2 \,[\eta_o m - \eta_1 m^2] \ .$$

Only the velocity in z direction is taken into account.

I.3.2 <u>The influence of the electric field</u>

The longitudinal electric space charge field influences the particle momentum as ([8], §17, pp. 46)

$$\frac{dm}{dt} = - \frac{e}{p_o} \frac{\partial \phi}{\partial z} - \frac{e}{p_o} \frac{1}{c} \frac{\partial A}{\partial t} \ .$$

The two potentials are defined as (G: Green function)

$$A = \int \frac{v}{c} \tilde{\rho} \ GdV' \quad \text{for the vector potential,}$$

$$\phi = \int \tilde{\rho} \ GdV' \quad \text{for the scalar potential.}$$

The volume density $\tilde{\rho}$ describes the particle distribution in the transverse and z directions.

Using these expressions for the potentials we get

$$\frac{dm}{dt} = - \frac{e}{p_o} \int G \left[\frac{1}{c^2} \frac{\partial}{\partial t} (\tilde{\rho} v) + \frac{\partial \tilde{\rho}}{\partial z'} \right] dV' .$$

Because inside the bunch the relative velocities v should be small compared with the speed of light c, the relation

$$\left| \int G \frac{\partial}{\partial t} (\tilde{\rho} v) \ dV' \right| << c^2 \left| \int G \frac{\partial \tilde{\rho}}{\partial z'} dV' \right|$$

should hold. Therefore we skip the term due to the vector potential and finally find

$$\frac{dm}{dt} = - \frac{e}{p_o} \frac{\partial \phi}{\partial z} \ .$$

The term $-e\partial\phi/\partial z$ is just the longitudinal space charge force in the bunch system (see Fig. II). We do not discuss here the

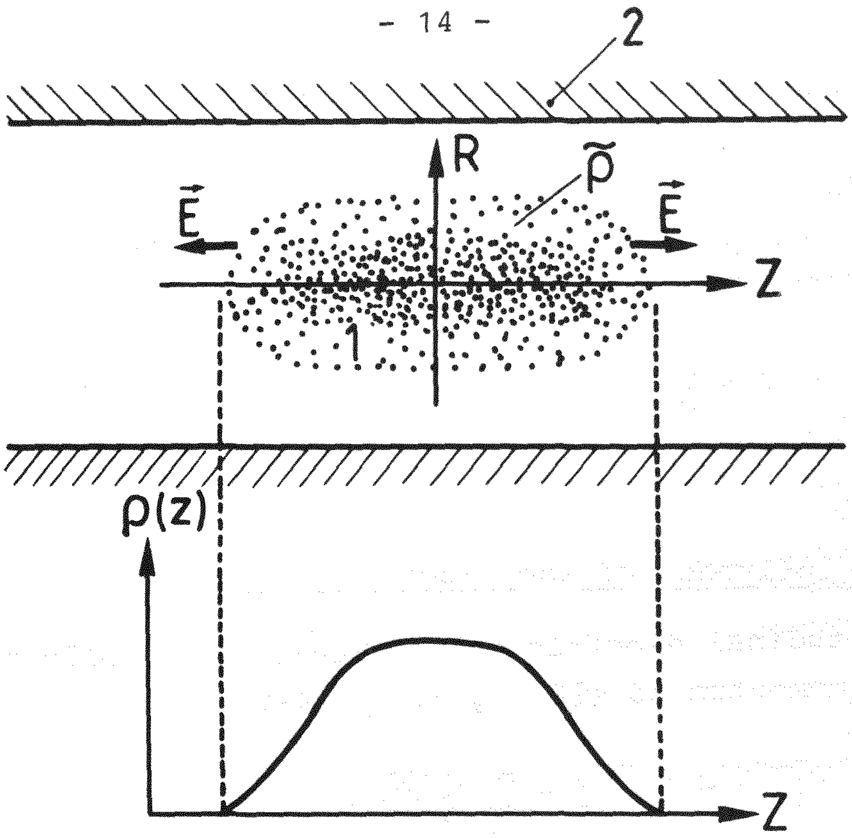

Fig. II : Particle cloud (1) inside a conductive chamber (2) and the density distribution $\tilde{\rho}$. Here $\vec{E}$ indicates the direction of the electric field and $\rho(z,t)$ the line density of the bunch.

case that ϕ includes the additional, external RF field [9].

For the potential ϕ we shall use an expression which holds as long as the typical bunch length z_B is large compared to the diameter 2b of the chamber $(z_B^2 \gg b^2)$. In the case of a circular superconducting chamber we get a linear relation between ϕ and ρ (see for example [2]),

$$\phi = \phi_o \rho = \phi_o \lambda / \gamma_o \ ,$$

where ρ and λ are the line densities in the bunch and in the laboratory system, respectively (see Section I.4).

Looking at a coaxial uniform beam of radius a , the potential ϕ_o on the beam center is given by [1o] (for more realistic walls, like elliptic cross sections, see [3])

$$\phi_o = \frac{Q}{2\pi} Z_o \ c \left[\frac{1}{2} + \ln \frac{b}{a}\right] ,$$

with $Z_o = \frac{1}{\varepsilon_o c} \cong 377\,\Omega$ and Q the total charge of the bunch.

I.3.3 The action function

Connecting the space charge force with the particle velocity, we get

$$\frac{dm}{dt} = \frac{\partial m}{\partial t} + v\,\frac{\partial m}{\partial z} = -\,\frac{e}{p_o}\,\frac{\partial \phi}{\partial z}\ ,$$

or

$$\frac{\partial m}{\partial t} = -\,\frac{\partial}{\partial z}\left[\frac{e}{p_o}\,\phi + \beta_o c \gamma_o^2 m^2 \left(\frac{\eta_o}{2} - \frac{\eta_1}{3}\,m\right)\right] .$$

From this result we can generate a kind of potential function, which is just the action function ([8], §16, pp. 44)

$$S = -\int_o^t \left[\frac{e}{p_o}\,\phi + \beta_o c \gamma_o^2 m^2 \left(\frac{\eta_o}{2} - \frac{\eta_1}{3}\,m\right)\right] dt' + \int_{z_o}^{z} m(z',t{=}o)\,dz'\ ,$$

and we choose as an initial condition $m(z,t{=}o) = o$.

The gradient of this function is just the momentum variable

$$\frac{\partial S}{\partial z} = m = -\int_o^t \left[\frac{e}{p_o}\,\frac{\partial \phi}{\partial z} + v\,\frac{\partial m}{\partial z}\right] dt'\ ,$$

and the time derivative is the Hamiltonian function for one particle

$$-\,\frac{\partial S}{\partial t} = H = \frac{e}{p_o}\,\phi + \beta_o c \gamma_o^2 m^2 \left(\frac{\eta_o}{2} - \frac{\eta_1}{3}\,m\right) .$$

From the Hamiltonian we get the further relations

$$\frac{\partial H}{\partial z} = \frac{e}{p_o}\,\frac{\partial \phi}{\partial z} = -\,\frac{dm}{dt}\ ,$$

$$\frac{\partial H}{\partial m} = \beta_o c \gamma_o^2 m(\eta_o - \eta_1 m) = \frac{dz}{dt}\ .$$

These relations between the two canonical variables will be used in the next section.

I.4 Particle bunch dynamics

I.4.1 The integrated Vlasov equation

In order to describe the development of the bunch, we introduce the density distribution in the (z,m) phase space of the bunch,

$$\psi = \psi(t,z,m) \ .$$

From this function we get the line density ρ by integrating over the m variable

$$\rho(z,t) = \int \psi \, dm \ ,$$

with the property $\int \rho \, dz = 1$. The density ρ describes the projection of the particle distribution onto the z axis.

The time development of the function ψ is as usual expressed by the Vlasov equation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{dz}{dt}\frac{\partial \psi}{\partial z} + \frac{dm}{dt}\frac{\partial \psi}{\partial m} = o \ .$$

By integrating the Vlasov equation over the m variable we get

$$\int \frac{d\psi}{dt} \, dm = \int \left[\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial z} \left(\frac{dz}{dt} \psi \right) - \psi \frac{\partial}{\partial z} \frac{dz}{dt} + \frac{\partial \psi}{\partial m} \frac{dm}{dt} \right] dm$$

$$= \int \left[\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial z} \left(\frac{dz}{dt} \psi \right) \right] dm - \int \left[\psi \frac{\partial}{\partial z} \frac{\partial H}{\partial m} + \frac{\partial \psi}{\partial m} \frac{\partial H}{\partial z} \right] dm \ .$$

By partial integration and interchange of the partial derivatives with respect to m and z , the second integral is found to be zero and we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} \int \frac{dz}{dt} \psi \, dm = o \ .$$

To simplify the velocity dependent integral, we cut the bunch at a given point (z,t) along the m axis. Approximately we assume that we have always a symmetric distribution ψ around a point $m = \bar{m}(z,t)$,

$$\psi(\mu) = \psi(-\mu) \ ,$$

where $\mu = m - \bar{m}$ is the deviation from the mean value $\bar{m}$.

To each value of $\bar{m}(z,t)$ we can relate an averaged velocity $\bar{v}(z,t)$ via

$$\frac{dz}{dt} = \bar{v}(z,t) + s(\mu,z,t)$$

with

$$\bar{v} = \beta_o' c\gamma_o^2 \bar{m}(\eta_o - \eta_1\bar{m}) \ ,$$

$$s = \beta_o c\gamma_o^2 \mu[\eta_o - \eta_1(2\bar{m} + \mu)] \ .$$

Now the integrated Vlasov equation becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho\bar{v}) + \frac{\partial}{\partial z}\int s\psi d\mu = o \ .$$

Because of the symmetry property of ψ , only the μ^2 term stays in the integral, leading to

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho\bar{v}) - \eta_1\beta_o c\gamma_o^2 \frac{\partial}{\partial z}\int \mu^2\psi d\mu = o \ .$$

This result yields in the case of a line bunch (i.e. no momentum spread, which means that $\psi \sim \delta(\mu)$)

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho\bar{v}) = o \ .$$

This way of integrating puts on a firm basis the limitation of the method of introducing the mean velocity by definition as given in [11].

In this one dimensional case the Liouville theorem is reduced to the continuity equation. But more important is that we gain the same equation if we use the generally good approximation η_1=o . Then, ignoring the m dependence, the time development of a symmetric bunch with a finite spread in momentum is completely described by a one dimensional continuity equation, where ρ is the longitudinal density and $\bar{v}$ the velocity of the symmetry points $m = \bar{m}(z,t)$.

In the following discussion we shall, instead of the more complicated Liouville equation, only treat the development of a line bunch by using the continuity equation. Taking into account the assumption made above, the results can also be applied to the development of a more realistic bunch.

Now the development of the line bunch is given by the three coupled equations

1) $\frac{dz}{dt} = \beta_o c\gamma_o^2 m(\eta_o - \eta_1 m)$,

2) $\frac{dm}{dt} = - \frac{\partial}{\partial z} \frac{e}{p_o} \phi$,

3) $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho\bar{v}) = o$.

These equations will be further reduced in the next section. Just to show the method, we shall assume $\eta_o >> \eta_1 m$ and choose $\eta_1 = o$ [1].

I.4.2 The scaling behaviour of the differential equations

For the potential ϕ we choose now the proposed relation

$$\phi = \phi_o \rho \ .$$

Introducing the averaged velocity from the first of the previous three equations into the two others yields (without bars)

$$\frac{\partial m}{\partial t} = - \frac{e\phi_o}{p_o} \frac{\partial\rho}{\partial z} - \beta_o c\gamma_o^2 \eta_o m \frac{\partial m}{\partial z} ,$$

$$\frac{\partial\rho}{\partial t} = -\beta_o c\gamma_o^2 \eta_o \frac{\partial}{\partial z} (\rho m) \ .$$

Due to the capacitive or inductive reactance of the chamber walls, ϕ_o can be positive or negative and, due to the particle energy above or below transition energy, η_o can be negative or positive. Therefore we have four pairs of differential equations with different sign combinations (see Fig. III).

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[1] This poorly linear relation between velocity and momentum is exactly fulfilled in the straight section of a linear accelerator [12]!

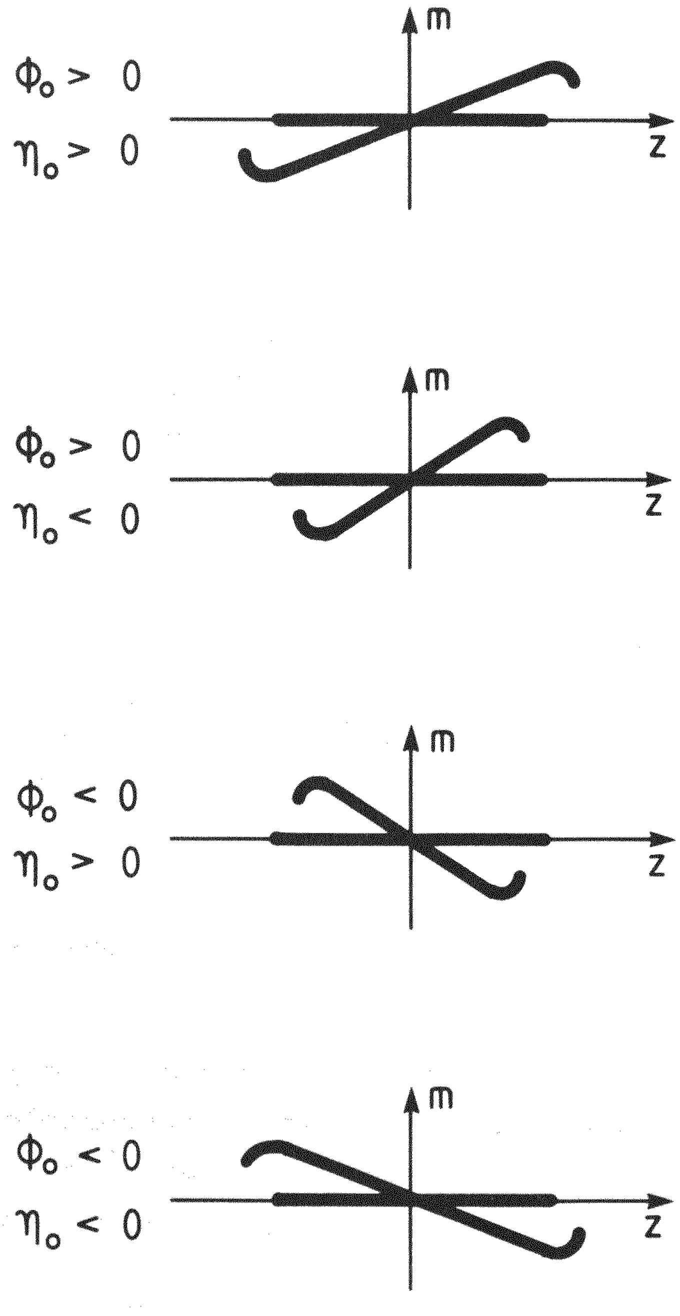


Fig. III : Qualitative deforming of a bunch due to space charge forces. The initial momentum is chosen as $m=o$ and is indicated on the z axis. For $\phi_o \eta_o > o$ we have de-bunching, which is described by a hyperbolic system of differential equations. For $\phi_o \eta_o < o$ we have self-bunching and an elliptic system.

Defining now ϕ_o and η_o as

$$\phi_o = \sigma|\phi_o| \ , \qquad \sigma = \pm 1 \ ,$$

$$\eta_o = \pi|\eta_o| \ , \qquad \pi = \pm 1 \ ,$$

and

$$z = \bar{z} \ ,$$

$$t = \bar{t}\cdot T \ , \quad T = \sqrt{\frac{p_o}{e|\phi_o|\beta_o c\gamma_o^2|\eta_o|}} \ ,$$

$$m = \bar{m}\cdot M \ , \quad M = \pi\sqrt{\frac{e|\phi_o|}{p_o\beta_o c\gamma_o^2|\eta_o|}}$$

$$\rho = \bar{\rho}\cdot R \ ,$$

the equations become (without the bars)

$$\frac{\partial m}{\partial t} = -R\sigma\pi\frac{\partial\rho}{\partial z} - m\frac{\partial m}{\partial z} \ ,$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial z}(\rho m) \ .$$

Excluding the "unphysical" situation of a negative density we choose $R = +1$.

The product $\sigma\pi$ can assume the two values $\sigma\pi = \pm 1$. Finally the two pairs of equations, which in their unscaled form depend on different machine parameters, are reduced to two fundamental cases, independent of any machine parameter:

case I :

$$\frac{\partial m}{\partial t} = -\frac{\partial\rho}{\partial z} - m\frac{\partial m}{\partial z} \ ,$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial z}(\rho m) \ ,$$

case II:

$$\frac{\partial m}{\partial t} = \frac{\partial\rho}{\partial z} - m\frac{\partial m}{\partial z} \ ,$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial z}(\rho m) \ .$$

The first pair of these equations is of hyperbolic character (see e.g. [13]) and describes a de-bunching situation. These two equations are very well-known in fluid dynamics. They describe the one dimensional isentropic flow of a polytropic gas with an adiabatic exponent equal to 2 ([13], pp. 80). Especially in the "shallow water" theory, an analogous nonlinear wave motion is studied [14]. In the following only this hyperbolic case will be discussed.

The second pair is of elliptic character and is more difficult to handle. In this case the self-bunching situation is described.

Deriving from the first equation of the hyperbolic system a scaled action function

$$\bar{S} = - \int_0^t (\rho + m^2/2)\, dt'$$

with the properties

$$\frac{\partial \bar{S}}{\partial t} = -H = -(\rho + \frac{m^2}{2}) \ ,$$

$$\frac{\partial \bar{S}}{\partial z} = m \ ,$$

we can write the continuity equation as

$$\frac{\partial}{\partial t} (\bar{S}_t + \bar{S}_z^2/2) = -\frac{\partial}{\partial z} [\bar{S}_z (\bar{S}_t + \bar{S}_z^2/2)] \ ,$$

where the index means partial derivation, or

$$\bar{S}_{tt} = -\frac{\partial}{\partial z} (\bar{S}_t \, \bar{S}_z) - \bar{S}_z \frac{\partial}{\partial z} (\bar{S}_t + \frac{3}{4} \bar{S}_z^2) \ .$$

This second order partial differential equation is equivalent to the hyperbolic system of two coupled equations of first order. An approximate equation, which is more amenable for a detailed discussion, will be derived in Section I.4.4 .

I.4.3 Conservation of energy and momentum

For the whole line bunch we have conservation of energy, as will be shown now. Taking the scaled Hamiltonian for one particle, multiplying with the particle density ρ and integrating over the whole bunch, we get for the total energy

$$E_{tot} = \int \rho\left(\frac{\rho}{2} + \frac{m^2}{2}\right) dz \ .$$

In this expression, ρ has a double meaning: in the first factor as the density distribution and in the second one as the scaled potential function. In order to avoid a double counting of the particles, we must divide ρ^2 by 2, as usual in potential theory (see [15], p. 21).

The total rate of change of energy in time is given by

$$\frac{d}{dt} E_{tot} = \int \frac{\partial}{\partial t} \rho\left(\frac{\rho}{2} + \frac{m^2}{2}\right) dz$$

$$= \int \left[\rho \frac{\partial \rho}{\partial t} + \frac{m^2}{2} \frac{\partial \rho}{\partial t} + \rho m \frac{\partial m}{\partial t}\right] dz \ .$$

Taking the two fundamental differential equations for m and ρ to express the time derivative via a position derivative yields

$$\frac{d}{dt} E_{tot} = \int \left[-\rho\frac{\partial}{\partial z} (\rho m) - \frac{m^2}{2} \frac{\partial}{\partial z} (\rho m) - \rho m \frac{\partial \rho}{\partial z} - \rho m^2 \frac{\partial m}{\partial z}\right] dz$$

$$= \int \left[- \frac{\partial}{\partial z} (\rho^2 m) - \frac{\partial}{\partial z} \left(\rho\frac{m^3}{2}\right)\right] dz \ .$$

This integral is zero at the bunch boundary.

Similarly, we find for the total momentum P of the line bunch, given by

$$P = \int \rho m dz \ ,$$

a total rate of change

$$\frac{dP}{dt} = \int \frac{\partial}{\partial t} (\rho m)\ dz = \int \left[m\frac{\partial \rho}{\partial t} + \rho\frac{\partial m}{\partial t} \right] dz$$

$$= -\int \left[m\frac{\partial}{\partial z} (\rho m) + \rho\frac{\partial \rho}{\partial z} + \rho m\ \frac{\partial m}{\partial z} \right] dz$$

$$= -\int \frac{\partial}{\partial z} (\rho m^2 + \frac{\rho^2}{2})\ dz\ .$$

Again, this integral is zero at the boundary of the line bunch.

As a result from these two conservation laws, we can treat the line bunch as a closed system. The maximum value of the kinetic energy is limited by the maximum value of the potential energy. A transfer of the potential energy

$$\frac{dE_p}{dt} = \frac{1}{2} \int \frac{\partial}{\partial t}\ \rho^2\ dz = \int \rho m\ \frac{\partial \rho}{\partial z}\ dz$$

into kinetic energy is possible, as long as somewhere inside the bunch both the density and the gradient of the density are different from zero.

I.4.4 The approximate equation

In order to simplify the hyperbolic system of differential equations we shall use an approximation. We always assume for the initial condition $m=o$ and, as long as the value of m and all longitudinal changes of m are small, we can neglect the term $m\ \partial m/\partial z$.

For the hyperbolic system we get now the following truncated system

$$\frac{\partial m}{\partial t} = - \frac{\partial \rho}{\partial z}$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial z}\ (\rho m)\ .$$

The new action function $\bar{\Gamma}$ is given by

$$\bar{\Gamma} = - \int_{o}^{t} \rho \, dt$$

with

$$\frac{\partial \bar{\Gamma}}{\partial z} = m \ ,$$

$$\frac{\partial \bar{\Gamma}}{\partial t} = -\rho \ .$$

A comparison of the particle energy H, which we get from the approximated form

$$H = - \frac{\partial \bar{\Gamma}}{\partial t} = \rho \ ,$$

with the exact expression

$$H = - \frac{\partial \bar{S}}{\partial t} = \rho + m^2/2$$

shows, that the approximation holds, if the kinetic energy is negligible compared with the potential energy.

Inserting the density and the momentum into the continuity equation yields

$$\bar{\Gamma}_{tt} = - \frac{\partial}{\partial z} (\bar{\Gamma}_t \, \bar{\Gamma}_z) ,$$

or, with $\bar{\Gamma}$ replaced by $-\Gamma$,

$$\Gamma_{tt} = \frac{\partial}{\partial z} (\Gamma_t \Gamma_z) \ .$$

This equation shows a much simpler form than the corresponding equation for $\bar{S}$ and will be discussed in great detail in Part II.

If the same approximation is applied to the elliptic system, and if we set

$$\bar{\Gamma}_z = m \ , \quad \bar{\Gamma}_t = \rho \ ,$$

we obtain the same equation for $\bar{\Gamma}$. This only indicates that it is not sufficient to speak of a hyperbolic equation for $\bar{\Gamma}$, but, since the density ρ must always be positive, we must in addition require that for all times $\bar{\Gamma}_t < o$, and therefore $\bar{\Gamma} < o$, in the hyperbolic case, and $\bar{\Gamma} > o$ in the elliptic case.

I.5 Miscellaneous

One of the most interesting results which we have obtained is the simple scaling law. On this level of approximation, the scaled equations are independent of any machine parameter. Therefore, a study of these two coupled fundamental differential equations is of very general character. In the following chapters we shall discuss only the truncated form of these two equations. The complete form of the fundamental equations is investigated at present.

The derivation of the continuity equation from the Liouville law shows that the use of the continuity equation is also very general. The further condition, i.e. to have a poorly linear relation between the velocity and the momentum, is exact in a straight section of a linear accelerator and a rather good approximation in a circular accelerator. However, to work close to transition energy is of special interest for the compressor ring IKOR, and to take only a linear relation should be justified.

The special choice of the potential function $\phi = \phi_o \rho$ is not so general, but a very common approximation. The condition, to have a smooth bunch, fails just in the case of a shock, where a jump appears in the line charge. As a first approach one might go on with this potential also in this case to study general shock properties. A more realistic choice of a potential should also take into account the resistive properties of the wall. Of special interest in a shock wave are the extremely high electric fields which could cause a momentum blow up in this region.

The physical problems when studying a shock also appear as numerical problems. Maybe, a better choice in the physical variables (ρ,m) could be done to avoid these numerical problems. For example, an integral expression like the scalar and the vector potentials (ϕ,A), or a combination of both, could have some advantage in the numerical treatment. A potential function does not show a jump, if there is a jump in the density.

PART II

Mathematical and Numerical Methods for the Nonlinear Hyperbolic Propagation Problem

$$\frac{\partial^2 \Gamma}{\partial t^2} = \frac{\partial}{\partial z} \left\{ \frac{\partial \Gamma}{\partial t} \, \frac{\partial \Gamma}{\partial z} \right\}$$

1 Analytic Results for the Equation $r-pt-qs=o$

1.1 Statement of the problem

In the following chapters we shall be concerned with a detailed discussion of the initial value problem

$$\frac{\partial^2\Gamma}{\partial t^2} = 2\,\frac{\partial}{\partial z}\left[\frac{\partial\Gamma}{\partial t}\,\frac{\partial\Gamma}{\partial z}\right] \tag{1.1}$$

with initial conditions

$$\Gamma(z,o) = o\;, \tag{1.2a}$$

$$\Gamma_t(z,o) = \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}\;. \tag{1.2b}$$

The factor 2 in eq. (1.1) could have been avoided by scaling the time according to $\bar{t} = 2t$. Its occurrence originates from the initial phase of this work, when a modified equation was considered, which contained eq. (1.1) as a special case. Since all numerical results to be presented later have been obtained with this factor, it was retained in eq. (1.1).

The initial conditions (1.2) may seem to be rather restrictive. In fact the more general case, when eq. (1.2b) is replaced by

$$\Gamma_t(z,o) = f(z)\;,$$

will be considered at various places, where $f(z)$ assumes one of the following forms

$$f(z) = \{z^{2/3}\;,\; a-bz^2\;,\; \cos^2 cz\}\;.$$

The essential features, however, can all be deduced from the choice (1.2).

The initial value problem (1.1/2) can be classified mathematically as a Cauchy problem for the second order partial differential equation (1.1), and physically as a nonlinear hyperbolic propagation problem. The condition for hyperbolicity will be derived in Chapter 4. In the following four sections some general properties of problem (1.1/2) will be discussed.

1.2 Some general remarks

1.2.1 Symmetry properties

We observe that eq. (1.1) remains invariant if z is replaced by $-z$. Since the initial conditions (1.2) are even functions of z , we conclude that

$$\Gamma(z,t) = \Gamma(-z,t) \ , \tag{1.3}$$

and therefore

$$\Gamma_t(z,t) = \Gamma_t(-z,t) \ , \tag{1.4a}$$

$$\Gamma_z(z,t) = -\Gamma_z(-z,t) \ . \tag{1.4b}$$

Here Γ_t is related to the density, and Γ_z to the momentum distribution (see eqs. (1.7a/b) of the following section).

If both t and Γ are replaced by $-t$ and $-\Gamma$, eq. (1.1) again remains invariant. The initial conditions (1.2) remain invariant as well, therefore

$$\Gamma(z,t) = -\Gamma(z,-t) \tag{1.5}$$

must hold. To be more general we can say that the relation (1.3) requires even, but otherwise arbitrary, initial conditions, whereas (1.5) is based on the fact that $\Gamma(z,o) = o$, but arbitrary $\Gamma_t(z,o)$.

1.2.2 Conservation laws

In Part I it was shown that the complete set of equations for the density ρ and the momentum distribution m satisfies the conservation of mass, momentum and energy. Eq. (1.1) results from the truncated system

$$\frac{\partial m}{\partial t} = -2 \frac{\partial \rho}{\partial z} \ , \tag{1.6a}$$

$$\frac{\partial \rho}{\partial t} = -2 \frac{\partial}{\partial z} (\rho m) \ , \tag{1.6b}$$

by setting

$$m = - \frac{\partial \Gamma}{\partial z} \, , \tag{1.7a}$$

$$\rho = \frac{1}{2} \frac{\partial \Gamma}{\partial t} \, . \tag{1.7b}$$

Eq. (1.6b) guarantees the conservation of mass, and since the initial condition (1.2b) is normalized to 1, it follows from (1.7b) that

$$\int_{-\infty}^{\infty} \Gamma_t(z,t)\,dz = 1$$

for all times t .

On the other hand, the conservation of momentum requires that

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho \; m \; dz = o \; .$$

With (1.7) we find

$$\begin{aligned}
\int_{-\infty}^{\infty} (\rho m_t + m\rho_t)\,dz &= -\frac{1}{2} \int_{-\infty}^{\infty} (\Gamma_t \Gamma_{tz} + \Gamma_z \Gamma_{tt})\,dz \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{2} \frac{\partial}{\partial z} (\Gamma_t^2) + 2\Gamma_z \frac{\partial}{\partial z} (\Gamma_t \Gamma_z) \right] dz \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} \Gamma_z^2 \Gamma_{tz} \; dz \, ,
\end{aligned}$$

where we have used the identity

$$2\Gamma_z \frac{\partial}{\partial z} (\Gamma_t \Gamma_z) = \frac{\partial}{\partial z} (\Gamma_t \Gamma_z^2) + \Gamma_z^2 \Gamma_{tz} \; .$$

The last integral does not vanish in general. If, however, Γ_t is even in z and Γ_z odd (eqs. (1.4a/b)). the integrand is odd and therefore the integral is equal to zero.

The conservation of momentum thus follows whenever the initial conditions $\Gamma(z,o)$, $\Gamma_t(z,o)$ are even in z .

1.2.3 An alternative equation

If we set

$$u = \int_{-\infty}^{z} \Gamma(\zeta,t)\,d\zeta$$

and integrate eq. (1.1) over z, we obtain an equation for u which reads (note that $\Gamma=u_z$)

$$u_{tt} = 2u_{tz}\, u_{zz} \, .$$

This form shows a striking similarity to the linear wave equation with a velocity of sound $V(z,t)$ given by

$$V^2 = 2u_{tz} = 4\rho > o \, ,$$

where the inequality follows from the physical meaning of ρ as given by (1.7b).

This already indicates that eq. (1.1) with initial conditions (1.2) describes a nonlinear wave propagation problem. This aspect will be studied in great detail in Chapter 4.

1.2.4 Non-existence of the general solution

It would be extremely valuable to know if the general solution of eq. (1.1) does already exist (the general solution allows to express Γ by means of two arbitrary functions with one argument each).

Setting

$$x = 2t \, , \quad y = z \, ,$$

$$w(x,y) = \Gamma(z,t) \, ,$$

and using the standard notation

$$p = w_x \, , \; q = w_y \, , \; r = w_{xx} \, , \; s = w_{xy} \, , \; t = w_{yy} \, ,$$

eq. (1.1) becomes

$$r - pt - qs = o \; . \tag{1.8}$$

For the very similarly looking equation

$$r + pt - qs = o \tag{1.9}$$

the general solution is in fact known (see Forsyth [16], pp. 342, 343, and Ames [17], p. 68).

In Appendix A1 we shall give a new derivation, which is well-known in gas dynamics, for the general solution of eq. (1.9). The basic key to this solution is the solubility of a linear second order partial differential equation for the time t as a function of the Riemann invariants ξ and η, in this case

$$t_{\xi\eta} = o \; ,$$

which can be solved immediately.

When this method is applied to eq. (1.8), the corresponding linear differential equation becomes (see Appendix A2)

$$t_{xx} - t_{yy} - \frac{1}{3xy}\left(x^{2/3} - y^{2/3}\right)\left[y^{1/3}t_x + x^{1/3}t_y\right] = o \; , \tag{1.1o}$$

where x and y are related to the Riemann invariants via

$$x = \xi + i\eta \; ,$$

$$y = \xi - i\eta \; .$$

It seems impossible to find the general solution (if this exists at all) of eq. (1.1o), and therefore of eq. (1.8).

1.3 Particular solutions for $r-pt-qs = 0$

As stated before, the general solution of eq. (1.1) is not available. Still a few particular integrals can be found which, of course, do not satisfy the initial condition (1.2b), but rather special conditions.

In this section we consider the problem

$$\frac{\partial^2 \Gamma}{\partial t^2} = 2 \frac{\partial}{\partial z} \left[\frac{\partial \Gamma}{\partial t} \frac{\partial \Gamma}{\partial z}\right] \tag{1.1}$$

with initial conditions

$$\Gamma(z,o) = o \ , \tag{1.11a}$$

$$\Gamma_t(z,o) = f(z) \ , \tag{1.11b}$$

where $f(z)$ depends on the particular integral, but still contains certain parameters which allow to imply the conditions

$$f(z) > o \ , \tag{1.12a}$$

$$f(z) = f(-z) \ . \tag{1.12b}$$

Inequality (1.12a) must be fulfilled for the case that $\Gamma(z,t)$ corresponds to a hyperbolic propagation problem (see Appendix A2.1). Condition (1.12b) is merely introduced in order to simplify the discussion.

1.3.1 Separation of variables

The most direct approach to obtain particular integrals consists in separating the variables,

$$\Gamma(z,t) = \varphi(t) \ \psi(z) \ . \tag{1.13}$$

Insertion into (1.1) then leads to the ordinary differential equations

$$\ddot{\varphi} = 2\lambda\dot{\varphi}\varphi \ , \tag{1.14}$$

$$\psi'' + \frac{1}{\psi}\,\psi'^2 = \lambda \ , \tag{1.15}$$

where dots and primes mean differentiation with respect to the corresponding arguments, and λ is an arbitrary parameter.

The initial conditions (1.11) require

$$\varphi(o) = o \ ,$$

$$\dot{\varphi}(o) = a \ ,$$

with

$$f(z) = a\psi(z) > o \ .$$

We shall assume that $a>o$ and $\psi(z) > o$.

A first integration of (1.14) yields

$$\dot{\varphi} = a + \lambda\varphi^2 \ ,$$

and this equation together with the initial condition $\varphi(o) = o$ is solved by

$$\varphi(t) = \sqrt{\frac{a}{\lambda}}\,\tan(\sqrt{a\lambda}\ t) \ , \qquad \lambda>o \ , \tag{1.16a}$$

$$\varphi(t) = \sqrt{-\frac{a}{\lambda}}\,\tanh\,(\sqrt{-a\lambda}\ t), \qquad \lambda<o \ . \tag{1.16b}$$

For $\lambda=o$ both solutions tend to $\varphi(t) = at$.

Introducing $p(\psi) = \psi'$, eq. (1.15) becomes

$$\frac{dp}{d\psi} + \frac{1}{\psi}\,p = \frac{\lambda}{p} \ ,$$

which is of the Bernoulli type (see Kamke [18], p. 19) and has the solution

$$\frac{d\psi}{dz} = \pm \frac{1}{\psi} \sqrt{c + \frac{2}{3} \lambda \psi^3} \; .$$

The solution for ψ , consistent with (1.12) and $\psi > o$, is

$$z(\psi) = \pm \int_{\psi_c}^{\psi} \frac{u \; du}{\sqrt{c + \frac{2}{3} \lambda u^3}} \; , \qquad (1.17)$$

where

$$\psi \geq \psi_c = o \; , \qquad \text{for} \quad c > o \; , \qquad (1.18a)$$

$$\psi \geq \psi_c = \left[- \frac{3c}{2\lambda} \right]^{1/3} , \qquad \text{for} \quad c < o \; . \qquad (1.18b)$$

Note that for (1.18b) only $\lambda > o$ can be accepted.

From eq. (1.17) it follows that

a) for small values of z ,

$$\psi(z) \cong [2\sqrt{c}|z|]^{1/2} , \quad \text{for} \quad c > o \; ,$$

$$\psi(z) \cong \psi_c + \frac{\lambda}{2} z^2 , \quad \text{for} \quad c < o \; ,$$

b) for large values of z , $\psi(z)$ behaves asymptotically like z^2 . This limiting case, however, only applies for $\lambda > o$.

Let us now consider the curvature of $\psi(z)$. For $\lambda < o$, eq. (1.15) already reveals that $\psi(z)$ must always be concave. For $\lambda > o$ we must determine the curvature explicitly. With

$$\psi'' = \frac{1}{3}\lambda - \frac{c}{\psi^3}$$

it follows that $\psi(z)$ is always convex if $c < o$, and for $c > o$ it becomes convex as soon as $\psi^3 > 3c/\lambda$.

In Fig. 1.1 the three possible types of solutions $\psi(z)$ are shown for a particular choice of the constants λ and c .

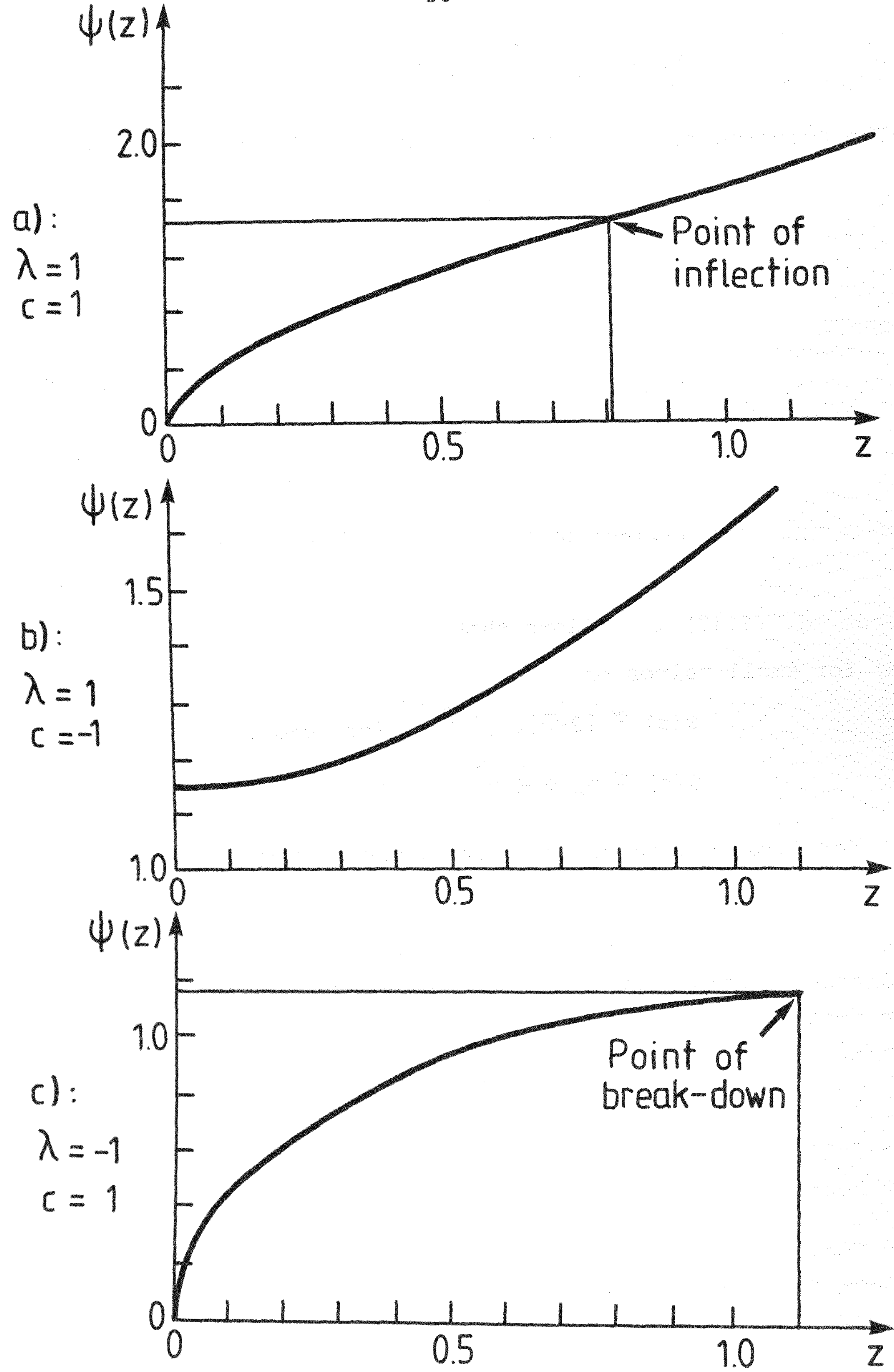


Fig. 1.1 : The three different types of solutions $\psi(z)$.

The influence of the initial distribution on the appearance of singularities in $\Gamma(z,t)$ will be briefly discussed at the end of Section 1.3.3, where a quite similar situation holds.

1.3.2 Similarity solutions

Similarity solutions result from the algebraic methods of Lie infinitesimal contact transformation group theory [19]. Invariance of a differential equation against an infinitesimal transformation of all variables (dependent and independent) leads to an equation with a reduced number of independent variables by one. Eq. (1.1) then becomes an ordinary differential equation. Here we follow the discussion of Bluman and Cole ([19], pp. 147).

For eq. (1.1) the function H, eq. (2.1-1) of [19], is given by

$$H(u_{tt},u_{zt},u_{zz},u_t,u_z) = u_{tt} - 2u_t u_{zz} - 2u_z u_{zt} = o \; , \qquad (1.19)$$

and eq. (2.1-25), which follows from the invariance of (1.19) against the infinitesimal transformation

$$\begin{aligned} u^* &= u + \varepsilon\eta(z,t,u) + O(\varepsilon^2) \; , \\ z^* &= z + \varepsilon\zeta(z,t,u) + O(\varepsilon^2) \; , \\ t^* &= t + \varepsilon\tau(z,t,u) + O(\varepsilon^2) \; , \end{aligned} \qquad (1.2o)$$

leads to

$$\bar{\eta}_{zz}u_t + \bar{\eta}_{zt}u_z - \frac{1}{2}\bar{\eta}_{tt} + \bar{\eta}_z u_{zt} + \bar{\eta}_t u_{zz} = o \; , \qquad (1.21)$$

where the rather lengthy expressions for the first and second extensions $\bar{\eta}_t,\ldots,\bar{\eta}_{zt}$ (eqs. (2.1-19) to (2.1-23)) are not given explicitly.

Eq. (1.21) can be satisfied in an overdetermined manner by equating to zero the 26 coefficients c_i of the resulting expression

$$c_1 + c_2 u_t + c_3 u_z + c_4 u_t^2 + \ldots + c_{26} u_z u_t u_{zz} = o \ ,$$

where the coefficients c_i are linear homogeneous combinations of first and second derivatives of the three infinitesimal elements η, ζ, τ with respect to z, t, u .

It turns out that all derivatives must vanish identically except for the two relations

$$\eta_u = 2\tau_t \ ,$$

$$\zeta_z = \frac{3}{2}\tau_t \ ,$$

and therefore the Lie group leaving eq. (1.19) invariant is given by (1.2o) with

$$\begin{aligned} \eta(z,t,u) &= 2c(u-u_o) \ , \\ \zeta(z,t,u) &= \frac{3}{2}c(z-z_o) \ , \\ \tau(z,t,u) &= c(t-t_o) \ , \end{aligned} \tag{1.22}$$

with arbitrary constants u_o, z_o, t_o, c . Since (1.19) does not depend explicitly on u, z, t , we may choose $u_o = z_o = t_o = o$.

In order to obtain the similarity solutions, one has to solve a first order partial differential equation (the "invariant surface condition" (2.1-8)) which by the method of characteristics leads to

$$\frac{dz}{\zeta} = \frac{dt}{\tau} = \frac{du}{\eta} \ . \tag{1.23}$$

With eqs. (1.22), integration of (1.23) yields the two relations

$$t = k_1 \ z^{2/3} \ ,$$

$$u = k_2 \ t^2 \ .$$

The constant k_1 is usually referred to as the *similarity variable* [19] which will be denoted by

$$x = t\, z^{-2/3} \, , \qquad (1.24)$$

and the constant k_2 will in general be a function of x .

