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H. J. Belitz and O. Rakova

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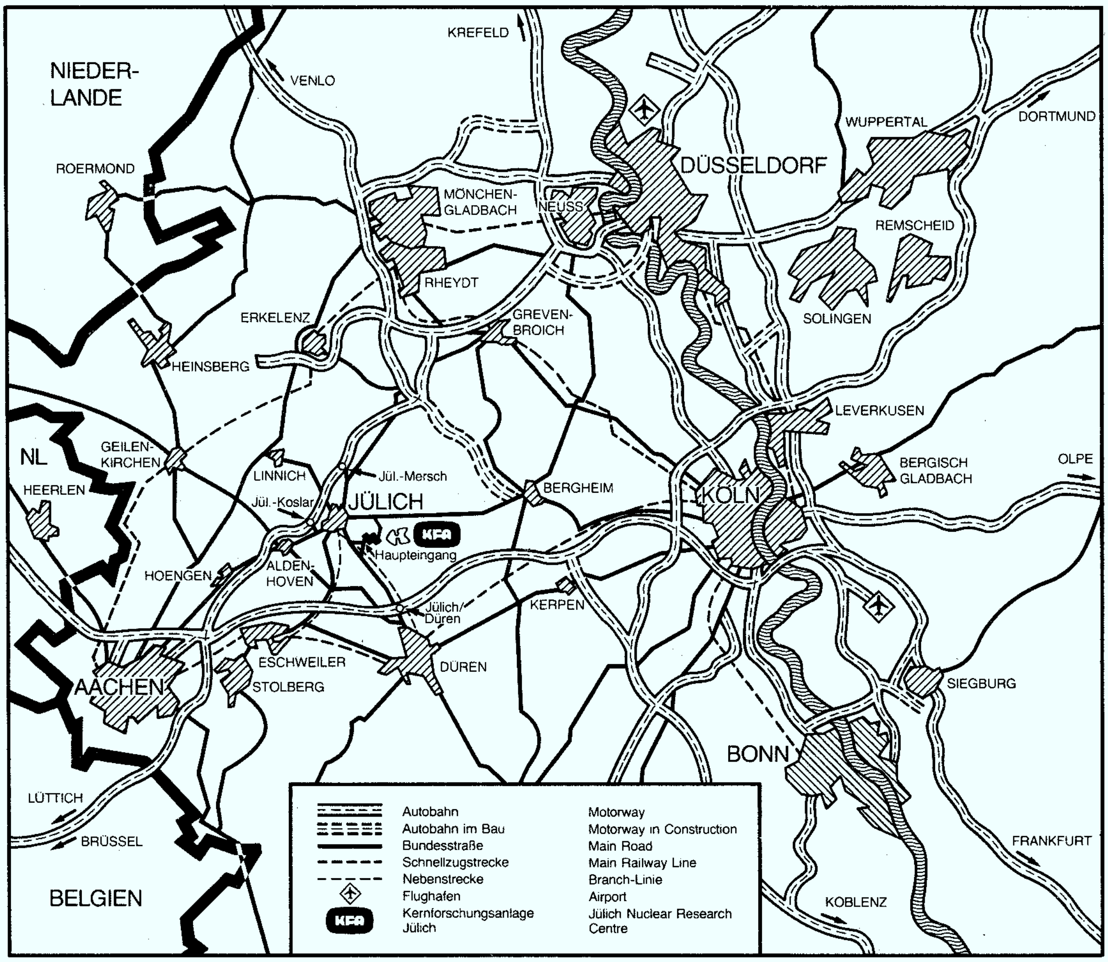

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Abstract
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A method is described to construct a polynome approximation of SHAFRANOWs equilibrium conditions from given distributions of the current density.

For the flux function a polynomial of a certain degree is assumed. The partial differential equation which connects the flux function and the current distribution is used to determine the coefficients of the polynomial. An accuracy test shows whether the assumed degree of this polynomial is sufficiently high.

The plasma pressure and the current function are related to the flux function. For both a polynomial with respect to the flux is assumed, and the given current density is splitted into the pressure part and the current flux part. This can be done, if the aspect ratio (major radius / minor radius of the plasma) is not too great. From the calculated total plasma force and the elctromagnetic force it can be seen, whether the degree of these polynomials is sufficiently high.

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Introduction

In TOKAMAKs the plasma current and the magnetic field form the containing vessel. The plasma shall be in an equilibrium. That leads to certain relations between the components of the currents and the magnetic field, and to a connection between these and the plasma pressure. SHAFRANOW [1] has given the conditions for the equilibrium. Here the magnetic flux function and the current function play an important roll. They are connected together and with the plasma pressure by a partial differential equation. The solution of this equation then allows to calculate a number of important physical quantities of the plasma.

Here a method will be described how one can construct an approximate computer solution which is satisfactory for practical purposes. An elliptical or circular cross section of the plasma is assumed.

The philosophy is the following. If we would try to solve the differential equation for the flux function using an infinite series in the coordinates, we had to do two things:

- to determine the coefficients of the series in a very general way and

- to investigate the convergence of the series.

We really only use a finite polynomial. Then we can determine the coefficients in a very special way from a finite number of equations. Of course, this is not quite right, since there are more equations than unknown coefficients, and a number of these equations are to be neglected. Therefore at that time we cannot say anything about the accuracy of the so found solution, even if the coefficients of the polynomial decrease. But from the flux function we can derive the magnetic induction and can see whether it corresponds to the total plasma current up to a sufficient high degree of exactness. If this is not the case one may increase the degree of the polynomial. This way we every time got a better approximation. We think that this criterion is able to decide on the convergence of our method.

It follows from the equilibrium conditions that the current density consists of two parts, one of them is related to the plasma pressure, while the other is related to the electromagnetic field or to the so called current function. The plasma pressure and the current function depend on the magnetic flux function, but we do not know in what a way. If we write these functions as polynomials of the flux, we can determine their coefficients by splitting the corresponding expression of the current density into its two parts (this can be done at least if the aspect ratio of the plasma is not too high). Now we can calculate the total force due to the plasma pressure and the total electromagnetic force. They must be equal in value but opposite in sign. Of course, since this method is only an approximation, the calcuated forces usually will differ. Then one may increase the degree of these polynomials and may see whether the agreement becomes better. This is an accuracy test for the second part of the calculations.

The equilibrium conditions

The equlibrium conditions for a plasma are [1]

$$\operatorname{div} \vec{B} = 0 \qquad \operatorname{rot} \vec{B} = \mu_0 . \vec{j} \qquad \operatorname{grad} p = \vec{j} \times \vec{B} \tag{1}$$

(B = magnetic induction, j = current density, p = plasma pressure, $\mu_0 = 4\pi . 10^{-7}$ Henry/m; MKS units are used).
These equations are discussed in detail e. g. by SHAFRANOW [2] and by ROSE and CLARK [3]. A cylindrical system of coordinates (r, φ, z) is used. The z-axis represents the main axis of the torus, the position of the plasma axis is r = R = RMAJ[1], z = 0. A circular plasma cross section with radius a = RMIN[1] in each φ = const plane is assumed. Then the plasma region is given by $(r-R)^2 + z^2 \leq a^2$. For an elliptical plasma cross section one has to replace z by $\varepsilon . z$ with $\varepsilon \neq 1$.

The plasma configuration is assumed to be axisymmetric. Then none of the physical quantities depends on the angle φ. Introducing the magnetic flux function $\psi = r . A_\varphi$ (A_φ is the φ-component of the electrodynamic potential A) the divergence of the magnetic induction vanishes identically since

$$B_r = -\frac{1}{r} \cdot \frac{\partial \psi}{\partial z} \qquad \text{and} \qquad B_z = \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} \tag{2}$$

From the second of the Eqs. (1) it follows for the φ - component

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} = -\mu_0 r j_\varphi \tag{3}$$