We have thus found the class of similarity solutions for eq. (1.1),

$$\Gamma(z,t) = g(x)t^2 \, , \qquad (1.25)$$

with x given by (1.24), and $g(x)$ satisfying the ordinary differential equation

$$L[g] - Q[g] = o \, , \qquad (1.26a)$$

with

$$L[g] = x^2 g'' + 4xg' + 2g \, , \qquad (1.26b)$$

$$Q[g] = \tfrac{4}{9}x^4[4xg''(g+xg') + g'(1og+11xg')] \, . \qquad (1.26c)$$

Next we want to investigate the behaviour of (1.26) for small x in order to find the initial distributions which can be realized by (1.25). Note that x is even in z , therefore (1.12b) is already satisfied.

For small x we make the assumption

$$g(x) \cong cx^{\alpha} \, .$$

Insertion into (1.26) yields

$$\alpha^2 + 3\alpha + 2 - \tfrac{16}{9}\alpha c x^{3+\alpha}(\alpha^2 + \tfrac{11}{4}\alpha + \tfrac{3}{2}) = o \, . \qquad (1.27)$$

Three possible outcomes for α will be distinguished:

a) $\underline{\alpha < -3}$

In this case the term proportional to $x^{3+\alpha}$, which becomes dominant for small x , must vanish. But

$$\alpha^2 + \tfrac{11}{4}\alpha + \tfrac{3}{2} = (\alpha+2)(\alpha+\tfrac{3}{4}) \, ,$$

which does not vanish for $\alpha < -3$.

b) $\underline{\alpha > -3}$

In this case the constant term in (1.27), which now becomes dominant for small x , must vanish. We find from

$$\alpha^2 + 3\alpha + 2 = (\alpha+1)(\alpha+2) ,$$

that the two values $\alpha=-1$ and $\alpha=-2$ are consistent with $\alpha > -3$.

c) $\underline{\alpha = -3}$

In this case (1.27) leads to $c = -\frac{1}{6}$.

We thus have found three possibilities:

$$1) \quad g(x) = \frac{c}{x} h(x) , \quad h(o) = 1 , \quad c \text{ arbitrary} , \qquad (1.28a)$$

$$2) \quad g(x) = \frac{c}{x^2} h(x) , \quad h(o) = 1 , \quad c \text{ arbitrary} , \qquad (1.28b)$$

$$3) \quad g(x) = \frac{1}{x^3} h(x) , \quad h(o) = -\frac{1}{6} . \qquad (1.28c)$$

Only the case (1.28a) needs to be considered, since then

$$\Gamma(z,t) = ct\, z^{2/3} [1+O(t)] ,$$

which satisfies condition (1.11a), $\Gamma(z,o) = o$. The initial distribution (1.11b) is given by

$$f(z) = cz^{2/3} , \qquad (1.29)$$

which is of course only of moderate interest - unfortunately a general feature of similarity solutions.

The differential equation (1.26) with $g(x)$ as given by (1.28a) will be examined now. If we insert (1.28a) into (1.26), the resulting differential equation for $h(x)$ will depend on c . Appropriate scaling, however, leads to a parameter free differential equation.

We set

$$\xi = \frac{2}{3}\sqrt{c}\, x \ , \qquad y(\xi) = h(x) \ ,$$

and obtain

$$(1 - 4\xi^3 y')y'' = 3\xi^2 y'^2 - 4\xi y y' + y^2 - \frac{2}{\xi} y' \ , \qquad (1.3o)$$

with $y(o) = 1$.

Because of the last term in (1.3o), the solution which is regular in $\xi = o$ must be even in ξ , starting with

$$y(\xi) = 1 + \frac{1}{6}\xi^2 - \frac{1}{2o}\xi^4 + O(\xi^6) \ .$$

The similarity solution is therefore reduced to the unique initial value problem (1.3o) with $y(o)=1$, $y'(o)=o$.

Let us consider the regularity properties of this solution. A critical point, where (1.3o) may lead to a singularity in $y(\xi)$, is determined by

$$1 - 4\xi_o^3 y_o' = o \ ,$$

where y_o'' will be infinite, unless the right hand side of (1.3o) vanishes at ξ_o . Note that y_o' will be finite.

In order to facilitate the following discussion, we want to make use of some results which we obtained when solving (1.3o) numerically. It turned out that y_o'' is finite, namely zero, but that $y_o^{(3)}$ no longer exists.

This leads us to the following expansion around the unknown point ξ_o :

$$y(\xi) = y_o - y_o'\varepsilon + \frac{1}{2}y_o''\varepsilon^2 + d\varepsilon^3 + O(\varepsilon^4) + c\varepsilon^\alpha + O(\varepsilon^{\alpha+1}) \ , \qquad (1.31)$$

where $\varepsilon = \xi_o - \xi > o$, and α should lie in the interval $2 < \alpha < 3$.

When we insert the ansatz (1.31) and its derivatives into

$$(\xi - 4\xi^4 y')y'' = 3\xi^3 y'^2 - 4\xi^2 yy' + \xi y^2 - 2y' ,$$

the following relations must hold (note that $y_o' = 1/(4\xi_o^3)$):

1) $\sim\varepsilon^o$: $\quad o = 3\xi_o^3 y_o'^2 - 4\xi_o^2 y_o y_o' + \xi_o y_o^2 - 2y_o' ,$

2) $\sim\varepsilon^1$: $\quad [-1 + 4\xi_o^4 y_o'' + 16\xi_o^3 y_o']y_o'' =$

$$[-6\xi_o^3 y_o' + 4\xi_o^2 y_o + 2]y_o'' - 5\xi_o^2 y_o'^2 + 6\xi_o y_o y_o' - y_o^2 ,$$

3) $\sim\varepsilon^{\alpha-1}$: $4c\alpha\xi_o^4 y_o'' + c\alpha(\alpha-1)\,[-1 + 4\xi_o^4 y_o'' + 16\xi_o^3 y_o'] =$

$$-6c\alpha\xi_o^3 y_o' + 4c\alpha\xi_o^2 y_o + 2c\alpha .$$

If we solve 1) for y_o , we obtain the two solutions

$$y_o = \frac{5}{4}\frac{1}{\xi_o^2} \quad \text{and} \quad y_o = -\frac{1}{4}\frac{1}{\xi_o^2} .$$

We want to show that the negative value must be excluded. For $y(\xi)$ to have an extreme value, we deduce from the differential equation that with $y'=o$ also $y''=y^2>o$ must be satisfied. Therefore $y(\xi)$ can only possess local minima. Since $y'(o)=o$, $y(\xi)$ is a monotonously increasing function which with $y(o)=1$ will always be positive.

Next we can determine y_o'' from 2), since the values for y_o and y_o' are known. Again we find two solutions,

$$y_o'' = o \quad \text{and} \quad y_o'' = \frac{5}{8}\frac{1}{\xi_o^4} .$$

Assuming that $c\alpha \neq o$, i.e. that a singularity does exist, this product cancels in the third relation, and α can be determined uniquely from 3). If we try the value $y_o''>o$, it follows that $\alpha = 17/11 < 2$, which is in contradiction to the existence of y_o'' . On the other hand, $y_o'' = o$ yields $\alpha = 17/6$, which lies in the required interval.

Whereas the coefficient c cannot be determined analytically, the value for d is easily obtained from the terms proportional to ε^2 and is equal to $-1/(3\xi_0^5)$. We have calculated c and ξ_0 numerically in the following way.

As already stated above, we first solved eq. (1.30) numerically, using Subroutine DVERK from IMSL [20] with TOL = 10^{-10}, from which ξ_0 was roughly estimated as $\bar{\xi}_0 = 1.04566$. Next we solved eq. (1.30) on the interval $0 \leq \xi \leq 0.8$, yielding two highly accurate values for $y(0.8)$ and $y'(0.8)$. At the - yet unknown - starting point $\xi_1 = \xi_0 - 10^{-4}$, eq. (1.30) had to be solved backwards with the initial values $y(\xi_1)$, $y'(\xi_1)$ according to formula (1.31) such that the function and its derivative should coincide with the given values at $\xi = 0.8$.

This amounts to solving a system of two nonlinear equations for the unknowns c and ξ_0. We minimized the corresponding sum of squares using Subroutine VA05AD from Harwell Subroutine Library [21], where $\bar{\xi}_0$ was taken as initial value for ξ_0. The variable c is rather insensitive because of the factor ε^α. The result was

$$\xi_0 = 1.0456667o559 \;,$$

$$c = 0.3279 \;,$$

where c may be accurate only within two decimal places. At $\xi = 0.8$, $y(\xi)$ and $y'(\xi)$ coincided with the exact values within 8 decimal places and, as a check, $y''(\xi)$ was still accurate up to 7 decimal places.

We summarize our results. For $\xi < \xi_0$, eq. (1.31) becomes

$$y(\xi) \cong \frac{5}{4}\frac{1}{\xi_0^2} - \frac{1}{4}\frac{1}{\xi_0^3}(\xi_0 - \xi) - \frac{1}{3}\frac{1}{\xi_0^5}(\xi_0 - \xi)^3 + c(\xi_0 - \xi)^{17/6} \;,$$

where higher terms have been omitted. Since c cannot be determined analytically, a continuation of the solution beyond ξ_0 is not possible in accordance with the violation of the Lipschitz condition at ξ_0.

1.3.3 The parabolic solution

Because of the rather special structure of the initial distributions , the problems considered so far are of theoretical interest only. Quite amazingly the case of an important initial distribution can also be solved analytically [22] which will be presented in this section.

If we try to find solutions of the type

$$\Gamma(z,t) = T_1(t) - T_2(t)\, g(z) \ , \tag{1.32}$$

these functions must satisfy the equation

$$\ddot{T}_1 - \ddot{T}_2 g = -2\dot{T}_1 T_2 g'' + 2\dot{T}_2 T_2 (gg''+g'^2) \ , \tag{1.33}$$

which allows solutions only, if the two equations

$$g'' = \mu g + \nu \ ,$$

$$gg'' + g'^2 = \bar{\mu} g + \bar{\nu}$$

hold simultaneously, where $\mu,\nu,\bar{\mu}$ and $\bar{\nu}$ are constants. For $\mu \neq 0$ the first equation has the general solution

$$g(z) = c_1 \cosh\sqrt{\mu} z + c_2 \sinh\sqrt{\mu} z - \frac{\nu}{\mu} \ ,$$

which, after insertion into the second equation, yields $c_1 = c_2 = 0$, i.e. an obviously trivial result.

On the other hand, for $\mu = 0$ we obtain a parabolic solution for $g(z)$, which is consistent with the second equation.

Now we can try to solve eq. (1.1) with initial conditions

$$\Gamma(z,0) = 0 \ , \tag{1.34a}$$

$$\Gamma_t(z,0) = a - bz^2 \ , \tag{1.34b}$$

by setting $g(z) = z^2$ in (1.32). Then (1.12b) is satisfied, and (1.12a) requires $a > 0$. In addition we first assume $b > 0$, the case $b < 0$ will be discussed later.

The conditions (1.34) are satisfied if we require

$$T_1(o) = o\ , \quad \dot{T}_1(o) = a\ ,$$

$$T_2(o) = o\ , \quad \dot{T}_2(o) = b\ ,$$

and from (1.33) we obtain the two equations

$$\ddot{T}_1 = -4\dot{T}_1 T_2\ , \tag{1.35a}$$

$$\ddot{T}_2 = -12\dot{T}_2 T_2\ . \tag{1.35b}$$

Eq. (1.35b) is of the type (1.14) and has the solution

$$T_2(t) = \sqrt{\frac{b}{6}}\ \tanh(\sqrt{6b}t)\ , \tag{1.36}$$

and from (1.35a) we obtain

$$\dot{T}_1(t) = \frac{a}{[\cosh(\sqrt{6b}t)]^{2/3}}\ ,$$

$$T_1(t) = \frac{a}{b}\int\limits_o^{T_2(t)} \frac{dT}{[1-6T^2/b]^{2/3}}\ ,$$

with $T_2(t)$ from (1.36).

The physical quantities of interest are the density and the momentum,

$$\frac{\partial\Gamma}{\partial t} = \frac{a}{[\cosh(\sqrt{6b}t)]^{2/3}} - \frac{b}{[\cosh(\sqrt{6b}t)]^{2}}\ z^2\ , \tag{1.37a}$$

$$\frac{\partial\Gamma}{\partial z} = -2\sqrt{\frac{b}{6}}\ \tanh(\sqrt{6b}t)\,z\ , \tag{1.37b}$$

where $\partial\Gamma/\partial z$ is, up to a constant factor, equal to the relative momentum distribution $\Delta p/p$.

We already note here that the advantage of having found this nontrivial analytic result will become apparent in Section 4.4, when the characteristics in the (t,z) plane for the parabolic case will be studied.

Let us now turn to the convex initial distribution, i.e. $b=-c$ with $c>o$. Substituting $b=-c$ in (1.37) then leads to

$$\frac{\partial\Gamma}{\partial t} = \frac{a}{[\cos(\sqrt{6c}t)]^{2/3}} + \frac{c}{[\cos(\sqrt{6c}t)]^{2}} z^2 , \qquad (1.38a)$$

$$\frac{\partial\Gamma}{\partial z} = 2\sqrt{\frac{c}{6}} \tan(\sqrt{6c}t) z . \qquad (1.38b)$$

This solution becomes singular at $t=\pi/(2\sqrt{6c})$, which is independent of z .

The results (1.37/38) reveal a striking similarity to the results (1.16a/b). The initial distribution, which becomes unbounded like z^2 , leads again to a break-down in Γ for finite times, whereas for the bounded and concave initial distribution, the solution remains regular for all times. Both situations correspond to $\lambda>o$ and $\lambda<o$ in (1.16), see also Fig. 1.1 .

In Chapter 4 we shall encounter a different type of singularity, where the second derivatives of Γ become infinite at a finite time. The initial distribution is bounded in that case, but it has a convex decreasing tail which is responsible for the appearance of this so-called "shock discontinuity".

As already mentioned in the introduction, it is very remarkable that for the unrestricted equation, which was derived for $\bar{S}$ in Part I, the important case of a parabolic initial distribution can also be solved analytically.

1.3.4 Concluding remarks

1) Solitons. One immediate question which nowadays would be asked, when nonlinear equations such as (1.1) are to be investigated, concerns the possibility whether these equations allow solitary waves as particular solutions. The usual procedure then is to try travelling waves as possible solutions. With

$$\Gamma(z,t) = f(z-Vt) ,$$

where V is a constant velocity, we obtain from (1.1)

$$(V + 4f')f'' = o .$$

This shows that only solutions $f(u)$ which are linear in $u = z-Vt$ are possible, thus eq. (1.1) is not a soliton carrying system.

2) Bäcklund transformations. The seemingly simple structure of eq. (1.1) suggests another approach. If we introduce a new function $\bar{\Gamma}(z,t)$ by requiring that

$$\begin{aligned} \frac{\partial\bar{\Gamma}}{\partial z} &= \frac{\partial\Gamma}{\partial t} , \\ \frac{\partial\bar{\Gamma}}{\partial t} &= 2\,\frac{\partial\Gamma}{\partial t}\,\frac{\partial\Gamma}{\partial z} , \end{aligned} \tag{1.39}$$

it follows immediately that these equations are compatible with (1.1) since cross differentiation leads to equal expressions for the second derivative $\bar{\Gamma}_{zt}$.

The transformation (1.39) is a special case of a Bäcklund transformation (see e.g. Forsyth [16], pp. 425, or Lamb [23], pp. 243), which has the general form

$$\begin{aligned} \bar{p} &= \varphi(z,\bar{z},p,q) , \\ \bar{q} &= \psi(z,\bar{z},p,q) , \end{aligned}$$

where the common notation $p=z_x$, $q=z_y$, $\bar{p}=\bar{z}_x$, $\bar{q}=\bar{z}_y$ has been used.

The advantage of having found a Bäcklund transformation shows up in the equation for the transformed quantity, which in the most optimistic case may be soluble, then one can hope to find the general solution of the original equation as well, since the Bäcklund transformation contains only first derivatives . Regardless thereof, however, any particular solution - even a trivial one - of the transformed equation may lead to a nontrivial particular solution of the original equation.

With (1.39) the transformed equation is given by

$$\frac{\partial^2\bar{\Gamma}}{\partial t^2} = \frac{\frac{\partial\bar{\Gamma}}{\partial t}}{\frac{\partial\bar{\Gamma}}{\partial z}} \frac{\partial^2\bar{\Gamma}}{\partial z\partial t} + 2\,\frac{\partial\bar{\Gamma}}{\partial z}\,\frac{\partial^2\bar{\Gamma}}{\partial z^2}\ , \qquad (1.4o)$$

which reveals at least the same degree of difficulty as the original equation. Therefore no real attempt was made to find any particular solutions of (1.4o). We only mention as an example that analogous to (1.32) the assumption

$$\bar{\Gamma}(z,t) = T_1(t) + T_2(t)\ g(z)$$

now leads to one equation only, namely

$$g'g'' = \mu g + \nu\ ,$$

which can be integrated once yielding

$$g' = \left[\tfrac{3}{2}\mu g^2 + 3\nu g + c\right]^{1/3}\ .$$

For the special case $\mu=o$ we find

$$g(z) = \frac{1}{3\nu}\,\{(2\nu z + k)^{3/2} - c\}\ .$$

Finally we obtain from (1.39)

$$\frac{\partial\Gamma}{\partial t} = T_2(t)\ g'(z)$$

$$\frac{\partial\Gamma}{\partial z} = \frac{1}{2T_2(t)g'(z)}\ [\dot{T}_1(t) + \dot{T}_2(t)g(z)]\ ,$$

which is still a rather special type of solution.

2 Taylor Series Expansion

2.1 A recurrence relation

For the special choice of initial conditions (1.2) we can derive a very powerful Taylor series expansion with respect to the time.

We assume that the series

$$\Gamma(z,t) = \sum_{\ell=o}^{\infty} t^{\ell} Q_{\ell}(z) \tag{2.1}$$

is convergent for some $t>o$. If we set

$$E(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} , \tag{2.2}$$

we obtain from the initial conditions (1.2):

$$Q_o(z) = o$$

$$Q_1(z) = E(z) .$$

Inserting (2.1) into (1.1), we have

$$\sum_{\ell=2}^{\infty} \ell(\ell-1)t^{\ell-2} Q_{\ell}(z) = 2\frac{\partial}{\partial z} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} k\, t^{k+m-1} Q_k(z) Q'_m(z) . \tag{2.3}$$

Comparison of equal powers in t yields $\ell = k+m+1$, and the new Q_{ℓ} on the left hand side are related to Q_k , Q_m with $k,m \leq \ell-1$.

The terms Q_{ℓ} with even ℓ result from a sum of products which involve Q_k , Q_m with $(k+m)$ odd, which means that k even, m odd, or vice versa. In particular, $Q_2 = o$ since $Q_o = o$, $Q_4 = o$ since $Q_2 = Q_o = o$, and in general $Q_{2\ell} = o$.

When calculating $Q_3(z)$, $Q_5(z)$, ... from (2.3) , it turns out that $Q_{2\ell+1}$ has the general form

$$Q_{2\ell+1}(z) = P_\ell(z)\ E^{\ell+1}(z) \ , \qquad \ell \geq o \ ,$$

where $P_\ell(z)$ is a polynomial of degree ℓ in z^2 . This can be shown by inserting

$$\Gamma(z,t) = \sum_{\ell=o}^{\infty} t^{2\ell+1}\ P_\ell(z)\ E^{\ell+1}(z) \qquad (2.4)$$

into (1.1):

$$\sum_{\ell=o}^{\infty} (2\ell+1)2\ell\ t^{2\ell-1}\ P_\ell(z)\ E^{\ell+1}(z) =$$

$$= 2 \sum_{k=o}^{\infty} \sum_{m=o}^{\infty} (2k+1)t^{2(k+\ell)+1} \frac{d}{dz}\left[P_k(P_m' - (m+1)zP_m)E^{k+m+2}\right] .$$

Comparison of equal powers in t finally leads to

$$P_\ell(z) = \frac{1}{(2\ell+1)\ell} \sum_{\substack{k\geq o \\ k+m=\ell-1}} \sum_{m\geq o} (2k+1)\ [R_{km}' - (\ell+1)\ z\ R_{km}] \qquad (2.5)$$

with

$$R_{km}(z) = P_k\left[P_m' - (m+1)\ z\ P_m\right] . \qquad (2.6)$$

From $Q_1 = E$ we have

$$P_o(z) = 1 .$$

Let $P_k(z)$, $k \leq \ell-1$, be polynomials of order k in z^2 . In equ. (2.6) we must have $k \leq \ell-1$, $m \leq \ell-1$, because $k+m = \ell-1$. Therefore, by assumption, $R_{km}(z)$ is an odd polynomial of order $[2(k+m)+1] = 2\ell-1$. Then it follows immediately from (2.5) , that $P_\ell(z)$ is a polynomial of degree ℓ in z^2 , which completes the proof by induction.

The sequence of polynomials starts with $P_o=1$,

$$P_1(z) = \frac{1}{3}\ (-1+2z^2)$$

$$P_2(z) = \frac{1}{1o}(3-19z^2+1oz^4)$$

$$P_3(z) = \frac{1}{63o}(-246+3192z^2-4544z^4+128oz^6) .$$

We observe that the coefficients are strictly alternating in sign. This will be shown to hold in general (see Section 2.4).

Remarks

1) The solution (2.4) with $P_\ell(z)$ from (2.5) satisfies the differential equation (1.1) with initial conditions (1.2). Since (1.1) is analytic and the initial conditions (1.2) are analytic, too, the solution is unique.

2) Nothing can be said about the radius of convergence of the solution (2.4). But the numerical evaluation of (2.4/5) will clearly show the limits of the representation (2.4) (Sections 2.3 and 2.4).

Next we derive a recurrence relation for the coefficients of the polynomial

$$P_\ell(z) = \sum_{\lambda=o}^{\ell} a_\ell^{(\lambda)} \; z^{2(\ell-\lambda)} \quad . \tag{2.7}$$

Equ. (2.6) then becomes

$$R_{mn} = \sum_{\mu,\nu} [2(n-\nu) - (n+1) \; z^2] \; a_m^{(\mu)} \; a_n^{(\nu)} \; z^{2(m+n-\mu-\nu)-1}$$

$$= \sum_{\mu,\nu} [2(n-\nu) \; a_m^{(\mu-1)} - (n+1) \; a_m^{(\mu)}] \; a_n^{(\nu)} \; z^{2(m+n-\mu-\nu)+1} \quad .$$

With

$$R'_{mn} = \sum_{\mu,\nu} [2(m+n-\mu-\nu)+3] \; [2(n-\nu) a_m^{(\mu-2)} - (n+1) a_m^{(\mu-1)}] \cdot a_n^{(\nu)} \; z^{2(m+n-\mu-\nu)+2}$$

we obtain

$$R'_{mn} - (\ell+1)\, z\, R_{mn} =$$

$$\sum_{\mu,\nu} \Big\{ [2(m+n-\mu-\nu)+3]\, [2(n-\nu)\, a_m^{(\mu-2)} - (n+1)\, a_m^{(\mu-1)}]$$

$$-(\ell+1)\, [2(n-\nu)\, a_m^{(\mu-1)} - (n+1)\, a_m^{(\mu)}] \Big\}\, a_n^{(\nu)}\, z^{2(m+n-\mu-\nu)+2} \; .$$

Comparison of equal powers in z of equ. (2.5) leads to the condition

$$2(\ell-\lambda) = 2(m+n-\mu-\nu)+2 \; ,$$

or, with the restriction $\ell = m+n+1$, to

$$\lambda = \mu + \nu \; .$$

The final result is

$$a_\ell^{(\lambda)} = \frac{1}{(2\ell+1)\ell} \sum_{\substack{m\geq o\\ m+n=\ell-1}} \sum_{n\geq o} (2m+1) \sum_{\substack{\mu\geq o\\ \mu+\nu=\lambda}} \sum_{\nu\geq o} C_{mn}^{(\mu\nu)} \;\; a_n^{(\nu)} \; , \tag{2.8}$$

$$C_{mn}^{(\mu\nu)} = [2(\ell-\lambda)+1]\, [2(n-\nu)\, a_m^{(\mu-2)} - (n+1)\, a_m^{(\mu-1)}]$$

$$-(\ell+1)\, [2(n-\nu)\, a_m^{(\mu-1)} - (n+1)\, a_m^{(\mu)}] \; .$$

This recurrence relation must be solved with the initial condition

$$a_o^{(o)} = 1 \; .$$

From (2.7) it is obvious that the coefficients $a_\ell^{(\lambda)}$ must satisfy

$$a_\ell^{(\lambda)} = o \quad \text{for} \quad \lambda < o \; , \; \lambda > \ell \; ,$$

which has to be regarded in the definition of $C_{mn}^{(\mu\nu)}$.

Next we introduce new coefficients $\bar{a}_\ell^{(\lambda)}$ defined by

$$a_\ell^{(\lambda)} = c^{\ell+1}\,\bar{a}_\ell^{(\lambda)} \,, \tag{2.9}$$

where c is an arbitrary constant to be chosen later. Since

$$a_m^{(\mu)}\, a_n^{(\nu)} = c^{m+n+2}\,\bar{a}_m^{(\mu)}\,\bar{a}_n^{(\nu)}$$

$$= c^{\ell+1}\,\bar{a}_m^{(\mu)}\,\bar{a}_n^{(\nu)} \,,$$

independent of μ,ν, the recurrence relation (2.7) remains invariant if all $a_r^{(\rho)}$ are replaced by $\bar{a}_r^{(\rho)}$. The initial condition must then be chosen as

$$\bar{a}_o^{(o)} = 1/c \,,$$

which follows from (2.9) for $\ell=o$.

A particular choice of c is suggested by the form of expression (2.4):

$$\Gamma(z,t) = \sum_{\ell=o}^{\infty} t^{2\ell+1}\,\bar{P}_\ell(z)\,e^{-(\ell+1)z^2/2} \tag{2.1o}$$

with

$$\bar{P}_\ell(z) = \frac{1}{(\sqrt{2\pi})^{\ell+1}} \sum_{\lambda=o}^{\ell} a_\ell^{(\lambda)}\, z^{2(\ell-\lambda)} \,,$$

leading in a natural way to

$$c = \sqrt{2\pi} \,.$$

From (2.1o), $\Gamma(o,t)$ is given by

$$\Gamma(o,t) = \sum_{\ell=o}^{\infty} t^{2\ell+1}\,\bar{P}_\ell(o) \,.$$

For this value of z, convergence of (2.4) will cease to exist first, since for $|z| \neq o$ the exponentially decreasing factor in (2.1o) will enable convergence of the series even for larger times. This will be confirmed by the numerical results of Sections 2.3 and 2.4 .

2.2 More general initial conditions

The results obtained in the previous section are strictly limited to the particular choice of initial conditions (1.2). It is, however, possible to derive similar results for more general initial conditions. This will be shown in this section.

The reason for this generalization originated from the following observation: When the series (2.4) was calculated numerically it was found that for $t=1$ this series ceased to be meaningful for z values in a certain interval around $z=o$. This will be discussed in more detail in Section 2.4. Nevertheless, for larger times the Taylor series approach may be continued by fitting suitable initial conditions to the results obtained for some time $t<1$, say $t=o.9$.

Let the initial conditions be given by

$$\Gamma(z,o) = P_o(z)\, E^{1/2}(z) , \tag{2.11a}$$

$$\Gamma_t(z,o) = P_1(z)\, E(z) , \tag{2.11b}$$

where E is again the normalized Gaussian.

P_o and P_1 are arbitrary, but even polynomials of z, and P_1 must in addition satisfy the normalization condition (1.4):

$$\int_{-\infty}^{\infty} P_1(z)\, E(z)\, dz = 1 . \tag{2.12}$$

We note that the normalization of E is no longer necessary once (2.12) is fulfilled.

The Taylor series

$$\Gamma(z,t) = \sum_{\ell=o}^{\infty} t^{\ell}\, P_{\ell}(z)\, E^{(\ell+1)/2}(z) \tag{2.13}$$

satisfies the initial conditions (2.11). When this series is introduced into (1.1), it is easily seen that (2.13) is compatible with (1.1), i.e. equal powers in t also lead to equal powers in E . Therefore, the P_ℓ follow again from a recurrence relation for their coefficients.

Omitting the details we obtain for $\ell \geq 2$:

$$P_\ell(z) = \frac{2}{\ell(\ell-1)} \sum_{\substack{m\geq o \\ m+n=\ell-1}} \sum_{n\geq o} m[R'_{mn} - \frac{\ell+1}{2} z R_{mn}] , \qquad (2.14)$$

$$R_{mn}(z) = P_m [P'_n - \frac{n+1}{2} z P_n] .$$

In the same way as before, we derive from (2.14) a recurrence relation for the coefficients of the polynomials $P_\ell(z)$.

Since the initial conditions (2.11) are assumed to be even, and since eq. (1.1) is even in z , we may write

$$P_\ell(z) = \sum_{\lambda\geq o} a_\ell^{(\lambda)} z^{2\lambda} . \qquad (2.15)$$

The recurrence relation for the coefficients of this polynomial is given by

$$a_\ell^{(\lambda)} = \frac{2}{\ell(\ell-1)} \sum_{\substack{m\geq o \\ m+n=\ell-1}} \sum_{n\geq o} m \sum_{\substack{\mu\geq o \\ \mu+\nu=\lambda}} \sum_{\nu\geq o} A_n^{(\nu)} a_m^{(\mu)} , \qquad (2.16)$$

$$A_n^{(\nu)} = 2(2\lambda+1)(\nu+1) a_n^{(\nu+1)} - [(2\lambda+1)\frac{n+1}{2} + (\ell+1)\nu] a_n^{(\nu)}$$

$$+ \frac{1}{4}(\ell+1)(n+1) a_n^{(\nu-1)} .$$

The upper limit of λ in (2.15) will be derived next.

Let us assume that P_o and P_1 have highest powers according to

$$P_o(z) \sim z^{2r} ,$$

$$P_1(z) \sim z^{2s} .$$

If α_ℓ denotes the highest exponent of P_ℓ, we have from (2.14), using the relation

$$R_{mn} \sim z\, P_m\, P_n \; ,$$

that

$$\alpha_2 = 2(r+s+1) \; ,$$

$$\alpha_3 = \max\,\{4r+2s+4,\ 4s+2\} \; ,$$

which already indicates that the inequality $2r \geq s-1$ will be of importance.

We want to treat the two different possibilities $2r \geq s-1$ and $2r < s-1$ separately.

a) $2r \geq s-1$.

We show by induction that

$$\alpha_\ell = 2\ell(r+1) + 2(s-r-1) \; , \qquad \ell \geq 1 \; . \tag{2.17}$$

For $\ell=1$ this is obviously true; assuming the validity for ℓ, we get from (2.14):

$$P_{\ell+1}(z) \sim z^2\, P_m\, P_n \quad , \quad m+n=\ell \; .$$

Now we must distinguish between $m=o$, $n=\ell$ (or vice versa) and $m\geq 1$, $n\geq 1$.

Then

$$\alpha_{\ell+1} = \max\,\{\alpha_\ell + 2(r+1) \;,\; 2(m+n)(r+1) + 4(s-r-1) + 2\} \; .$$

Requiring that the first argument be greater or equal to the second, we just find the condition $2r \geq s-1$ to hold. Therefore, (2.17) is also valid for $\ell+1$.

b) $2r < s-1$.

We show by induction that

$$\begin{aligned} \alpha_{2\ell} &= 2\ell\ (s+1) + 2r\ , \qquad \ell = o,1,2,\ldots \\ \alpha_{2\ell+1} &= 2\ell\ (s+1) + 2s\ , \qquad \ell = o,1,2,\ldots \end{aligned} \tag{2.18}$$

For $\ell=o$, (2.18) is satisfied. Assuming (2.18) to be valid for ℓ , we have

$$P_{2\ell+2}(z) \sim z^2\, P_{2i}\, P_{2j+1}\ , \quad i+j = \ell\ ,$$

$$\begin{aligned} \alpha_{2\ell+2} &= 2(j+i)(s+1) + 2(r+s+1) \\ &= 2(\ell+1)(s+1) + 2r\ . \end{aligned}$$

This proves the first part of (2.18).

Next we distinguish between

$$P_{2\ell+3}(z) \sim z^2\, P_{2i}\, P_{2j}\ , \quad i+j = \ell+1 :$$

$$\alpha^{(1)}_{2\ell+3} = 2(\ell+1)(s+1) + 4r + 2\ ,$$

and

$$P_{2\ell+3}(z) \sim z^2\, P_{2i+1}\, P_{2j+1}\ , \quad i+j = \ell\ :$$

$$\alpha^{(2)}_{2\ell+3} = 2(\ell+1)(s+1) + 2s\ .$$

Since $2r < s-1$ also implies $\alpha^{(2)}_{2\ell+3} > \alpha^{(1)}_{2\ell+3}$, we have $\alpha_{2\ell+3} = \alpha^{(2)}_{2\ell+3}$, which completes the proof of (2.18).

2.3 Results for times $t \leq 1$

In this section we present results which have been obtained from a numerical evaluation of the series (2.1o) which can be approximated by

$$\Gamma(z,t) \cong \sum_{\ell=o}^{\ell_m} t^{2\ell+1}\, \bar{P}_\ell(z)\, e^{-(\ell+1)z^2/2} , \qquad (2.19)$$

$$\bar{P}_\ell(z) = \sum_{\lambda=o}^{\ell} \bar{a}_\ell^{(\lambda)}\, z^{2(\ell-\lambda)} , \qquad (2.2o)$$

$\bar{a}_\ell^{(\lambda)}$ from (2. 8) and $\bar{a}_o^{(o)} = 1/\sqrt{2\pi}$.

The FORTRAN programme of Appendix A3 was used to calculate $\bar{a}_\ell^{(\lambda)}$ in fourfold precision choosing LMAX = 7o which corresponds to $\ell_m = 69$ in (2.19). The fourfold precision (for IBM) guarantees a maximal accuracy of about 31 decimal places. In order to achieve also numerically a high accuracy, the sum in (2.2o) was split into an even and an odd part, because the $\bar{a}_\ell^{(\lambda)}$ are strictly alternating in sign (see Section 2.4).

In Tables 2.1-2.1o results are given for $t=o.1(o.1)1.o$. Because of the symmetry of $\Gamma(z,t)$ in z , the z values can be limited to negative values only, in our case $z = -4.o(o.1)o.o$.

The meaning of the columns is as follows:

GAMMA : $\Gamma(z,t)$
DG DT : $\Gamma_t(z,t)$
DG DZ : $\Gamma_z(z,t)$
D2G DT2 : $\Gamma_{tt}(z,t)$
PDE : error $| \Gamma_{tt} - 2\frac{\partial}{\partial z}(\Gamma_t\, \Gamma_z) |$
of the partial differential equation (1.1).

The following property is quite remarkable: For times $t \leq o.5$ the differential equation is satisfied exactly within fourfold precision. The results which are presented in Tables 2.1-2.1o have a length of 15 decimal places and are therefore exact up to $t=o.7$. These numbers will be important whenever analytic or numerical results for the initial value problem (cont. on p. 69)

INPUT:

D = 0.0

T = 0.10

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.1338320765737320-04	0.1338357782997300-03	0.5353347544609700-04	0.1110561122300230-06	0.4780-38
-3.90	0.1986593416543000-04	0.1986670824952830-03	0.7747844747894650-04	0.2322379473212090-06	0.1160-37
-3.80	0.2919548472303200-04	0.2919706910108400-03	0.1109454203675540-03	0.4753495302469580-06	0.2830-37
-3.70	0.4247961383295620-04	0.4248278763665240-03	0.1571795524120620-03	0.9522402927844170-06	0.3560-37
-3.60	0.6119330355564040-04	0.6119952530252890-03	0.2203052943133200-03	0.1866787448177430-05	0.7930-37
-3.50	0.8727423602088280-04	0.8728617074340750-03	0.3054771567687190-03	0.3581092847821500-05	0.1940-36
-3.40	0.1232331143471970-03	0.1232555135388550-02	0.4190237814550560-03	0.6721434656232420-05	0.3970-36
-3.30	0.1722774520346830-03	0.1723185783384070-02	0.5685703875333690-03	0.1234191044114280-04	0.6700-36
-3.20	0.2384457390696440-03	0.2385196001616730-02	0.7631202754396090-03	0.2216762971489810-04	0.9400-36
-3.10	0.3267467468314310-03	0.3268764811809440-02	0.1013071854576100-02	0.3894108632886810-04	0.1790-35
-3.00	0.4432961977030910-03	0.4435190226357900-02	0.1330144181213870-02	0.6689227459956290-04	0.3670-35
-2.90	0.5954402063101310-03	0.5958143675601250-02	0.1727181943149280-02	0.1123414835508840-03	0.7150-35
-2.80	0.7918519797036080-03	0.7924660883073400-02	0.2217811002261090-02	0.1844190793276660-03	0.1040-34
-2.70	0.1042585507386780-02	0.1043570457484250-01	0.2815918831093700-02	0.2958447297238300-03	0.1200-34
-2.60	0.1359067723037200-02	0.1360610988574140-01	0.3534941217773160-02	0.4636470489301680-03	0.1880-34
-2.50	0.1754009290382190-02	0.1756370742363120-01	0.4386948041693190-02	0.7096254306242260-03	0.4060-34
-2.40	0.2241214210644390-02	0.2244741652485760-01	0.5381536986645030-02	0.1060267958876830-02	0.6090-34
-2.30	0.2835270324611360-02	0.2840411746743440-01	0.6524564518889340-02	0.1545761629099300-02	0.1140-33
-2.20	0.3551106890525130-02	0.3558415129555030-01	0.7816767441615760-02	0.2197693507474170-02	0.1450-33
-2.10	0.4403412037353000-02	0.4413536347634100-01	0.9252354220611520-02	0.3045082604806910-02	0.2050-33
-2.00	0.5405912096940720-02	0.5419570490789960-01	0.1081767063412180-01	0.4108542788037040-02	0.1690-33
-1.90	0.6570526402655430-02	0.6588452971427240-01	0.1249006596274450-01	0.5392687001539060-02	0.3250-33
-1.80	0.7906424554490190-02	0.7929287832751100-01	0.1423710019113970-01	0.6877344497250360-02	0.4210-33
-1.70	0.9419027456307230-02	0.9447319581730800-01	0.1601623567385280-01	0.8508731680324480-02	0.4810-33
-1.60	0.1110900732339490-01	0.1114290930027150+00	0.1777514496775110-01	0.1019229136546230-01	0.5780-33
-1.50	0.1297135373581210-01	0.1301058905988490+00	0.1945273772581330-01	0.1178933698658530-01	0.7340-33
-1.40	0.1499458092825610-01	0.1503827711136210+00	0.2098096323222720-01	0.1311971946577830-01	0.9510-33
-1.30	0.1716015418951780-01	0.1720673772609540+00	0.2228738338698420-01	0.1397230583140700-01	0.1040-32
-1.20	0.1944220916692880-01	0.1948936230362940+00	0.2329843855490220-01	0.1412403542626100-01	0.1140-32
-1.10	0.2180762632215860-01	0.2185233185839480+00	0.2394325306956120-01	0.1336676294198420-01	0.1180-32
-1.00	0.2421650388740970-01	0.2425519607068420+00	0.2415775743769470-01	0.1153924649854490-01	0.1100-32
-0.90	0.2662304749043440-01	0.2665187300177230+00	0.2388884988989470-01	0.8559888036406760-02	0.9510-33
-0.80	0.2897686511388900-01	0.2899203978872050+00	0.2309828904641650-01	0.4454649139628650-02	0.8190-33
-0.70	0.3122462487804720-01	0.3122285221650200+00	0.2176600746316840-01	-0.6256405261438540-03	0.6860-33
-0.60	0.3331200328225260-01	0.3329090391951460+00	0.1989256418426370-01	-0.6403369357222190-02	0.5060-33
-0.50	0.3518582594840220-01	0.3514431708708200+00	0.1750051053878140-01	-0.1249026980208580-01	0.2890-33
-0.40	0.3679628407259020-01	0.3673484749926470+00	0.1463452115022830-01	-0.1842128770659970-01	0.4810-34
-0.30	0.3809909946310610-01	0.3801988746807100+00	0.1136023306930910-01	-0.2370272172009880-01	0.1080-33
-0.20	0.3905751008701830-01	0.3896425940973790+00	0.7761830925061540-02	-0.2786839541227940-01	0.1810-33
-0.10	0.3964395643084700-01	0.3954170809031210+00	0.3938506478040840-02	-0.3053584998033820-01	0.3490-33
0.00	0.3984136589043410-01	0.3973601863340290+00	-0.3957887303050360-21	-0.3145413672458350-01	0.2770-33

Table 2.1 : Results from the Taylor series expansion for t=0.1 .