Introducing the current function $I(\psi)$

$$I(\psi) = r . B_\varphi / \mu_0 \tag{4}$$

the corresponding r- and z-components of this equation lead to

$$\left.\begin{aligned} j_r &= -\frac{1}{r} \cdot \frac{\partial I}{\partial z} = -\frac{1}{r} \cdot \frac{dI}{d\psi} \frac{\partial \psi}{\partial z} \\ j_z &= \frac{1}{r} \cdot \frac{\partial I}{\partial r} = \frac{1}{r} \cdot \frac{dI}{d\psi} \cdot \frac{\partial \psi}{\partial r} \end{aligned}\right\} \tag{5}$$

Finally from the third of the Eqs. (1)

$$j = r \frac{dp}{d\psi} + \frac{\mu_0}{r} \cdot I \cdot \frac{dI}{d\psi} \tag{6}$$

If the functions $\psi(r,z)$, $I(\psi)$, $p(\psi)$ are known then a number of important physical quantities of the plasma can be calculated.

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[1] This notation is used in the plotter pictures since the plotter only writes capital letters.

Additional assumptions for the solutions

We are only looking for simple solutions of the Eqs. (2) to (6), which can be easily compared with experimental observations. The solutions shall satisfy a few conditions. They are, however, fulfilled in the most cases of practical interest. We briefly summarize these points:

(a) Symmetry relative to the plane z = 0, that means that $\psi(r,-z) = \psi(r,z)$.

(b) On the plasma surface the plasma pressure p shall vanish, while the flux function $\psi(r,z)$ must be constant. (We choose this constant equal zero).

(c) The component of the magnetic induction normal to the plasma surface vanishes at each point of this surface. The corresponding poloidal component, however, must not vanish at any point of this surface.

(d) There exists one and only one magnetic axis in the plasma region. The magnetic axis is defined by $\partial\psi/\partial r = 0$ and $\partial\psi/\partial z = 0$.

(e) The plasma pressure is positive everywhere within the plasma region.

The conditions (a) to (c) are fulfilled if the function ψ is of the form ([4], [5])

$$\left.\begin{aligned} \psi(x,z) &= (a^2 - x^2 - z^2) \cdot F(x,z) \\ F(x,z) &= \sum_{k=0}^{KE} \sum_{i=0}^{[k/2]} C(k-2i,2i) \cdot x^{k-2i} \cdot z^{2i} \end{aligned}\right\} \qquad (7)$$

Here $x = r - R$, and the plasma region is given by $x^2 + z^2 \le a^2$. $[k/2]$ is the greatest integer number $\le k/2$. KE is a given integer number, the degree of the polynomial.

Whether the conditions (d) and (e) are fulfilled, that depends on the coefficients C(k-2i,2i).

The calculation of the magnetic flux function

We assume that the current density j_φ is given by a polynomial in x and z of a certain degree. From the Eqs. (3) and (7) the coefficients C(k-2i,2i) are determined by comparing equal terms $x^m . z^n$ of the left and of the right side of Eq. (3). This procedure is described in more detail in the Appendix. Really it neglects a lot of terms and it is not clear a priori whether the so found flux function is sufficiently exact or not. Of course, one can compare the calculated coefficients and may see whether they decrease or increase. But a general proof of the convergence of this method is not possible in this way.

Therefore every time the following accuracy test is made. From the current density j_φ the total current J_φ can be calculated with very high accuracy by integration. On the other hand the poloidal magnetic field on the plasma surface is given by B_r and B_z, which are determined according Eqs. (2). We think that the so found flux function is a useful solution if for a given small value ε the calculated expressions satisfy

$$\left| \frac{1}{\mu_0} \cdot \int_{(S)} B_{pol} \cdot ds \; - \; J_\varphi \right| \le \varepsilon \tag{8}$$

The integral is taken over a closed path on the plasma surface in the plane φ = const. If the condition (8) is not fulfilled, than the degree KE of the flux function must be increased. Using $\varepsilon = 10^{-6}$ in all our calculations the condition (8) was fulfilled with KE between 6 and 12.

The pressure function and the current function

We assume that the functions p and I can be approximated by a power expansion with respect to ψ with a small number of terms. Therefore we write ([4], [5], [7])

$$p = \sum_{K=1}^{JE} a(k) \cdot \psi^K \tag{9}$$

$$I^2 = I_0^2 + \sum_{K=1}^{JE} b(k) \cdot \psi^K \tag{10}$$

According to Eqs. (9) and (7) the pressure on the plasma surface is zero. From Eq. (10) in connection with Eqs. (7) and (4) it follows that I_0 represents the vacuum case.

The unknown a(k) and b(k) appear only linearly in the Eqs. (9) and (10). Therefore Eq. (6) may be written in the general form [1]

$$r.j = r^2 \sum_{K=1}^{JE} A(k) \cdot \psi^{K-1} + \sum_{K=1}^{JE} B(k) \cdot \psi^{K-1} \tag{11}$$

If this equation is taken at N = 2.JE plasma points the coefficients can be calculated from a system of N linear equations with N unknowns. It seems, however, better to use more equations at more plasma points and to determine the unknown coefficients by a linear least square program. In this case one immediately can see how good the approximation is; if necessary the degree of the polynomial JE must be increased.

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[1] For too great aspect ratios A = R/a this method may fail. In this case r is nearly R, and then it is not possible to separate the two parts of the right side of Eq. (11).

The a(k) and b(k) are given by

$$a(k) = \frac{A(k)}{k} \qquad b(k) = \frac{2}{\mu_o} \cdot \frac{B(k)}{k} \tag{12}$$

It should be mentioned that this procedure would lead to a system of non linear equations if we put the current function in the form

$$I = I_o + \sum_{k=1}^{JE} c(k) \cdot \psi^k \tag{13}$$

Here we will not treat this case.

The ratio

$$\delta = \frac{(B_{\varphi,v})^2 - (B_{\varphi,m})^2}{(B_{\varphi,v})^2} = - \frac{1}{I_o^2} \cdot \sum_{k=1}^{JE} b(k) \cdot \psi_m^k \tag{14}$$

shows whether the plasma is diamagnetic or paramagnetic.

$$B_{\varphi,m} = \mu_o \cdot I_m / r_m \tag{15}$$

is the magnetic induction on the magnetic axis ($r = r_m$, $z = 0$),

$$B_{\varphi,v} = \mu_o \cdot I_o / r_m \tag{16}$$

is the magnetic induction of the vacuum field on the magnetic axis. ψ_m is the value of function ψ on the magnetic axis.

With this notation the toroidal beta β_t on the magnetic axis is given by

$$\beta_t = 2 \mu_o p_m / (B_{\varphi,m})^2 \tag{17}$$

The poloidal beta β_{pol} is

$$\beta_{pol} = 2 \mu_o \bar{p} / (\bar{B}_{pol})^2 \tag{18}$$

$\bar{p}$ is the mean plasma pressure, that is

$$\bar{p} = \frac{\int_0^{\psi_m} p(\psi) \cdot d\psi}{\int_0^{\psi_m} d\psi} = \frac{1}{\psi_m} \cdot \int_0^{\psi_m} p(\psi) d\psi = \sum_{k=1}^{JE} \frac{a(k)}{(k+1)} \cdot \psi_m^k \tag{19}$$

$\overline{B}_{pol}$ is the mean poloidal field on the plasma surface, that is

$$\overline{B}_{pol} = \frac{1}{2\pi a} \int B_{pol} \cdot ds = \mu_0 \cdot J_\varphi \, / \, 2\pi \, a \tag{20}$$

Therefore

$$\beta_{pol} = \frac{8\pi^2 a^2}{\mu_0 J_\varphi^2} \cdot \sum_{K=1}^{JE} \frac{a(k)}{(K+1)} \cdot \psi_m^K \tag{21}$$