INPUT:

D = 0.0

T = 0.20

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.267675259300340D-04	0.133852439153999D-03	0.107075263154338D-03	0.222177228036287D-06	0.110D-37
-3.90	0.397341908693528D-04	0.198701925343263D-03	0.154973780818599D-03	0.464666739979128D-06	0.231D-37
-3.80	0.583957233929839D-04	0.292042013763157D-03	0.221924383265270D-03	0.951241087180760D-06	0.411D-37
-3.70	0.849687513093307D-04	0.424970768232516D-03	0.314424248243380D-03	0.190596949714825D-05	0.999D-37
-3.60	0.122405278280516D-03	0.612275419406443D-03	0.440734261320368D-03	0.373752976104747D-05	0.197D-36
-3.50	0.174584291429888D-03	0.873399252045304D-03	0.611183782397560D-03	0.717234124860499D-05	0.359D-36
-3.40	0.246533464033298D-03	0.123356429524267D-02	0.838463596713043D-03	0.134680731514553D-04	0.796D-36
-3.30	0.344678373502240D-03	0.172503933473239D-02	0.113787763472981D-02	0.247442557220553D-04	0.893D-36
-3.20	0.477113270971251D-03	0.238852639014553D-02	0.152751515179493D-02	0.444752223025029D-04	0.249D-35
-3.10	0.653883165026700D-03	0.327461769730120D-02	0.202829629904284D-02	0.781951166513675D-04	0.428D-35
-3.00	0.887261880060029D-03	0.444524935346057D-02	0.266383640337838D-02	0.134459764991322D-03	0.785D-35
-2.90	0.119200499636617D-02	0.597504749928891D-02	0.346006909976877D-02	0.226088035651024D-03	0.136D-34
-2.80	0.158555049371406D-02	0.795242920363244D-02	0.444456726886820D-02	0.371656160627825D-03	0.196D-34
-2.70	0.208813398663192D-02	0.104802848910554D-01	0.564550526124205D-02	0.597132284898684D-03	0.331D-34
-2.60	0.272227803205847D-02	0.136760351276601D-01	0.709021795083534D-02	0.937408060531611D-03	0.519D-34
-2.50	0.351512983475217D-02	0.176708262328741D-01	0.880333347951511D-02	0.143731047291352D-02	0.813D-34
-2.40	0.449305678824076D-02	0.226076126693740D-01	0.108044885060731D-01	0.215147682595452D-02	0.122D-33
-2.30	0.568604039522190D-02	0.286378775864887D-01	0.131056780000454D-01	0.314228893296135D-02	0.236D-33
-2.20	0.712425673145350D-02	0.359167760775162D-01	0.157083455089126D-01	0.447495835878401D-02	0.217D-33
-2.10	0.883737341748263D-02	0.445965592198803D-01	0.186003818488688D-01	0.620893040654908D-02	0.229D-33
-2.00	0.108530490343096D-01	0.548182600136298D-01	0.217532651865961D-01	0.838514114568787D-02	0.373D-33
-1.90	0.131951655980916D-01	0.667017993868099D-01	0.251196352195003D-01	0.110094090681138D-01	0.734D-33
-1.80	0.158818540374328D-01	0.803348967284196D-01	0.286316372128305D-01	0.140333858941848D-01	0.903D-33
-1.70	0.189234071421853D-01	0.957614263306728D-01	0.322003844205904D-01	0.173359060644511D-01	0.128D-32
-1.60	0.223202081697202D-01	0.112970113377404D+00	0.357168564184779D-01	0.207089534941349D-01	0.141D-32
-1.50	0.260608312910845D-01	0.131884660172315D+00	0.390544663879449D-01	0.238533265524321D-01	0.170D-32
-1.40	0.301204868221278D-01	0.152356477325367D+00	0.420733908477202D-01	0.263888810914860D-01	0.194D-32
-1.30	0.344599846158796D-01	0.174161115220201D+00	0.446265710787291D-01	0.278825580497178D-01	0.211D-32
-1.20	0.390253696902747D-01	0.196999223858249D+00	0.465670899909138D-01	0.278942306297985D-01	0.220D-32
-1.10	0.437483446269156D-01	0.220502431430313D+00	0.477564350819393D-01	0.260362589426707D-01	0.221D-32
-1.00	0.485475365840357D-01	0.244243987728686D+00	0.480730139760372D-01	0.220385492471902D-01	0.224D-32
-0.90	0.533305993926952D-01	0.267753467421809D+00	0.474202233923832D-01	0.158081266351570D-01	0.202D-32
-0.80	0.579970718477403D-01	0.290534379635008D+00	0.457333990084231D-01	0.747174473664306D-02	0.172D-32
-0.70	0.624418510372375D-01	0.312083271366803D+00	0.429850861249127D-01	-0.260778590291487D-02	0.136D-32
-0.60	0.665590918625462D-01	0.331908882522436D+00	0.391882449098327D-01	-0.138465432620588D-01	0.903D-33
-0.50	0.702463152427776D-01	0.349550092680969D+00	0.343972047109135D-01	-0.254853180150375D-01	0.602D-33
-0.40	0.734084987827094D-01	0.364591730380693D+00	0.287063748092542D-01	-0.366607836328277D-01	0.265D-33
-0.30	0.759619327242640D-01	0.376677707207427D+00	0.222468778779115D-01	-0.464926601976507D-01	0.249D-32
-0.20	0.778376466269347D-01	0.385521305798064D+00	0.151813841508434D-01	-0.541755678556668D-01	0.229D-32
-0.10	0.789842436076435D-01	0.390912729100898D+00	0.769748864960612D-02	-0.590641839130139D-01	0.241D-33
0.00	0.793700147778108D-01	0.392724174643978D+00	-0.773338285058750D-21	-0.607417279440575D-01	0.107D-32

Table 2.2 : Results from the Taylor series expansion for t=0.2 .

INPUT:

D = 0.0

T = 0.30

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.4015406608339400-04	0.1338802180822100-03	0.1606336824965680-03	0.3334284571098980-06	0.1620-37
-3.90	0.5960709455907850-04	0.1987600285798910-03	0.2325029061232650-03	0.6974776601835800-06	0.3450-37
-3.80	0.8760547537847460-04	0.2921609754168730-03	0.3329704869197650-03	0.1428218416948410-05	0.7930-37
-3.70	0.1274769509670450-03	0.4252091697278940-03	0.4717993789411720-03	0.2862682992316990-05	0.1440-36
-3.60	0.1836546349118660-03	0.6127430233579780-03	0.6614109536369770-03	0.5616205044648420-05	0.2850-36
-3.50	0.2619660872370780-03	0.8742968552008880-03	0.9173502870283250-03	0.1078398051089400-04	0.5320-36
-3.40	0.3699685366530760-03	0.1235250438397780-02	0.1258737729852560-02	0.2026538056788690-04	0.1030-35
-3.30	0.5173268388633330-03	0.1728138688897100-02	0.1708663789625800-02	0.3726828704205470-04	0.1970-35
-3.20	0.7162257901888540-03	0.2394100452532880-02	0.2294470760751630-02	0.6706517283470380-04	0.3710-35
-3.10	0.9818020611300450-03	0.3284424912596300-02	0.3047850607716580-02	0.1180819385638010-03	0.6210-35
-3.00	0.1332573299383540-02	0.4462127773163090-02	0.4004675566925690-02	0.2033963388675650-03	0.1540-34
-2.90	0.1790833044394590-02	0.6003456441919620-02	0.5204467440060870-02	0.3426926392846700-03	0.1730-34
-2.80	0.2382970353358180-02	0.7999183610039340-02	0.6689405752874520-02	0.5646461197543980-03	0.2860-34
-2.70	0.3139663061955750-02	0.1055550227196040-01	0.8502776569032960-02	0.9095757232222130-03	0.4810-34
-2.60	0.4095884296482330-02	0.1379428409195210-01	0.1068677615160720-01	0.1431987990552250-02	0.7450-34
-2.50	0.5290654492560230-02	0.1785241128553850-01	0.1327961088650220-01	0.2202333808427190-02	0.1140-33
-2.40	0.6766467463773270-02	0.2287984784196660-01	0.1631188141609190-01	0.3306930437264550-02	0.1810-33
-2.30	0.8568321154985800-02	0.2903609227782900-01	0.1980230911783450-01	0.4844629384332790-02	0.1810-33
-2.20	0.1074229413896690-01	0.3648467076556700-01	0.2375295977965970-01	0.6918566876678440-02	0.3130-33
-2.10	0.1333363032585450-01	0.4538541159490050-01	0.2814424174640220-01	0.9621425048249810-02	0.4210-33
-2.00	0.1638432901114700-01	0.5588441631149900-01	0.3293009625159140-01	0.1301331972664330-01	0.5420-33
-1.90	0.1993028628884110-01	0.6810192918580700-01	0.3803393808187730-01	0.1709295030458570-01	0.1180-32
-1.80	0.2399809557794160-01	0.8211870334551410-01	0.4334601455280430-01	0.2176505471823900-01	0.1790-32
-1.70	0.2860168448222470-01	0.9796192963043440-01	0.4872288840323700-01	0.2681018377665600-01	0.2010-32
-1.60	0.3373903295043750-01	0.1155922442139050+00	0.5398967117898490-01	0.3186549032141090-01	0.2140-32
-1.50	0.3938927029366330-01	0.1348936297546730+00	0.5894540643961750-01	0.3642630812069860-01	0.2680-32
-1.40	0.4551047094849820-01	0.1556680254152900+00	0.6337162821273550-01	0.3987644200535740-01	0.4890-32
-1.30	0.5203844835171280-01	0.1776360606420170+00	0.6704364961935270-01	0.4154971606299590-01	0.1040-32
-1.20	0.5888677780498880-01	0.2004445307425800+00	0.6974366708716490-01	0.4081730286885840-01	0.1290-32
-1.10	0.6594816884270280-01	0.2236801847298290+00	0.7127441729993470-01	0.3718711993568860-01	0.3130-33
-1.00	0.7309717285559690-01	0.2468883567269720+00	0.7147199693339170-01	0.3039626849348560-01	0.3730-33
-0.90	0.8019407722752360-01	0.2695942243335240+00	0.7021658736559010-01	0.2047742478225350-01	0.1200-34
-0.80	0.8708972752126740-01	0.2913242161488280+00	0.6744017689560200-01	0.7785510748713630-02	0.2720-32
-0.70	0.9363095135359470-01	0.3116253502961470+00	0.6313083899164970-01	-0.7020128514904530-02	0.3830-32
-0.60	0.9966623696481460-01	0.3300809404829000+00	0.5733358493661510-01	-0.2303359283092190-01	0.2730-32
-0.50	0.1050513402967620+00	0.3463219381274520+00	0.5014816586106480-01	-0.3918913882725460-01	0.1960-32
-0.40	0.1096545435097910+00	0.3600339695749310+00	0.4172440533954890-01	-0.5436258329527310-01	0.2010-32
-0.30	0.1133613500358120+00	0.3709607242893070+00	0.3225570247712200-01	-0.6747383951001820-01	0.5780-33
-0.20	0.1160784633675240+00	0.3789046907663390+00	0.2197129306325600-01	-0.7757989188514360-01	0.3850-33
-0.10	0.1177369501088000+00	0.3837263339274920+00	0.1112774121604050-01	-0.8395114251019530-01	0.0
0.00	0.1182945283590880+00	0.3853427147513040+00	-0.1117541451119810-20	-0.8612729132432370-01	0.2120-32

Table 2.3 : Results from the Taylor series expansion for t=0.3 .

INPUT:

D = 0.0

T = 0.40

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.535439406841747D-04	0.133919131359584D-03	0.214217067079292D-03	0.444875121718660D-06	0.224D-37
-3.90	0.794869735039225D-04	0.198841440016685D-03	0.310082770045330D-03	0.930862865892779D-06	0.364D-37
-3.80	0.116829510908726D-03	0.292327711873065D-03	0.444117441182411D-03	0.190682871553262D-05	0.112D-36
-3.70	0.170013781197750D-03	0.425543454564576D-03	0.629370532851433D-03	0.382388914900375D-05	0.209D-36
-3.60	0.244960163555576D-03	0.613399058931210D-03	0.882460186646179D-03	0.750683816077837D-05	0.379D-36
-3.50	0.349455748494995D-03	0.875557074107640D-03	0.122420897303089D-02	0.144264067060976D-04	0.843D-36
-3.40	0.493606326929969D-03	0.123761992586005D-02	0.168026899595164D-02	0.271393498830657D-04	0.155D-35
-3.30	0.690348140816092D-03	0.173249914239566D-02	0.228168136833414D-02	0.499769103362061D-04	0.273D-35
-3.20	0.956009318902701D-03	0.240195373965245D-02	0.306529655611777D-02	0.900848251219985D-04	0.480D-35
-3.10	0.131090258133469D-02	0.329826628576183D-02	0.407396213021229D-02	0.158935510582960D-03	0.111D-34
-3.00	0.177992043067474D-02	0.448599859756687D-02	0.535636410850249D-02	0.274434194404773D-03	0.128D-34
-2.90	0.239309168742219D-02	0.604373285788385D-02	0.696638887554623D-02	0.463707358495869D-03	0.256D-34
-2.80	0.318604410071327D-02	0.806565570260426D-02	0.896185698096116D-02	0.766567211827753D-03	0.361D-34
-2.70	0.420030238660356D-02	0.106627802243285D-01	0.114024715164552D-01	0.123945515557443D-02	0.632D-34
-2.60	0.548333537947598D-02	0.139635229386230D-01	0.143468277037242D-01	0.195934192938882D-02	0.100D-33
-2.50	0.708825165785369D-02	0.181132621359536D-01	0.178483548375744D-01	0.302657354746365D-02	0.163D-33
-2.40	0.907303261361183D-02	0.232724102180597D-01	0.219501179771863D-01	0.456500825300497D-02	0.602D-34
-2.30	0.114991893247146D-01	0.296124566727621D-01	0.266785086611539D-01	0.671708105273966D-02	0.289D-33
-2.20	0.144297400578128D-01	0.373094176499036D-01	0.320360157082295D-01	0.963089281935784D-02	0.506D-33
-2.10	0.179264352805355D-01	0.465342183089261D-01	0.379934976591265D-01	0.134364795205729D-01	0.927D-33
-2.00	0.220462134079380D-01	0.574398029980450D-01	0.444826705147208D-01	0.182096801600109D-01	0.160D-32
-1.90	0.268369577568745D-01	0.701452744850307D-01	0.513898409402594D-01	0.239250875161678D-01	0.284D-32
-1.80	0.323327421691022D-01	0.847181151743396D-01	0.585521761765687D-01	0.304045855090110D-01	0.333D-32
-1.70	0.385488881355428D-01	0.101156444459393D+00	0.657578856961309D-01	0.372740293298651D-01	0.578D-32
-1.60	0.454772854336679D-01	0.119374078059955D+00	0.727514613643724D-01	0.439452797818149D-01	0.178D-32
-1.50	0.530825151221406D-01	0.139191520615086D+00	0.792444988924195D-01	0.496405684906049D-01	0.166D-32
-1.40	0.612993196566379D-01	0.160335595713846D+00	0.849316515067825D-01	0.534682990421840D-01	0.927D-33
-1.30	0.700318648649151D-01	0.182449069985200D+00	0.895101708631288D-01	0.545444498089649D-01	0.130D-32
-1.20	0.791550371321911D-01	0.205109619450471D+00	0.927006211308796D-01	0.521369394265521D-01	0.831D-32
-1.10	0.885177564272146D-01	0.227855440874415D+00	0.942660372829511D-01	0.457995425646174D-01	0.797D-32
-1.00	0.979480258674737D-01	0.250213449642861D+00	0.940271680888217D-01	0.354630619115127D-01	0.667D-32
-0.90	0.107259244783848D+00	0.271725781665742D+00	0.918723537340218D-01	0.214642466555118D-01	0.609D-32
-0.80	0.116257222658595D+00	0.291971197463036D+00	0.877616976922604D-01	0.451119905463839D-02	0.496D-32
-0.70	0.124747347434041D+00	0.310579525375887D+00	0.817261455958844D-01	-0.144002953288113D-01	0.111D-32
-0.60	0.132541455354695D+00	0.327238894058041D+00	0.738626607739642D-01	-0.340971469316699D-01	0.183D-32
-0.50	0.139464079943634D+00	0.341696746444334D+00	0.643268527712696D-01	-0.533440539218651D-01	0.156D-32
-0.40	0.145357888171722D+00	0.353756315134160D+00	0.533242722756457D-01	-0.709471353751604D-01	0.289D-32
-0.30	0.150088218753542D+00	0.363270417529799D+00	0.411012815528047D-01	-0.858374932835318D-01	0.963D-33
-0.20	0.153546711368423D+00	0.370134260622753D+00	0.279360709833439D-01	-0.971326122633627D-01	0.770D-33
-0.10	0.155654057368541D+00	0.374278606609655D+00	0.141300974503376D-01	-0.104177732702474D+00	0.116D-32
0.00	0.156361918440221D+00	0.375664273924269D+00	-0.141843660250491D-20	-0.106571191277523D+00	0.289D-32

Table 2.4 : Results from the Taylor series expansion for t=0.4 .

INPUT:

D = 0.0

T = 0.50

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.669382642537194D-04	0.133969201808663D-03	0.267833771899216D-03	0.556582863864721D-06	0.318D-37
-3.90	0.993761617171535D-04	0.198946226743177D-03	0.387730380211812D-03	0.116501583370841D-05	0.470D-37
-3.80	0.146072616546980D-03	0.292542413997787D-03	0.555399123923167D-03	0.238762438107499D-05	0.144D-36
-3.70	0.212588855336062D-03	0.425974148217043D-03	0.787203745784526D-03	0.479111596646560D-05	0.253D-36
-3.60	0.306340773954841D-03	0.614244926001122D-03	0.110400792723173D-02	0.941352479295822D-05	0.470D-36
-3.50	0.437089708328937D-03	0.877183517084580D-03	0.153199500546640D-02	0.181102602123240D-04	0.658D-36
-3.40	0.617515596876997D-03	0.124068175583606D-02	0.210348708348600D-02	0.341167796286721D-04	0.193D-35
-3.30	0.863869420090608D-03	0.173814240163286D-02	0.285769882044367D-02	0.629355696873327D-04	0.353D-35
-3.20	0.119669418266504D-02	0.241213689531014D-02	0.384133763747338D-02	0.113689161873724D-03	0.630D-35
-3.10	0.164159355758370D-02	0.331625606925502D-02	0.510893513344856D-02	0.201111739477853D-03	0.133D-34
-3.00	0.223001431173993D-02	0.451711097235200D-02	0.672276443506181D-02	0.348365687790487D-03	0.188D-34
-2.90	0.299999249410499D-02	0.609640164103145D-02	0.875216566760676D-02	0.590841382883828D-03	0.293D-34
-2.80	0.399679391318135D-02	0.815291308624395D-02	0.112720683568764D-01	0.980988182729484D-03	0.451D-34
-2.70	0.527335673763838D-02	0.108042130488001D-01	0.143604690463714D-01	0.159397216815021D-02	0.843D-34
-2.60	0.689041874346209D-02	0.141877086538885D-01	0.180946022937727D-01	0.253348379952533D-02	0.139D-33
-2.50	0.891618547801585D-02	0.184605664265877D-01	0.225455475204535D-01	0.393624598622381D-02	0.241D-34
-2.40	0.114253719435896D-01	0.237978207241833D-01	0.277710664305801D-01	0.597266357037222D-02	0.602D-34
-2.30	0.144974357665639D-01	0.303878205734501D-01	0.338066008754897D-01	0.883971440302769D-02	0.674D-33
-2.20	0.182138246658095D-01	0.384240629260312D-01	0.406546232315318D-01	0.127410193497337D-01	0.101D-32
-2.10	0.226541002243676D-01	0.480925504810211D-01	0.482729612550634D-01	0.178489412747092D-01	0.182D-32
-2.00	0.278908910938162D-01	0.595542579377667D-01	0.565633242370055D-01	0.242459777907702D-01	0.224D-32
-1.90	0.339837864671656D-01	0.729232488857172D-01	0.653619614787784D-01	0.318491396689500D-01	0.550D-32
-1.80	0.409725056013076D-01	0.882424871827860D-01	0.744349764954536D-01	0.403317421235849D-01	0.692D-32
-1.70	0.488699434504423D-01	0.105461164772236D+00	0.834809473252573D-01	0.490691950750139D-01	0.124D-32
-1.60	0.576559296571572D-01	0.124418747506641D+00	0.921427792092949D-01	0.571418268906489D-01	0.223D-32
-1.50	0.672726518421123D-01	0.144840985517748D+00	0.100028978447354D+00	0.634196904473181D-01	0.847D-32
-1.40	0.776225951988466D-01	0.166351167329498D+00	0.106742128876060D+00	0.667283335985371D-01	0.963D-32
-1.30	0.885695126472494D-01	0.188496159027153D+00	0.111910161627884D+00	0.660597579478582D-01	0.732D-32
-1.20	0.999424376612435D-01	0.210782632958233D+00	0.115215058799225D+00	0.607686271448368D-01	0.746D-32
-1.10	0.111542241427042D+00	0.232716193648336D+00	0.116414427209593D+00	0.506963107933394D-01	0.636D-32
-1.00	0.123149879714377D+00	0.253835995276035D+00	0.115353532170816D+00	0.361937165853325D-01	0.554D-32
-0.90	0.134535360487061D+00	0.273739667759897D+00	0.111967878644567D+00	0.180508571003033D-01	0.477D-32
-0.80	0.145466573343416D+00	0.292096622705888D+00	0.106278291728640D+00	-0.263273532138237D-02	0.313D-33
-0.70	0.155717365809180D+00	0.308650683224778D+00	0.983812416397803D-01	-0.245958736013968D-01	0.674D-33
-0.60	0.165074525025460D+00	0.323214684669159D+00	0.884370135776064D-01	-0.465497433664637D-01	0.125D-32
-0.50	0.173343552652646D+00	0.335660179907157D+00	0.766576584721558D-01	-0.672809002007972D-01	0.231D-32
-0.40	0.180353275058931D+00	0.345905040431949D+00	0.632958800660444D-01	-0.857194461191154D-01	0.270D-32
-0.30	0.185959412137474D+00	0.353901037362469D+00	0.486353606299775D-01	-0.100977094635746D+00	0.597D-32
-0.20	0.190047254394199D+00	0.359622752593423D+00	0.329825845733087D-01	-0.112363465728607D+00	0.597D-32
-0.10	0.192533589980633D+00	0.363058581670053D+00	0.166599693200030D-01	-0.119389237509904D+00	0.385D-33
0.00	0.193367997756157D+00	0.364204193014406D+00	-0.167163549552225D-20	-0.121763331332183D+00	0.250D-32

Table 2.5 : Results from the Taylor series expansion for t=0.5 .

INPUT:

D = 0.0

T = 0.60

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.8033815392450320-04	0.1340304588366760-03	0.3214921805448120-03	0.6686177534907800-06	0.3990-37
-3.90	0.1192770008911340-03	0.1990744752811670-03	0.4654628258349270-03	0.1400131807839290-05	0.5580-37
-3.80	0.1753396007159720-03	0.2928053282312490-03	0.6668496381272780-03	0.2871164799414780-05	0.1790-36
-3.70	0.2552118469452260-03	0.4265019302239600-03	0.9453656541457380-03	0.5765917881459180-05	0.3230-36
-3.60	0.3678155364677430-03	0.6152824385856250-03	0.1326181691451420-02	0.1134045635341180-04	0.7140-36
-3.50	0.5249048144135400-03	0.8791808723269560-03	0.1840947412300120-02	0.2184651314723440-04	0.9400-36
-3.40	0.7417661440273410-03	0.1244447658852740-02	0.2528830991216740-02	0.4122556854200250-04	0.2490-35
-3.30	0.1038020320683200-02	0.1745096883696100-02	0.3437506280723100-02	0.7621320086760900-04	0.4040-35
-3.20	0.1438516562691880-02	0.2424716565572810-02	0.4623987900872540-02	0.1380437680820810-03	0.9450-35
-3.10	0.1974297073986830-02	0.3338545566711880-02	0.6155179919327290-02	0.2449973216487910-03	0.6770-35
-3.00	0.2683594987770050-02	0.4555797322475520-02	0.8107961314727180-02	0.4260686367260670-03	0.2330-34
-2.90	0.3612808414185850-02	0.6162169289322300-02	0.1056857964279040-01	0.7260303884277940-03	0.4360-34
-2.80	0.4817367441553130-02	0.8262407145080560-02	0.1363106232736060-01	0.1212049935253020-02	0.6240-34
-2.70	0.6362378406914390-02	0.1098267960580140-01	0.1739428445191090-01	0.1981702837629700-02	0.1020-33
-2.60	0.8322890031691660-02	0.1447234567140850-01	0.2195726093640640-01	0.3171556497402510-02	0.1720-33
-2.50	0.1078357998068030-01	0.1890444576559080-01	0.2741218071270360-01	0.4964283684668330-02	0.7220-34
-2.40	0.1383761208314770-01	0.2447392119604800-01	0.3383471500284450-01	0.7590346918137050-02	0.6140-33
-2.30	0.1758437390929480-01	0.3139220316334850-01	0.4127128909292290-01	0.1131773943306560-01	0.9150-33
-2.20	0.2212579143213200-01	0.3987653651320000-01	0.4972342344176700-01	0.1642078515888300-01	0.1730-32
-2.10	0.2756096475671490-01	0.5013246800414000-01	0.5913005826786020-01	0.2311851760933890-01	0.1910-32
-2.00	0.3397902007124210-01	0.6232872617365620-01	0.6935004461878210-01	0.3147813153521380-01	0.5770-32
-1.90	0.4145036897391930-01	0.7656565622719570-01	0.8014855620084610-01	0.4129326460771760-01	0.7460-33
-1.80	0.5001701240522500-01	0.9284150771881920-01	0.9119246691791570-01	0.5197055309598290-01	0.1440-32
-1.70	0.5968304549884000-01	0.1110243771449450+00	0.1020597122963580+00	0.6248125510407640-01	0.7700-32
-1.60	0.7040692479478850-01	0.1308395174972170+00	0.1122652773183110+00	0.7143664110380240-01	0.8670-32
-1.50	0.8209710120431480-01	0.1518797510635570+00	0.1213017011038810+00	0.7730736001872710-01	0.7320-32
-1.40	0.9461213660142000-01	0.1736402202299850+00	0.1286866339412630+00	0.7873659804991730-01	0.7700-32
-1.30	0.1077654937951200+00	0.1955702373418170+00	0.1340069709696700+00	0.7483873357346460-01	0.4810-32
-1.20	0.1213341696587760+00	0.2171291643114740+00	0.1369503844526900+00	0.6537569843921130-01	0.2890-32
-1.10	0.1350696464460090+00	0.2378332313771370+00	0.1373199226656590+00	0.5076406135039280-01	0.2910-32
-1.00	0.1487094871797750+00	0.2572852670428050+00	0.1350328761472120+00	0.3194147726164740-01	0.7220-34
-0.90	0.1619882229216850+00	0.2751859364243420+00	0.1301086936755890+00	0.1016426266517270-01	0.9630-34
-0.80	0.1746467205972160+00	0.2913299903019850+00	0.1226515588763660+00	-0.1319359666763790-01	0.1390-31
-0.70	0.1864397361038050+00	0.3055930166739790+00	0.1128321224140700+00	-0.3679713571948620-01	0.1200-30
-0.60	0.1971417183446820+00	0.3179138700352460+00	0.1008710923604800+00	-0.5944971472242010-01	0.7030-30
-0.50	0.2065511171613100+00	0.3282765469124610+00	0.8702580484942850-01	-0.8014138571657680-01	0.1320-29
-0.40	0.2144935037857440+00	0.3366937376893290+00	0.7157986124841460-01	-0.9806077139396970-01	0.5780-29
-0.30	0.2208237913118170+00	0.3431930823945730+00	0.5483539662206470-01	-0.1125852879724790+00	0.3210-28
-0.20	0.2254277884533600+00	0.3478063944357920+00	0.3710737399176600-01	-0.1232615362491520+00	0.4050-28
-0.10	0.2282232592551150+00	0.3505617193210450+00	0.1871931836462790-01	-0.1297840164099700+00	0.3190-28
0.00	0.2291606072991510+00	0.3514779493147340+00	-0.1877461011456050-20	-0.1319772292449880+00	0.9420-28

Table 2.6 : Results from the Taylor series expansion for t=0.6 .

INPUT:

D = 0.0

T = 0.70

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.9374473010092410-04	0.1341029384838060-03	0.3752007124595460-03	0.7810463980933450-06	0.4590-37
-3.90	0.1391918423501440-03	0.1992262917842640-03	0.5432972985696010-03	0.1636408257229550-05	0.6170-37
-3.80	0.2046352992933100-03	0.2931167573776750-03	0.7785033699377390-03	0.3358018174469450-05	0.2090-36
-3.70	0.2978925053673520-03	0.4271276371818240-03	0.1103923651528820-02	0.6749882810325150-05	0.3670-36
-3.60	0.4294037240030690-03	0.6165138357102600-03	0.1549110954333010-02	0.1329194603742730-04	0.8520-36
-3.50	0.6129384387712040-03	0.8815549456681840-03	0.2151310125182720-02	0.2564656214724420-04	0.1080-35
-3.40	0.8664290806866090-03	0.1248932223463820-02	0.2956751594848880-02	0.4849503380401540-04	0.2730-35
-3.30	0.1212933680089490-02	0.1753398117176640-02	0.4021921717521880-02	0.8988328339631470-04	0.6770-35
-3.20	0.1681720154501190-02	0.2439776622851580-02	0.5414704347934880-02	0.1633281838774180-03	0.1160-34
-3.10	0.2309452342342600-02	0.3365326712505560-02	0.7215245171320190-02	0.2910200410252130-03	0.7520-35
-3.00	0.3141440314802530-02	0.4602483369758710-02	0.9516332849041030-02	0.5085367709473840-03	0.2630-34
-2.90	0.4232893207020040-02	0.6241950706839330-02	0.1242301311352090-01	0.8715230534242120-03	0.4660-34
-2.80	0.5650079414094950-02	0.8396041482121270-02	0.1605104665284540-01	0.1464700967574540-02	0.7820-34
-2.70	0.7471253467182370-02	0.1120200967376400-01	0.2052368376126370-01	0.2413208547781000-02	0.1380-33
-2.60	0.9787148025441850-02	0.1482486929106820-01	0.2596606142036500-01	0.3895332711478500-02	0.1220-33
-2.50	0.1270075125317810-01	0.1945878885813140-01	0.3249635735763410-01	0.6153795895087020-02	0.1810-33
-2.40	0.1632599717742820-01	0.2532556709860770-01	0.4021273491785350-01	0.9499396929300540-02	0.7460-33
-2.30	0.2078490365716120-01	0.3266797059084690-01	0.4917524404782350-01	0.1429585604351790-01	0.1470-32
-2.20	0.2620263575789710-01	0.4173506576047560-01	0.5938249337484530-01	0.2090938373998260-01	0.1950-32
-2.10	0.3270002239633440-01	0.5275665442816060-01	0.7074446293049130-01	0.2960510706357510-01	0.4040-32
-2.00	0.4038331803954990-01	0.6590548330373010-01	0.8305556385709220-01	0.4038377681224170-01	0.1080-33
-1.90	0.4933158740578700-01	0.8125008450396570-01	0.9597560523887020-01	0.5278592716313580-01	0.7220-33
-1.80	0.5958300868085200-01	0.9870788839629520-01	0.1090289785815400+00	0.6574398234588650-01	0.8090-32
-1.70	0.7112240205150970-01	0.1180148434230480+00	0.1216310368534100+00	0.7760103030497340-01	0.6740-32
-1.60	0.8387285689401580-01	0.1387284805627400+00	0.1331429400123150+00	0.8637855845891020-01	0.4430-32
-1.50	0.9769386517762520-01	0.1602716473176600+00	0.1429441360935670+00	0.9024379974962680-01	0.9240-32
-1.40	0.1123867653927240+00	0.1820070542929140+00	0.1505024642600010+00	0.8798473522330670-01	0.1390-30
-1.30	0.1277062461764210+00	0.2033192748069690+00	0.1554227232300900+00	0.7927971338075770-01	0.4800-29
-1.20	0.1433752121476520+00	0.2236802309188700+00	0.1574650107266800+00	0.6467560371044820-01	0.6620-28
-1.10	0.1591000843850850+00	0.2426851975054470+00	0.1565364948707830+00	0.4534740123336480-01	0.3180-26
-1.00	0.1745843833852880+00	0.2600597604153440+00	0.1526672581766300+00	0.2278382951785900-01	0.7870-25
-0.90	0.1895395640263140+00	0.2756463499840520+00	0.1459810591407140+00	-0.1484281460320510-02	0.1590-23
-0.80	0.2036929933293300+00	0.2893805652370050+00	0.1366682799077990+00	-0.2606240045218630-01	0.2390-22
-0.70	0.2167934821198070+00	0.3012651676268140+00	0.1249643221096360+00	-0.4977932380080760-01	0.2390-21
-0.60	0.2286149565089870+00	0.3113463239713010+00	0.1111339364395840+00	-0.7170800779809520-01	0.1320-20
-0.50	0.2389588248727950+00	0.3196940235190680+00	0.9546060012157860-01	-0.9114788694554980-01	0.2360-20
-0.40	0.2476554812572330+00	0.3263869710479940+00	0.7823965083544670-01	-0.1075899141294280+00	0.1150-19
-0.30	0.2545652587989980+00	0.3315014751160140+00	0.5977397207681480-01	-0.1206777445601640+00	0.6130-19
-0.20	0.2595790388206790+00	0.3351035926840860+00	0.4037128865714910-01	-0.1301723611453530+00	0.7630-19
-0.10	0.2626186413696750+00	0.3372438062473150+00	0.2034241111843070-01	-0.1359235189133550+00	0.6140-19
0.00	0.2636370689677250+00	0.3379536419060370+00	-0.2039469515328780-20	-0.1378492300523400+00	0.1790-18

Table 2.7 : Results from the Taylor series expansion for t=0.7 .

INPUT:

D = 0.0

T = 0.80

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.107159117125470D-03	0.134186683482495D-03	0.428967830282639D-03	0.893936054109399D-06	0.514D-37
-3.90	0.159123049025457D-03	0.199401802283637D-03	0.621251113682319D-03	0.187404534301108D-05	0.852D-37
-3.80	0.233964581295087D-03	0.293477061580953D-03	0.890395047893956D-03	0.384876341556997D-05	0.238D-36
-3.70	0.340640671608685D-03	0.427852266457156D-03	0.126294604970837D-02	0.774463948891224D-05	0.514D-36
-3.60	0.491124855175198D-03	0.617941794949313D-03	0.177292757515552D-02	0.152724563547881D-04	0.423D-36
-3.50	0.701228591821490D-03	0.884312707953116D-03	0.246333307242929D-02	0.295223280967936D-04	0.113D-35
-3.40	0.991577127395355D-03	0.125415305526557D-02	0.338771438219055D-02	0.559562603191123D-04	0.339D-35
-3.30	0.138874626508436D-02	0.176308925447888D-02	0.461179760213971D-02	0.104025024248602D-03	0.804D-35
-3.20	0.192655796654282D-02	0.245741975048279D-02	0.621502293744982D-02	0.189739787629481D-03	0.126D-34
-3.10	0.264751999021815D-02	0.339683677709024D-02	0.829185467273374D-02	0.339661094559537D-03	0.113D-34
-3.00	0.360437592839871D-02	0.465770154195547D-02	0.109526332926207D-01	0.596917536842906D-03	0.354D-34
-2.90	0.486170402102173D-02	0.633690584008066D-02	0.143236304425283D-01	0.102999282604091D-02	0.594D-34
-2.80	0.649746138417455D-02	0.855626770991563D-02	0.185458000920739D-01	0.174501462256542D-02	0.880D-34
-2.70	0.860430809634045D-02	0.114672228847345D-01	0.237714738396987D-01	0.290189555206411D-02	0.158D-33
-2.60	0.112904541896275D-01	0.152552114795939D-01	0.301579084563624D-01	0.473341962143113D-02	0.481D-34
-2.50	0.146796428911005D-01	0.201425141059670D-01	0.378561752794594D-01	0.756335915015606D-02	0.144D-33
-2.40	0.189097118725997D-01	0.263872537095034D-01	0.469934929870516D-01	0.118137176843746D-01	0.831D-33
-2.30	0.241289773673034D-01	0.342748448101549D-01	0.576470658535559D-01	0.179813581893702D-01	0.159D-32
-2.20	0.304895292156517D-01	0.440967170783064D-01	0.698084821773491D-01	0.265526348359137D-01	0.315D-32
-2.10	0.381365631950713D-01	0.561108982350761D-01	0.833408371919938D-01	0.378214882236246D-01	0.142D-32
-2.00	0.471933751453467D-01	0.704824477821079D-01	0.979369021449254D-01	0.516050715336594D-01	0.520D-32
-1.90	0.577428803403247D-01	0.872114423324666D-01	0.113094841518635D+00	0.669359995505829D-01	0.539D-32
-1.80	0.698084728681630D-01	0.106071176695950D+00	0.128132695599842D+00	0.819283885495360D-01	0.156D-31
-1.70	0.833389083651554D-01	0.126589647565042D+00	0.142255026248450D+00	0.940418494700710D-01	0.582D-29
-1.60	0.982021561872449D-01	0.148097890943387D+00	0.154661510805762D+00	0.100764321868921D+00	0.104D-26
-1.50	0.114190777393090D+00	0.169836525258217D+00	0.164661551046928D+00	0.100397213466826D+00	0.117D-25
-1.40	0.131037300227776D+00	0.191079535893451D+00	0.171753947543302D+00	0.925053849575080D-01	0.644D-23
-1.30	0.148434870633782D+00	0.211229598391676D+00	0.175651997000276D+00	0.778484681486755D-01	0.450D-21
-1.20	0.166057949922967D+00	0.229862476463792D+00	0.176262536635300D+00	0.579583564713814D-01	0.902D-20
-1.10	0.183579553299880D+00	0.246725577802057D+00	0.173641706691513D+00	0.346517790345341D-01	0.367D-18
-1.00	0.200683789677016D+00	0.261709303394696D+00	0.167947899002517D+00	0.966809531016638D-02	0.911D-17
-0.90	0.217074058720203D+00	0.274809101587517D+00	0.159403023600466D+00	-0.155222908370864D-01	0.181D-15
-0.80	0.232477940088038D+00	0.286089400407839D+00	0.148265115348443D+00	-0.397763165995924D-01	0.267D-14
-0.70	0.246649864818179D+00	0.295654195154533D+00	0.134810970842217D+00	-0.622593564759962D-01	0.261D-13
-0.60	0.259372440840463D+00	0.303625117097331D+00	0.119326129776028D+00	-0.823928459133366D-01	0.141D-12
-0.50	0.270457034589355D+00	0.310126012702080D+00	0.102099633673388D+00	-0.997945554492523D-01	0.237D-12
-0.40	0.279743983178003D+00	0.315272549455621D+00	0.834216089938164D-01	-0.114224140627120D+00	0.129D-11
-0.30	0.287102648942626D+00	0.319165443070593D+00	0.635823590111249D-01	-0.125538584045206D+00	0.665D-11
-0.20	0.292431423131580D+00	0.321886179662030D+00	0.428721535212656D-01	-0.133658162772697D+00	0.814D-11
-0.10	0.295657723339042D+00	0.323494406472772D+00	0.215812607639755D-01	-0.138541966685066D+00	0.672D-11
0.00	0.296737996289759D+00	0.324026422598351D+00	-0.216296574456582D-20	-0.140171610502241D+00	0.194D-10

Table 2.8 : Results from the Taylor series expansion for t=0.8 .

INPUT:

D = 0.0

T = 0.90

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.1205824439513800-03	0.1342817433279960-03	0.4828020472835830-03	0.1007354740452700-05	0.3820-37
-3.90	0.1790729974573560-03	0.1996011529811180-03	0.6993417315974740-03	0.2113246398520870-05	0.8820-37
-3.80	0.2633323546666060-03	0.2938866594923930-03	0.1002559803626600-02	0.4343992094487380-05	0.2760-36
-3.70	0.3834662946333920-03	0.4286769806504380-03	0.1422502245015450-02	0.8751865179549250-05	0.5910-36
-3.60	0.5529987389536470-03	0.6195694488087940-03	0.1997766241812020-02	0.1728662849843250-04	0.3760-36
-3.50	0.7898140416802770-03	0.8874623528056950-03	0.2777273336886450-02	0.3348636460550850-04	0.1460-35
-3.40	0.1117284924101810-02	0.1260130972746010-02	0.3822202400125360-02	0.6364248965462830-04	0.4180-35
-3.30	0.1565599564918910-02	0.1774221710703920-02	0.5208028242210640-02	0.1187247129067990-03	0.7990-35
-3.20	0.2173294297797090-02	0.2477769449268980-02	0.7026576436590290-02	0.2174983848711780-03	0.1530-34
-3.10	0.2988986885459510-02	0.3433364436401920-02	0.9387949808332620-02	0.3914701901948670-03	0.1350-34
-3.00	0.4073286758084400-02	0.4722108651554660-02	0.1242209192410410-01	0.6925602786288950-03	0.3910-34
-2.90	0.5500828251353350-02	0.6448489220489180-02	0.1627960227545730-01	0.1204686622444740-02	0.6850-34
-2.80	0.7362322829101090-02	0.8746218752714990-02	0.2113116634098860-01	0.2060630847502910-02	0.1080-33
-2.70	0.9766443675689900-02	0.1178487182729880-01	0.2716455310226840-01	0.3465269624038670-02	0.2410-34
-2.60	0.1284122152219520-01	0.1577664051135400-01	0.3457749356241450-01	0.5724631255939370-02	0.3610-34
-2.50	0.1673441870902950-01	0.2098150463902220-01	0.4356384662003960-01	0.9275469365155260-02	0.5540-33
-2.40	0.2161203619741130-01	0.2770725430939730-01	0.5428940353232940-01	0.1469909210207280-01	0.1130-32
-2.30	0.2765370461463120-01	0.3629793074106280-01	0.6685307643277310-01	0.2268295363546430-01	0.2730-32
-2.20	0.3504333069341030-01	0.4710100898644690-01	0.8123069098813880-01	0.3386736327513180-01	0.2680-32
-2.10	0.4395336473918460-01	0.6040296706444440-01	0.9720521047155320-01	0.4850914438350060-01	0.4570-32
-2.00	0.5452210048318470-01	0.7633128646918320-01	0.1143018632421890+00	0.6597522140489190-01	0.2040-31
-1.90	0.6682623088502730-01	0.9474497856681590-01	0.1317656293411080+00	0.8430405786668930-01	0.5480-28
-1.80	0.8085509364733030-01	0.1151678538403370+00	0.1486230920028540+00	0.1003040621127630+00	0.1040-24
-1.70	0.9649589204838340-01	0.1368247465275670+00	0.1638362570310350+00	0.1104795477200650+00	0.5480-22
-1.60	0.1135368138916640+00	0.1587893914484580+00	0.1764927647616110+00	0.1123582327947490+00	0.7420-20
-1.50	0.1316872944532360+00	0.1801764302221660+00	0.1859452597828370+00	0.1053130228713600+00	0.2030-18
-1.40	0.1506073479988290+00	0.2002870667845270+00	0.1918510618104610+00	0.9042835940821590-01	0.6280-16
-1.30	0.1699363731218320+00	0.2186654925284390+00	0.1941289411470980+00	0.6976007607825000-01	0.4780-14
-1.20	0.1893154982810580+00	0.2350816671959030+00	0.1928812465559910+00	0.4557485025881470-01	0.1210-12
-1.10	0.2084021091407130+00	0.2494787272837930+00	0.1883199214572100+00	0.1987575774639440-01	0.4710-11
-1.00	0.2268779235655890+00	0.2619153646299160+00	0.1807125069566170+00	-0.5780921447250620-02	0.1170-09
-0.90	0.2444527306731590+00	0.2725178041353130+00	0.1703488047207330+00	-0.3028852183869550-01	0.2280-08
-0.80	0.2608656190170610+00	0.2814449057580450+00	0.1575229929593670+00	-0.5291462259539380-01	0.3300-07
-0.70	0.2758849483830740+00	0.2888648344307940+00	0.1425255340252090+00	-0.7320739723872890-01	0.3170-06
-0.60	0.2893078166302960+00	0.2949403312457400+00	0.1256405958232400+00	-0.9091414812478940-01	0.1680-05
-0.50	0.3009594234376320+00	0.2998199442533450+00	0.1071461557086310+00	-0.1059051640277560+00	0.2660-05
-0.40	0.3106925149914800+00	0.3036329746085140+00	0.8731557169381320-01	-0.1181264691287910+00	0.1620-04
-0.30	0.3183869677665020+00	0.3064859958193190+00	0.6641821641592330-01	-0.1276858829063840+00	0.8050-04
-0.20	0.3239495424355510+00	0.3084649690516780+00	0.4472185164191400-01	-0.1342971171421470+00	0.9720-04
-0.10	0.3273136853161850+00	0.3096277674326150+00	0.2249476944613660-01	-0.1383322496064280+00	0.8200-04
0.00	0.3284394887098940+00	0.3100097466303570+00	-0.2253000421320960-20	-0.1399239876891310+00	0.2340-03

Table 2.9 : Results from the Taylor series expansion for t=0.9 .