The total forces

In this axisymmetric system the physical quatities do not depend on the coordinate φ , and further the function ψ is symmetric with respect to the plane z = 0. Therefore the only non vanishing components of the total force due to the plasma pressure and of the total electromagnetic force are their radial components K_r^{pl} and K_r^{em}

$$\left.\begin{aligned} K_r^{pl} &= - \int (\, \text{grad } p \,)_r \; . \; dv \\ K_r^{em} &= \int (j_\varphi B_z - j_z B_\varphi) \; . \; dv \end{aligned}\right\} \tag{22}$$

The integrals are taken over the whole plasma volume. Due to the third of the Eqs. (1) the sum of these two forces is zero, if the plasma is in equilibrium. On the other hand, with the exception of j_φ all the quatities of the right sides of Eqs. (22) are derived by approximation as shown above. Therefore usually the two forces K_r^{pl} and K_r^{em} will differ. We can write the forces in the following form

$$\left.\begin{aligned} K_r^{pl} &= - \int \left(\frac{dp}{d\psi} \cdot \frac{\partial \psi}{\partial r} \right) \cdot dv \\ &= - \int \frac{\partial \psi}{\partial r} \cdot \left(\sum_{K=1}^{JE} K \cdot a(k) \cdot \psi^{K-1} \right) \cdot dv \end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} K_r^{em} &= \int \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} \left[j_\varphi - \frac{\mu_0}{r} \cdot I \cdot \frac{dI}{d\psi} \right] \cdot dv \\ &= \int \frac{1}{r} \cdot \frac{\partial \psi}{\partial r} \left[j_\varphi - \frac{\mu_0}{2r} \left(\sum_{K=1}^{JE} K \cdot b(k) \cdot \psi^{K-1} \right) \right] dv \end{aligned}\right\} \tag{24}$$

From these formulas we conclude that the reason for the difference between the calculated values K_r^{pl} and K_r^{em} lies in a non appropriate separation of the current density into the pressure and the current function part according to Eq. (11). That means that the degree JE of the polynomials of the Eqs. (9) and (10) are too small. Indeed we always got a better agreement between these two forces if we increased JE.

The expression

$$\delta K = (K_r^{pL} + K_r^{em}) / K_r^{em} \tag{25}$$

shows quantitatively the quality of the separation of the given current density into the pressure part and the current function part. It seems to be a good accuracy criterion for the second part of the calculations.

Solutions for a few special current distributions

Various types of current distributions has been investigated. Here we will discuss only current distributions of the form

$$j_\varphi = j_o \cdot (a^2 - x^2 - z^2) \cdot G(x,z) \tag{26}$$

where G(x,z) is a polynomial with respect to x and z.

Then the current density j_φ on the plasma surface vanishes identically. Up to now measurements (e. g. at TEXTOR) only show that the current density on the plasma surface is very small, may be that it is really zero. Then Eq. (26) seems to be an appropriate formula.

Two other cases can also be treated by this method; the current density on the plasma surface

- is constant (but not zero),
- is not constant, but depends on the poloidal angle.

Then the current density may be of the form

$$j_\varphi(x,z) = g(x,z) / h(x,z) \tag{27}$$

where g(x,z) and h(x,z) are polynomials. We will discuss such current distributions in a later paper. But it may be mentioned already that the calculation of the coefficients C(k-2i,2i) becomes more complicate, since the left side of Eq. (3) at first is to multiply by the polynomial h(x,z).

Of course, in general the structure of the polynomial G(x,z) in Eq. (26) may be similar to that of F(x,z) in Eq. (7). But it is also interesting to investigate very simple distributions of the plasma current. Here we will discuss only three special types of current distributions:

(A) $$G(x,z) = 1 \tag{28}$$

(B) $$G(x,z) = a + b.x \tag{29}$$

(C) $$\left.\begin{aligned} G(x,z) = {} & a(1) + a(2).x + \dots + a(7).x^6 \\ & +b(3).z^2 + b(5).z^4 + b(7).z^6 \end{aligned}\right\} \tag{30}$$

The given datas of the TOKAMAK are RMAJ, RMIN, the vacuum magnetic induction at the centre of the plasma B_o, and the total plasma current in φ-direction J_φ. We always used the datas of TEXTOR RMAJ = 1.75 m, RMIN = 0.45 to 0.5 m, B_o = 2 Tesla, J_φ = 50 to 500 kA.