INPUT:

D = 0.0

T = 1.00

RESULTS:

POSITION	GAMMA	DG DT	DG DZ	D2G DT2	PDE
-4.00	0.1340158448229250D-03	0.1343881743604040D-03	0.5367119349069750D-03	0.1121371354587990D-05	0.353D-37
-3.90	0.1990440798288750D-03	0.1998245105910390D-03	0.7775867798895670D-03	0.2354218424223690D-05	0.999D-37
-3.80	0.2927435721997950D-03	0.2943460296382290D-03	0.1115033234280430D-02	0.4844310485657210D-05	0.356D-36
-3.70	0.4263794481429660D-03	0.4296031128521950D-03	0.1582662891140980D-02	0.9773293819830370D-05	0.652D-36
-3.60	0.6150455211013300D-03	0.6214004041545700D-03	0.2223764938141610D-02	0.1933931394909630D-04	0.705D-36
-3.50	0.8787344397951540D-03	0.8910133657658440D-03	0.3093396392725250D-02	0.3755197728118910D-04	0.150D-35
-3.40	0.1243629362942710D-02	0.1266890244426430D-02	0.4260719462631120D-02	0.7158955878965370D-04	0.588D-35
-3.30	0.1743640655525270D-02	0.1786855948584540D-02	0.5811557946938530D-02	0.1340772963172810D-03	0.931D-35
-3.20	0.2422206940417200D-02	0.2500972547297490D-02	0.7851114827755210D-02	0.2468517272584550D-03	0.451D-35
-3.10	0.3334371648938010D-02	0.3475257529911830D-02	0.1050673963704870D-01	0.4470844145369250D-03	0.150D-34
-3.00	0.4549130572888770D-02	0.4796509082367690D-02	0.1393053353206750D-01	0.7970790559013290D-03	0.451D-34
-2.90	0.6152016177191300D-02	0.6578516916345460D-02	0.1830138912784550D-01	0.1399627848844070D-02	0.700D-34
-2.80	0.8247828073231180D-02	0.8969895216731500D-02	0.2382570412675440D-01	0.2421389864918750D-02	0.181D-33
-2.70	0.1096331340977900D-01	0.1216353999898180D-01	0.3073535311639910D-01	0.4126831025912290D-02	0.120D-34
-2.60	0.1444940710406130D-01	0.1640702834779310D-01	0.3928035515737040D-01	0.6923352600331770D-02	0.963D-34
-2.50	0.1888229926270800D-01	0.2201164310616620D-01	0.4971181545353570D-01	0.1141064734565490D-01	0.951D-33
-2.40	0.2446203074254150D-01	0.2935432157034870D-01	0.6224815161346050D-01	0.1840661735380180D-01	0.219D-32
-2.30	0.3140649479328000D-01	0.3886092650423940D-01	0.7701539577567860D-01	0.2887862667559860D-01	0.173D-32
-2.20	0.3993784236371500D-01	0.5095189527050130D-01	0.9395447884393610D-01	0.4364788140595120D-01	0.564D-31
-2.10	0.5025818728848380D-01	0.6593013495903380D-01	0.1127037128666850D+00	0.6273248544592210D-01	0.860D-28
-2.00	0.6251400591097160D-01	0.8381451461302620D-01	0.1325018453169830D+00	0.8445019984479160D-01	0.433D-25
-1.90	0.7675545020068200D-01	0.1041846721928380D+00	0.1521995938941040D+00	0.1050017961786780D+00	0.103D-21
-1.80	0.9290552088465960D-01	0.1261614856362110D+00	0.1704492640799950D+00	0.1194966224964000D+00	0.194D-18
-1.70	0.1107554508405690D+00	0.1485959118887020D+00	0.1860165819802260D+00	0.1241787265254420D+00	0.932D-16
-1.60	0.1299901313770950D+00	0.1703726774818460D+00	0.1980382539165180D+00	0.1180492741416590D+00	0.115D-13
-1.50	0.1502305894619000D+00	0.1906458343158680D+00	0.2060968854070040D+00	0.1026775563261140D+00	0.330D-12
-1.40	0.1710753854151290D+00	0.2089144697626470D+00	0.2101393430557970D+00	0.8088996562582910D-01	0.922D-10
-1.30	0.1921303275434770D+00	0.2249764583257270D+00	0.2103414993261480D+00	0.5556456651538610D-01	0.948D-08
-1.20	0.2130252929325710D+00	0.2388377639726970D+00	0.2069930876878060D+00	0.2902498398907480D-01	0.280D-06
-1.10	0.2334215955696100D+00	0.2506277601351030D+00	0.2004241694112350D+00	0.2922151237726210D-02	0.106D-04
-1.00	0.2530138354537140D+00	0.2605351381207580D+00	0.1909657729920580D+00	-0.2197858318686490D-01	0.262D-03
-0.90	0.2715294288023530D+00	0.2688113904005150D+00	0.1789479558856330D+00	-0.3963465988636020D-01	0.506D-02
-0.80	0.2887224662564450D+00	0.2751308959268500D+00	0.1645677779233670D+00	-0.1243165577761830D+00	0.722D-01
-0.70	0.3044156225098790D+00	0.2846990726420700D+00	0.1478638621011330D+00	0.4195295786194040D+00	0.684D+00
-0.60	0.3182209031447050D+00	0.2694666515630250D+00	0.1420015314283700D+00	-0.2319314557405960D+01	0.355D+01
-0.50	0.3304646852078860D+00	0.2970916938281060D+00	0.2850954167456600D-01	0.9666416837466160D+00	0.346D+01
-0.40	0.3421934273376160D+00	0.5347726589964260D+00	0.3278021503295400D+00	0.3374513662778300D+02	0.265D+02
-0.30	0.3416314853516860D+00	-0.6532011465083900D+00	0.7951135729411930D-03	-0.1317068590264990D+03	0.379D+02
-0.20	0.3613643224368780D+00	0.1306633164264750D+01	-0.9717060191209930D+00	0.1402063597724390D+03	0.961D+02
-0.10	0.3649625671695630D+00	0.1329247926785690D+01	0.1690643398203620D+01	0.1436106888156060D+03	0.446D+03
0.00	0.3395861297069740D+00	-0.2385221051511630D+01	0.2143670283191920D-17	-0.3727006514950360D+03	0.650D+03

Table 2.1o : Results from the Taylor series expansion for t=1.0 .

(1.1/2) are to be compared with the (unknown) exact solution (see also Section 3.3).

When the time is increased beyond $t=0.6$, a rapid increase of the error around $z=0$ can be observed. At $z=0$, for instance, this error grows by 9 orders of magnitude between $t=0.6$ and $t=0.7$, and there is still an increase of error of 7 orders of magnitude between $t=0.8$ and $t=0.9$. For $t=0.9$ the error at $z=0$ is less than 0.001 , for $t=1.0$, however, the results obviously are no longer meaningful within the range $|z| < 0.8$.

Fig. 2.1 shows $\Gamma(z,t)$ for $t = 0.1(0.1)1.0$, where the lowest curve corresponds to $t=0.1$. The oscillations for $t=1.0$ indicate that (2.19) does no longer represent the solution of (1.1/2).

Fig. 2.2 shows $\Gamma_t(z,t)$ for $t = 0.0(0.1)1.0$, where the highest curve corresponds to the initial condition (1.2b) at $t=0.0$. The central part of the curve at $t=1.0$ has been omitted on account of the strong oscillations. Note that the areas under all curves must be equal to one.

Fig. 2.3 finally shows $\Gamma_z(z,t)$ for $t = 0.1(0.1)1.0$, with the largest amplitude at $t=1.0$. Here again the central part of this curve has been omitted.

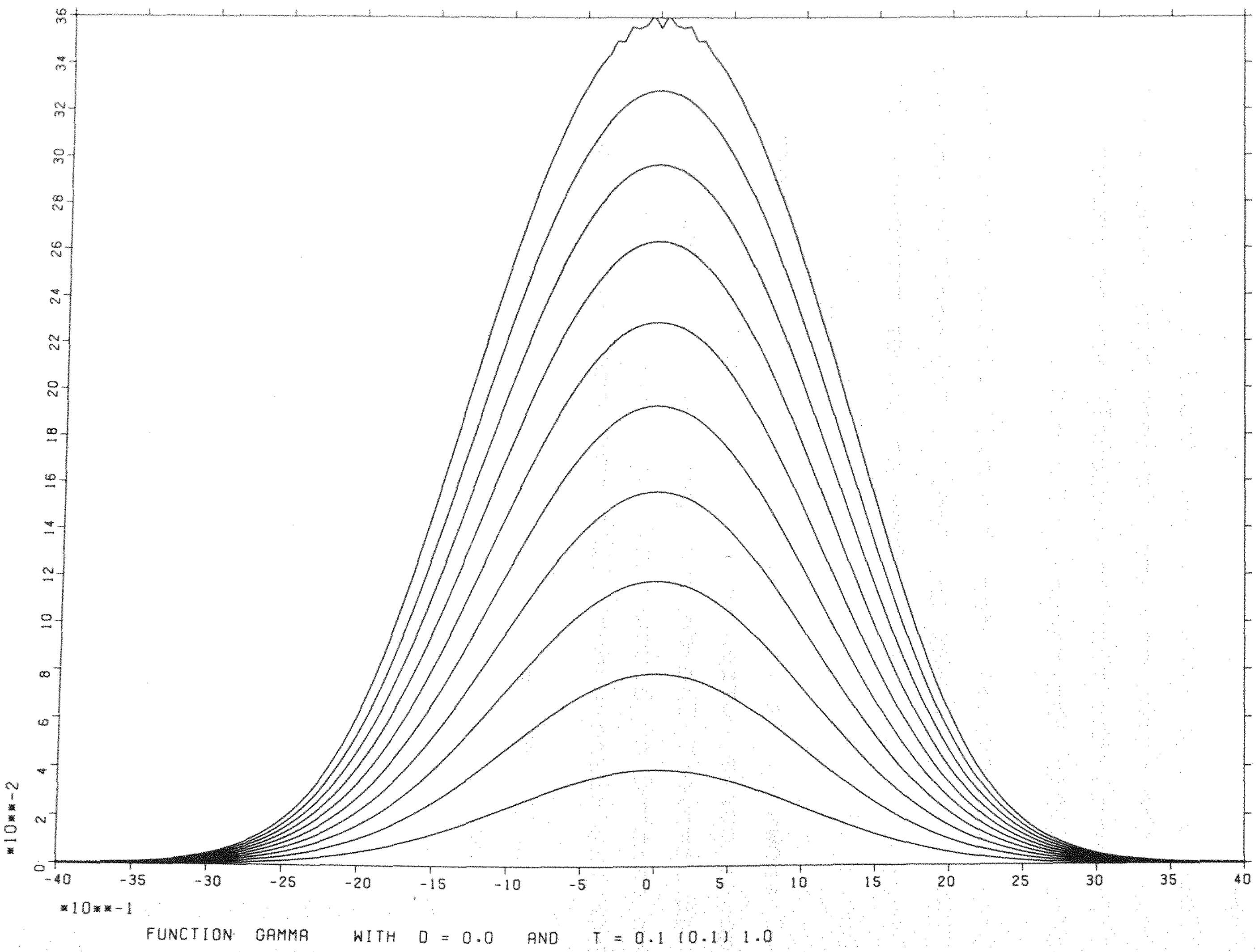


Fig. 2.1 : $\Gamma(z,t)$ as function of z for $t = 0.1, 0.2, \ldots, 1.0$.

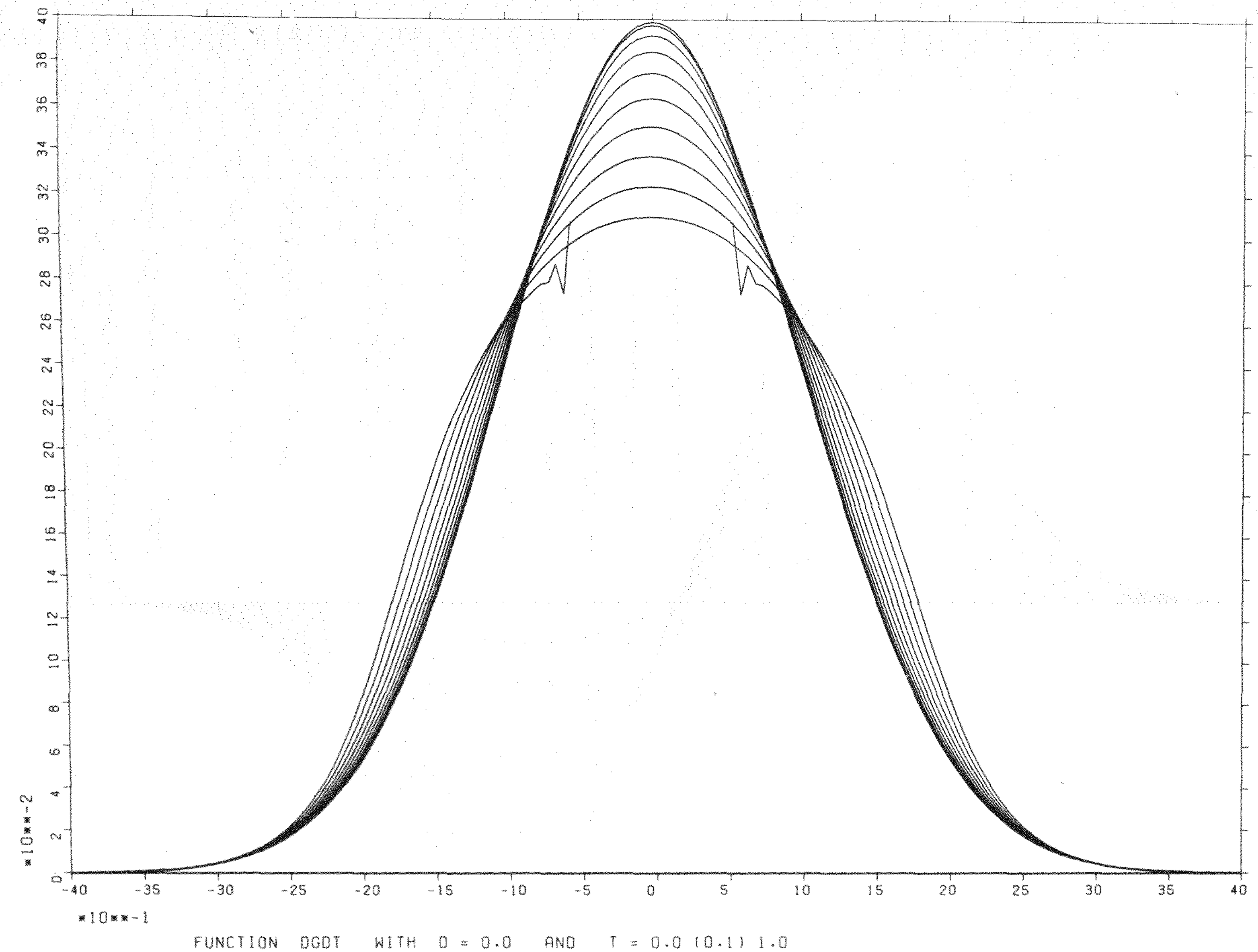


Fig. 2.2 : $\frac{\partial}{\partial t}\,\Gamma(z,t)$ as function of z for $t = 0.0, 0.1, \ldots, 1.0$.

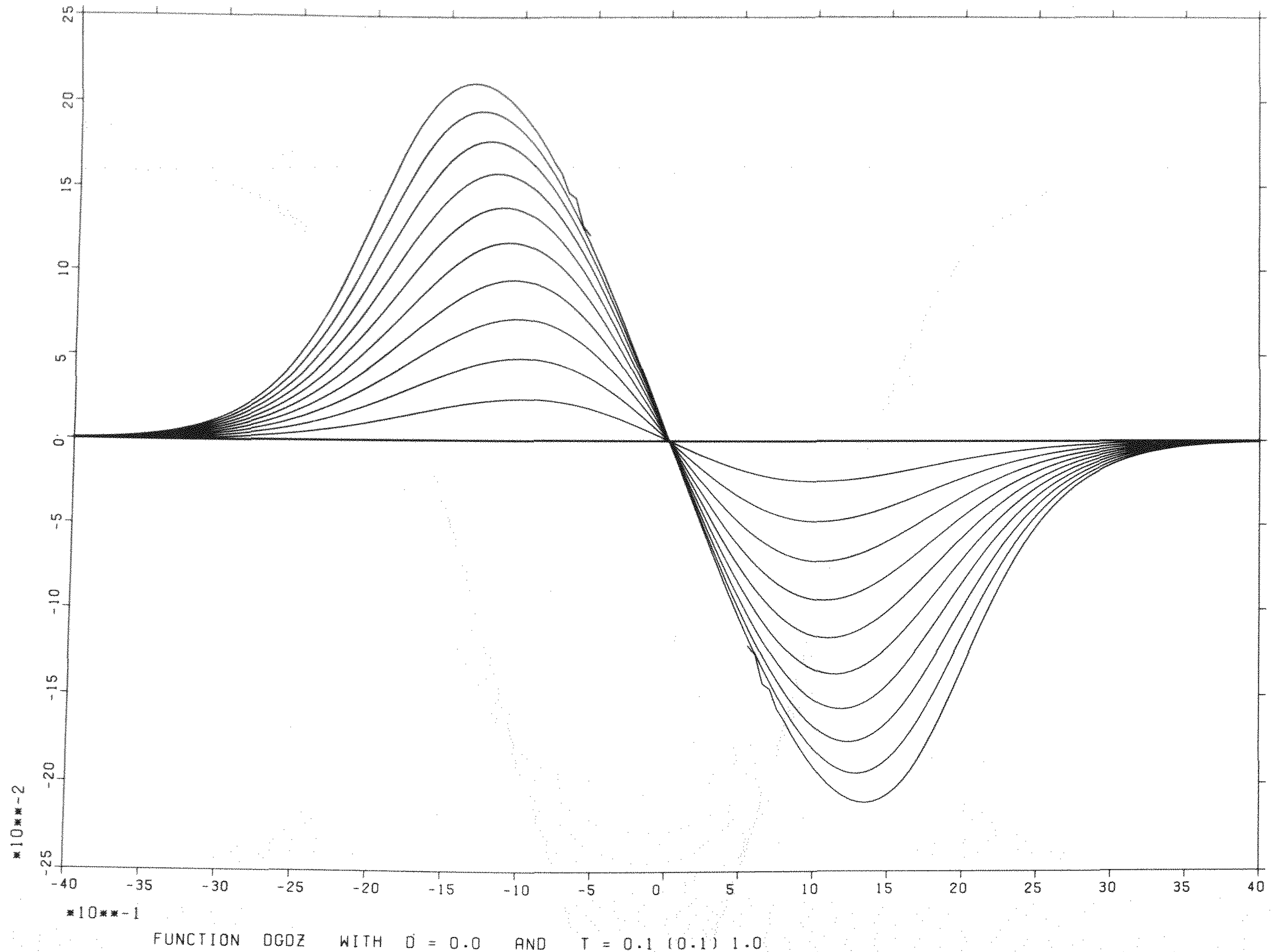


Fig. 2.3 : $\frac{\partial}{\partial z}\ddot{\Gamma}(z,t)$ as function of z for $t = 0.1, 0.2, \ldots, 1.0$.

2.4 Convergence of the Taylor series

In this section we briefly discuss the problem of convergence of the series (2.1o). Since our conclusions will mainly be based on computational results, this is a very difficult subject because the distinction whether the breakdown of the solution as given by (2.19) is due to failure of convergence or merely caused by numerical effects is nearly impossible.

We begin the discussion with an attempt to determine the radius of convergence of (2.1o) for $z=o$. Let us first consider the "past", i.e. reverse the time in (2.1o). We have

$$\Gamma(z,-t) = -\Gamma(z,t) \ , \qquad (2.21)$$

therefore the convergence behaviour along the negative time axis is equal to that for positive times.

At this point a short comment should be made as to what concerns "past" in connection with the initial value problem (1.1/2). If we want to know the solution which, after a certain time t_o has elapsed, will assume those values used as initial conditions (1.2), this solution must satisfy the initial value problem

$$\frac{\partial^2 w}{\partial t^2} = 2 \frac{\partial}{\partial z} \left[\frac{\partial w}{\partial t} \frac{\partial w}{\partial z}\right] , \qquad (2.22)$$

$$w(z,o) = -\Gamma(z,t_o) \ ,$$

$$w_t(z,o) = \Gamma_t(z,t_o) \ ,$$

where t_o may be one of the times of Tables 2.1-2.9, t_o=o.5 say, and Γ, Γ_t are the corresponding values at this time. The solution of (2.22) then satisfies

$$w(z,t_o) = o \ ,$$

$$w_t(z,t_o) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \ .$$

It follows from (2.21) that replacing t and Γ by $-t$ and $-\Gamma$ in (1.1/2) we just obtain (2.22).

In order to locate the singularity on the radius of convergence, it will be important to investigate the behaviour of (2.1o) along the imaginary axis, $t=i\tau$. For this purpose we must first derive a property of the coefficients $a_\ell^{(\lambda)}$, which obey the recurrence relation (2.8), namely the property

$$a_\ell^{(\lambda)} = (-1)^\lambda \, \alpha_\ell^{(\lambda)} , \tag{2.23}$$

where $\alpha_\ell^{(\lambda)} > o$.

Eq. (2.23) inserted into (2.8) yields the relation

$$\alpha_\ell^{(\lambda)} = \frac{1}{(2\ell+1)\ell} \sum_{\substack{m\geq o \\ m+n=\ell-1}} \sum_{n\geq o} (2m+1) \sum_{\substack{\mu\geq o \\ \mu+\nu=\lambda}} \sum_{\nu\geq o} C_{mn}^{(\mu\nu)} \, \alpha_n^{(\nu)} ,$$

$$C_{mn}^{(\mu\nu)} = [2(\ell-\lambda)+1]\left[2(n-\nu)\, \alpha_m^{(\mu-2)} + (n+1)\, \alpha_m^{(\mu-1)}\right]$$
$$+(\ell+1)\left[2(n-\nu)\, \alpha_m^{(\mu-1)} + (n+1)\, \alpha_m^{(\mu)}\right] .$$

Since only positive terms are involved in this expression, supposed that all $\alpha_k^{(\kappa)}$, $k \leq \ell-1$, are positive, it follows by induction that $\alpha_\ell^{(\lambda)} > o$ if $\alpha_o^{(o)} > o$.

With (2.23) and $P_\ell(z)$ from (2.7), we find

$$P_\ell(o) = (-1)^\ell \, \alpha_\ell^{(\ell)} ,$$

and therefore from (2.1o)

$$\Gamma(o,i\tau) = \frac{i\tau}{\sqrt{2\pi}} \sum_{\ell=o}^{\infty} \left\{\frac{\tau^2}{\sqrt{2\pi}}\right\}^\ell \alpha_\ell^{(\ell)} .$$

Since the $\alpha_\ell^{(\ell)}$ are positive, we conclude that, irrespective of the series representation (2.1o), we always have

$$|\Gamma(o,t)| < |\Gamma(o,i\tau)|, \; (-\tfrac{\pi}{2} < \arg t < \tfrac{\pi}{2}) ,$$

where, because of (2.21), only $\mathrm{Re}\ t > o$ needs to be considered.

This result shows that any singular behaviour of $\Gamma(o,t)$ must first occur on the imaginary time axis. Except for this result, no further conclusions concerning the convergence of the series (2.1o) can be drawn at present. In Fig. 2.4 the functions $i\,\Gamma(z,i\tau)$, $\tau = o.1(o.1)1.o$, which are calculated from (2.19), are plotted versus z . Apparently the oscillations around $z=o$, which occur for $\tau=1.o$, are much stronger than the corresponding ones of Fig. 2.1 for $t=1.o$.

Let us introduce the curve $\zeta(t)$ which separates two regions in the (t,z) plane where the quantity R , defined by

$$R = PDE/|\Gamma_{tt}| \ ,$$

is less than ε , and greater or equal to ε , respectively. Fig. 2.5 shows $\zeta(t)$ for $\varepsilon=o.1$, the upper part corresponds to the region where (2.19) represents the solution of (1.1/2) within the given limit. Point A in the lower part has a special meaning and will be explained in Section 4.7 .

It should be stressed again that the curve $\zeta(t)$ does not delimit regions of convergence from those of divergence, but gives the range for which the finite series (2.19) yields an acceptable solution within numerical accuracy.

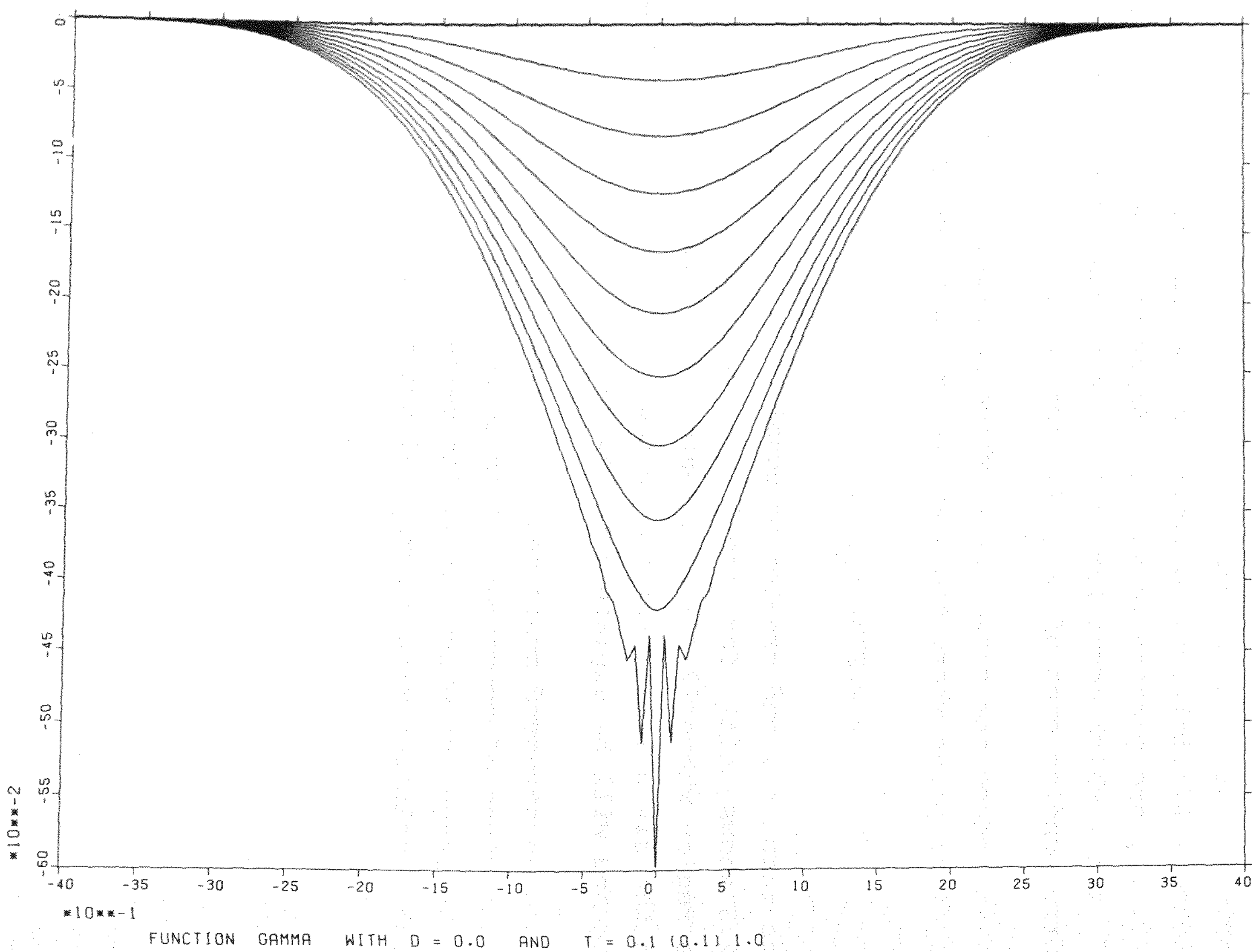


Fig. 2.4 : $i\Gamma(z,i\tau)$ as function of z for $\tau = 0.1, 0.2, \ldots, 1.0$.

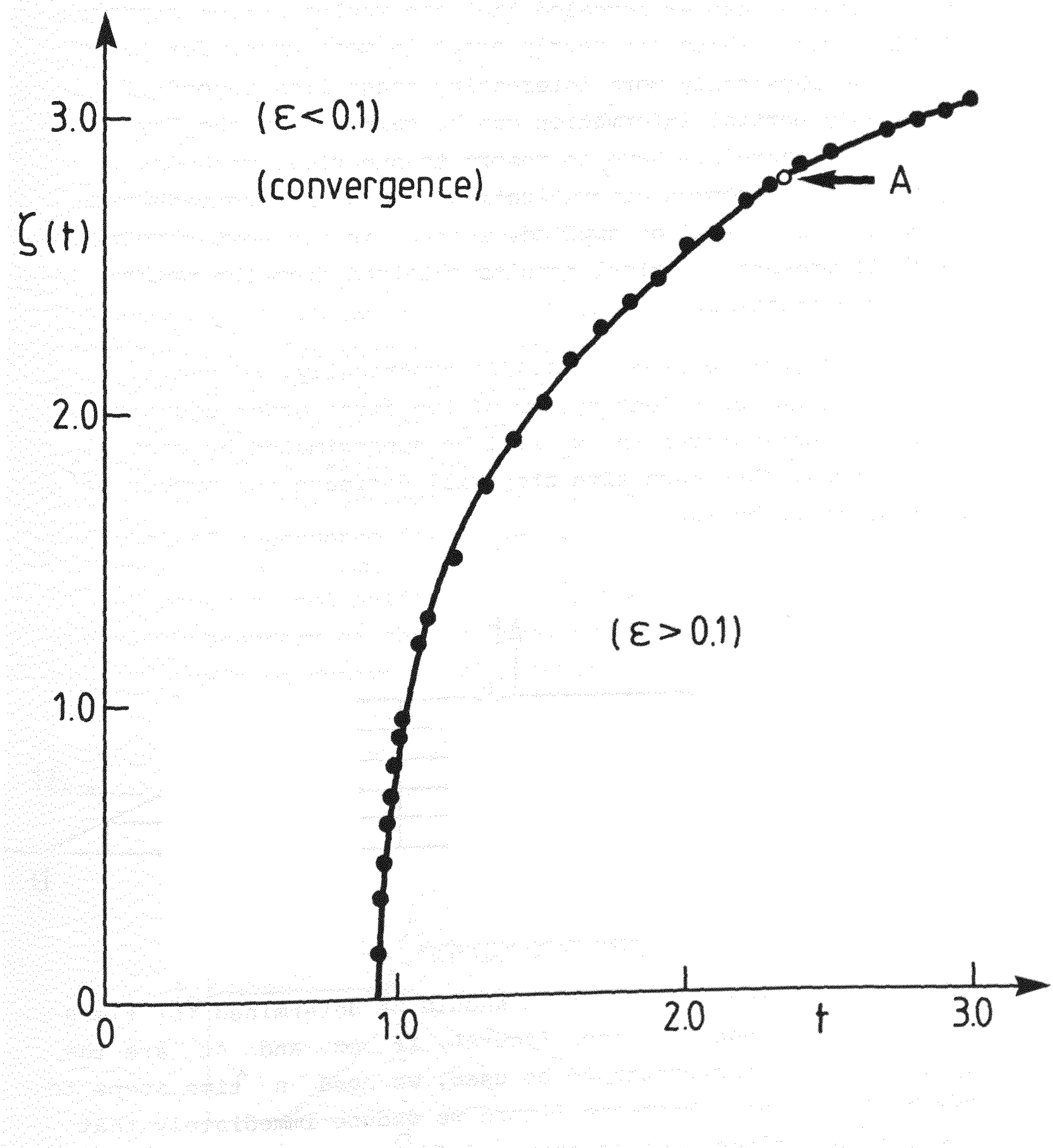


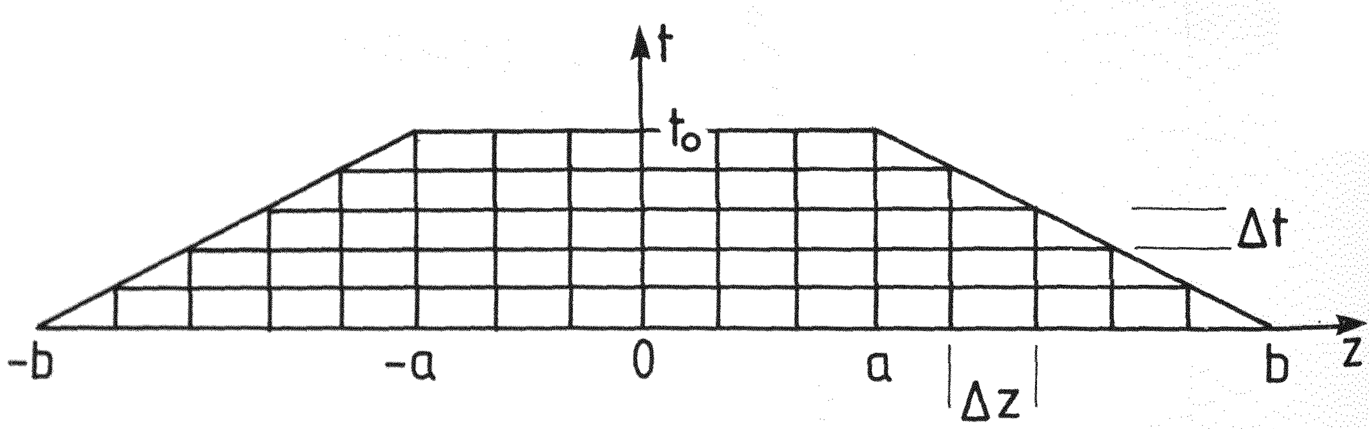
Fig. 2.5 : Approximate range of convergence.

3 Numerical Results from Finite Difference Methods

3.1 The initial boundary value problem

The preceding chapter revealed that the Taylor series approach yields results which are nearly exact in most cases for times $t<1$. The physically more interesting range lies beyond $t=1$, where only partial information can be gained from the Taylor series. We therefore have to resort to numerical methods. In this chapter a common discretization, using uniform meshes in space and time, will be applied, whereas in the next chapter we shall present numerical results obtained from the method of characteristics.

Instead of solving problem (1.1/2) numerically, we shall discretize an equivalent system of two first order equations, where the derivatives in z will be approximated by central differences. Then each time step will decrease the number of spatial steps by two.



Let us assume that the solution should be determined for times $t \leq t_o$ over a range $2a$ (see figure). If Δz and Δt are the space and time increments to be used, we need n time steps to reach $t_o = n\,\Delta t$. From the figure we deduce immediately that the initial values must be supplied for $z \in [-b,b]$, where

$$b = a + n\,\Delta z \ .$$

Let us illustrate this situation by inserting some typical numbers.

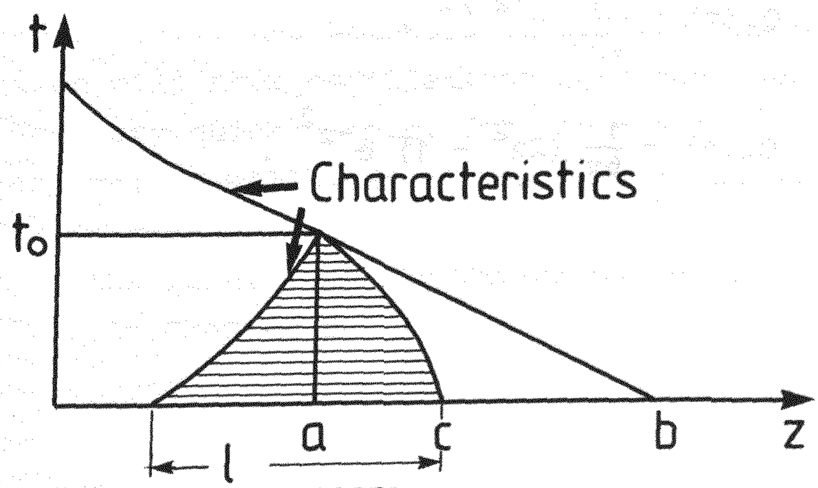
We shall later use the following values:

$$a=4 \ , \ t_o=3 \ , \ \Delta t=10^{-4} \ , \ \Delta z = 5\cdot 10^{-3} \ .$$

Then $n = 30\,000$, and $b = 154$. This value of b would require to provide initial values for $\Gamma_t(z,t)$ over a range of z values, where $\Gamma_t(z,o)$ according to (1.2b) is almost always negligibly small.

A second complication arises which is characteristic for hyperbolic problems [1)] (see e.g. Sauer [24], pp. 85,86). The solution at a given point (t_o,z_o) of the (t,z) plane is determined uniquely by prescribing initial values on the so-called "interval of dependence" for this particular point. In order to obtain the solution on the whole interval 2a at time t_o, one must ensure that the intervals of dependence for the boundary points $(t_o,\pm a)$ also lie within the interval $[-b,b]$ on the initial line $t=o$. In the schematic graph below, ℓ denotes the interval of dependence for (t_o,a).

It is, however, not sufficient to choose $b \geq c$, where c is the right boundary of the interval of dependence ℓ, but the whole "domain of determinacy" belonging to ℓ (the shaded region of the graph) must be covered by the discretization net, which is a necessary condition for stability (Courant-Friedrichs-Lewy, see Meis and Marcowitz [25], p. 68).



1) In Chapter 4 we shall derive the condition which must be satisfied in order that problem (1.1/2) is hyperbolic.

The increments Δt and Δz are still subject to further constraints. The slope of the characteristics, e.g. the slope of the boundaries of the domain of determinacy, is equal to the local propagation speed $V = dz/dt$, and we must additionally require that

$$\frac{\Delta z}{\Delta t} \geq |V(z,t)| \tag{3.1}$$

for all t,z in the domain of determinacy (see Ames [17], p. 454).

All the difficulties as described above (except the last one) can be avoided if the original problem can be converted into an initial boundary value problem with prescribed boundary values along the lines $z = \pm a$. As a consequence the solution of the original problem must be known for $z = \pm 4$, say. This is indeed the case for problem (1.1/2), as can be seen from the tables obtained by the Taylor series approach.

It is sufficient to replace the exact boundary values by a few terms of the Taylor series (2.1o). With $Q_\ell(z)$ from (2.1), the series (2.1o) becomes

$$\Gamma(z,t) = Q_1(z)t + Q_3(z)t^3 + O(t^5) \quad ,$$

where

$$Q_1(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad ,$$

$$Q_3(z) = \frac{1}{6\pi} (2z^2 - 1) e^{-z^2} \quad .$$

The following table shows how accurate these boundary values are in comparison with the exact values as obtained from the tables of Chapter 2.[1)]

t	Γ_t^* [* 10^{-4}]	Γ_t [* 10^{-4}]	Γ_z^* [* 10^{-3}]	Γ_z [* 10^{-3}]
0.5	1.339690	1.339692	0.267833	0.267834
1.0	1.343855	1.343882	0.536706	0.536712
1.5	1.350795	1.350934	0.807656	0.807701
2.0	1.360511	1.360954	1.081722	1.081916
2.5	1.373004	1.374099	1.359944	1.360541
3.0	1.388273	1.390579	1.643360	1.644868

In this table Γ_t and Γ_z denote the exact values,

$$\begin{aligned} \Gamma_t^*(z,t) &= Q_1(z) + 3\,Q_3(z)t^2 , \\ \Gamma_z^*(z,t) &= Q_1'(z)t + Q_3'(z)t^3 , \end{aligned} \tag{3.2}$$

and z assumes its left boundary, $z = -4$.

If instead of (3.2) the boundary values are set equal to zero, the solution will show oscillations near the boundary, which will be smoothed very quickly, e.g. for $\Delta z = 5 \cdot 10^{-3}$ both choices of boundary values will lead to the same solution for $|z| \leq 3.9$.

---

1) For $t>1$ the results of the Taylor series are not given explicitly in Chapter 2.

3.2 The discretization method

Instead of discretizing eq. (1.1), it is more convenient to solve numerically the original system of two equations, from which eq. (1.1) was derived in Section I.4.4 of Part I.

In accordance with the commonly used notation u and v for the dependent variables, we set $u=\rho$ and $v=-m$, where ρ and m are the density and the momentum. Then we have

$$u(z,t) = \Gamma_t(z,t) ,$$
$$v(z,t) = \Gamma_z(z,t) ,$$

and the following system

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}2v & 2u\\1 & o\end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix}u\\v\end{pmatrix} \tag{3.3}$$

is equivalent to the original equation (1.1). The initial values are given by

$$u(z,o) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} ,$$

$$v(z,o) = o ,$$

and the boundary values follow immediately from (3.2).

For the time derivatives we shall simply use forward differences. If we also use forward differences for the space derivatives, the method will break down immediately. Even with central differences for the derivatives in z, the resulting method is known to be unstable (Richtmyer and Morton [26], p. 262, or Meis and Marcowitz [25], p. 63). We have still chosen this unstable method, because the results turned out to be highly accurate (see Section 3.3, and the end of this section). The break-down which occurred for times $t \approx 3.5$ did not have its origin in the numerically unstable method but was due to the physical behaviour of the solution, as will be seen later.

Let $U_k^{(n)}$ be the discretized value approximating $u(z,t)$ at $z = k\,\Delta z$, $t = n\,\Delta t$, and similarly for $V_k^{(n)}$, then the following discretization method was used:

$$U_k^{(n+1)} = U_k^{(n)} + \frac{\Delta t}{\Delta z}\left\{V_k^{(n)}\left[U_{k+1}^{(n)} - U_{k-1}^{(n)}\right] + U_k^{(n)}\left[V_{k+1}^{(n)} - V_{k-1}^{(n)}\right]\right\} , \tag{3.4a}$$

$$V_k^{(n+1)} = V_k^{(n)} + \frac{1}{2}\,\frac{\Delta t}{\Delta z}\left[U_{k+1}^{(n)} - U_{k-1}^{(n)}\right] , \tag{3.4b}$$

$$G_k^{(n+1)} = G_k^{(n)} + \Delta t\; U_k^{(n+1)} , \tag{3.4c}$$

where $G_k^{(n)}$ approximates $\Gamma(z,t)$.

Since eq. (3.3) can be written in the conservative form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\partial}{\partial z}\begin{pmatrix} 2\,u\,v \\ u \end{pmatrix} ,$$

the discretization (3.4a) can be replaced by the formula

$$U_k^{(n+1)} = U_k^{(n)} + \frac{\Delta t}{\Delta z}\left\{U_{k+1}^{(n)}\,V_{k+1}^{(n)} - U_{k-1}^{(n)}\,V_{k-1}^{(n)}\right\} , \tag{3.5}$$

which will certainly be much faster. The numerical accuracy, however, turned out to be the same as with eq. (3.4a). Also the oscillations as shown in Fig. 3.3 occurred with this formula, though reduced by 25%.

As stated before, the numerical solution based on eqs. (3.4) will break down at about $t=3.5$. Anticipating results of Chapter 4, this is due to the occurrence of a shock discontinuity. This break-down can be avoided if the original system (3.3) is modified by introducing dissipative terms which are of second order in the spatial derivatives (see e.g. Whitham [27], where Burgers' equation is discussed in great detail).

Instead of modifying (3.3), one can also apply a different discretization for the time derivative which introduces artificially a dissipative term, the so-called "numerical viscosity" (Meis and Marcowitz [25], or Ames [17], pp. 458,459). If we replace $U_k^{(n)}$ in (3.5) (or in (3.4a)) by

$$\frac{1}{2}\left[U_{k+1}^{(n)} + U_{k-1}^{(n)}\right] ,$$

i.e. by the mean of two neighbouring values, eq. (3.5) becomes

$$U_k^{(n+1)} - U_k^{(n)} = \frac{1}{2}\left[U_{k+1}^{(n)} + U_{k-1}^{(n)} - 2U_k^{(n)}\right] + \frac{\Delta t}{\Delta z}\{\ldots\} , \qquad (3.6)$$

which is an approximation to the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial}{\partial z} (uv)$$

with the "diffusion constant"

$$D = \frac{(\Delta z)^2}{2\Delta t} .$$

The discretization (3.6), with a similar formula for $V_k^{(n)}$, is known as the "Friedrichs method" (Meis and Marcowitz [25], p. 63). When we tried (3.6), the breakdown disappeared indeed and the oscillations as shown in Fig. 3.3 did not occur, but the solution coincided with the one obtained from (3.4) only by one significant figure for all times.

The reason for this poor agreement lies in the discretization chosen here, i.e. $\Delta t = 10^{-4}$, $\Delta z = 5 \cdot 10^{-3}$. Then $D=0.125$, which is by no means negligible. In order to reduce the influence of the diffusive term, a smaller value for Δz should be chosen. But this implies, according to (3.1), a smaller value for Δt. For reasons of computation time this becomes prohibitive very soon.

A final remark should be made concerning the use of the unstable method (3.4). The stability property of a method can be rather easily established for linear problems, the generalization to nonlinear problems such as (3.3) is not straightforward but generally accepted due to lack of better knowledge.

As a check we solved the linear wave equation by replacing $U_\ell^{(n)}$ within the brackets of (3.4a) by $V^2/2$, where V is the propagation speed. With $\Delta t = 10^{-4}$ and $\Delta z = 5 \cdot 10^{-3}$, the results of this method were exact up to 5 decimal places for $V = \sqrt{2}$ and $t \leq 3$. For $V = 10$ a break-down occurred at about $t=1$.

This behaviour can be explained investigating the amplification matrix which is obtained by applying a discrete Fourier transform to the spatial discretization of this method (Richtmyer and Morton [26]).

With

$$\bar{U}_\ell^{(n)} = \sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N} k\ell} \, U_k^{(n)} \quad ,$$

where N is the number of equidistant points $z_k = k\Delta z + z_0$, and a corresponding expression for $\bar{V}_\ell^{(n)}$, the transformation of the system

$$U_k^{(n+1)} = U_k^{(n)} + \frac{V^2}{2} \frac{\Delta t}{\Delta z} \left[V_{k+1}^{(n)} - V_{k-1}^{(n)} \right] ,$$

$$V_k^{(n+1)} = V_k^{(n)} + \frac{1}{2} \frac{\Delta t}{\Delta z} \left[U_{k+1}^{(n)} - U_{k-1}^{(n)} \right]$$

leads to the equation

$$\begin{pmatrix} \bar{U}_\ell^{(n+1)} \\ \bar{V}_\ell^{(n+1)} \end{pmatrix} = G_\ell \begin{pmatrix} \bar{U}_\ell^{(n)} \\ \bar{V}_\ell^{(n)} \end{pmatrix} \quad ,$$

where

$$G_\ell = \begin{pmatrix} 1 & iV^2 \frac{\Delta t}{\Delta z} \sin \frac{2\pi}{N} \ell \\ i \frac{\Delta t}{\Delta z} \sin \frac{2\pi}{N} \ell & 1 \end{pmatrix}$$

is the amplification matrix.

In order that the method is stable, it is necessary that G_ℓ^n remains bounded, which follows if the eigenvalues $\lambda_\ell^{(j)}$ of G_ℓ satisfy $|\lambda_\ell^{(j)}| \leq 1$. But

$$\lambda_\ell^{(j)} = 1 \pm iV \frac{\Delta t}{\Delta z} \sin \frac{2\pi}{N} \ell \quad ,$$

and therefore $|\lambda_\ell^{(j)}|>1$. However, with $V=\sqrt{2}$ and the discretization chosen here, it follows that

$$|\lambda_\ell^{(j)}| \leq 1.0004 ,$$

which is very weakly unstable.

3.3 Results for the Gaussian initial distribution

In order to demonstrate the accuracy of the numerical method (3.4), results for $t=0.9$ and $t=1.0$ are listed in Table 3.1. Comparing these results with Tables 2.9 and 2.10 of the previous chapter, we find an agreement between 5 and 6 decimal places. Note that the values of Tables 2.9/10 lying close to $z=0$ are no longer accurate.

Figs. 3.1 and 3.2, the continuations of Figs. 2.2 and 2.3, show $\Gamma_t(z,t)$ and $\Gamma_z(z,t)$ for $t=1.0(0.1)2.0$. We observe that a dip begins to develop in $\Gamma_t(z,t)$ for times between $t=1.3$ and $t=1.4$. Moreover, both figures clearly exhibit a steepening of the slopes with increasing time.

This steepening is the reason why the solution cannot be extended to arbitrarily large times. Fig. 3.3 shows $\Gamma_t(z,t)$ for $t=2.1(0.1)2.6$ over the critical z region. Already for $t=2.4$ strong oscillations occur within the range $z\in[-2.83,-2.74]$. Indeed we shall see in Chapter 4 that the solution breaks down at $t=2.34$ with an infinite slope at $z=-2.79$.

Finally the surface $\Gamma_t(z,t)$ is illustrated in Fig. 3.4 over the domain $z\in[-4,4]$, $t\in[0,2.3]$. It can be observed that the initial boundary values at $z=\pm4$ hardly change during time (that they do change follows from the table of the previous section).

INPUT :

DELT = 0.10D-04
DELZ = 0.50D-02

RESULTS :

	T = 0.90			T = 1.00		
POSITION	GAMMA	DG DT	DG DZ	GAMMA	DG DT	DG DZ
-4.00	0.1205821232D-03	0.1342799592D-03	0.4827985368D-03	0.1340153012D-03	0.1343854522D-03	0.5367059840D-03
-3.90	0.1790730240D-03	0.1996012316D-03	0.6993777134D-03	0.1990441163D-03	0.1998246094D-03	0.7776269035D-03
-3.80	0.2633324054D-03	0.2938868087D-03	0.1002608329D-02	0.2927436421D-03	0.2943462178D-03	0.1115087419D-02
-3.70	C.3834663895D-03	0.4286772574D-03	0.1422566918D-02	0.4263795789D-03	0.4296034629D-03	0.1582735236D-02
-3.60	0.5529989119D-03	0.6195699495D-03	0.1997851425D-02	0.6150457599D-03	0.6214010404D-03	0.2223860447D-02
-3.50	0.7898143494D-03	0.8874632364D-03	0.2777384214D-02	0.8787348656D-03	0.8910144951D-03	0.3093521079D-02
-3.40	0.1117285458D-02	0.1260132493D-02	0.3822345004D-02	0.1243630104D-02	0.1266892201D-02	0.4260880431D-02
-3.30	0.1565600469D-02	0.1774224257D-02	0.5208209416D-02	0.1743641914D-02	0.1786859256D-02	0.5811763409D-02
-3.20	0.2173295787D-02	0.2477773596D-02	0.7026803674D-02	0.2422209023D-02	0.2500977992D-02	0.7851374006D-02
-3.10	0.2988989272D-02	0.3433370981D-02	0.9388230895D-02	0.3334375004D-02	0.3475266243D-02	0.1050706244D-01
-3.00	0.4073290469D-02	0.4722118616D-02	0.1242243423D-01	0.4549135822D-02	0.4796522577D-02	0.1393092980D-01
-2.90	0.5500833831D-02	0.6448503734D-02	0.1628001150D-01	0.6152024124D-02	0.6578536988D-02	0.1830186711D-01
-2.80	0.7362330895D-02	0.8746238664D-02	0.2113164430D-01	0.8247839647D-02	0.8969923477D-02	0.2382626761D-01
-2.70	0.9766454786D-02	0.1178489676D-01	0.2716509415D-01	0.1096332946D-01	0.1216357655D-01	0.3073599621D-01
-2.60	0.1284123587D-01	0.1577666682D-01	0.3457807893D-01	0.1444942789D-01	0.1640706866D-01	0.3928105331D-01
-2.50	0.1673443553D-01	0.2098152154D-01	0.4356443626D-01	0.1888232347D-01	0.2201167140D-01	0.4971251127D-01
-2.40	0.2161205281D-01	0.2770723723D-01	0.5428992586D-01	0.2446205363D-01	0.2935429863D-01	0.6224873535D-01
-2.30	0.2765371501D-01	0.3629783188D-01	0.6685341910D-01	0.3140650559D-01	0.3886077137D-01	0.7701568682D-01
-2.20	0.3504332405D-01	0.4710074819D-01	0.8123070139D-01	0.3993782145D-01	0.5095146890D-01	0.9395422956D-01
-2.10	0.4395332480D-01	0.6040243328D-01	0.9720472135D-01	0.5025810590D-01	0.6592925360D-01	0.1127026746D+00
-2.00	0.5452200760D-01	0.7633036672D-01	0.1143007565D+00	0.6251383273D-01	0.8381303388D-01	0.1324999196D+00
-1.90	0.6682606823D-01	0.9474362246D-01	0.1317639237D+00	0.7675516669D-01	0.1041826305D+00	0.1521969899D+00
-1.80	0.8085485631D-01	0.1151661430D+00	0.1486209933D+00	0.9290513753D-01	0.1261591773D+00	0.1704464810D+00
-1.70	0.9649559377D-01	0.1368229036D+00	0.1638341074D+00	0.1107550099D+00	0.1485937685D+00	0.1860141705D+00
-1.60	0.1135364859D+00	0.1587877001D+00	0.1764909089D+00	0.1299896923D+00	0.1703710535D+00	0.1980365691D+00
-1.50	0.1316869763D+00	0.1801751246D+00	0.1859439270D+00	0.1502302081D+00	0.1906448882D+00	0.2060960212D+00
-1.40	0.1506070766D+00	0.2002862672D+00	0.1918503302D+00	0.1710751009D+00	0.2089141835D+00	0.2101392116D+00
-1.30	0.1699361748D+00	0.2186652074D+00	0.1941287714D+00	0.1921301584D+00	0.2249767154D+00	0.2103419356D+00
-1.20	0.1893153859D+00	0.2350818310D+00	0.1928815363D+00	0.2130252402D+00	0.2388384199D+00	0.2069939189D+00
-1.10	0.2084020837D+00	0.2494792418D+00	0.1883205489D+00	0.2334216472D+00	0.2506286013D+00	0.2004252222D+00
-1.00	0.2268779770D+00	0.2619161277D+00	0.1807133558D+00	0.2530139869D+00	0.2605380246D+00	0.1909676626D+00
-0.90	0.2444528494D+00	0.2725187257D+00	0.1703497753D+00	0.2715293944D+00	0.2687792131D+00	0.1789384844D+00
-0.80	0.2608657872D+00	0.2814459143D+00	0.1575240045D+00	0.2887257912D+00	0.2755630501D+00	0.1646360589D+00
-0.70	0.2758851501D+00	0.2888658752D+00	0.1425265233D+00	0.3043900362D+00	0.2810859411D+00	0.1483412411D+00
-0.60	0.2893080380D+00	0.2949413796D+00	0.1256415100D+00	0.3183363404D+00	0.2855233484D+00	0.1303208735D+00
-0.50	0.3009596532D+00	0.2998209570D+00	0.1071469989D+00	0.3304050607D+00	0.2890269779D+00	0.1108316672D+00
-0.40	0.3106927449D+00	0.3036338590D+00	0.8731615358D-01	0.3404618643D+00	0.2917239520D+00	0.9012364995D-01
-0.30	0.3183871973D+00	0.3064873610D+00	0.6641877269D-01	0.3483972022D+00	0.2937169807D+00	0.6844287010D-01
-0.20	0.3239497614D+00	0.3084654428D+00	0.4472260190D-01	0.3541260131D+00	0.2950849526D+00	0.4603330736D-01
-0.10	0.3273139004D+00	0.3096282305D+00	0.2249429235D-01	0.3575875811D+00	0.2958836162D+00	0.2313806666D-01
0.0	0.3284397126D+00	0.3100118164D+00	0.0	0.3587454840D+00	0.2961461642D+00	0.0

Table 3.1 : Results from the discretization method for t=0.9 and t=1.0 .

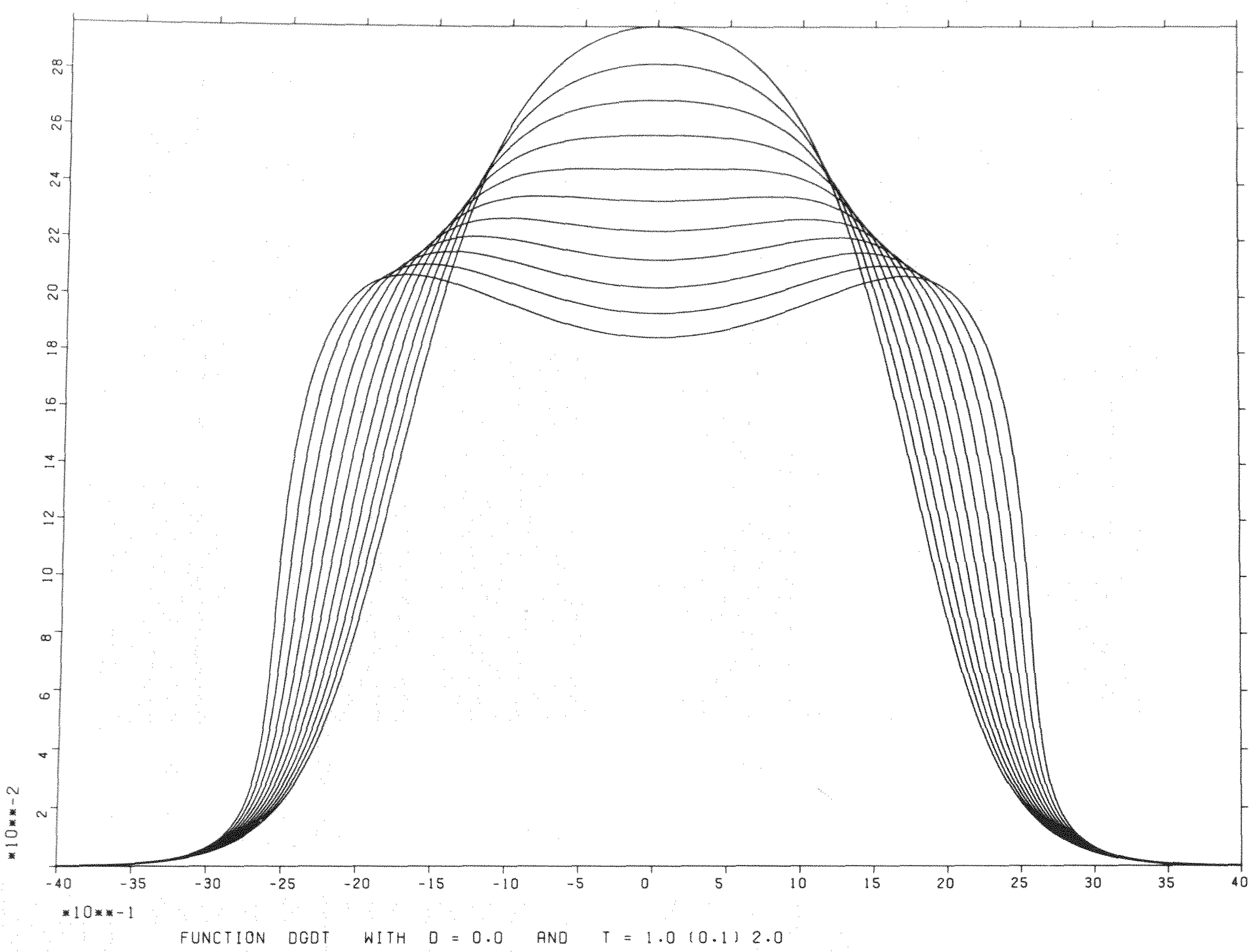


Fig. 3.1 : $\frac{\partial}{\partial t}\,\Gamma(z,t)$ as function of z for $t = 1.0, 1.1, \ldots, 2.0$.

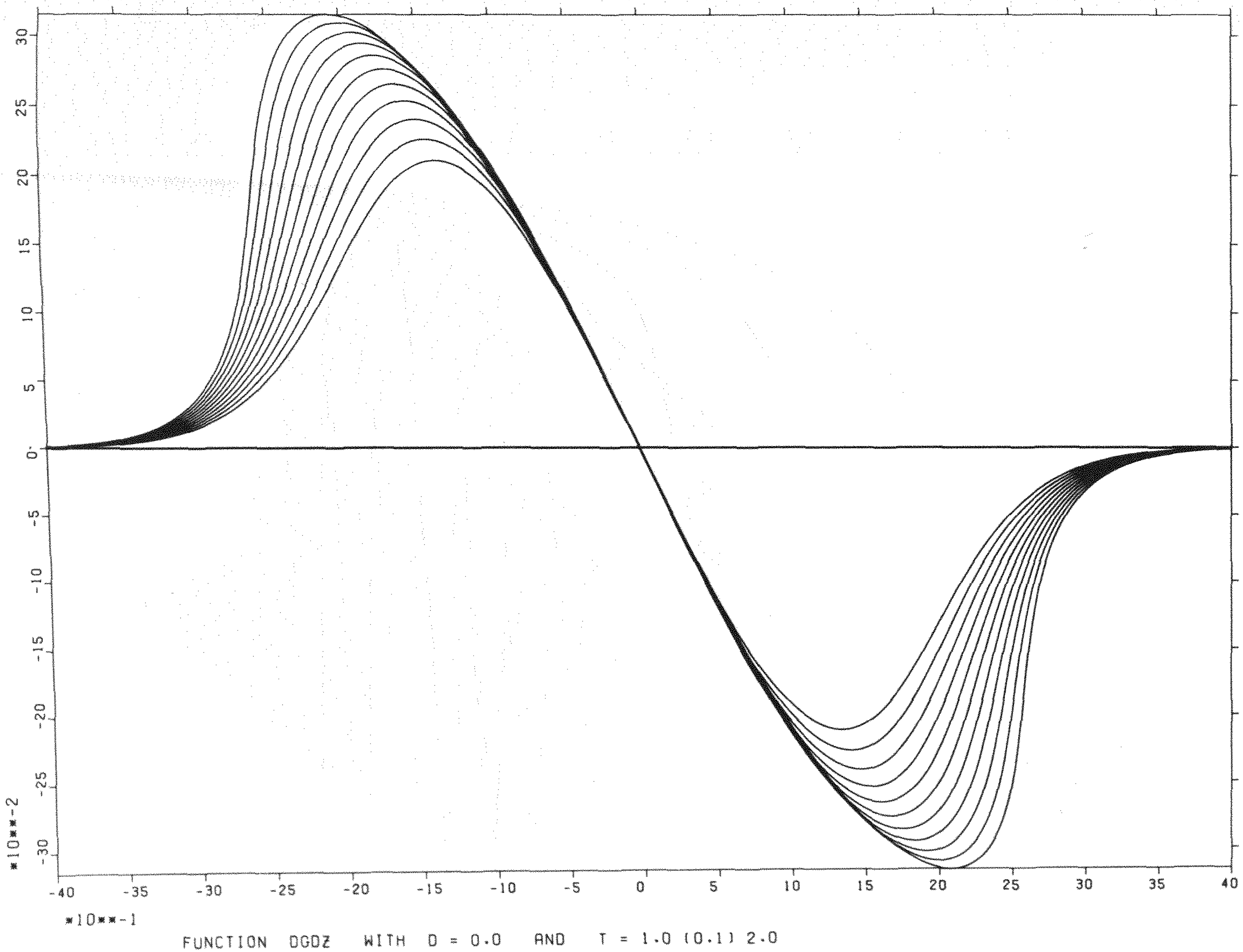


Fig. 3.2 : $\frac{\partial}{\partial z}\,\Gamma(z,t)$ as function of z for $t = 1.0, 1.1, \ldots, 2.0$.

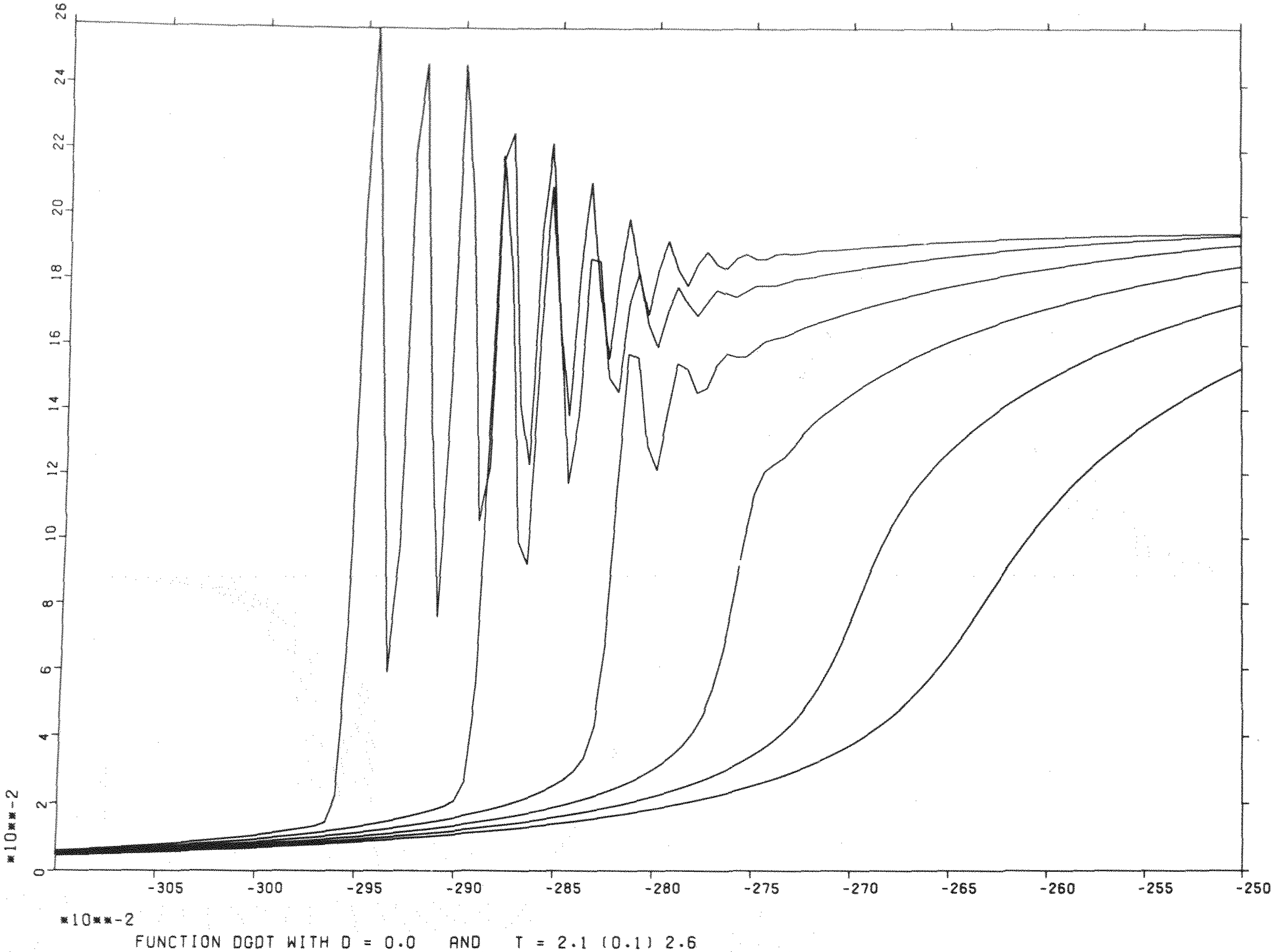


Fig. 3.3 : $\frac{\partial}{\partial t}\,\Gamma(z,t)$ as function of z for $t = 2.1, 2.2, \ldots, 2.6$.

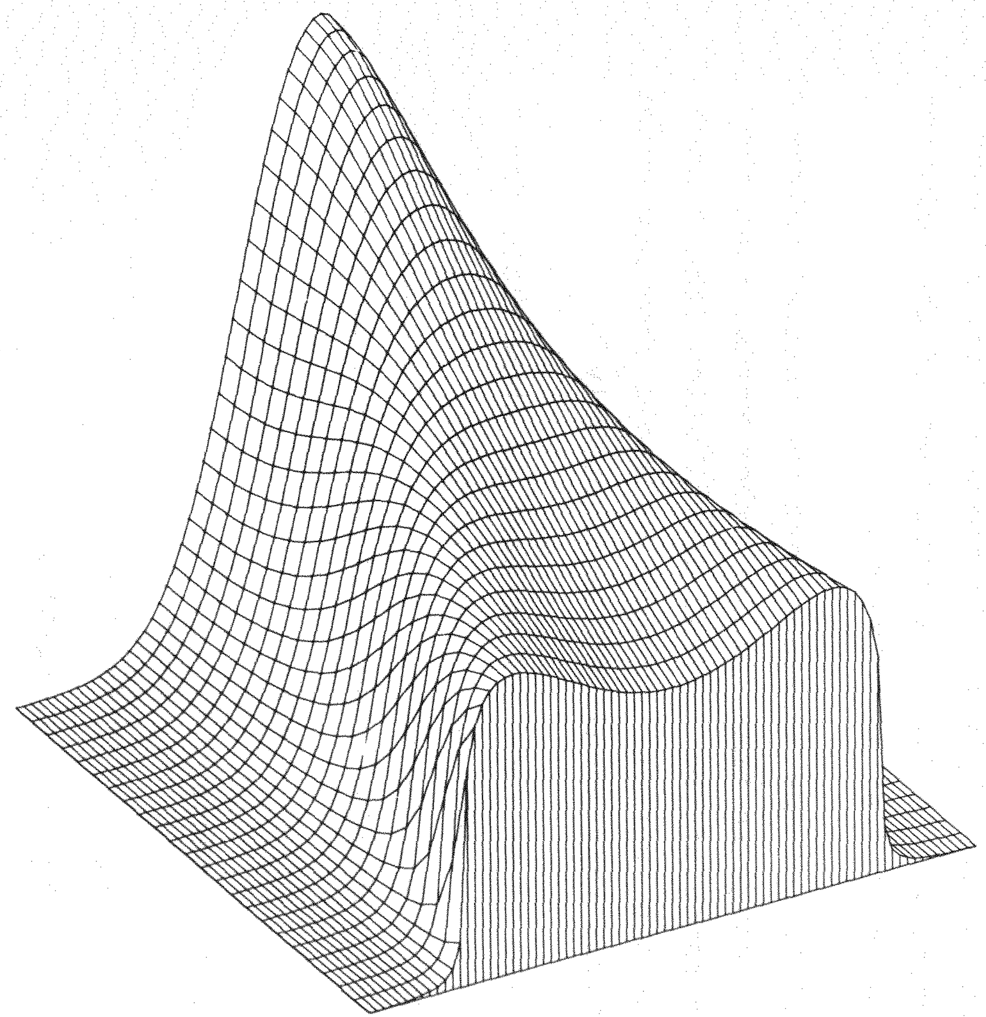


Fig. 3.4 : $\frac{\partial}{\partial t}\,\Gamma(z,t)$ for the Gaussian initial distribution.

3.4 Results for the initial distribution $\cos^2 \frac{\pi}{8} z$

The final remark of the previous section led to a different choice of the initial distribution. We solved eq. (1.1) over $z \in [-6,6]$ with initial condition (1.2b) replaced by

$$\Gamma_t(z,o) = \frac{1}{4} \cos^2 \frac{\pi}{8} z \,, \qquad z \in [-4,4] \,,$$
$$\Gamma_t(z,o) = o \qquad\qquad , \qquad |z| > 4 \,. \tag{3.7}$$

The resulting surface $\Gamma_t(z,t)$ is shown in Fig. 3.5 over the domain $z \in [-6,6]$, $t \in [o,1.6]$. A breakdown of the solution occurred between $t=1.6$ and $t=1.8$.

From Fig. 3.5 it becomes apparent that the initial values at $z=\pm 4$ do not change in time, or, in other words, that the nonvanishing range of the initial distribution (the "bunch length") does not increase in time. We shall show now under which conditions this result holds rigorously.

In the notation of Section 3.2 we want to show that, for a fixed value $z = z_o$, $u(z_o,t)$ does not change in time if certain conditions are satisfied at z_o for $t=o$. In order to stress that only the time dependence is of interest, we write

$$y_1(t) = u(z_o,t) \,.$$

If we further introduce

$$y_2(t) = v(z_o,t) \,,$$
$$y_3(t) = u_z(z_o,t) \,,$$

we can set up three differential equations for the $y_i(t)$, where two follow immediately from (3.3), and the third is obtained by differentiating the first equation of (3.3) with respect to z .

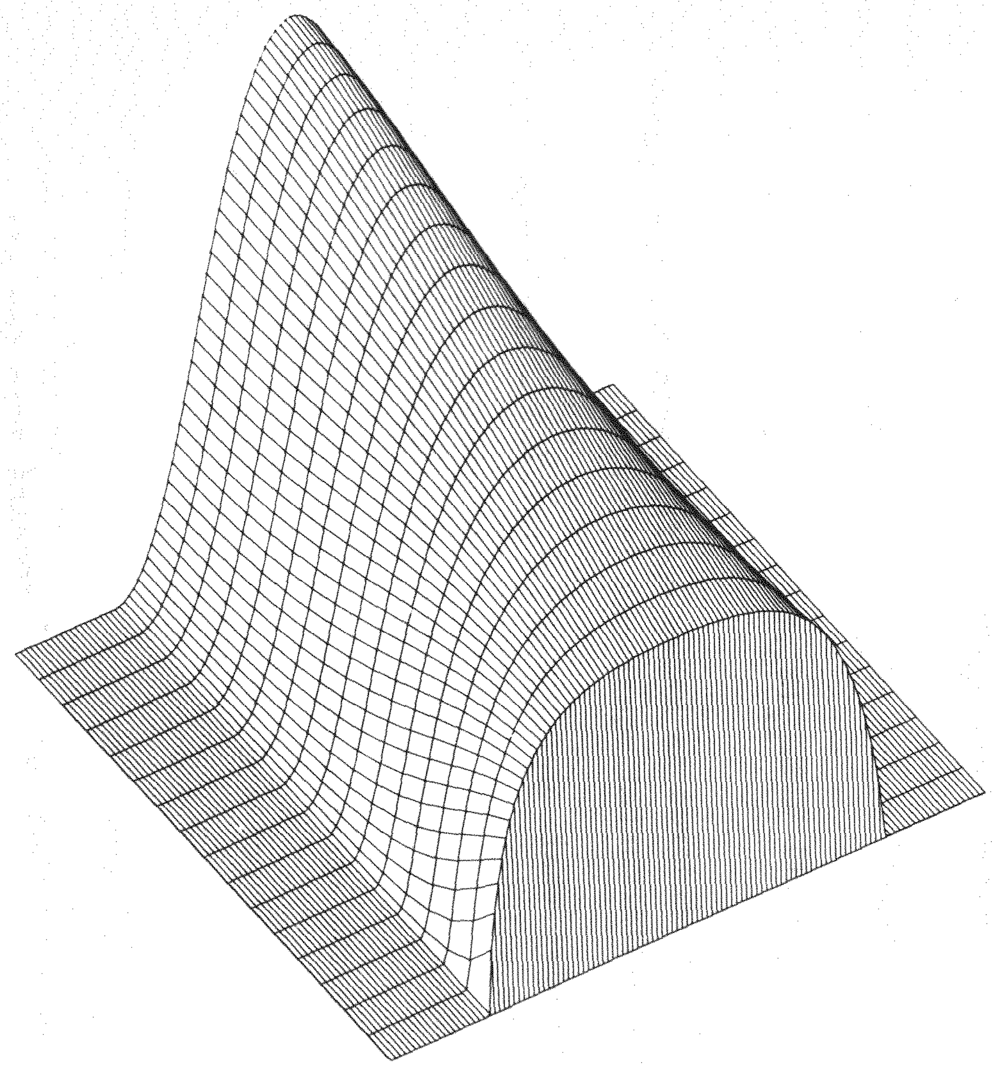


Fig. 3.5 : $\frac{\partial}{\partial t}\,\Gamma(z,t)$ for the initial distribution $\cos^2 \frac{\pi}{8}z$.

We then obtain

$$\begin{aligned} \dot{y}_1 &= f_1(z_o,t)y_1 + 2y_2y_3 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= f_2(z_o,t)y_1 + f_3(z_o,t)y_2 + 2f_1(z_o,t)y_3 \ , \end{aligned} \tag{3.8}$$

where

$$f_1(z_o,t) = 2v_z(z_o,t)$$

$$f_2(z_o,t) = 2v_{zz}(z_o,t)$$

$$f_3(z_o,t) = 2u_{zz}(z_o,t)$$

are to be considered as known coefficients for the system of ordinary differential equations (3.8).

If we require that $y_i(o) = o$, $i=1,2,3$, the system (3.8) has the unique solution

$$y_i(t) = o \ , \quad i = 1,2,3 \ .$$

Applied to the initial condition (3.7), we have for $z_o = \pm 4$:

$$u(z_o,o) = o \ ,$$

$$v(z_o,o) = o \ ,$$

$$u_z(z_o,o) = o \ ,$$

thereby proving that $u(z_o,t)$ remains zero for all times.

4 The Method of Characteristics

4.1 Description of the method

In this section we want to give a short description of the method of characteristics for the case of a quasi-linear system of two first order equations in two independent variables. We have already seen in the previous chapter that a quasi-linear partial differential equation of second order in two independent variables can always be transformed into such a system, the reverse, however, is in general not possible, therefore our discussion, which mainly follows the one of Busch et al. [28], includes the more general case.

A quasi-linear system of the kind described above can always be written as

$$\begin{aligned} a_1 u_x + a_2 u_y + a_3 v_x + a_4 v_y &= h_1 \,, \\ b_1 u_x + b_2 u_y + b_3 v_x + b_4 v_y &= h_2 \,, \end{aligned} \tag{4.1}$$

where a_i, b_i, h_i depend on x, y, u, v. If (4.1) represents an initial value problem, with x corresponding to the time t, initial values

$$u(o,y) = u_o(y) \,,$$
$$v(o,y) = v_o(y) \,,$$

must be prescribed.

The system (4.1) can be expressed in the matrix form

$$M_1 \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} + M_2 \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} , \tag{4.2}$$

where

$$M_1 = \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} a_2 & a_4 \\ b_2 & b_4 \end{pmatrix} .$$

Let us assume that the generalized eigenvalue problem

$$e^T(M_2 - \lambda M_1) = o \tag{4.3}$$

has two real eigenvalues satisfying $\lambda_1\lambda_2 < o$, and that the corresponding left eigenvectors e_i^T are linearly independent. Then the system (4.2) is said to be hyperbolic.

Multiplying (4.2) from left by e_i^T yields

$$e_i^T \left[M_1 \frac{\partial}{\partial x} + M_2 \frac{\partial}{\partial y}\right] \begin{pmatrix} u \\ v \end{pmatrix} = e_i^T \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} ,$$

and with (4.3)

$$e_i^T M_1 \left[\frac{\partial}{\partial x} + \lambda_i \frac{\partial}{\partial y}\right] \begin{pmatrix} u \\ v \end{pmatrix} = e_i^T \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} . \qquad (4.4)$$

This expression allows to gain a deeper insight into the system (4.2) and to construct a most natural method to find its solution.

Consider a curve $y = k(x)$ in the (x,y) plane. If $F(x,y)$ is an arbitrary function of x and y , then along the curve $k(x)$, this function depends on x only. The derivative with respect to x along this curve is given by

$$\frac{d}{dx} F(x,k(x)) = \frac{\partial F}{\partial x} + k'(x) \frac{\partial F}{\partial y} .$$

The operator

$$\frac{d}{dx} = \frac{\partial}{\partial x} + k'(x) \frac{\partial}{\partial y}$$

is called the *directional derivative* along the curve $k(x)$.

We now apply this concept to (4.4). Let two curves $y = k_i(x)$ be defined by their slopes

$$\frac{dy}{dx} = \lambda_i(x,y,u,v) , \quad i = 1,2 , \qquad (4.5)$$

and their initial values

$$k_i(o) = y_o ,$$

which means that

$$k_i'(o) = \lambda_i(o,y_o,u_o(y_o),v_o(y_o))$$

must hold. These curves will be called the *characteristics* of the system (4.2). Since $\lambda_1\lambda_2 < o$, these two curves will cross each other with a positive and a negative slope at any point (x,y) of the region of hyperbolicity.

The differential operator occurring in (4.4) then expresses the fact that the directional derivative along the curve $k_i(x)$, or, as it is usually known, along the characteristic C_i , must be taken. Thus the partial derivatives can be replaced by total derivatives along the characteristics.

We may then write (4.4) as

$$e_i^T M_1 \left(\frac{d}{dx}\right)_i \begin{pmatrix} u \\ v \end{pmatrix} = e_i^T \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} , \tag{4.6}$$

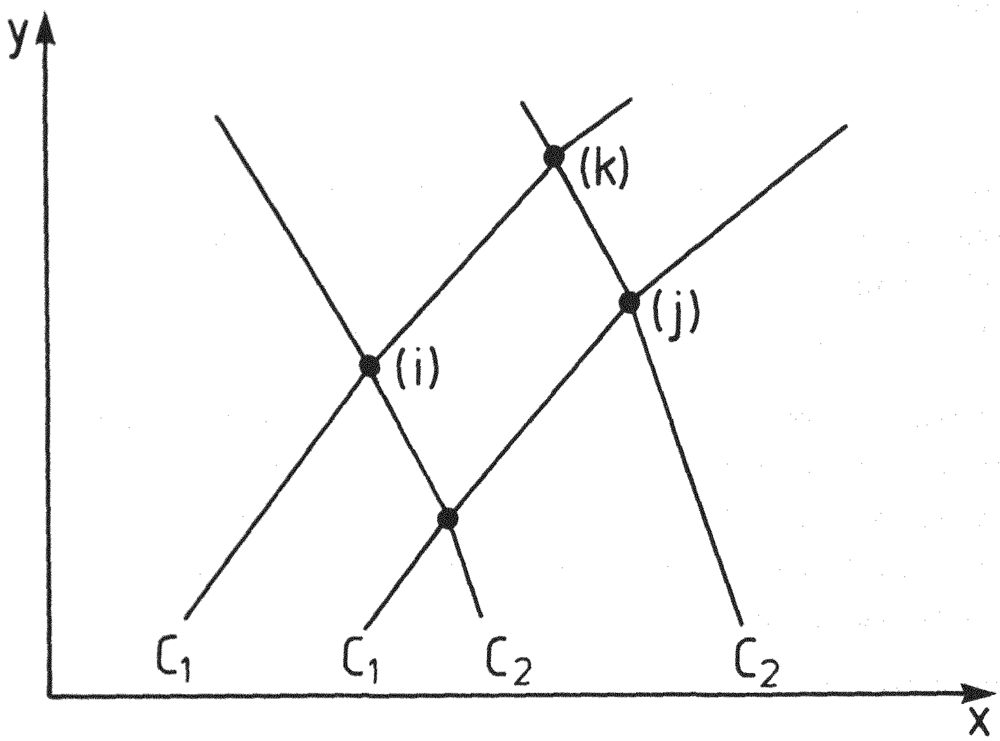
or explicitly

$$c_i\left(\frac{du}{dx}\right)_i + d_i\left(\frac{dv}{dx}\right)_i = f_i , \quad i = 1,2 , \tag{4.7}$$

where c_i,d_i,f_i are known functions of x,y,u,v .

Eqs. (4.5) and (4.7) will serve to construct a most natural algorithm in order to solve the system (4.1) numerically. This is the so-called MASSAU method ([29], see also Sauer [24], pp. 9o, Meis and Marcowitz [25], pp. 23, Lister [3o]).

Let two adjacent points be given in the (x,y) plane, (x_i,y_i) and (x_j,y_j) say, or briefly (i) and (j), with corresponding values u_i,v_i,u_j,v_j . Further, let two different pairs of characteristics from the two families C_1 and C_2 pass through both points (see Figure).



Here, "adjacent" means that the characteristics can be approximated linearly between two neighbouring vertices of the characteristic net. It should be stressed that the points (i) and (j) must not belong to one and the same characteristic.

Eq. (4.5) then allows to calculate, within linear approximation, the new point (k) of the characteristic net to which the points (i) and (j) already belong.

Let us assume for convenience that the characteristics of family C_1 have the positive slope, and those of C_2 the negative slope, and that $x_i < x_j$. Then x_k and y_k follow by setting equal the two straight lines

$$y(x) = y_i + (x - x_i)\ \lambda_1(x_i,y_i,u_i,v_i) \ ,$$

$$y(x) = y_j + (x - x_j)\ \lambda_2(x_j,y_j,u_j,v_j) \ .$$

It follows that

$$x_k = \frac{1}{(\lambda_1 - \lambda_2)} [y_j - y_i + x_i\lambda_1 - x_j\lambda_2] \quad ,$$

$$y_k = \frac{1}{(\lambda_1 - \lambda_2)} [y_j\lambda_1 - y_i\lambda_2 + (x_i - x_j)\lambda_1\lambda_2] \quad . \tag{4.8}$$

It remains to determine u_k and v_k from (4.7).

Now the operator $\left(\frac{d}{dx}\right)_i$ appearing in (4.7) means to take the directional derivative along the characteristic C_i . Therefore, we have approximately

$$\left(\frac{du}{dx}\right)_1 \cong \frac{u_k - u_i}{x_k - x_i} \quad ,$$

$$\left(\frac{du}{dx}\right)_2 \cong \frac{u_k - u_j}{x_k - x_j} \quad ,$$

and, with analogous expressions for the derivatives of v , we obtain from (4.7)

$$c_1(u_k - u_i) + d_1(v_k - v_i) = f_1(x_k - x_i) \quad ,$$

$$c_2(u_k - u_j) + d_2(v_k - v_j) = f_2(x_k - x_j) \quad , \tag{4.9}$$

which is a system of two linear equations for the two unknowns u_k , v_k .

With (4.8/9) we have gained a new point of the characteristic net from two known points. This procedure can be applied to n successive points on the initial curve, thus gaining (n-1) points on a second row, which can be continued until the n-th row consists of a single point. The characteristic net thus obtained will be called the *characteristic cone*.

4.2 Two special cases

Until now our considerations were quite generally valid for the quasi-linear system (4.1). If, however, special conditions are satisfied, we are able to get more information about the solution. In particular, two special cases will be of great importance:

a) The coefficients a_i, b_i of (4.1) depend on x and y only,

b) the coefficients a_i, b_i of (4.1) depend on u and v only, and the system is homogeneous, i.e. $h_1 = h_2 = o$.

Both cases will be treated in more detail now.

4.2.1 The semi-linear case

If the coefficients a_i, b_i of (4.1) depend on x and y only, the system becomes semi-linear. As an immediate consequence, the eigenvalues and eigenvectors of (4.3) will also depend on x and y only. Therefore (4.5) reduces to two ordinary differential equations which can be solved *independently of the initial values*. Thus the net of (x,y) characteristics is solely determined by the semi-linear system or, correspondingly, by the second order equation under consideration.

As an example, let us consider the semi-linear second order equation

$$a\, z_{xx} + 2b\, z_{xy} + c\, z_{yy} = h(x,y,z_x,z_y) \;, \qquad (4.1o)$$

where a,b,c are functions of x and y only. Then, with $u = z_x$, $v = z_y$, the matrices in (4.2) are given by

$$M_1 = \begin{pmatrix} a & o \\ o & 1 \end{pmatrix} \;, \qquad M_2 = \begin{pmatrix} 2b & c \\ -1 & o \end{pmatrix} .$$

The eigenvalues of the generalized eigenvalue problem (4.3) follow from $\det(M_2 - \lambda M_1) = o$ and are given by

$$\lambda_i = \frac{1}{a}\left[b \pm \sqrt{b^2 - ac}\right] , \quad i = 1,2 , \tag{4.11}$$

the corresponding left eigenvectors are

$$e_i^T = \left[1, b \mp \sqrt{b^2 - ac}\right] , \quad i = 1,2 . \tag{4.12}$$

(4.11) immediately gives the condition for hyperbolicity of (4.1o), namely $ac < o$. Moreover, eqs. (4.5) become two ordinary differential equations, which can be written as

$$ay'^2 - 2by' + c = o , \tag{4.13}$$

and which determine the net of (x,y) characteristics.