(A) The symmetric parabolic distribution, Eqs. (26), (28)

Fig. 1 shows the following curves:

- J-PHI - the given current distribution j_φ according to Eqs. (26) and (28)
- FUNCTION PSI - the calculated flux function ψ (KE = 6)
- J-SHAFRANOW - the approximation of the given current distribution according Eq. (6) or Eq. (11), using the Eqs. (9) and (10) with JE = 3
- P(PSI) - the plasma pressure calculated from Eq. (9).

The curves are normalized to 100. For J-PHI and J-SHAFRANOW the same normalizing factor 100/Max(J-PHI) is used. These two curves are nearly identical, the approximation is very good. The abscissa is x = (r-RMAJ)/RMIN. The position of the magnetic axis is x = 0,07676, that cooresponds to 3,838 cm for RMIN = 0,5 m.

For this case (with KE = 6 and JE = 3):

$$j_o = 509{,}3\ \mathrm{kA/m^2} \quad \text{or} \quad 50{,}93\ \mathrm{A/cm^2}$$

$$\psi_m = 5{,}23.10^{-2}\ \text{Weber}$$

$$p_m = 1{,}79.10^{3}\ \text{Newton/m}^2$$

$$\delta = -0{,}4\ \% \qquad (\text{ see Eq. (14) })$$

$$\beta_t = 1{,}2.10^{-3}$$

$$\beta_{pol} = 0{,}246$$

While these values are nearly independent as well from KE as from JE, the total forces depend on JE. From the Eqs. (23), (24) and (25) the following values are obtained (forces in kNewton and δK in %):

	K_r^{pl}	K_r^{em}	δK
JE = 3	2.603	-2.416	-7.76
4	2.535	-2.441	-3.847
5	2.516	-2.448	-2.76
6	2.509	-2.450	-2.402

Exactly the same curves are obtained also for other parameters than indicated at the picture. This is understandable, - the differential operator in Eq. (3) and the distribution of the given current only depend on the aspect ratio, the flux function depends linearly on j_φ, and the curves are normalized with respect to the ordinate and to the abscissa. The abscissa of the magnetic axis is a fixed value, there is no free parameter to shift the magnetic axis.

Fig. 2 shows that the plasma pressure really remains positive, if the current changes its direction. As can be seen from the printed results, that there is no difference between the pressure values of Fig. 1 and Fig. 2. Fig. 3 shows the corresponding magnetic field lines. In Fig. 4 a typical distribution of the poloidal magnetic field B-POL in the plane z = 0 is plotted. (It is similar to a corresponding curve given by SOLTWISCH [6], while other curves, e. g. his current distribution, are less comparable).

(B) The non symmetric distribution, Eqs. (26), (29)

It is a certain disadvantage of the current distribution just discussed, that there is no possibility to influence the position of the magnetic axis. That, however, should be possible in order to compare the calculations with experimental observations.

Using a formula as Eq. (29) this can be done, at least indirectly. The coefficients a and b are determined such way that

- the current density has its maximum at a given position
- the maximum of the function $(1-x^2).(a+b.x)$ is 1.

With KE = 10 in Eq. (7), in all cases the accuracy test Eq. (8) is fulfilled up to a high degree; the relative error is $< 10^{-3}$ %.

With b = 0 the curves of Fig. 1 are obtained. If the position of the maximum of the current density (say xc) is shifted to the left, the abscissa of the magnetic axis (xm) is shifted to the left too. Hereby the shape of the pressure curve is changed in a remarkable manner, - see Fig. 5 to 7.