Using (4.12), we obtain from (4.6) with $h_1 = h$, $h_2 = 0$

$$\begin{aligned} a\left(\frac{du}{dx}\right)_1 + \left[b - \sqrt{b^2 - ac}\right]\left(\frac{dv}{dx}\right)_1 &= h , \\ a\left(\frac{du}{dx}\right)_2 + \left[b + \sqrt{b^2 - ac}\right]\left(\frac{dv}{dx}\right)_2 &= h . \end{aligned} \tag{4.14}$$

Eqs. (4.11/14) coincide with (19.4/5) of Sauer [24], p. 94. They have been rederived in order to demonstrate that this derivation, based on an eigenvalue analysis of the system (4.1), is very transparent and short.

We note that the equations (4.14) are *not* ordinary differential equations and that they must be solved using a Massau type method. We may then ask for the advantages offered by the special character of the coefficients.

First we can say that the Massau net in the (x,y) plane is uniquely given by the differential equation and thus can be constructed once, independent of initial values, moreover this can be done exactly, just solving two ordinary differential equations, either analytically or numerically.

If the equations (4.13) can be solved analytically, a further advantage is most remarkable. Let the solutions of (4.13) be solved for the integration constants $c_{1,2}$, then the

$$c_i = f_i(x,y) \ , \quad i = 1,2$$

can be used as new independent variables, ξ and η say. After this transformation the equation (4.1o) will have normal form (see Courant-Hilbert, [31], pp. 123), i.e. in the hyperbolic case

$$z_{\xi\eta} + g(\xi,\eta,z_\xi,z_\eta) = o \ . \qquad (4.15)$$

This equation may be soluble by standard techniques, e.g. if the original equation (4.1o) is linear, that is, if h depends linearly on z_x, z_y, eq. (4.15) will also be linear and the Riemann integration formula may be applied.

4.2.2 The reducible case

We now consider the other extreme case where the coefficients a_i, b_i of the system (4.1) are functions of the dependent variables u,v only. Whereas in Section 4.2.1 the equations (4.5) reduced to ordinary differential equations, we now attempt to gain ordinary differential equations from (4.7). For this to be possible we must additionally require that (4.7) is homogeneous, or $h_1 = h_2 = o$. Then the common denominator dx can be eliminated, and (4.7) will define two first order ordinary differential equations for $u(v)$. This case is known as *reducible* (see Ames [17], p. 35).

Here we meet the situation that the net of (u,v) characteristics is solely determined by the system (4.1) and thus independent of the initial values.

Again let us consider the quasi-linear second order equation

$$a\ z_{xx} + 2b\ z_{xy} + c\ z_{yy} = o\ , \tag{4.16}$$

where a,b,c depend on z_x, z_y only. Then, with $u = z_x$ and $v = z_y$ as in Section 4.2.1 , we obtain the same eigenvalues and eigenvectors, eqs. (4.11/12), and therefore the same expressions (4.14) except that these are zero on the right hand side.

Eqs. (4.14) can then be written as

$$\left(\frac{du}{dv}\right)_{1,2} = -\frac{1}{a}\left[b \mp \sqrt{b^2 - ac}\right]\ , \tag{4.17}$$

which with (4.11) become

$$\left(\frac{du}{dv}\right)_{1,2} = -\lambda_{2,1}\ . \tag{4.18}$$

Since

$$\left(\frac{dy}{dx}\right)_{1,2} = \lambda_{1,2}\ , \tag{4.19}$$

we have derived the result

$$\left(\frac{du}{dv}\right)_{1,2} = -\left(\frac{dy}{dx}\right)_{2,1}\ ,$$

which means that the two characteristic nets are "orthogonal-reciprocal" to each other (see Sauer [24], pp. 95). In other words, the slopes of the C_1 characteristics in the (x,y) plane are orthogonal to the corresponding slopes of the C_2 characteristics in the (u,v) plane (note that here u and v are abscissa and ordinate, respectively, we shall later choose the u axis as vertical axis).

From (4.17) we easily obtain the ordinary differential equations

$$au'^2 + 2bu' + c = o\ , \tag{4.2o}$$

which determine the net of (u,v) characteristics. These equations reveal a great similarity to eqs. (4.13) of the semi-linear case.

As in Section 4.2.1, we briefly discuss the advantages which are offered by this special choice of coefficients. For a given quasi-linear equation of reducible type, the characteristic net can now be constructed once in the (u,v) plane, independent of initial values. The (x,y) net follows from eqs. (4.19), which are *not* ordinary differential equations, therefore again the Massau method or a related method has to be applied.

If the ordinary differential equations (4.2o) can be solved analytically, the integration constants will play a fundamental role. Let again the solutions of (4.2o) be solved for these constants,

$$c_i = f_i(u,v) \ , \quad i = 1,2 \ ,$$

then, along a characteristic C_i in the (u,v) plane, the function $f_i(u,v)$ will remain constant. This has led to the name *Riemann invariants* for the integration constants. They can be used as new independent variables replacing u and v, once the original equation (4.16) has been transformed such that the independent variables x,y and the dependent variables u,v have changed their role. This transformation is known as *hodograph transformation* (see Ames [17], pp. 35).

In Appendix A1 and A2 we shall demonstrate this procedure, and it will become clear why the explicit knowledge of the Riemann invariants is so important.

4.3 The (u,v) characteristics for eq. (1.1)

We want to apply the results of the preceding section to eq. (1.1). If we set $u = \Gamma_t$, $v = \Gamma_z$, eq. (1.1) becomes

$$\Gamma_{tt} - 2v\Gamma_{tz} - 2u\Gamma_{zz} = o \ .$$

Therefore this equation is reducible. Identifying t and z with x and y , a comparison with eq. (4.16) yields

$$a = 1 \ , \ b = -v \ , \ c = -2u \ ,$$

and (4.2o) becomes

$$u'^2 - 2vu' - 2u = o \ . \tag{4.21}$$

This differential equation for $u(v)$ is of the d'Alembert type and can be solved analytically (see Appendix A2).

The solution in parametric form is given by the two families

$$\begin{aligned} u(s,\xi) &= -\xi s + \frac{1}{6}s^4 \ , \\ v(s,\xi) &= \frac{\xi}{s} + \frac{1}{3}s^2 \ , \end{aligned} \tag{4.22a}$$

and

$$\begin{aligned} u(s,\eta) &= \eta s + \frac{1}{6}s^4 \ , \\ v(s,\eta) &= \frac{\eta}{s} - \frac{1}{3}s^2 \ , \end{aligned} \tag{4.22b}$$

where ξ and η , the two integration constants of (4.21), represent the Riemann invariants, and the parameter s can be restricted to $o \leq s < \infty$ (see Appendix A2).

We want to discuss the structure of the two families (4.22), and consider first the hyperbolic case. The hyperbolicity condition $\lambda_1\lambda_2 < o$ then requires that with (4.11), or

$$\lambda_{1,2} = -v \pm \sqrt{v^2 + 2u} \ , \tag{4.23}$$

the condition $u>o$ must hold.

The initial conditions (1.2) will, for convenience, serve to select values for the Riemann invariants ξ and η [1]. Although these conditions define a fixed sign combination (see below), we assume for the moment that all combinations are possible, and obtain

$$\begin{aligned} \xi_j &= \pm \frac{1}{3} s_j^3 , \qquad j = 1,\dots,N , \\ \eta_k &= \pm \frac{1}{3} s_k^3 , \qquad k = 1,\dots,N , \end{aligned} \tag{4.24}$$

with

$$s_\ell = \left(\frac{2}{\sqrt{2\pi}}\right)^{1/4} e^{-z_\ell^2/8} , \quad \ell = 1,\dots,N ,$$

and the signs of (4.24) have to be chosen appropriately.

If we choose the signs such that $\xi < o$, $\eta > o$, the two families (4.22) satisfy $u > o$ for all admissible s values. In Fig. 4.1 these curves are shown for

$$z_\ell = -4 + (\ell-1)o.1 , \qquad \ell = 1,\dots,41 .$$

Note that for large s eqs. (4.22) yield

$$u(v) = \frac{3}{2} v^2$$

and $v \Rightarrow +\infty$ for (4.22a), $v \Rightarrow -\infty$ for (4.22b). Therefore the family with positive slopes in Fig. 4.1 corresponds to $\xi < o$, and the one with negative slopes to $\eta > o$.

Next we consider the case $\xi > o$. Then hyperbolic solutions are possible only for

$$(6\,\xi)^{1/3} < s < \infty ,$$

for the remaining s interval we have $u \leq o$, the envelope

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[1] Any other choice would represent the characteristics as well, only their density depends on the actual initial distribution.

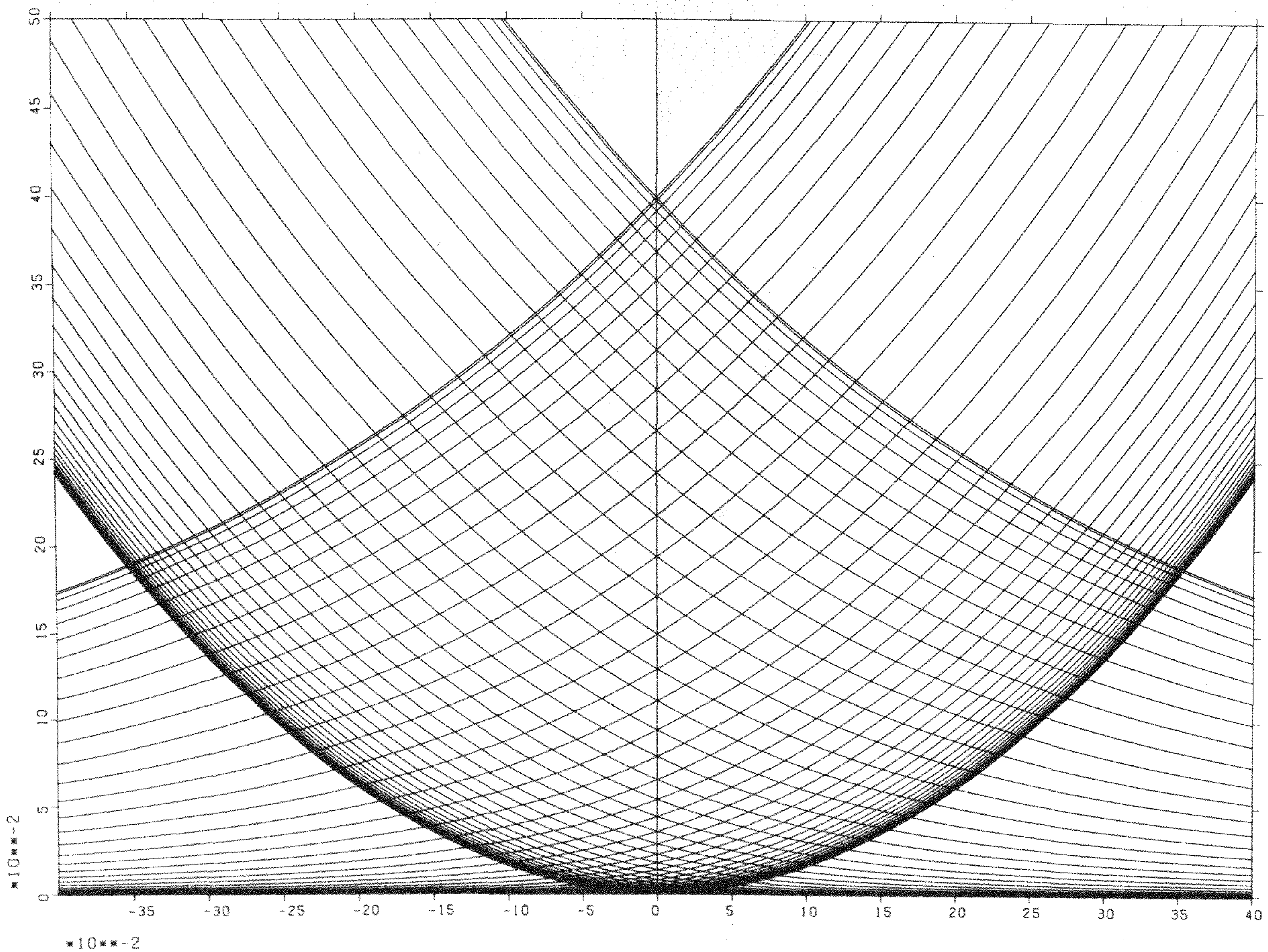

Fig. 4.1 : Characteristics in the (u,v) plane for u>o . The abscissa represents the v axis.

being given by (see Appendix A2)

$$u(v) = -\frac{1}{2} v^2 .$$

In any case we have $v>o$ according to (4.22a). Analogous results hold for $\eta<o$, where now $v<o$. Fig. 4.2 shows again the curves of Fig. 4.1 and, in addition, these new curves, where for reasons of transparency only the range $u\leq o.1$ has been plotted.

To summarize, the following remarks will be useful when the (u,v) characteristics of eq. (1.1) are to be applied to a given hyperbolic problem.

1) Whenever the initial conditions are such that $v=o$ for $t=o$ (as in (1.2)), it follows from (4.22) that $\xi<o$ and $\eta>o$ are the only possible choices for the signs of the Riemann invariants (Fig. 4.1).

2) The case $\xi>o$, $\eta<o$ must be excluded since the two families do not intersect each other ($v>o$ and $v<o$, respectively, in Fig. 4.2). Note that for $u<o$ each family intersects itself due to violation of the Lipschitz condition for eq. (4.21) along the discriminant $u = -v^2/2$.

3) The five different (ξ,η) regions of Fig. 4.2,

$\{(-,-), (+,+), (-,+)\}$ for $u>o$, and $\{v<o , v>o , v$ arbit.$\}$
$\{(\cdot,-), (+,\cdot)\}$ for $u<o$, and $\{v<o , v>o\}$,

are separated by the two limiting characteristics

$$u(v) = \frac{3}{2} v^2 H(v) , \quad \text{belonging to the } \xi \text{ family} ,$$

$$u(v) = \frac{3}{2} v^2 H(-v) , \quad \text{belonging to the } \eta \text{ family} ,$$

where $H(x)$ is the Heaviside function, i. e. $H=1$ for $x\geq o$, $H=o$ for $x<o$.

The meaning of the limiting characteristics will be elucidated in the next section, where a particular example will be discussed.

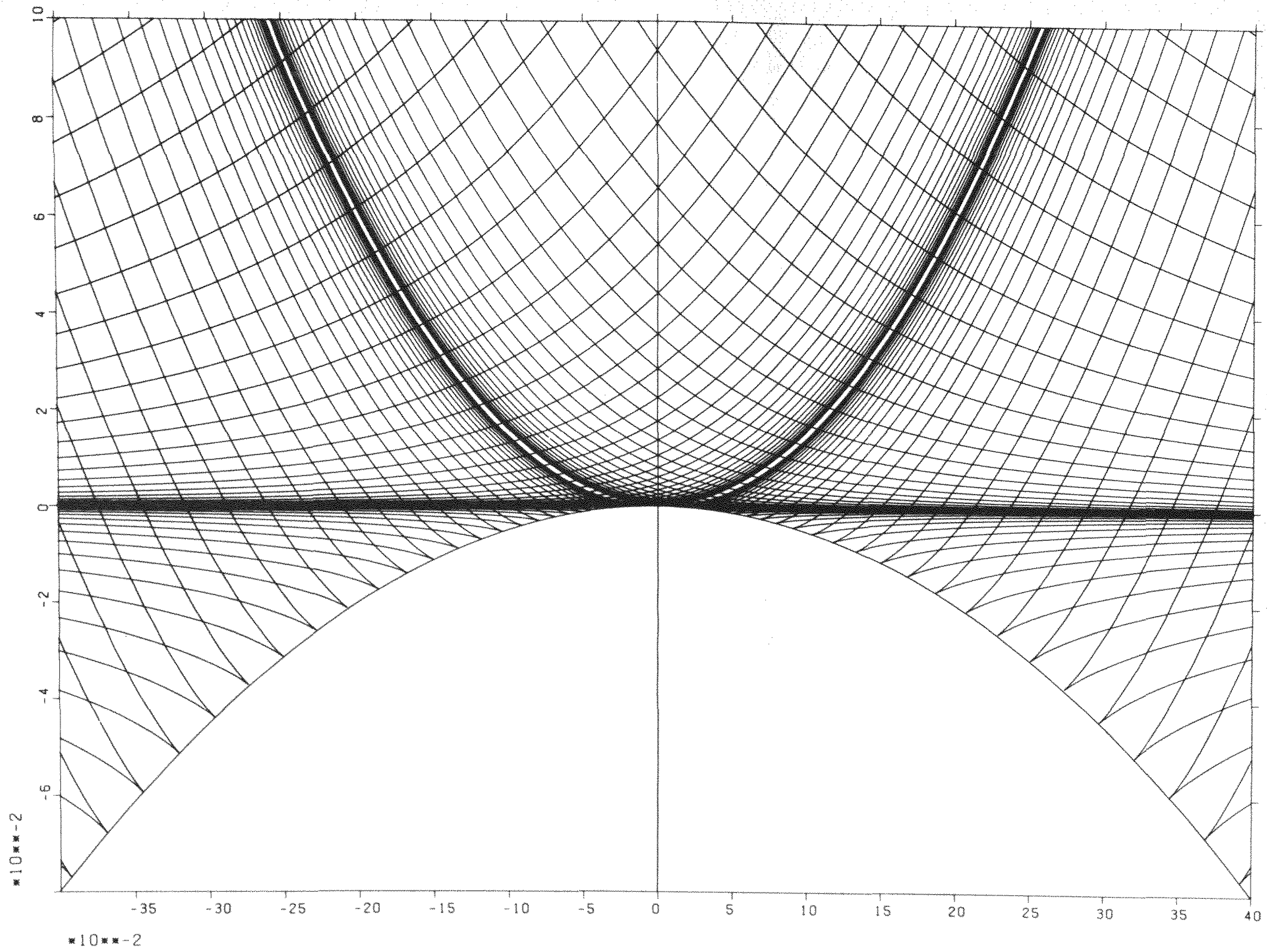

Fig. 4.2 : The full set of curves (4.22). The abscissa represents the v axis.

4.4 The (t,z) characteristics for the parabolic solution

The meaning which the limiting characteristics in the (u,v) plane exhibit for the characteristics in the (t,z) plane becomes especially clear for the parabolic solution which was introduced in Section 1.3.3 . This will be discussed now.

In order to simplify the notation, we set $b=1/6$ in (1.37). The initial condition (1.34b) then becomes

$$u(z,o) = \frac{8}{3} - \frac{1}{6} z^2 , \qquad |z| \leq 4 ,$$

$$u(z,o) = o \qquad , \qquad |z| > 4 .$$

Note that now $\int u dz = 128/9$, which, however, is completely irrelevant for the present considerations.

With $a=8/3$ and $b=1/6$, eqs. (1.37a/b) become

$$u(z,t) = \frac{8}{3} (\cosh t)^{-2/3} - \frac{1}{6} (\cosh t)^{-2} z^2 , \qquad (4.25a)$$

$$v(z,t) = -\frac{1}{3} z \tanh t \qquad . \qquad (4.25b)$$

According to (4.5) and (4.23), the characteristics in the (t,z) plane obey

$$\frac{dz}{dt} = -v \pm \sqrt{v^2 + 2u} , \qquad (4.26a)$$

or, with (4.25),

$$3(\dot{z} \cosh t)^2 - 2z\dot{z} \sinh t \cosh t + z^2 = 16(\cosh t)^{4/3} . \qquad (4.26b)$$

This ordinary differential equation must in general be solved numerically. However, for the limiting characteristics, analytic results can be obtained, as will be shown now.

1) $u=o$:

In this case eq. (4.25a) leads to

$$z(t) = \pm 4 (\cosh t)^{2/3} . \qquad (4.27)$$

2) $u = \frac{3}{2} v^2$:

In this case eqs. (4.25a/b) lead to

$$z(t) = \pm 4 \ (\cosh t)^{-1/3} \ . \tag{4.28}$$

On the other hand, if we assume that

$$z(t) = c(\cosh t)^{\alpha} \ ,$$

eq. (4.26b) leads to

$$c^2 (\cosh t)^{2\alpha}[\alpha(3\alpha-2)(\cosh t)^2 - (\alpha-1)(3\alpha+1)] = 16(\cosh t)^{4/3}.$$

In fact, this equation has exactly two solutions, $\alpha=2/3$ and $\alpha=-1/3$, with $c=\pm 4$ in both cases, which shows that (4.27/28) are particular solutions of (4.26b).

In Fig. 4.3 the (t,z) characteristics for this problem are shown. At each point on the z axis, given by

$$z_\ell = -4 + (\ell-1)o.1 \ , \quad \ell=1,\dots,81 \ ,$$

two characteristics issue, which are obtained by integrating (4.26a) numerically with $z(o) = z_\ell$. Only at $z(o) = \pm 4$ this integration is not necessary since the corresponding characteristics are given by (4.27/28) (note that in Fig. 4.3 only the range $z \geq o$ is plotted).

The outer characteristics (4.27) describe the range over which the initial distribution (1.34b) is spread at time t (the "bunch length"). The characteristics (4.28) separate the inner characteristic cone, where two characteristics pass through each point, from the outer characteristic cones with only one characteristic per point (see Fig. 4.3).

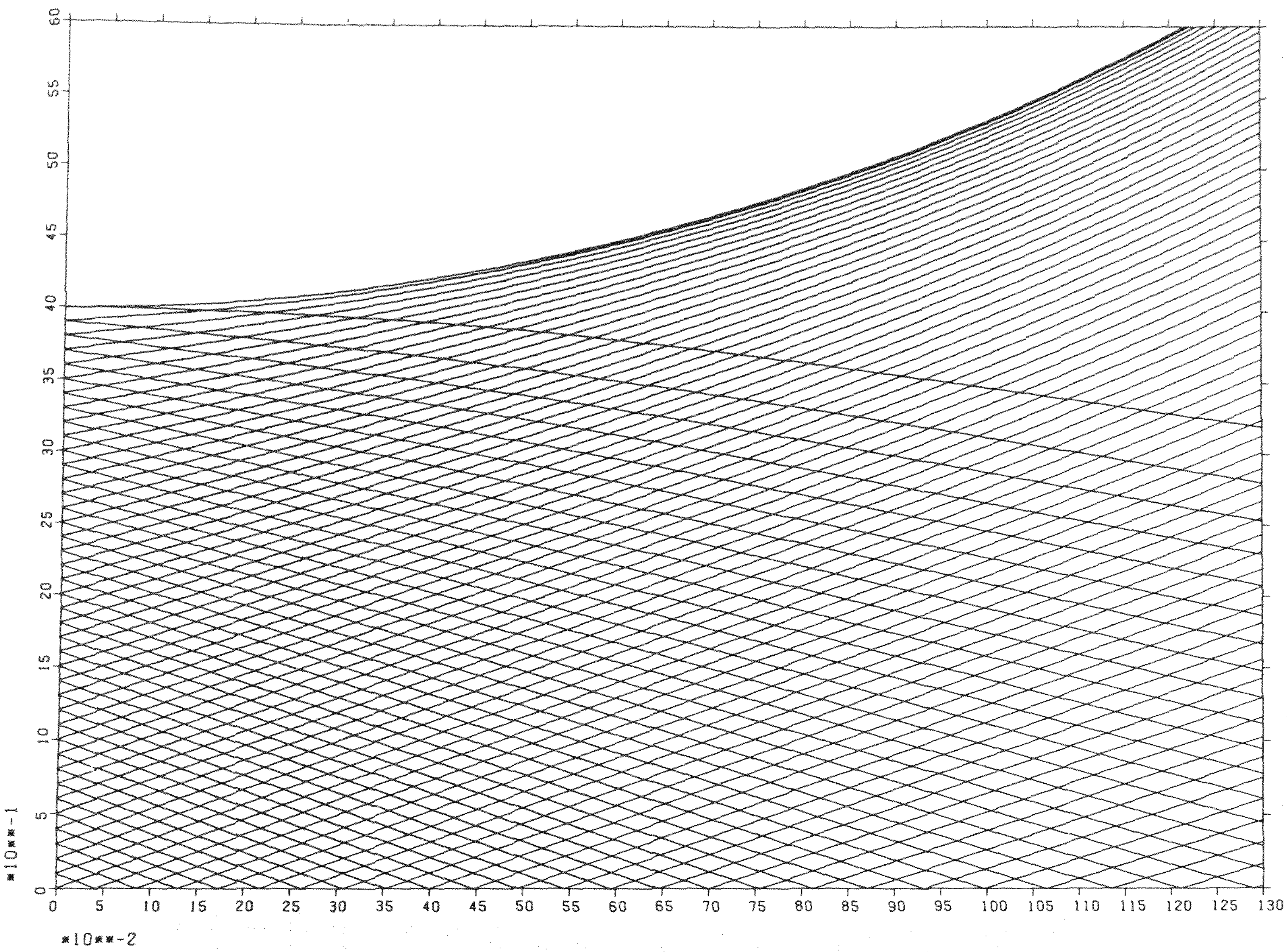

Fig. 4.3 : Characteristics for the parabolic case in the (t,z) plane for $z \geq o$.
The abscissa represents the t axis.

4.5 Application of the Massau method to problem (1.1/2)

In this section we want to apply the Massau construction as described in Section 4.1 to the initial value problem (1.1/2). Although the net of (u,v) characteristics is known analytically, as shown in Section 4.3, we shall still construct both nets numerically by approximating linearly the characteristics between two adjacent points.

Initially there are N equidistant points given on the z axis in the interval [-4,o],

$$(z_k,t_k) = [-4 + (k-1)h,o] \;, \qquad\qquad k = 1,\ldots,N \;, \quad (4.29)$$

with $(N-1)h = 4$. Here use will be made of the fact that problem (1.1/2) is symmetric in z . However, the choice (4.29) will modify slightly the original Massau construction which one would apply for $z \in [-4,4]$ (see below).

The points in the (u,v) plane which correspond to the initial conditions (1.2) are then given by

$$(u_k,v_k) = \left[\frac{1}{\sqrt{2\pi}}\, e^{-z_k^2/2} \;,\; o\right] \;, \qquad k = 1,\ldots,N \,. \qquad (4.3o)$$

With (4.29/3o) the (N-1) points of the second row can now be determined using eqs. (4.8) for the (t,z) plane and eqs. (4.9), or eqs. (4.17) in the reducible case, for the (u,v) plane.

When this procedure is continued until all initial points have been consumed, we obtain two nets of N(N+1)/2 points each, which for $N=41$ are shown in Fig. 4.4 for the (u,v) plane and in Fig. 4.5 for the (t,z) plane. Note that Fig. 4.4 represents identically (except for discretization errors) that region of Fig. 4.1 which shows the two families $\xi<o$ and $\eta>o$ of Section 4.3 intersecting in the range $v\geq o$.

As can be seen from Fig. 4.5, the characteristics of one family of the (t,z) characteristics cross each other at a certain point. This critical point represents a shock singularity and will be discussed in more detail in Section 4.7 .

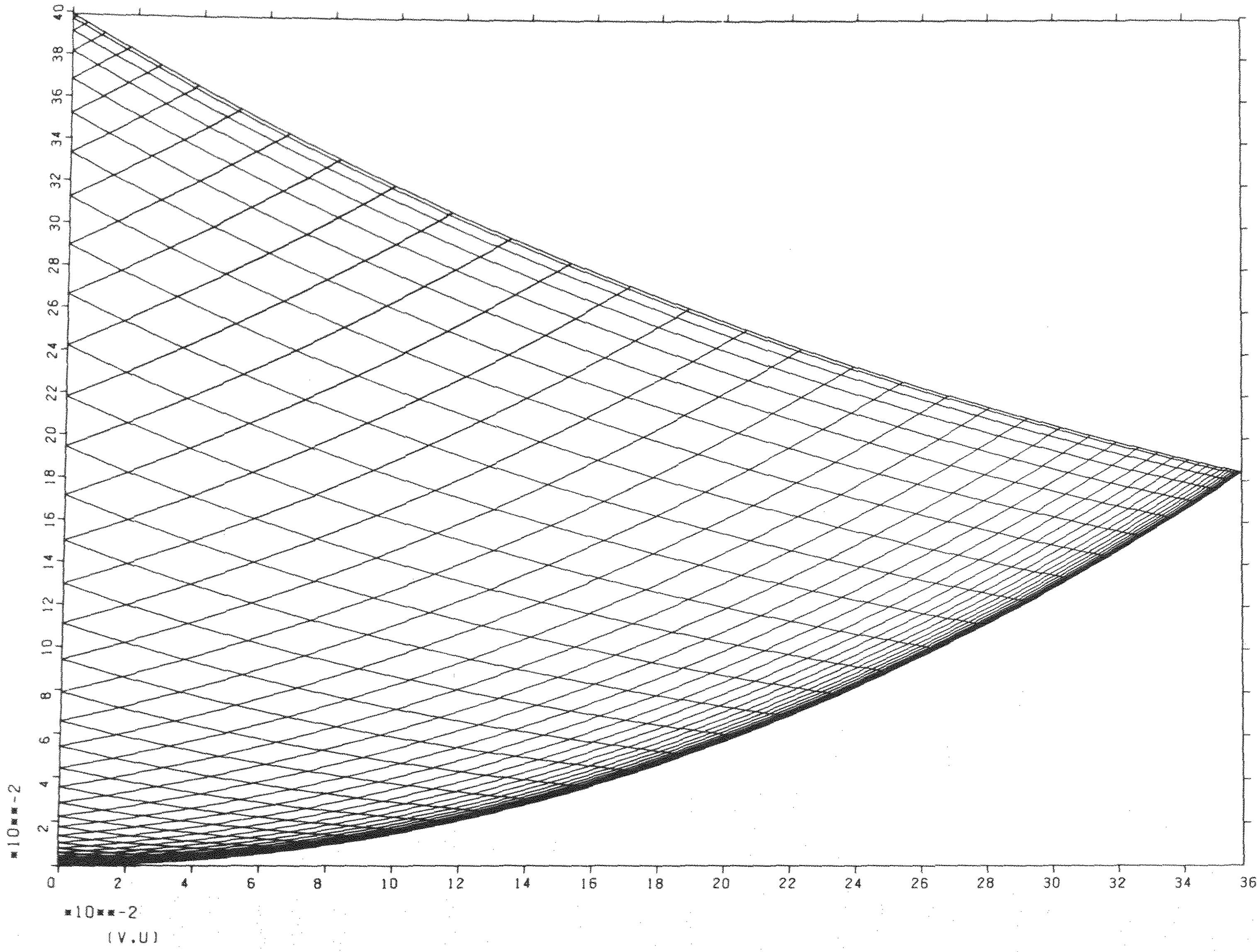

Fig. 4.4 : Massau construction in the (u,v) plane. The abscissa represents the v axis.

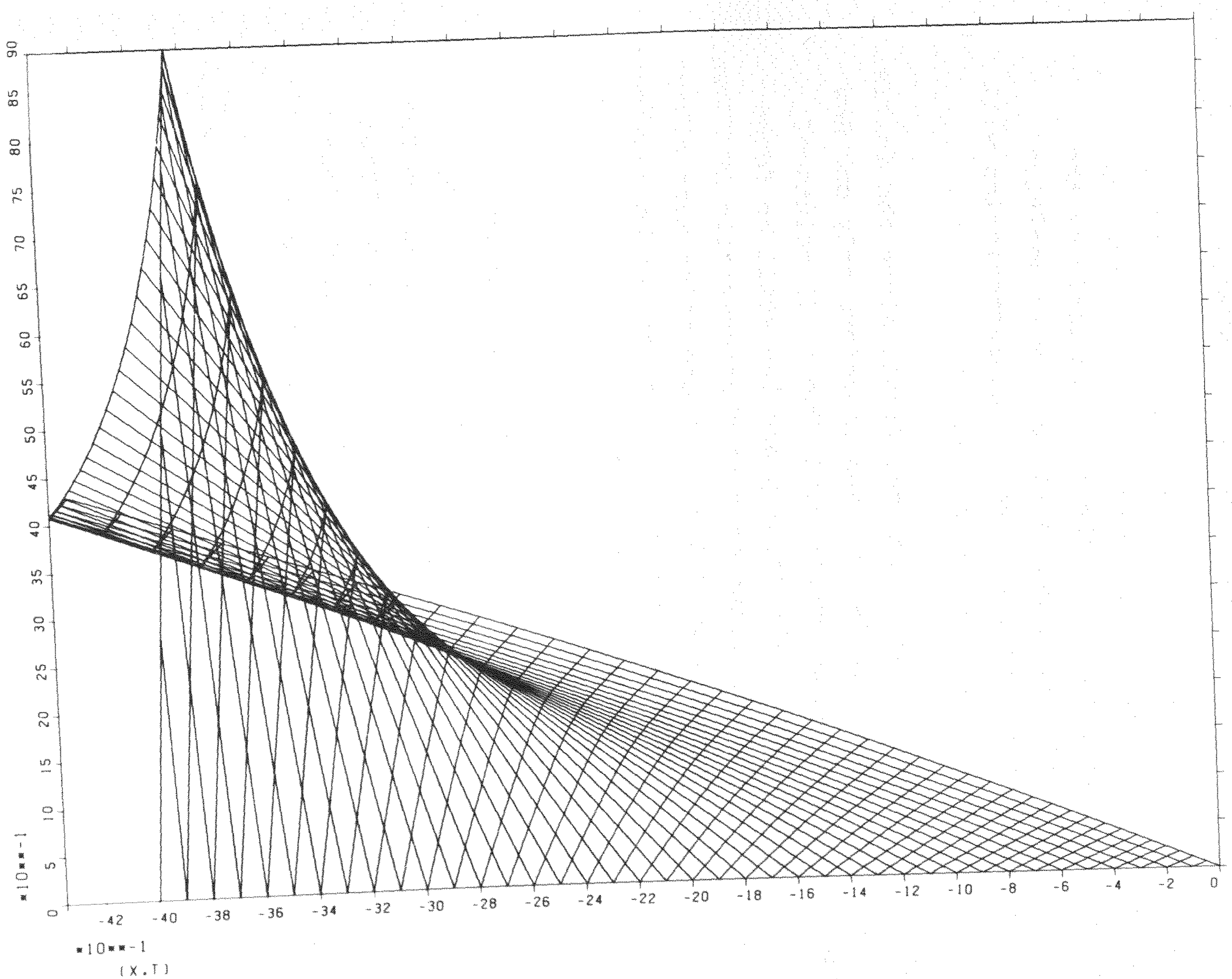

Fig. 4.5 : Massau construction in the (t,z) plane. The abscissa represents the negative z axis.

The construction as described above gives already the *complete* net of (u,v) characteristics belonging to the initial values (4.3o), but only *one half* of the net of (t,z) characteristics. In order to obtain the complete (t,z) net, a second mapping of the (u,v) net onto the (t,z) plane is necessary.

First we observe that the lower characteristic of the (u,v) net in Fig. 4.4 - in the notation of Section 4.3 the ξ_1 characteristic - corresponds to the characteristic of the (t,z) net in Fig. 4.5 which issues at (o,o) with negative slope. The N points on each of these two characteristics will serve as initial data for the second mapping.

However, this mapping differs from the first, since the initial points now lie on characteristics, and therefore only one new direction per point can be constructed. The (N-1) points of the new characteristic in the (t,z) plane can still be found successively, if a single point on this characteristic is known. From the symmetry property we deduce that one point must always lie on the t axis. Thus the complete second net can be constructed. It contains (N-1)N/2 points, in total N^2 points for the (t,z) net which is shown in Fig. 4.6 .

Finally we note that the numerical accuracy was considerably increased, when for the second mapping the u values along the u axis were not obtained from the Massau construction but chosen according to the analytic representation (4.3o). That these values have to be assumed again follows from the fact that we have v=o along the z axis and along the t axis.

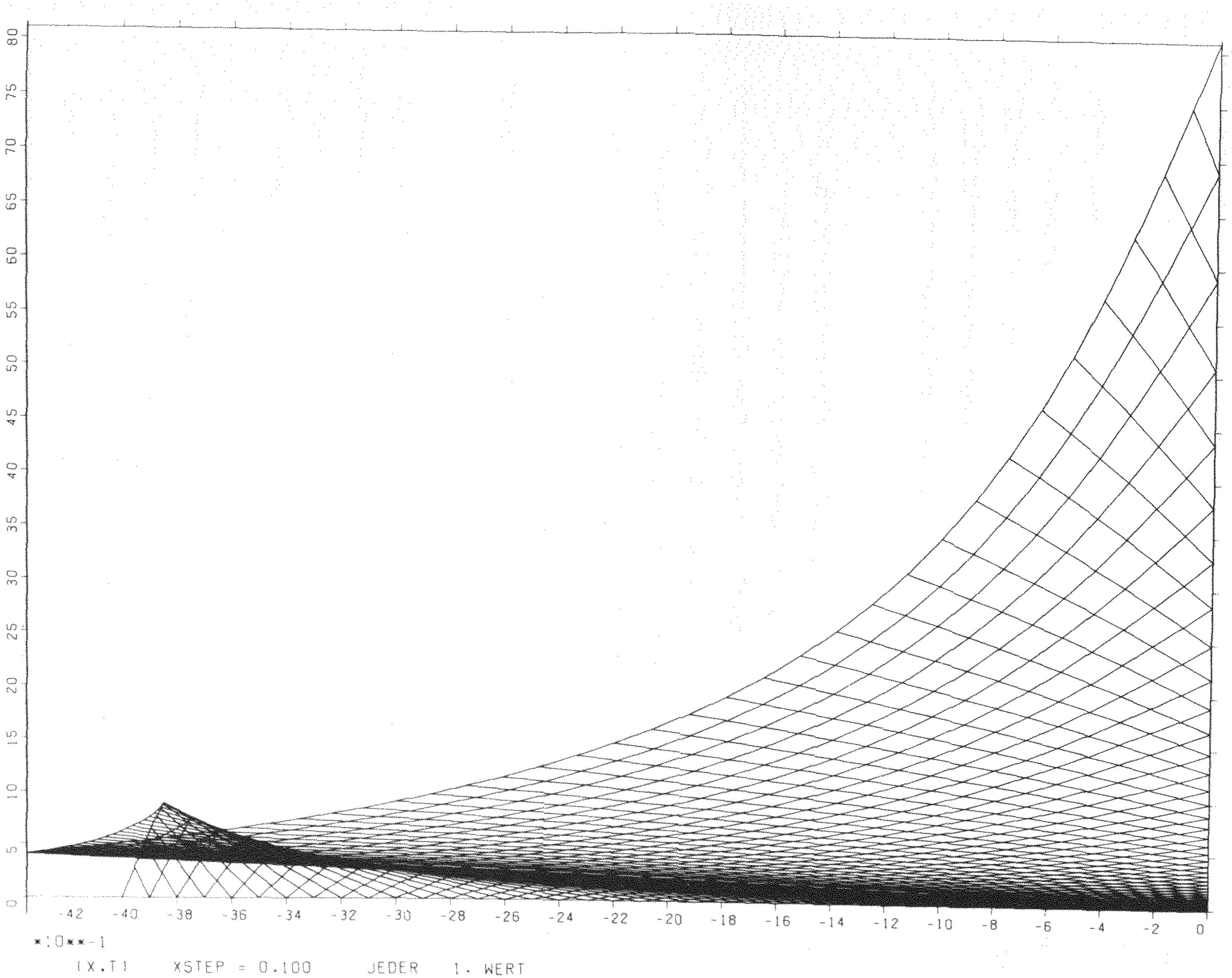

Fig. 4.6 : The complete characteristic cone in the (t,z) plane. The abscissa represents the negative z axis.

4.6 Some algorithms using the method of characteristics

Let us first make some general remarks concerning the method of characteristics. Although this method is the most natural access to hyperbolic problems, certain disadvantages should not be overlooked. Among these, the nonuniform distribution of grid points along the characteristics is only a minor drawback which, as the solution will usually be required for fixed times, makes interpolation, preferably by splines, necessary. More serious, however, is the density of the grid points which may vary strongly along a given characteristic so that regions with sparsely distributed solution points can occur.

The greatest advantage of the method of characteristics, as concerns the present problem, is obviously its ability to predict and locate shock singularities. The difference method of Chapter 3 also showed the onset of the shock (Fig. 3.3), but its location was rendered more difficult by the rapid increase of the discretization error near the shock singularity.

In the following we want to present three Massau type algorithms available from the literature and discuss some results which have been obtained by these codes.

The first algorithm (Smith and McCall [32], here referred to by SMITH), is a simple Massau method which in addition performs an extrapolation to the limit with step sizes $4h$, $2h$ and h , where h is the increment of the initial data. Thus only $N/4$ of the originally N rows are available. Whereas the Massau method has an error of order $O(h)$, this method has an error of $O(h^3)$. If shock singularities are encountered, the algorithm will omit the range of influence belonging to the shock (see e.g. Courant and Friedrichs [13], pp. 48, for the meaning of range of influence, and Fig. 4.7 showing the rows in the (t,z) plane when SMITH is applied to problem (1.1/2) with $h=o.1$).

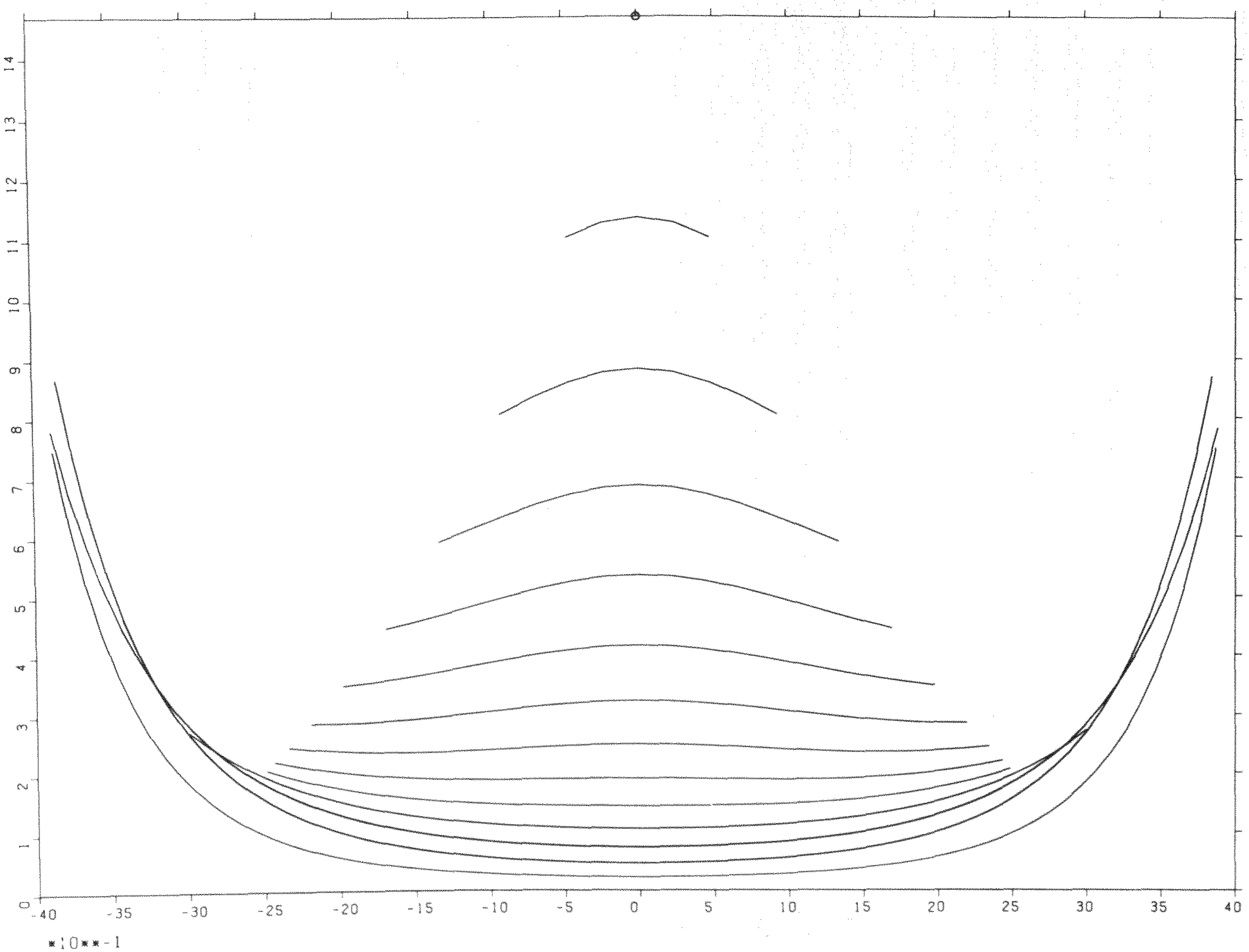

Fig. 4.7 : Rows of the characteristic cone obtained from the SMITH algorithm.

The second algorithm (Busch, Esser, Hackbusch, Herrmann [28]) is an extension and improvement of the SMITH method. Since this algorithm is published in ALGOL, we used its original FORTRAN version [1], supplied by one of the authors, and will denote this code by ESSER (see Appendix A4 for a listing). It mainly allows to perform higher extrapolation steps which are controlled by a parameter nex. The number of available rows will then be reduced to N/(2 nex). When shocks occurred, formal difficulties were met as to which part of the characteristic net should be omitted. We changed this code slightly to obtain the complete characteristic cone thereby allowing for a crossing of the rows (see Fig. 4.8 which shows the first 25 rows in the (t,z) plane for problem (1.1/2), with nex=1 and h=o.1).

The third algorithm (Meis and Marcowitz [25], pp. 377, denoted by MEIS), designed to be simple and modular, is a pure Massau method which, however, does not work in the (t,z) plane but instead in a (σ,τ) plane, where σ and τ are so-called characteristic coordinates. The net of characteristics in the (σ,τ) plane consists of straight lines. Like the SMITH method, this algorithm also avoids the range of influence of the shock singularity. Fig. 4.9 shows the characteristic net in the (σ,τ) plane when MEIS is applied to problem (1.1/2), in Fig. 4.1o the corresponding net in the (t,z) plane is presented [2].

The fourth algorithm included in this comparison is our own MASSAU method as described in Section 4.5 .

With these algorithms, problem (1.1/2) was solved numerically using different increments h for the initial data z_k in the interval $z \in [-4,4]$. The solution was compared at four points on the t axis, therefore z , and by symmetry

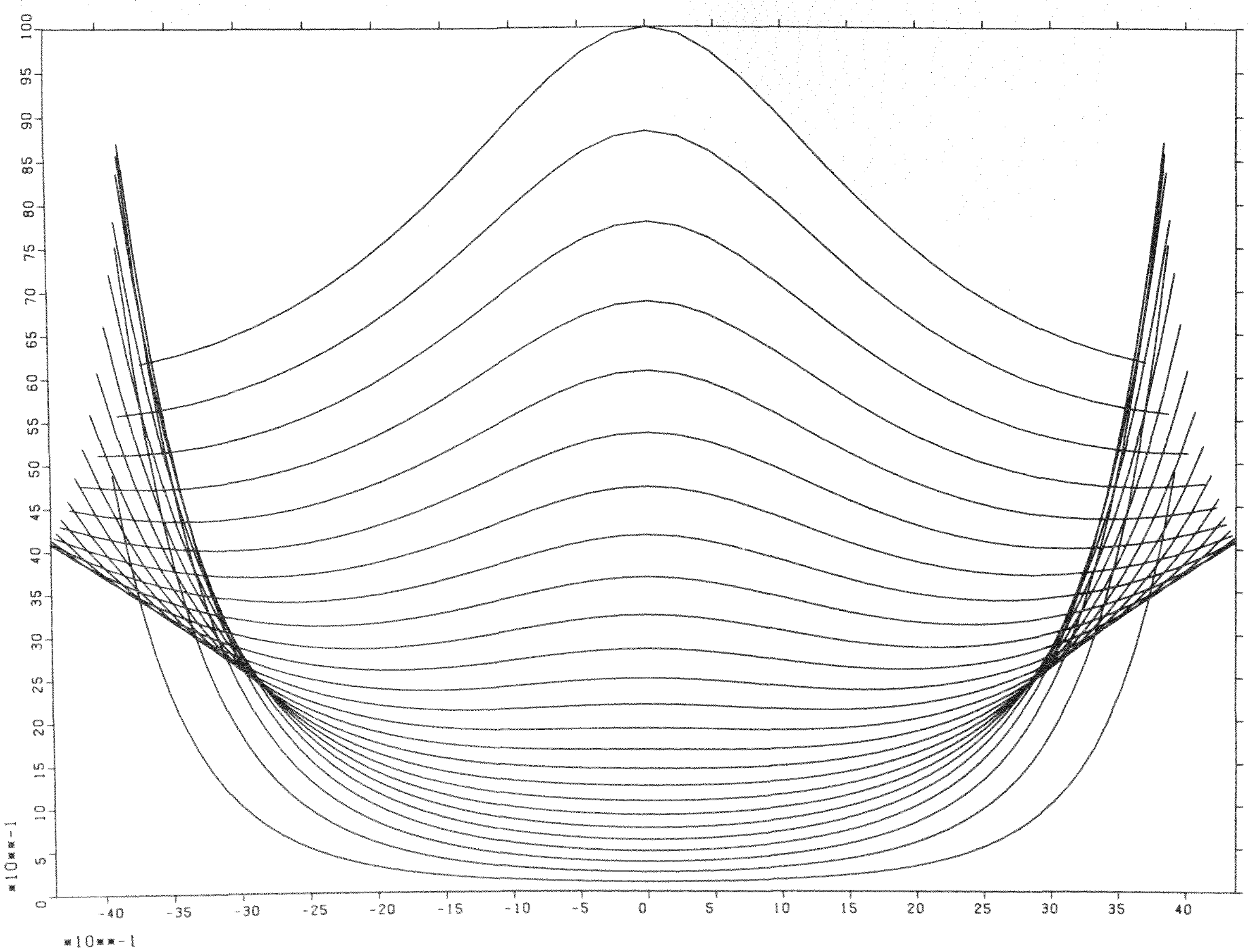

Fig. 4.8 : Rows of the characteristic cone obtained from the ESSER algorithm.

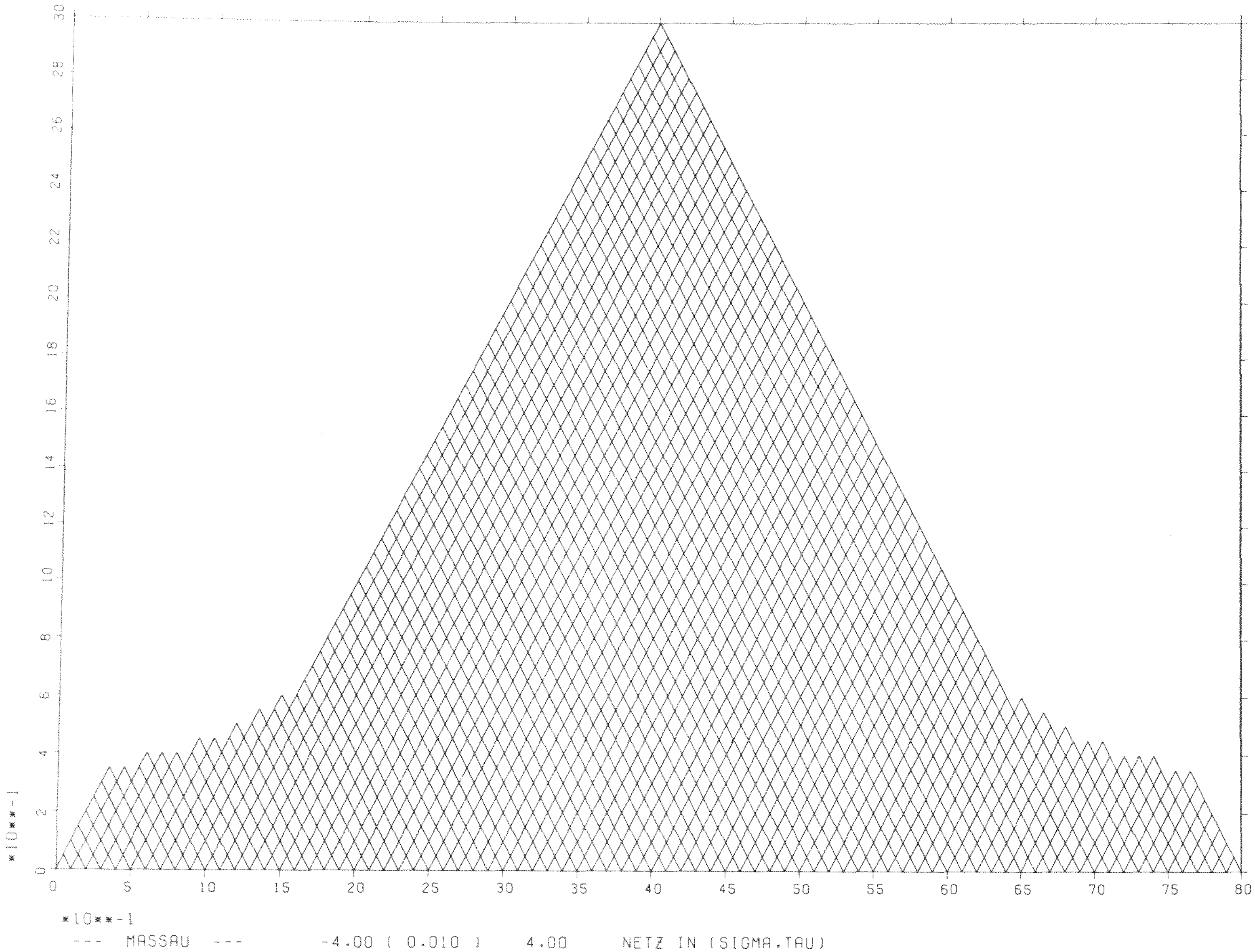

Fig. 4.9 : Characteristic cone in the (σ,τ) plane obtained from the MEIS algorithm.

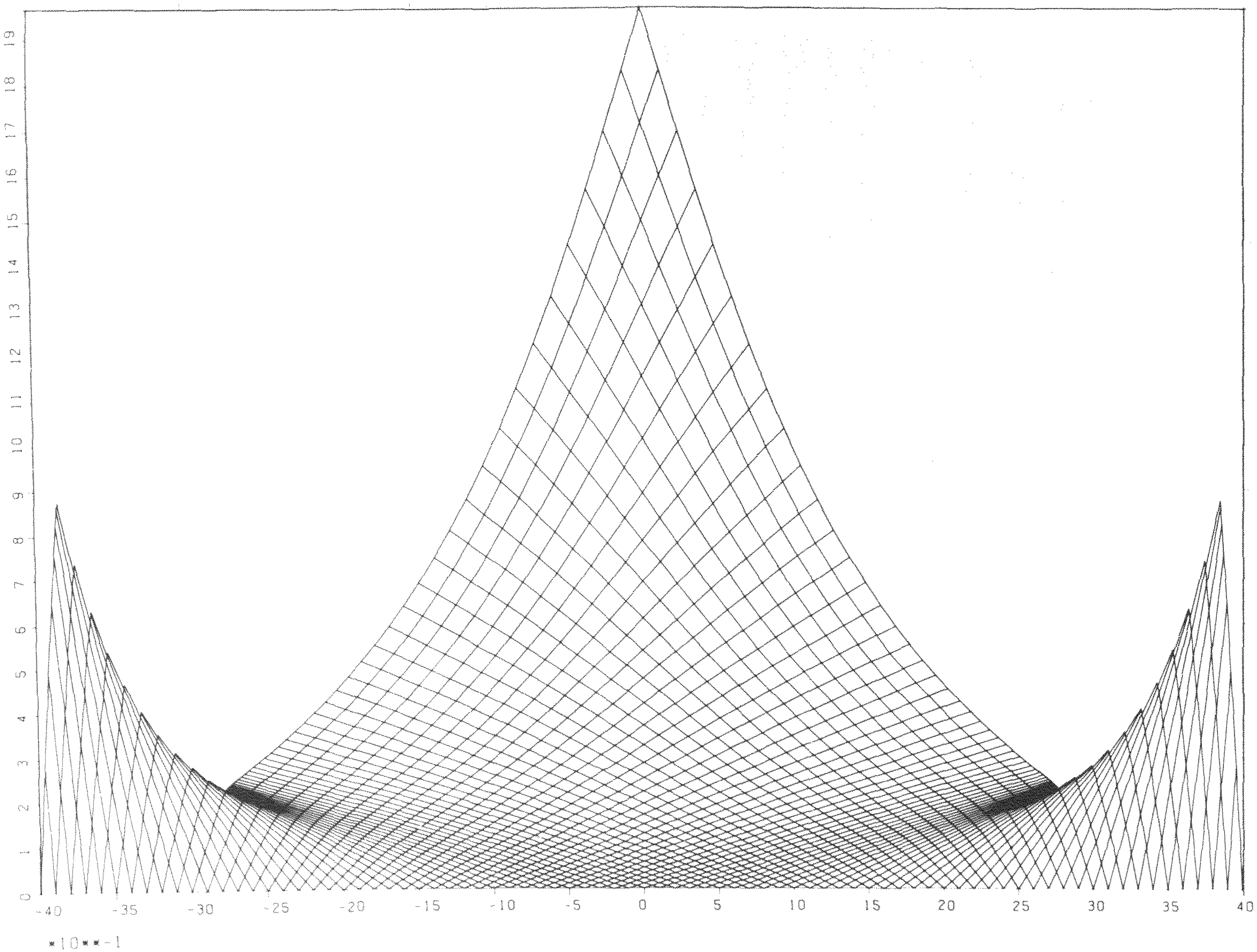

Fig. 4.1o : Characteristic cone in the (t,z) plane obtained from the MEIS algorithm.

also v, were equal to zero. The four points were selected from those characteristics, which issued on the initial line at z values equal to -1, -2, -3, -4 with positive slope thereby intersecting the t axis at the reference points.

Table 4.1 contains the results for the t values, and Table 4.2 for the u values. Since $v=o$ is valid on the initial line as well as on the t axis, the u values on the initial line, given by (4.3o), will again be assumed on the t axis if they are pursued along their characteristics. The algorithm MASSAU used this property, and therefore the exact values appear in Table 4.2 (see pp. 126/127 for Tables 4.1/2).

These tables allow a number of interesting observations. First we notice that the algorithms SMITH and MEIS do not cover the complete characteristic cone because of the shock singularity, therefore the values for $z=-4$ are omitted. Also values for $z=-3$ are omitted, if the increment is too large. In these cases the condition for hyperbolicity is violated on numerical grounds.

Further it can be seen that the pure Massau methods MEIS and MASSAU clearly exhibit linear convergence when $h \to o$, i.e. reducing h to $o.1\ h$ yields a gain in accuracy of one decimal place only.

More generally, if the order of a method is $O(h^p)$, the solution should have an error which is p decimals smaller for $o.1\ h$ than for h. From Table 4.2 we thus confirm that SMITH is of order 3, and ESSER for $nex=2$ of order 4, as it should be according to the authors' statements ([32], p. 568, [28], p. 347). For $nex=1$ the order of the latter algorithm should be 2, which also follows from Table 4.1; on the other hand, from Table 4.2 we observe the remarkable fact that this algorithm shows a behaviour of order 3, if not even better.

As concerns the execution time, the last column of Table 4.1 reveals a very rapid increase with N, the number of initial points. Since we have $N=8o1$ for $h=o.o1$, it is thus nearly prohibitive for the methods using extrapolation to go beyond N=1ooo , whereas the two Massau methods MEIS and MASSAU still worked at N=8oo1 in a reasonable time [1]. Although here the gain in accuracy would never justify these times, the increase in the number of rows, along which the solution is available, may very well be worthwhile the additional computational effort.

Finally we make the attempt to draw some general conclusions from these restricted results. Comparing MEIS and MASSAU it becomes evident that the additional information contained in the knowledge of the analytic solution of the (u,v) characteristics improves considerably the pure Massau method [2]. Here we have used this information only along the u axis, even better results should be expected from the application of the complete analytic (u,v) net.

Were it not for the larger number of rows available from the pure Massau methods (they alone provide the total net of characteristics determined by the initial data), these methods would be far inferior to the methods using extrapolation. Now SMITH as well as ESSER with nex=2 only supply every fourth of the original N rows which already seems to be rather sparse when for instance a location of the shock singularity is desired (see next section). One might compensate for this by increasing N , but, as we have seen, this becomes impossible very soon.

Comparing the corresponding algorithms SMITH and the nex=2 version of ESSER, the latter algorithm is significantly superior even with slightly more execution time.

As a compromise between accuracy and sufficient density of solution rows, we conclude that ESSER with nex=1 , which provides every second row, seems to be optimal.

---

[1] The difference in the execution times is mainly caused by the fact that MASSAU calculates the characteristic net in the (t,z) plane only for $z \leq o$ (see Section 4.5), whereas MEIS cannot take into account the symmetry of the problem.

[2] See the final remark of Section 4.5 .

Algorithm	h	-1.0	-2.0	-3.0	-4.0	CPU[sec]
SMITH	0.5	1.413	5.58	-	-	
	0.1	1.42483	5.317	-	-	0.57
	0.01	1.42494221	5.3210906	19.137334	-	46.53
ESSER	0.5	1.4299	5.43	21.40	-	
(NEX=1)	0.1	1.42515	5.3256	19.183	85.36	0.70
	0.01	1.4249445	5.321139	19.13776	84.931	1 m 2.92
	0.5	1.4271	5.28	12.34	-	
(NEX=2)	0.1	1.4249445	5.321106	19.1365	84.87	0.87
	0.01	1.4249423809	5.3210942 4	19.1373050	84.92731	1 m 19.21
MEIS	0.5	1.470	6.12	-	-	0.09
	0.1	1.4282	5.48	42.85	-	0.28
	0.01	1.42529	5.337	19.65	-	16.58
	0.001	1.424977	5.3227	19.186	-	27 m 6.21
MASSAU	0.5	1.435	5.51	18.88	66.80	0.10
	0.1	1.4271	5.356	19.06	80.95	0.19
	0.01	1.42516	5.3245	19.129	84.52	6.79
	0.001	1.424964	5.32143	19.1365	84.886	10 m 23.94

Table 4.1 : Values of t at four points on the t axis.

Algorithm	h	-1.0	-2.0	-3.0	-4.0
SMITH	0.5	0.26	0.05352	-	-
	0.1	0.2421	0.05410	-	-
	0.01	0.24197082	0.05399107	0.00432	-
ESSER (NEX=1)	0.5	0.241996	0.05380	0.0036	-
	0.1	0.241970753	0.05399036	0.00443109	0.00013370
	0.01	0.2419707245211	0.05399096645	0.00443184835	0.000133830217
(NEX=2)	0.5		0.05323	0.026	-
	0.1	0.24197060	0.0539936	0.0044359	0.0001346
	0.01	0.2419707245126	0.05399096677	0.00443184866	0.000133830261
MEIS	0.5	0.22	0.013	-	-
	0.1	0.239	0.046	0.0002	-
	0.01	0.24172	0.05323	0.0039	-
	0.001	0.241946	0.053915	0.00437	-
exact		0.2419707245191	0.05399096651	0.00443184841	0.000133830225

Table 4.2 : Values of u at four points on the t axis.

4.7 The shock discontinuity

In the preceding chapters three totally different methods have been applied to problem (1.1/2), and in each case the indication of a singularity in the solution could be established:

1) The Taylor series expansion of Chapter 2 ceased to converge beyond a curve $\zeta(t)$ in the (t,z) plane (Section 2.5, Fig. 2.5),

2) the discretization method of Chapter 3 revealed the onset of strong oscillations over a small range of z values (Section 3.3, Fig. 3.3),

3) the method of characteristics of this chapter exhibits the (forbidden) crossing of characteristics for one family of the (t,z) characteristics (Section 4.5, Fig. 4.5, and Section 4.6, Figs. 4.8/1o).

The location of this singularity is possible only from the third method. We used the programme ESSER of the previous section with parameters nex=1 and h=o.o1 in the following way: Through each row a spline was drawn in the interval $z \in [-3.0,-2.5]$ and the point of cross over for two neighbouring rows (if any) was determined (see Fig. 4.8 for the approximate location). That point which then emerged with the smallest time was chosen as the best possible approximation to the location of the singularity. We thus found $z = -2.787$, $t = 2.337$, which should be accurate within the given decimals.

It is rather remarkable that the two methods which *a priori* do not allow to locate the singularity still reproduced this value within the scope of possibility for each method. In the case of the Taylor series expansion, point A in Fig. 2.5 represents the true value, whereas the curve $\zeta(t)$ only designates the line for which the relative error is o.1 . However, if we would have plotted the line with relative error 1o say, this line would pass so closely to the line drawn that within 5% uncertainty we can state that *A lies on the radius of convergence*. In the case of the discretization method we refer to the remarks made in Section 3.3 in connection with Fig. 3.3 .

Still it is highly unsatisfactory that there seems to be no analytic or numerical method which allows to predict the precise location of this discontinuity. Another important question would be to know the conditions which the initial distribution must satisfy in order to develop (or not to develop) a singulartiy. To a certain extent the breakdown theory of Jeffrey [33] and Lax [34] gives at least a formal answer to the first question posed. Here we want to present only the basic idea following Ames [35], pp. 59.

The Riemann invariants ξ and η are constant along their corresponding characteristics in the (t,z) plane. The directional derivative of ξ along a ξ characteristic in the (t,z) plane must therefore vanish, or

$$\frac{d\xi}{dt} \equiv \frac{\partial \xi}{\partial t} + \lambda \frac{\partial \xi}{\partial z} = o , \tag{4.31}$$

where we have set $\lambda = \dot{z}_1$, with $z_1(t)$ defining the ξ characteristic.

Differentiation of (4.31) with respect to z yields

$$\xi_{tz} + \lambda \xi_{zz} + (\lambda_\xi \xi_z + \lambda_\eta \eta_z) \xi_z = o ,$$

or

$$\frac{d}{dt} \xi_z + \lambda_\xi \xi_z^2 + \lambda_\eta \eta_z \xi_z = o . \tag{4.32a}$$

Here all arguments have been suppressed, but it should be recalled that

$$\xi_z = \xi_z(t, z_1(t)) ,$$

$$\lambda_\xi = \lambda_\xi(\xi, \eta(t, z_1(t))) ,$$

and the same applies for η_z and λ_η .

With the analogous equation along the η characteristics

$$\frac{d}{dt}\,\eta_z + \mu_\eta\,\eta_z^2 + \mu_\xi\,\xi_z\,\eta_z = o\ , \qquad (4.32b)$$

where $\mu = \dot{z}_2$, and

$$\eta_z = \eta_z(t, z_2(t))\ ,$$

$$\lambda_\eta = \lambda_\eta(\xi(t, z_2(t)), \eta)\ ,$$

a coupled system of equations for ξ_z and η_z has been set up. It should be stressed, however, that eqs. (4.32a/b) are not ordinary differential equations.

Whenever the solution ξ_z (or η_z) becomes unbounded, this will indicate the existence of a shock discontinuity. Varying all ξ characteristics, the one which leads for the first time to a breakdown will locate the discontinuity.

This is a formal result since eqs. (4.32a/b) can in general not be solved. Even numerically one cannot expect a higher accuracy than it was possible with the Massau method. Only when appropriate bounds for η and η_z are introduced in (4.32a), this equation becomes an ordinary differential equation. Thus Jeffrey [33] succeeded to obtain upper and lower bounds for the location of the discontinuity applying comparison theorems known from the theory of ordinary differential equations.

5 Conclusions

A summary of the most relevant results of Part I was already given in Section I.5, therefore we want to restrict these conclusions to some general remarks concerning Part II.

The results of Part II revealed that a very simple, but non-linear, second order partial differential equation already can lead to a number of interesting effects, which are usually absent in linear problems. Here we only want to summarize the different types of singularities which were encountered in the solutions for different initial distributions $f(z)$.

Let us first consider the results of Chapter 1. The two particular solutions, for which asymptotically $f(z) \sim z^2$, i.e.,

a) eq. (1.16a), with $f(z)$ according to Figs. 1.1a/b ,

b) eq. (1.38), with $f(z) = a+cz^2$ and $a>o$, $c>o$,

revealed a very similar singular behaviour: Both solutions $\Gamma(z,t)$ became, independently of z , infinite at a finite time. The similarity solution, for which $f(z) \sim z^{2/3}$, showed a much weaker singularity expressed by the third derivatives of Γ which behaved like $(x_o-x)^{-1/6}$, where $x = t\ z^{-2/3}$ was the similarity variable.

On the other hand, the two particular solutions with a finite initial distribution $f(z)$, i. e. ,

a) eq. (1.16b), with $f(z)$ according to Fig. 1.1c ,

b) eq. (1.37), with $f(z) = a-bz^2 \geq o$ and $a>o$, $b>o$,

remained regular for all times.

A totally different situation appeared for the (finite) Gaussian initial distribution introduced in Chapter 2, which led to singularities in the second derivatives of Γ , the so-called shock discontinuities. This break-down can be explained if the rate of change of the speed is assumed to be proportional to the gradient of the density (this gradient is proportional to the longitudinal space charge force). Due to the inflection point present in $f(z)$, particles beyond this point have a smaller velocity than particles close to it. Since the density of the slower particles is smaller than that of the faster ones (the opposite is true in Fig. 1.1c), a steepening of the density must take place.

It is quite remarkable in which way the shock discontinuity could be observed in each of the three methods which were employed:

i) For the Taylor series expansion of Chapter 2 the curve $\zeta(t)$ deliminated the region of convergence from that of divergence. The shock discontinuity was found to lie on this curve (Fig. 2.5).

ii) The finite difference method of Chapter 3 revealed the onset of oscillations around the discontinuity, as shown in Fig. 3.3 . Recently, we confirmed this result by a different, but stable, numerical method, the two-step Lax-Wendroff method (Byers and Killeen [36], p. 264).

iii) The method of characteristics of Chapter 4 finally allowed to locate this singularity (Figs. 4.5-4.1o). A fairly good accuracy in its determination was rendered possible by using the algorithm ESSER, a FORTRAN version of the ALGOL code *sivp* of Busch et al. [28] (see Appendix A4).

In order to obtain analytic results for the shock discontinuity, it would have been useful to find the general solution of the equation $r-pt-qs = o$. Whereas in general a nonlinear second order partial differential equation cannot be solved by standard methods, there are still systematic ways to solve particular classes of equations. Here the equation is reducible and can therefore be transformed to the (linear) hyperbolic normal form, an equation for the time as a function of the Riemann invariants. This equation exhibited such a difficult structure, that gaining any analytic results from it was beyond all efforts.

6 Outlook

Although eq. (1.1) turned out to be most valuable for studying analytic and numerical methods for a nonlinear hyperbolic propagation problem, the physically relevant questions are based on the two fundamental systems as derived in Section I.4.2 of Part I.

Preliminary results which were obtained from an investigation of the complete equation for $\bar{S}$ of Part I showed that all the methods compiled here - and this includes methods such as the similarity transformation and the Taylor series expansion - can be applied to this equation as well.

Further, since the hyperbolic normal form of the equation for $\bar{S}$, although not soluble, still allows to use the Riemann integration formula, we expect interesting results from this approach, especially concerning the treatment of the shock.

A completely new situation arises with the nonlinear elliptic initial value problem of Part I. This was observed for the first time when we tried to apply the "naive" finite difference method of Chapter 3 for the known parabolic solution, where this method failed even with very small time increments. Only after introducing numerical diffusion the results became acceptable.

Since in the elliptic case the eigenvalues of the underlying system are not real, the application of the method of characteristics - and therefore all the algorithms of Chapter 4, which represent the natural access to the numerical solution of hyperbolic problems - is no longer possible.

The lack of characteristics has a further consequence for the elliptic system: Shock discontinuities cannot occur. This, however, does not exclude that other types of singularities will be encountered.

7 References

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APPENDIX

Appendix A1

A1.1 General solution of the equation $r+pt-qs=o$

Equ. (1.9) has been solved by a method known as "Hamburger's method" in Forsyth [16], pp. 342,343. In this section we want to give a new derivation of the general solution of this equation, which is entirely analogous to the way the Born-Infeld equation can be solved (see Whitham [27], pp. 617).

Instead of equ. (1.9) we shall consider

$$r + 2pt - 2qs = o \ , \tag{A1.1}$$

where now the factor 2 has again been introduced for convenience of comparison with equ. (1.1).

In the notation of Section 1.1, equ. (A1.1) represents the equation

$$\frac{\partial^2\Gamma}{\partial t^2} = 2\,\frac{\partial\Gamma}{\partial z}\,\frac{\partial^2\Gamma}{\partial t\partial z} - 2\,\frac{\partial\Gamma}{\partial t}\,\frac{\partial^2\Gamma}{\partial z^2}\ , \tag{A1.2}$$

or equivalently the system

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}-2v & 2u\\-1 & o\end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix}u\\v\end{pmatrix} = o\ , \tag{A1.3}$$

where

$$u = \frac{\partial\Gamma}{\partial t}\ , \qquad v = \frac{\partial\Gamma}{\partial z}$$

has been set.

Now the eigenvalues determining the type of (A1.3) are given by

$$\lambda_{1,2} = -v \pm \sqrt{v^2 - 2u}\ , \tag{A1.4}$$

which means that equ. (A1.2) is hyperbolic if $u<o$.

With eq. (4.18) of Section 4.2.2, the (u,v) characteristics follow from the ordinary differential equations

$$\left(\frac{du}{dv}\right)_{1,2} = -\lambda_{2,1} ,$$

or, with (A1.4),

$$u = vu' - \frac{1}{2} u'^2 . \qquad (A1.5)$$

This equation is of the Clairaut type (see Kamke [18], p. 31) and has the general integral

$$u(v) = cv - \frac{1}{2} c^2 \qquad (A1.6)$$

and the singular integral

$$u(v) = \frac{1}{2} v^2 .$$

Solving (A1.6) for c, we obtain the Riemann invariants of (A1.3).

We shall need the Riemann invariants as new variables. If we set $\xi = c_1$, $\eta = c_2$, with $c_{1,2}$ from (A1.6), we have the set of transformation equations

$$\begin{aligned} \xi(u,v) &= v + \sqrt{v^2 - 2u} , \\ \eta(u,v) &= v - \sqrt{v^2 - 2u} , \end{aligned} \qquad (A1.7)$$

$$\begin{aligned} u(\xi,\eta) &= \frac{1}{2} \xi\eta , \\ v(\xi,\eta) &= \frac{1}{2}(\xi+\eta) , \end{aligned} \qquad (A1.8)$$

and therefore

$$\begin{aligned} \xi_u &= -\frac{1}{w} , \quad & \xi_v &= 1 + \frac{v}{w} , \\ \eta_u &= \frac{1}{w} , \quad & \eta_v &= 1 - \frac{v}{w} , \end{aligned} \qquad (A1.9)$$

with $w = \sqrt{v^2 - 2u} = |\xi-\eta|/2$.

Since (A1.3) is a homogeneous quasi-linear system with coefficients which depend only on the dependent variables u,v (and not on t,z), this system can be transformed to a linear system applying the hodograph transformation (see e.g. Ames [17], pp. 35).

According to the rules of implicit differentiation, we have from

$$u = u(z,t) \; ,$$

$$v = v(z,t) \; ,$$

that

$$\begin{aligned} z_u &= \frac{1}{D} v_t \; , \quad t_u = -\frac{1}{D} v_z \; , \\ z_v &= -\frac{1}{D} u_t \; , \quad t_v = \frac{1}{D} u_z \; , \end{aligned} \tag{A1.1o}$$

with $D = u_z v_t - u_t v_z$.

With (A1.1o), the system (A1.3) can now be transformed to the linear system

$$\begin{aligned} z_v + 2v\, t_v + 2u\, t_u &= o \; , \\ z_u - t_v &= o \; . \end{aligned} \tag{A1.11}$$

An important step now consists in introducing the Riemann invariants (A1.7) as new independent variables [1].

The system (A1.11) then becomes, using (A1.8) for u,v and (A1.9) for the corresponding derivatives (we take $w = (\xi-\eta)/2$ assuming $\xi>\eta$, the final result is the same for $\xi<\eta$) :

---

[1] With this transformation one succeeds to transform a linear second order equation to its normal form (Courant-Hilbert [31], pp. 123). In our case elimination of t from (A1.11) leads to the second order equation

$$z_{vv} + 2v\, z_{uv} + 2u\, z_{uu} + 2z_u = o \; ,$$

and the same equation holds for $t(u,v)$. These equations will have normal form after the transformation.

$$\begin{aligned}\xi\, z_\xi - \eta\, z_\eta + \xi^2\, t_\xi - \eta^2\, t_\eta &= o\ ,\\ z_\xi - z_\eta + \xi\, t_\xi - \eta\, t_\eta &= o\ .\end{aligned} \tag{A1.12}$$

If we multiply the second equation by ξ, a comparison with the first leads to

$$z_\eta = -\eta\, t_\eta\ , \tag{A1.13a}$$

in the same way multiplication by η yields

$$z_\xi = -\xi\, t_\xi\ . \tag{A1.13b}$$

Cross differentiation of (A1.13) gives

$$z_{\eta\xi} = -\eta\, t_{\eta\xi}\ ,$$

$$z_{\xi\eta} = -\xi\, t_{\xi\eta}\ ,$$

and therefore

$$t_{\xi\eta} = o\ . \tag{A1.14}$$

This result is most desirable because the solution of (A1.14) can be written down immediately.

Since t_ξ is independent of η, we may write $t_\xi = f'(\xi)$ say, and thus the general solution of (A1.14) is

$$t(\xi,\eta) = f(\xi) + g(\eta)\ , \tag{A1.15}$$

where f and g are arbitrary functions.

The general expression for $z(\xi,\eta)$ follows after integrating the two equations (A1.13). It is then more convenient to write, instead of (A1.15),

$$t(\xi,\eta) = F'(\xi) + G'(\eta)\ . \tag{A1.16a}$$

From (A1.13), partial integration of

$$z_\xi = -\xi\, F''(\xi)\ ,$$

$$z_\eta = -\eta\, G''(\eta)\ ,$$

finally leads to

$$z(\xi,\eta) = F(\xi) - \xi F'(\xi) + G(\eta) - \eta G'(\eta) \ . \qquad \text{(A1.16b)}$$

Eqs. (A1.16) constitute the general solution of the system (A1.12).

The original aim, however, was to solve equ. (A1.2) for $\Gamma(z,t)$. In the new independent variables ξ,η , $\Gamma(z,t)$ must satisfy

$$\Gamma_\xi = \Gamma_z \, z_\xi + \Gamma_t \, t_\xi \ ,$$
$$\Gamma_\eta = \Gamma_z \, z_\eta + \Gamma_t \, t_\eta \ ,$$

or with

$$\Gamma_z = v(\xi,\eta) \ , \quad \Gamma_t = u(\xi,\eta) \ :$$

$$\Gamma_\xi = -\frac{1}{2}(\xi+\eta) \ \xi \ F''(\xi) + \frac{1}{2} \ \xi\eta \ F''(\xi)$$
$$= -\frac{1}{2} \ \xi^2 \ F''(\xi) \ , \qquad \text{(A1.17a)}$$

and similarly

$$\Gamma_\eta = -\frac{1}{2} \ \eta^2 \ G''(\eta) \ . \qquad \text{(A1.17b)}$$

In order to avoid indefinite integrals, we set

$$\varphi'(\xi) = F(\xi) \ ,$$
$$\psi'(\eta) = G(\eta) \ ,$$

and obtain from (A1.17)

$$\Gamma(z,t) = -\varphi(\xi) + \xi\varphi'(\xi) - \frac{1}{2} \ \xi^2 \ \varphi''(\xi) \qquad \text{(A1.18a)}$$
$$-\psi(\eta) + \eta\psi'(\eta) - \frac{1}{2} \ \eta^2 \ \psi''(\eta) \ ,$$

$$z(\xi,\eta) = \varphi'(\xi) - \xi\varphi''(\xi) + \psi'(\eta) - \eta\psi''(\eta) \ , \qquad \text{(A1.18b)}$$

$$t(\xi,\eta) = \varphi''(\xi) + \psi''(\eta) \ , \qquad \text{(A1.18c)}$$

which is the final result.

The solution (A1.18) must now be compared with the solution of eq. (1.9), as e.g. given in Ames [17], p. 68. eq. 21 . There is first of all the difference caused by the factor two occurring in (A1.1) as opposed to eq. (1.9), but also (A1.18a) and (A1.18c) have the opposite sign as compared to the corresponding equations in Ames. Since eq. (A1.2) remains invariant if Γ and t are both replaced by $-\Gamma$ and $-t$, we conclude that after all the solution (A1.18) coincides with the original Forsyth result.

A1.2 Application to initial value problems

Let us discuss how these results can be applied to a particular initial value problem. In analogy to eqs. (1.1 /11) we consider the problem given by eq. (A1.2) and the initial values

$$\Gamma(z,o) = o ,$$

$$\Gamma_t(z,o) = f(z) .$$

We want to determine $u = \Gamma_t$ and $v = \Gamma_z$ (note that eq. (A1.2) is hyperbolic for $u<o$, therefore (1.12a) becomes now $f(z) < o$).

For $t=o$ we have

$$u = \frac{1}{2}\, \xi\, \eta = f(z) , \tag{A1.19a}$$

$$v = \frac{1}{2}(\xi + \eta) = o . \tag{A1.19b}$$

These equations require that initially $\eta = -\xi$ and

$$z(\xi,\eta)\big|_{\eta=-\xi} = z_o(\xi) , \tag{A1.2oa}$$

where z_o is determined by

$$f(z_o) = -\frac{1}{2}\, \xi^2 , \tag{A1.2ob}$$

i.e. the inverse of $f(z_o)$.

For $t=o$ eq. (A1.16a) becomes

$$o = F'(\xi) - G'(\xi) ,$$

thus

$$G(\xi) = F(\xi) + k , \tag{A1.21}$$

where k is an arbitrary constant. With (A1.16b) then follows

$$z_o(\xi) = 2F(\xi) - 2\,\xi F'(\xi) + k\ , \qquad \text{(A1.22)}$$

where $z_o(\xi)$ is given by (A1.2o) . This ordinary differential equation determines $F(\xi)$.

Without explicit knowledge of $F(\xi)$ we shall first determine $z(\xi,\eta)$, and then $t(\xi,\eta)$. Equ. (A1.16b) becomes with (A1.21)

$$z(\xi,\eta) = F(\xi) - \xi F'(\xi) + F(\eta) - \eta F'(\eta) + k\ ,$$

and with (A1.22)

$$z(\xi,\eta) = \frac{1}{2}\left[z_o(\xi) + z_o(\eta)\right]\ , \qquad \text{(A1.23a)}$$

which satisfies $z(\xi,-\xi) = z_o(\xi)$, since (A1.2ob) is even in ξ .

Inserting (A1.23a) into (A1.13) leads to

$$t_\xi = -\frac{1}{2\xi}\, z_o'(\xi)\ ,$$

$$t_\eta = -\frac{1}{2\eta}\, z_o'(\eta)\ ,$$

and therefore

$$t(\xi,\eta) = -\frac{1}{2}\int^{\xi} \frac{1}{\zeta}\, z_o'(\zeta)\, d\zeta - \frac{1}{2}\int^{\eta} \frac{1}{\zeta}\, z_o'(\zeta)\, d\zeta + \text{const}\ .$$

If in the second integral ζ is replaced by $-\zeta$, if further the constant is chosen such that $t(\xi,-\xi) = o$, we obtain

$$t(\xi,\eta) = -\frac{1}{2}\int_{-\eta}^{\xi} \frac{1}{\zeta}\, z_o'(\zeta)\, d\zeta\ . \qquad \text{(A1.23b)}$$

This integral cannot become singular, because with $u = \xi\eta/2$ and $u<o$ we always have $\zeta \neq o$.

With (A1.23) we can construct the solution of the problem stated above: For a given pair of values ξ,η with $\xi\eta < o$,

$$u(\xi,\eta) = \frac{1}{2}\,\xi\eta\ ,$$

$$v(\xi,\eta) = \frac{1}{2}(\xi+\eta)\ ,$$

and eqs. (A1.23a/b) determine the complete set of unknowns t,z,u and v .

We now want to determine the net of (t,z) characteristics for this problem. As usual we shall assume that the characteristics issue at

$$z_n = -4 + (n-1)h \ , \qquad n = 1,\dots,N \ ,$$

and $(N-1)h=8$. The Riemann invariants ξ_n then follow from (A1.2o), or

$$z_o(\xi_n) = z_n \ , \qquad n = 1,\dots,N \ ,$$

where we want to agree that $\xi_n > o$ should be taken , and the η_n must satisfy (A1.19b), i.e. $\eta_n = -\xi_n$.

If we set $Z(n,m) = z(\xi_n,\eta_m)$, and correspondingly $T(n,m)$, eqs. (A1.23a/b) become

$$Z(n,m) = \frac{1}{2}(z_n + z_m) = \frac{h}{2}(n+m) - 4 - h \ , \qquad \text{(A1.24a)}$$

$$T(n,m) = -\frac{1}{2}\int_{\xi_m}^{\xi_n} \frac{1}{\zeta}\, z_o'(\zeta)\,d\zeta \ . \qquad \text{(A1.24b)}$$

We observe that the z coordinates of this net are equal for (n+m) held constant.

As an example we want to choose $f(z)$, in analogy to problem (1.1/2), as

$$f(z) = -c\, e^{-z^2/2} \ , \qquad c = \frac{1}{\sqrt{2\pi}} \ .$$

Then (A1.2o) leads to

$$z_o(\xi) = \pm\left[2\,\ell n(2c/\xi^2)\right]^{1/2} \ ,$$

and (A1.24b) can be written as

$$T(n,m) = \frac{1}{\sqrt{2c}} \int_{\frac{1}{2}z_m}^{\frac{1}{2}z_n} e^{x^2}\,dx \ ,$$

where $m \leq n$ will be assumed. This integral is easily evaluated using Dawson's integral (see e.g. FUNCTION MMDAS of IMSL); in particular we have

$$T(N,1) \cong 36.83793158538886$$

for the apex of the characteristic cone. Fig. A1.1 shows the net of (t,z) characteristics for $N=81$. Shock discontinuities are obviously missing.

Fig. A1.2 shows $-\Gamma_t(z,t)$ over the same range of the (t,z) plane as Fig. 3.4 for the corresponding surface resulting from eq. (1.1). Since eq. (A1.2), as opposed to eq. (1.1), does not satisfy a conservation law, the "density" now can decay thus avoiding the development of infinite slopes.

Finally we want to make a remark concerning the Jacobian of the solution considered here. One easily obtains

$$j \equiv \frac{\partial(u,v)}{\partial(t,z)} \equiv \frac{\partial u}{\partial t}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial t}$$

using the chain rule

$$\frac{\partial(u,v)}{\partial(t,z)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(t,z)} ,$$

where

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{1}{4}(\eta-\xi) , \qquad \text{(A1.25)}$$

$$\frac{\partial(t,z)}{\partial(\xi,\eta)} = \frac{1}{4\,\xi\eta}(\xi-\eta)\, z_o'(\xi)\, z_o'(\eta) \qquad \text{(A1.26)}$$

follow from (A1.8) and (A1.23), respectively.

With $u = \xi\eta/2 < o$ the functional determinant (A1.25) can never vanish. But from (A1.2ob) it follows after differentiation that

$$f'(z_o)\, z_o'(\xi) = -\xi ,$$

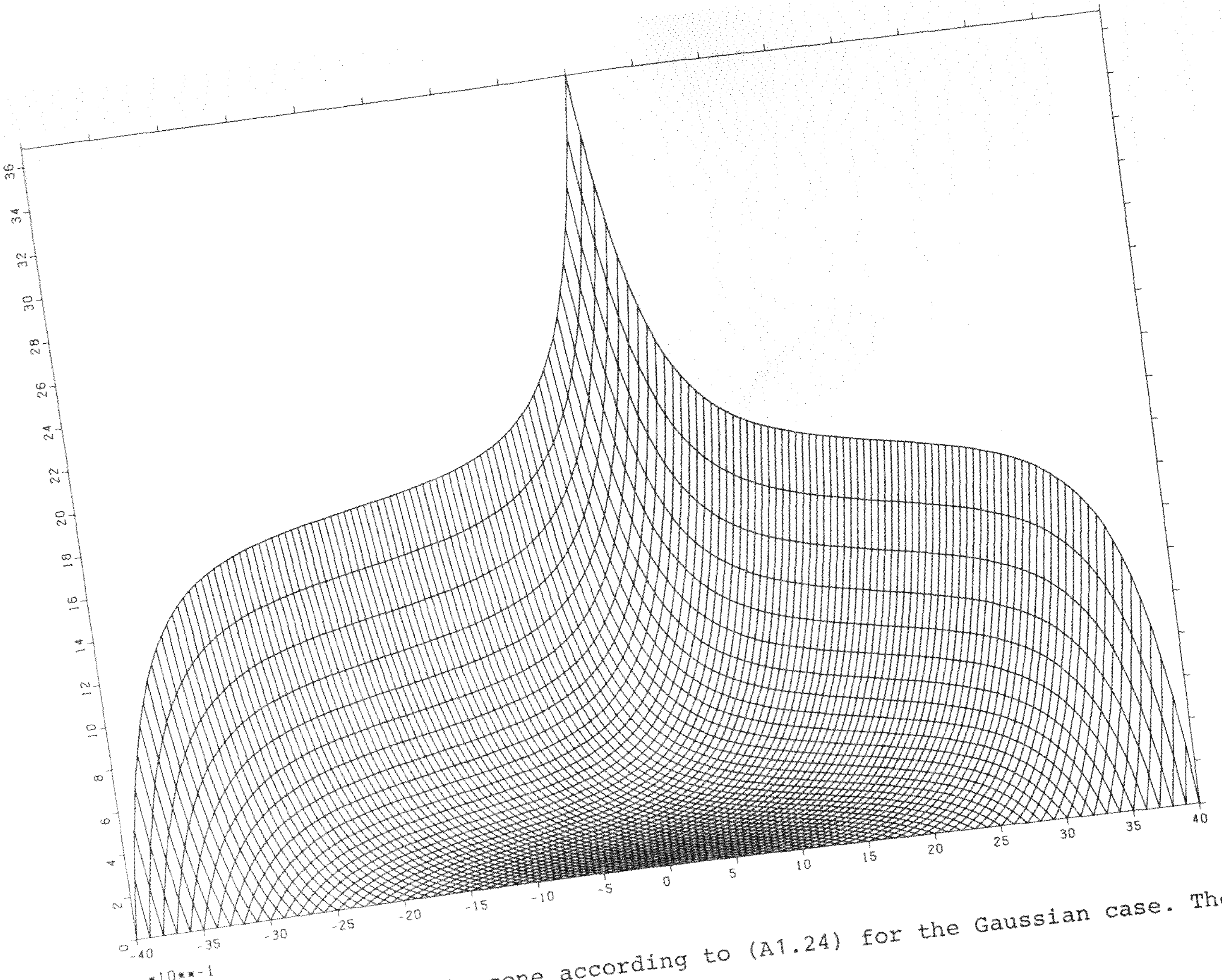
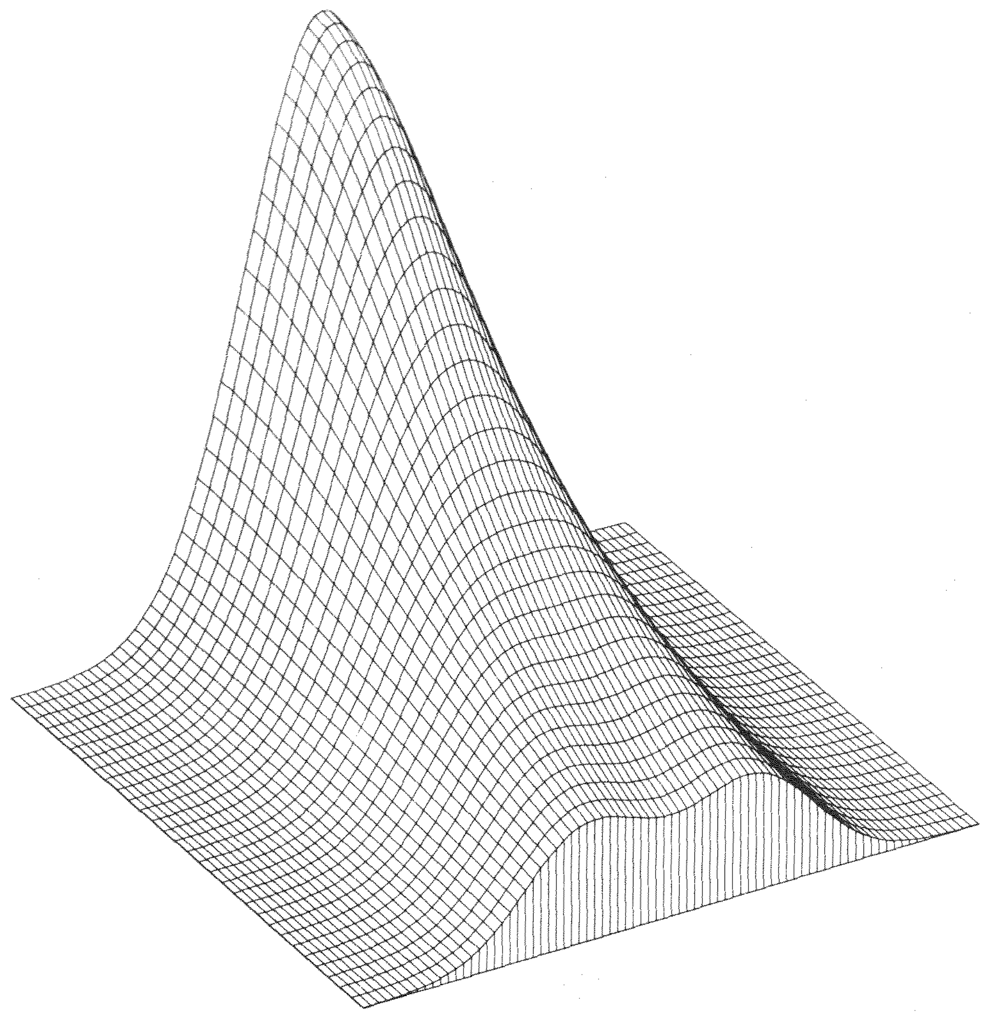


Fig. A1.1 : Characteristic cone according to (A1.24) for the Gaussian case. The abscissa represents the z axis.



FUNCTION DGDT WITH T = 0.0 (0.1) 3.0

Fig. A1.2 : $-\frac{\partial}{\partial t}\Gamma(z,t)$ for the Gaussian initial distribution.

therefore (A1.26) becomes infinite whenever $f'(z_o) = o$, or $j=o$ in this case. Let us denote by ξ_o the corresponding Riemann invariant.

The characteristic in the (t,z) plane which corresponds to ξ_o (or η_o) is known as the *transition characteristic* (Courant and Friedrichs [13], p. 69) or *branch line* (see e.g. Schiffer [37], p. 87), and one would expect that some irregular behaviour occurs along the corresponding characteristic in the (u,v) plane (Fig. 14 of [13], p. 68). Instead, the (u,v) characteristics are simply straight lines (see eq. (A1.6)).

The answer to this deviating behaviour follows if the derivation in [13] is critically examined. There it is argued that $j=o$ implies $(u_\xi v_\eta - v_\eta u_\xi) = o$, since $(t_\xi z_\eta - t_\eta z_\xi) \neq o$. In our case this latter expression is not only different from zero but becomes infinite thereby implying $j=o$, whereas $(u_\xi v_\eta - v_\eta u_\xi) \neq o$, which explains the regular behaviour of all (u,v) characteristics.

Appendix A2

Reduction of the equation r-pt-qs=o to normal form

In Appendix A1 equ. (1.9) was solved exactly. It seems most natural to apply the same procedure to equ. (1.8) which differs from (1.9) only in one single sign. We shall see that this sign causes a tremendous difference.

Let us briefly summarize the basic steps of Appendix A1 which can be applied to any homogeneous quasi-linear equation with two independent variables z,t , if the coefficients do not depend on z and t . More generally, we can assume that instead of the second order equation a system of two first order equations for the unknown variables u,v is given.

1) One has to find out first, if the ordinary differential equations for the (u,v) characteristics can be solved analytically. If this is the case, the integration constants - the so-called Riemann invariants - can be used later as new (independent) variables.
2) To the quasi-linear system in question a hodograph transformation can be applied thus interchanging the role of dependent and independent variables. As a result the new system will be linear.
3) The new independent variables u,v are replaced by the Riemann invariants. The resulting system for z,t is then equivalent to second order equations for z and t , which are transformed to their normal form. These equations may be soluble by standard techniques.

A2.1 The Riemann invariants

Instead of equ. (1.8) we shall again consider the original equation (1.1) which now reads

$$r - 2pt - 2qs = o \ , \qquad \text{(A2.1)}$$

or

$$\frac{\partial^2 \Gamma}{\partial t^2} = 2 \frac{\partial \Gamma}{\partial z} \frac{\partial^2 \Gamma}{\partial t \partial z} + 2 \frac{\partial \Gamma}{\partial t} \frac{\partial^2 \Gamma}{\partial z^2} \qquad \text{(A2.2)}$$

Setting

$$u = \frac{\partial \Gamma}{\partial t} \ , \quad v = \frac{\partial \Gamma}{\partial z} \ ,$$

we obtain the system

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -2v & -2u \\ -1 & o \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} u \\ v \end{pmatrix} = o \ , \qquad \text{(A2.3)}$$

with eigenvalues

$$\lambda_{1,2} = -v \pm \sqrt{v^2 + 2u} \ , \qquad \text{(A2.4)}$$

from which it follows that equ. (A2.2) is hyperbolic whenever $u > o$.

The (u,v) characteristics obey the ordinary differential equations

$$\left(\frac{du}{dv}\right)_{1,2} = - \lambda_{2,1} \ ,$$

or, with (A2.4),

$$u = -vu' + \frac{1}{2} u'^2 \ . \qquad \text{(A2.5)}$$

This equation is of the d'Alembert type (see Kamke [18], pp. 31,32 , also p. 357 , equ. (1.383)) and has the two general integrals, expressed in parametric form,

$$u(s,c) = \mp cs + \frac{1}{6} s^4 ,$$
$$v(s,c) = \frac{c}{s} \pm \frac{1}{3} s^2 , \qquad \text{(A2.6)}$$

with parameter s and an arbitrary constant c .

For $c=o$, (A2.6) leads to the particular solution

$$u(v) = \frac{3}{2} v^2 , \qquad \text{(A2.7)}$$

and the discriminant of eq. (A2.5) is given by

$$u(v) = -\frac{1}{2} v^2 , \qquad \text{(A2.8)}$$

which is *not* a solution of eq. (A2.5) [1)].

If c and s are both replaced by $-c$ and $-s$, eqs. (A2.6) remain invariant. Therefore we can restrict the range of s values to $s \geq o$.

The two families defined by (A2.6) are shown in Fig. 4.1 and Fig. 4.2 of Section 4.3 .

---

1) Let $F(x,y,p) = o$ with $p=y'$ be a first order differential equation. Then this equation and the equation $F_p(x,y,p) = o$ together determine, after elimination of p , the discriminant or envelope of $F=o$ (note that the regular integrals satisfy $F_p \neq o$). In order that the discriminant is a singular integral of $F=o$, the condition $F_x + pF_y = o$ must hold, which is true for eq. (A1.5), but not for eq. (A2.5). As a consequence, the envelope in the Clairaut case is always tangent to the regular solutions, whereas in our d'Alembert case the envelope consists of cusp loci not tangent to the regular solutions (Davis [38], pp. 53, cf. also Fig. 4.2).

The parametric representations

$$u(s,\xi) = -\xi s + \frac{1}{6} s^4 \,, \qquad v(s,\xi) = \frac{\xi}{s} + \frac{1}{3} s^2 \,, \tag{A2.9a}$$

and

$$u(s,\eta) = \eta s + \frac{1}{6} s^4 \,, \qquad v(s,\eta) = \frac{\eta}{s} - \frac{1}{3} s^2 \,, \tag{A2.9b}$$

define two surfaces

$$\xi(u,v) = F(u,v) \,, \qquad \eta(u,v) = G(u,v) \,, \tag{A2.1o}$$

which are the Riemann invariants of equ. (A2.2) .

For later purposes we shall need the derivatives ξ_u , ξ_v , η_u , η_v . This requires to determine the corresponding derivatives of the implicit functions F and G .

Let us first consider the surface $F(u,v)$ in its parametrized form

$$u(s,t) = -ts + \frac{1}{6} s^4 \,, \qquad v(s,t) = \frac{t}{s} + \frac{1}{3} s^2 \,, \qquad \xi(s,t) = t \,. \tag{A2.11}$$

With $s' = s + \Delta s$, $t' = t + \Delta t$ we obtain up to first order

$$u(s',t') - u(s,t) \cong u_s(s,t)\Delta s + u_t(s,t)\Delta t \,,$$

and similar expressions for the other differences.

Thus

$$\left(\frac{\partial \xi}{\partial u}\right)_v \cong \frac{\xi_s \; \Delta s + \xi_t \; \Delta t}{u_s \; \Delta s + u_t \; \Delta t} \,,$$

where the increments Δs , Δt are connected by the relation $\Delta v = o$, or

$$\Delta s = - \frac{v_t}{v_s} \Delta t \ .$$

Therefore,

$$\left(\frac{\partial \xi}{\partial u}\right)_v = \frac{v_s \xi_t - v_t \xi_s}{v_s u_t - v_t u_s} \ ,$$

and accordingly

$$\left(\frac{\partial \xi}{\partial v}\right)_u = \frac{u_s \xi_t - u_t \xi_s}{u_s v_t - u_t v_s} \ .$$

All required derivatives can now be determined using (A2.11) and a corresponding representation for $G(u,v)$.

We obtain

$$\left(\frac{\partial \xi}{\partial u}\right)_v = - \frac{1}{2s} \ , \quad \left(\frac{\partial \xi}{\partial v}\right)_u = \frac{s}{2} \ , \tag{A2.12a}$$

$$\left(\frac{\partial \eta}{\partial u}\right)_v = \frac{1}{2s} \ , \quad \left(\frac{\partial \eta}{\partial v}\right)_u = \frac{s}{2} \ . \tag{A2.12b}$$

We observe that the parameter t does not appear in these relations and that the parameter s refers to the actual surface under consideration, i.e. either to $F(u,v)$ or to $G(u,v)$ for (A2.12a) and (A2.12b), respectively.

A2.2 Reduction to normal form

Using the relations (A1.1o) we apply the hodograph transformation to the system (A2.3) and obtain the linear system

$$\begin{aligned} z_v + 2v\, t_v - 2u\, t_u &= o \ , \\ z_u - t_v &= o \ . \end{aligned} \tag{A2.13}$$

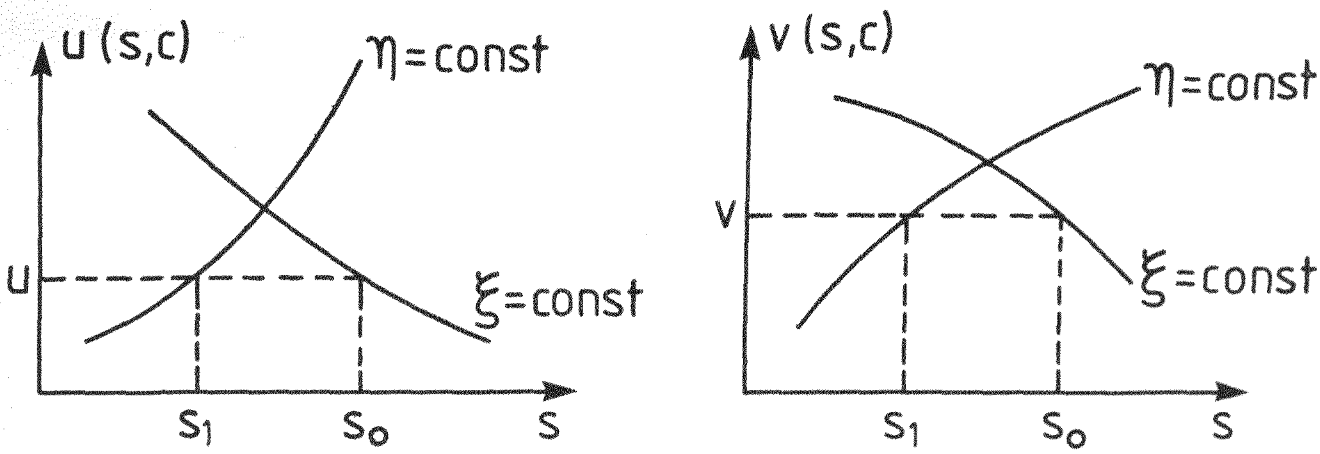
Let us assume that (A2.1o) can be solved for $u(\xi,\eta)$, $v(\xi,\eta)$. Then (A2.13) can be written in the new independent variables ξ and η , defined by (A2.1o), as

$$\xi_v \, z_\xi + \eta_v \, z_\eta + (2v \, \xi_v - 2u \, \xi_u) \, t_\xi + (2v \, \eta_v - 2u \, \eta_u) \, t_\eta = o \, ,$$

$$\xi_u \, z_\xi + \eta_u \, z_\eta - \xi_v \, t_\xi - \eta_v \, t_\eta = o \, .$$

Before we replace $\xi_u , \ldots , \eta_v$ by the expressions (A2.12), a short illustration will be helpful.

Consider the following schematic graph.



For a given pair of values ξ and η, solving (A2.1o) for u and v means to determine two parameters, s_o from (A2.9a) and s_1 from (A2.9b), such that the illustrated configuration is realized. Therefore u,v can be represented either as $u(s_o,\xi)$, $v(s_o,\xi)$ or as $u(s_1,\eta)$, $v(s_1,\eta)$.

Using (A2.9) and (A2.12) we obtain

$$2v \, \xi_v - 2u \, \xi_u = \left(\frac{\xi}{s_o} + \frac{1}{3} s_o^2\right) s_o + \left(-\xi \, s_o + \frac{1}{6} s_o^4\right)\frac{1}{s_o}$$

$$= \frac{1}{2} s_o^3 \, ,$$

$$2v \, \eta_v - 2u \, \eta_u = \left(\frac{\eta}{s_1} - \frac{1}{3} s_1^2\right) s_1 - \left(\eta \, s_1 + \frac{1}{6} s_1^4\right)\frac{1}{s_1}$$

$$= -\frac{1}{2} s_1^3 \, ,$$

and finally

$$s_o \, z_\xi + s_1 \, z_\eta + s_o^3 \, t_\xi - s_1^3 \, t_\eta = o \, ,$$

$$-\frac{1}{s_o} \, z_\xi + \frac{1}{s_1} \, z_\eta - s_o \, t_\xi - s_1 \, t_\eta = o \, .$$

If we multiply the second equation first with s_o^2 and then with s_1^2, we obtain the two relations

$$z_\eta = s_1^2\, t_\eta \;,$$

$$z_\xi = -s_o^2\, t_\xi \;,$$

and from these

$$t_{\xi\eta} + \frac{2s_o}{(s_o^2 + s_1^2)} \frac{\partial s_o}{\partial \eta} t_\xi + \frac{2s_1}{(s_o^2 + s_1^2)} \frac{\partial s_1}{\partial \xi} t_\eta = o \;, \qquad (A2.14)$$

which is the hyperbolic normal form of the equation

$$2u\, t_{uu} - 2v\, t_{uv} - t_{vv} + 2t_u = o \;,$$

as derived from (A2.13).

A2.3 Explicit representation of the normal form

Until now we can only say that the normal form (A2.14) has no longer a simple structure such as the normal form (A1.14). But before a more detailed investigation of equ. (A2.14) can be made, we must derive explicit expressions for the coefficients occurring in this equation.

The relations

$$u(s_o,\xi) = u(s_1,\eta) \;,$$

$$v(s_o,\xi) = v(s_1,\eta) \;,$$

when used in connection with (A2.9), provide a system of linear equations for ξ and η,

$$s_o\, \xi + s_1\, \eta = \frac{1}{6}(s_o^4 - s_1^4) \;,$$

$$-\frac{1}{s_o}\, \xi + \frac{1}{s_1}\, \eta = \frac{1}{3}(s_o^2 + s_1^2) \;,$$

or

$$\xi(s_o,s_1) = \frac{1}{6}(s_o^3 - 3s_o\, s_1^2) \ , \qquad \text{(A2.15a)}$$

$$\eta(s_o,s_1) = \frac{1}{6}(3s_o^2\, s_1 - s_1^3) \ . \qquad \text{(A2.15b)}$$

These equations will be used first to eliminate the derivatives $\partial s_o/\partial\eta$, $\partial s_1/\partial\xi$, which occur in (A2.14).

The formulae for implicit differentiation

$$\frac{\partial s_o}{\partial\xi} = \frac{1}{D}\frac{\partial\eta}{\partial s_1} \ , \qquad \frac{\partial s_1}{\partial\xi} = -\frac{1}{D}\frac{\partial\eta}{\partial s_o} \ ,$$

$$\frac{\partial s_o}{\partial\eta} = -\frac{1}{D}\frac{\partial\xi}{\partial s_1} \ , \qquad \frac{\partial s_1}{\partial\eta} = \frac{1}{D}\frac{\partial\xi}{\partial s_o} \ ,$$

$$D = \frac{\partial\xi}{\partial s_o}\frac{\partial\eta}{\partial s_1} - \frac{\partial\xi}{\partial s_1}\frac{\partial\eta}{\partial s_o} \ ,$$

allow to express the derivatives $\partial s_o/\partial\eta$, $\partial s_1/\partial\xi$ with the help of (A2.15). With

$$D = \frac{1}{4}\left(s_o^2 + s_1^2\right)^2 \ ,$$

these derivatives follow immediately, and equ. (A2.14) becomes

$$t_{\xi\eta} + \frac{8\, s_o\, s_1}{\left(s_o^2 + s_1^2\right)^3}\left[s_o\, t_\xi - s_1\, t_\eta\right] = o \ . \qquad \text{(A2.16)}$$

It remains to determine s_o and s_1 as functions of ξ and η .

From (A2.15) the following very useful relation

$$\xi^2 + \eta^2 = \frac{1}{36}\left(s_o^2 + s_1^2\right)^3 \qquad \text{(A2.17)}$$

can be deduced. When solved for

$$k = s_o^2 + s_1^2 \ , \qquad \text{(A2.18)}$$

we obtain

$$k = [36(\xi^2 + \eta^2)]^{1/3} \quad . \qquad \text{(A2.19)}$$

The roots proportional to $e^{2\pi in/3}$, $n = 1,2$, need not to be considered since only k^3 will enter in the final results.

If (A2.18) is inserted in (A2.15), the cubic equations

$$s_o^3 - \frac{3}{4} k s_o - \frac{3}{2} \xi = o \, , \qquad \text{(A2.2oa)}$$

$$s_1^3 - \frac{3}{4} k s_1 + \frac{3}{2} \eta = o \, , \qquad \text{(A2.2ob)}$$

can be set up for s_o and s_1 . We shall solve (A2.2oa) for s_o , and then obtain s_1 from (A2.18), where the correct choice of sign will be determined after insertion into (A2.2ob).

With $p = -k/4$, $q = -3\,\xi/4$, the solutions of (A2.2oa) can be expressed in terms of

$$a = (-q + w)^{1/3} \, ,$$

$$b = (-q - w)^{1/3} \, ,$$

$$w^2 = q^2 + p^3 = -\frac{9}{16} \eta^2 \, ,$$

or

$$a = \left(\frac{3}{4}\right)^{1/3} (\xi + i\eta)^{1/3} \, ,$$

$$b = \left(\frac{3}{4}\right)^{1/3} (\xi - i\eta)^{1/3} \, .$$

In the following we shall distinguish between the three roots for s_o .

1) $s_o^{(1)} = a + b$.

In this case it turns out to be more appropriate to use the complex combinations

$$x = \xi + i\eta \, ,$$

$$y = \xi - i\eta \, , \qquad \text{(A2.21)}$$

of the Riemann invariants as new independent variables. Then

$$s_o^{(1)} = \left(\frac{3}{4}\right)^{1/3} \left[x^{1/3} + y^{1/3}\right] ,$$

$$s_1^{(1)} = -i\left(\frac{3}{4}\right)^{1/3} \left[x^{1/3} - y^{1/3}\right] , \tag{A2.22}$$

where $s_1^{(1)}$ has been obtained as described above.

Now it was shown in Section 4.3 that not all sign combinations of ξ and η correspond to hyperbolic solutions. More precisely, the case $\xi > o$, $\eta < o$ had to be excluded. If we set

$$x = r\, e^{i\varphi} ,$$

it follows that $\varphi \in [o,\frac{3}{2}\pi]$ for the feasible region, and therefore

$$\arg(x^{1/3}) \in [o,\frac{\pi}{2}] .$$

This means that

$$x^{1/3} + y^{1/3} \equiv 2\ \mathrm{Re}(x^{1/3}) \geq o ,$$

$$-i(x^{1/3} - y^{1/3}) \equiv 2\ \mathrm{Im}(x^{1/3}) \geq o ,$$

or $s_o^{(1)} \geq o$, $s_1^{(1)} \geq o$. Thus the s values of (A2.22) fall into the admissible range $s \geq o$.

If we insert (A2.22) in (A2.16) and introduce the new variables (A2.21), we finally obtain

$$t_{xx} - t_{yy} - \frac{1}{3xy}\left(x^{2/3} - y^{2/3}\right)\left[y^{1/3}\, t_x + x^{1/3}\, t_y\right] = o . \tag{A2.23}$$

This equation is also represented in hyperbolic normal form (Courant-Hilbert [31], pp. 123) and reveals the full amount of difficulties as already indicated at the beginning of this appendix.

2) $s_o^{(2,3)} = -\frac{1}{2}(a+b) \pm \frac{i}{2}\sqrt{3}(a-b)$.

With (A2.22) we have

$$s_o^{(2)} = -\frac{1}{2}\left[s_o^{(1)} + s_1^{(1)}\right] \leq o ,$$

therefore $s_o^{(2)}$ is always inadmissible.

In the third case we have

$$s_o^{(3)} = -\frac{1}{2}\left[s_o^{(1)} - \sqrt{3}\, s_1^{(1)}\right] ,$$

which can assume either sign, but

$$s_1^{(3)} = -\frac{1}{2}\left[\sqrt{3}\, s_o^{(1)} + s_1^{(1)}\right] \leq o$$

rendering impossible this root, too.

We conclude that (A2.23) is the unique normal form to which eq. (A2.1) can be transformed.

Eq. (A2.23) suggests to introduce the new variables

$$\bar{x} = x^{2/3} ,$$

$$\bar{y} = y^{2/3} ,$$

which lead to the equation

$$\bar{y}\, t_{\bar{x}\bar{x}} - \bar{x}\, t_{\bar{y}\bar{y}} - \frac{1}{2}\, t_{\bar{x}} + \frac{1}{2}\, t_{\bar{y}} = o . \qquad \text{(A2.24)}$$

Although this equation looks far more amenable than eq. (A2.23), we still did not succeed to find any nontrivial particular solutions for either of these equations. Since eq. (A2.24) no longer possesses normal form, the only equation of interest for further investigations should be eq. (A2.23).

Appendix A3

FORTRAN programme for the recurrence relation (2.8)

```
      IMPLICIT REAL*16 (A-H,O-Z)
      DIMENSION A(50,51)
      DATA A/2550*0.D0/,ZERO/0.D0/,ONE/1.D0/,EIGHT/8.D0/
      CALL MASKE
      WZWPI = ONE/QSQRT(EIGHT*QATAN(ONE))
C
C        INITIALIZATION
C
      LMAX = 50
      IMAX = LMAX + 1
      A(1,1) = WZWPI
C
C        RECURSION
C
      DO 50 LL=1,LMAX
       L = LL-1
       IF(L.EQ.0) GOTO 50
       FAK = (2*L+1)*L
       FAK = ONE/FAK
       DO 40 II=1,LL
        I = II-1
        IA = 2*(L-I)+1
C
C        DETERMINATION OF THE AL
C
        AL = ZERO
        DO 30 I1=1,L
         M = I1-1
         NP = L-M
         FO = 2*M+1
        DO 30 I2 = 1,II
         MU = I2-1
         NU = I-MU
         IBM = 2*(NP-NU)-2
         A1 = ZERO
         IF (MU.LE.1) GOTO 10
         A1 = IA*IBM
         A1 = A1*A(I1,I2-2)
   10    A2 = ZERO
         IF(MU.EQ.0) GOTO 20
         A2 = IA*NP + IBM*LL
         A2 = A2*A(I1,I2-1)
   20    A3 = LL*NP
         A3 = A3*A(I1,I2)
         A4 = FO*(A1-A2+A3)*A(NP,NU+1)
         AL = AL+A4
   30   CONTINUE
C
C        COEFFICIENTS AL(I)
C
   40  A(LL,II)=FAK*AL
      WRITE(6,600)
   50 WRITE(6,610) (A(LL,II),II=1,LL)
      STOP
  600 FORMAT('-')
  610 FORMAT(' ',2D40.32)
      END
```

Appendix A4

FORTRAN version ESSER of ALGOL code *sivp* from [28]

```
      SUBROUTINE MESH(JJL,JJR,NQ,Q,INEU,NR,R,JL,JR,NP,P,ISSD,EPW,NORMAL,
     1                BSD,KONT,*)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Q(4,NQ),R(4,NR),P(12,NP)
      LOGICAL BSD,NORMAL
      ISD = ISSD
      IF (BSD) GOTO 100
      IIL = JJL
      IIR = JJR
      IL = JL
      IR = JR
      GO TO 110
 100  IIL = JJR
      IIR = JJL
      IL = JR
      IR = JL
C--------- COEFFICIENTS OF THE EQUATIONS FOR (X,Y)
 110  X1 = Q(1,IIL)
      Y1 = Q(2,IIL)
      X2 = Q(1,IIR)
      Y2 = Q(2,IIR)
      D1 = X1 - X2
      D2 = Y1 - Y2
      Z1 = P(1,IL)
      Z2 = P(7,IR)
      IF (.NOT.NORMAL) GOTO 130
      H = Z1 - Z2
      E1 = DABS(Z1)
      E2 = DABS(Z2)
      IF (DABS(H).LE.EPW*(E1+E2)) GOTO 220
      IF (E1.GE.E2) GOTO 120
      S2 = (Z1*D1-D2)/H
      Z2 = Z2*S2
      GO TO 140
 120  S1 = (Z2*D1-D2)/H
      Z1 = Z1*S1
      GO TO 160
 130  S1 = P(2,IL)
      S2 = P(8,IR)
C--------- SOLUTION (X,Y) OF THE SYSTEM
      E1 = S2*Z1
      E2 = S1*Z2
      H = E1-E2
      IF (DABS(H).LE.EPW*(DABS(E1)+DABS(E2))) GO TO 220
      IF(DMIN1(DABS(S1),DABS(Z1)).GT.DMIN1(DABS(S2),DABS(Z2))) GO TO 150
      H =(Z1*D1-S1*D2)/H
      S2 = S2*H
      Z2 = Z2*H
      Z1 = Z2 - D2
 140  S1 = S2 - D1
      R(1,INEU) = X2 + S2
      R(2,INEU) = Y2 + Z2
      GO TO 170
 150  H = (Z2*D1-S2*D2)/H
      S1 = S1*H
      Z1 = Z1*H
      Z2 = Z1 + D2
 160  S2 = S1 + D1
      R(1,INEU) = X1 + S1
      R(2,INEU) = Y1 + Z1
C--------- CHECK OF THE DIRECTION OF THE DISCRETIZATION
 170  H=S1*D2-D1*Z1
      IF (H.EQ.0.D0) GOTO 180
      IH = 1
      IF (H.LT.0.D0) IH = -1
      IF (IH.EQ.ISD) GOTO 200
      IF (ISD.EQ.0) GOTO 190
      IF (IH+IH.EQ.ISD) GOTO 190
 180  IF (IABS(ISD).EQ.1) GOTO 230
      GOTO 240
 190  ISD = IH
```

```
C--------- COEFFICIENTS OF THE EQUATIONS FOR (U,V)
200    X1 = Q(3,IIL)
       Y1 = Q(4,IIL)
       D1 = P(3,IL)
       E1 = P(4,IL)
       D2 = P(9,IR)
       E2 = P(10,IR)
       S1 = P(5,IL)*S1
       S2 = D2*(X1 - Q(3,IIR))  +  E2*(Y1 - Q(4,IIR))  -  P(11,IR)*S2
       IF (NORMAL) GO TO 210
       S1 = S1 + P(6,IL)*Z1
       S2 = S2 - P(12,IR)*Z2
C--------- SOLUTION (U,V) OF THE SYSTEM
210    Z1 = D1*E2
       Z2 = D2*E1
       H = Z1 - Z2
       IF (DABS(H).LE.EPW*(DABS(Z1)+DABS(Z2))) GO TO 220
       R(3,INEU) = X1 + (E1*S2 + E2*S1)/H
       R(4,INEU) = Y1 - (D1*S2 + D2*S1)/H
       RETURN
220    KONT = 2
       RETURN 1
230    KONT = 3
       RETURN 1
240    KONT = 4
       RETURN 1
       END
C
       SUBROUTINE COEFF(J,NQ,Q,I,NP,P,NORMAL,SYSTEM,KONT,*)
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION Q(4,NQ),P(12,NP)
       LOGICAL NORMAL
       B = Q(1,J)
       R = Q(2,J)
       W = Q(3,J)
       H = Q(4,J)
       CALL SYSTEM(B,R,W,H,KONT,NORMAL,
      1            D12,D13,D14,D23,D24,D34,D1H,D2H,D3H,D4H,
      2            P(1,I),P(3,I),P(4,I),P(5,I),
      3            P(7,I),P(9,I),P(10,I),P(11,I) )
       IF (KONT.GT.0) RETURN 1
       IF (NORMAL) RETURN
       B = 0.5D0*(D14 + D23)
       H = D13*D24
       R = B*B
       W = R - H
       IF (W.LE.0.D0) GO TO 130
       W = DSQRT(W)
       IF (B.GE.0.D0) GO TO 100
       R = B - W
       P(1,I) = R
       P(2,I) = D13
       P(7,I) = D24
       P(8,I) = R
       GO TO 110
100    R = B + W
       P(1,I) = D24
       P(2,I) = R
       P(7,I) = R
       P(8,I) = D13
110    R = B - D23
       IF (R.GE.0.D0) GO TO 120
       B = R - W
       P(3,I) = D12
       P(4,I) = B
       P(9,I) = B
       P(10,I) = D34
       P(5,I) = -D2H
       P(6,I) = D1H
       P(11,I) = -D4H
       P(12,I) = D3H
       RETURN
```

```
120   B = R + W
      P(3,I) = B
      P(4,I) = D34
      P(9,I) = D12
      P(10,I) = B
      P(5,I) = -D4H
      P(6,I) = D3H
      P(11,I) = -D2H
      P(12,I) = D1H
      RETURN
130   KONT = 2
      RETURN 1
      END
C
      SUBROUTINE SIVP (SYSTEM,ND,DATA,NEX,N,IROWS,EPS,SOUTD,KONT,
     1                 NP,P, NQ,Q, NF,F)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION DATA(4,ND),Q(4,NQ),P(12,NP),F(NF),E(4,9),X(9),FAC(8),
     1          IZ(9),IY(9),IM(9),KIS(9),KST(9),KSTA(9),JST(9),JSTA(9)
      LOGICAL BN
      EXTERNAL SYSTEM
      IF (N.LE.0 .OR. NEX.LE.0 .OR. NEX.GT.9 .OR. IROWS.GT.N .OR.
     1    IROWS.LE.0 .OR. IABS(KONT).GT.1 .OR. EPS.LE.0.D0) GO TO 410
      I = 2**NEX
      IF (ND.LT.I*N+1 .OR. NP.LT.NEX*(I+1)+1 .OR.
     1    NQ.LT.2*(NEX+1+(NEX-1)*I) .OR. NF.LT.1+I*(N-1) ) GOTO 410
      NEX1 = NEX - 1
      K = 1
      I = NEX
      JL = 1
      DO   100   M=1,NEX
      IY(I) = K
      K = K + K
      IF (M.EQ.NEX) GO TO 100
      I = I - 1
      JL = 4*JL
      VHK = JL-1
      FAC(I) = 1/VHK
100   IZ(M) = K
      NEY = K
      DO 110 M=1,NP
      P(2,M) = 1.D0
      P(8,M) = 1.D0
      P(6,M) = 0.D0
110   P(12,M) = 0.D0
      J = 1
      KLANG = NEY + 1
      K = -NEY
      DO 120 M = 1,NEX
      JSTA(M) = J
      K = K + KLANG
      KSTA(M) = K
      IMM = KLANG - IY(M)
      IM(M) = IMM
120   J = J + IMM + IMM
      EPW = 10.D0*EPS
      ISD = 2*KONT
      KONT = 0
      NL = ND
C--------- LOOP OVER THE ROWS
      DO   400   NZ=1,IROWS
      NL = NL - NEY
      NN = NL
C--------- LOOP OVER THE POINTS OF A ROW
      DO   370   IS=1,NN
      IF(IS.LE.1)   GO TO   130
      I = NEY - 1
      GO TO   150
130   I = -1
      DO   140   M=1,NEX
      K = KSTA(M)
      KIS(M) = K
```

```
      KST(M) = K
 140  JST(M) = JSTA(M)
C--------- COEFFICIENTS ON THE INITIAL CURVE
 150  I = I + 1
      ISS = IS + I
      KMAX = 0
      DO   190   M=1,NEX
      K = KIS(M)
      KMAX = KMAX + KLANG
      IF(M.GT.1)   GO TO   160
      CALL COEFF(ISS,ND,DATA,K,NP,P,BN,SYSTEM,KONT,&390)
      KL = K
      GO TO   180
 160  DO   170   J=1,110
 170  P(J,K) = P(J,KL)
 180  K = K + IZ(M)
      IF(K.GT.KMAX)   K = K - KLANG
 190  KIS(M) = K
      IF(I.LT.NEY)   GO TO   150
      KMAX = 0
      JMAX = 0
C--------- LOOP OVER DIFFERENT MESH WIDTHS
      DO    290     M=1,NEX
      IZM = IZ(M)
      IYM = IY(M)
      IMM = IM(M)
      JLANG = IMM + IMM
      KMAX = KMAX + KLANG
      JMAX = JMAX + JLANG
      KSTART = KST(M)
      JSTART = JST(M)
      ISS = IS
      K = 0
      IF(IS - IYM)   220,210,200
 200  K = IZM - 1
      ISS = ISS + NEY - IYM
      GO TO   220
 210  JL = KSTART + 1
      J = JSTART + IMM
      GO TO   230
 220  JL = KSTART + IZM
      J = JSTART + IZM - 1
 230  IF(JL.GT.KMAX)   JL = JL - KLANG
      KST(M) = JL
      IF(J.GT.JMAX)   J = J - JLANG
      JST(M) = J
C--------- FIRST MESH (MASSAU)
 240  K = K + 1
      KL = KSTART
      JR1 = JSTART
      JL = ISS
      KR = KL + NEY
      IF(KR.GT.KMAX)   KR = KR - KLANG
      JR = JL + IYM
      CALL MESH(JL,JR,ND,DATA,JR1,NO,O,KL,KR,NP,P,ISD,EPW,BN,.FALSE.,
     1          KONT,&390)
      CALL COEFF(JR1,NO,O,KL,NP,P,BN,SYSTEM,KONT,&390)
      IF(IS.GT.IYM)   GO TO   250
      KSTART = KR
      JSTART = JSTART + IMM - 1
      IF(JSTART.GT.JMAX)   JSTART = JSTART - JLANG
      ISS = JR
      IF(K.LE.1)   GO TO   240
C--------- SECOND MESH (GRAGG)
 250  KR = KL
      KL = KL + 1
      IF(KL.GT.KMAX)   KL = KL - KLANG
      JR2 = JR1 + 1
      IF(JR2.GT.JMAX)   JR2 = JR2 - JLANG
      JL = JL - IYM
      CALL MESH(JL,JR,ND,DATA,JR2,NO,O,KL,KR,NP,P,ISD,EPW,BN,.FALSE.,
     1          KONT,&390)
```

```
      IF(M.LE.1)   GO TO   270
      CALL COEFF(JR2,NO,Q,KL,NP,P,BN,SYSTEM,KONT,&390)
      IF(K.LE.2)   GO TO   240
C--------- REMAINING MESHES (GRAGG)
      DO   260   I=3,K
      KR = KL
      KL = KL + 1
      IF(KL.GT.KMAX)   KL = KL - KLANG
      JR = JR1
      JR1 = JR2
      JR2 = JR1 + 1
      IF(JR2.GT.JMAX)   JR2 = JR2 - JLANG
      CALL MESH(JR2,JR,NO,Q,JR2,NO,Q,KL,KR,NP,P,ISD,EPW,BN,.FALSE.,
     1          KONT,&390)
      IF(I.GE.IZM)   GO TO   270
260   CALL COEFF(JR2,NO,Q,KL,NP,P,BN,SYSTEM,KONT,&390)
      GO TO   240
C--------- LAST MESH (STABILIZATION)
270   CALL COEFF(JR2,NO,Q,NP,NP,P,BN,SYSTEM,KONT,&390)
      JL = JR2 + IMM
      IF(JL.GT.JMAX)   JL = JL - JLANG
      DO   280   J=1,4
      Q(J,JL)  = 0.5D0*(Q(J,JL)  + Q(J,JR2))
280   Q(J,JR2) = 0.5D0*(Q(J,JR1) + Q(J,JR2))
      CALL MESH(JL,JR2,NO,Q,M,NEX,E,NP,NP,NP,P,ISD,EPW,BN,.FALSE.,KONT,
     1          &390)
290    CONTINUE
C--------- EXTRAPOLATION
300   FX = 2-NEX
      DO   360   J=1,4
      IF (NEX.GT.1) GOTO 310
      DATA(J,IS) = E(J,1)
      GOTO 360
310   DO   320   K=1,NEX
320   X(K) = E(J,K)
      DO   350   K=1,NEX1
      JR = NEX - K
      VHK = FAC(JR)
      DO   330   I=1,JR
      XA = X(I+1)
      D = (XA - X(I))*VHK
      XN = XA + D
330   X(I) = XN
      VHK = DMAX1(DABS(XA),DABS(XN))
      FEH = DABS(D)
      IF (VHK.GT.FEH) GO TO 340
      FEH = 1.D0
      GO TO 350
340   FEH = FEH/VHK
350   CONTINUE
      DATA(J,IS) = XN
      IF (FEH.GT.FX) FX = FEH
360   CONTINUE
370   F(IS) = FX
380   JR = NZ
      CALL SOUTD (NL,ND,DATA,NF,F,JR)
      IF (KONT.GT.0) RETURN
      GO TO 400
390   NL = IS - 1
      GO TO 380
400   CONTINUE
      RETURN
410   KONT = 1
      RETURN
      END
```