In	Fig. 5	xc = -0.084	(or -4,2 cm)	xm = 2,435 cm
	Fig. 6	xc = -0.086	(or -4,3 cm)	xm = 2,400 cm
	Fig. 7	xc = -0.088	(or -4,4 cm)	xm = 2,365 cm

These are critical current distributions and for smaller xc the pressure becomes negativ everywhere within the plasma region. If xc goes from zero to negative values, then p_m, β_{pol} and the total force due to the plasma pressure decrease.

(C) Polynomial of higher degree, Eqs. (26), (30)

Observed current distributions often are more complex than those just discussed ([6]). Unfortunately such measurements are not yet complete, and additional assumptions are to be made with respect to the current distribution in the z - direction. We therefore tried to handle a current density, which may be described by Eqs. (26) and (30). An arbitrary current distribution in the plane z = 0 was supposed, and the coefficients a(k) were determined by least square method. Then we took b(3) = a(3), b(5) = a(5), b(7) = a(7). It was the aim of these calculations to see what values of KE and JE are needed for a good approximation.

Fig. 8 shows the given current distribution and the calculated curves. The degree of the polynomial for the flux function must be ≥ 12 to fulfill the accuracy test Eq. (8). According to Eq. (11) the function J-SHAFRANOW was calculted using JE = 3 as in all preceding cases. This approximation is not good, there are great differences especially at $x = \pm 1$. Fig. 9 shows this approximation with JE = 5. Here the given values of J-PHI and the calculated values of J-SHAFRANOW differ by less then 1 %, so these two curves coincide.

This fact, however, does not answer the question how good the approximation with JE = 5 really is. Using the Eqs. (23) to (25) for the forces the following results are obtained (forces in kNewton, δK in %):

JE = 3	K_r^{pl} = 6.124	K_r^{em} = -4.354	δK = -40.66
4	5.188	-4.625	-12.17
5	5.002	-4.666	-7.20
6	5.000	-4.673	-6.99

Conclusions

It is possible to construct an approximate polynome solution of SHAFRANOWs equilibrium conditions, sufficient for practical purposes. The degree of the polynomial of the flux function (KE ≤ 12) is not too high, for the pressure and the current function about three to six terms are needed. The program will be applicated to TEXTOR as soon as complete measurements are present, that means measurements of the current densities in the r- and z-direction.

Appendix: Some remarks on the program

(a) Transformation of the coordinates

The program always uses the following coordinates

$$\xi = x/a = (r-R)/a = (r-RMAJ)/RMIN \qquad \zeta = z/a \tag{A.1}$$

We will, however, continue to denote also the new variables by x and z, respectively, since confusion seems to be excluded. That means especially

$$\frac{\partial}{\partial x} \Rightarrow \frac{1}{a}\cdot\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial z} \Rightarrow \frac{1}{a}\cdot\frac{\partial}{\partial z} \tag{A.2}$$

Then the plasma region is defined by

$$x^2 + z^2 \leq 1 \tag{A.3}$$

The flux function ψ is written

$$\psi(x,z) = a^2 . (1 - x^2 - z^2) . F(x,z) \tag{A.4}$$

$$F(x,z) = \sum_{k=0}^{KE} \sum_{i=0}^{[k/2]} C(k-2i,2i) . x^{k-2i} . z^{2i} \tag{A.5}$$

Using the aspect ratio

$$A = R/a = RMAJ/RMIN \tag{A.6}$$

and therefore

$$r = a . (A + x) \tag{A.7}$$

the Eqs. (2) - (6) may immediately written in the new coordinates.

If we insert Eq. (A.5) into Eq. (3) on p. 2 we get

$$D\, F(x,z) = \mu_0 . a . (A+x)^2 . j_\varphi \tag{A.8}$$

Here all the derivatives are evaluated as far as possible. The left side of Eq. (A.8) is given by

$$\left.\begin{aligned} D\, F(x,z) = {} & 4.A.F + 2.x.F + Fx + 4.A.x.Fx + 3.x^2.Fx - \\ & - z^2.Fx + 4.A.z.Fz + 4.x.z.Fz - A.Fxx - \\ & - x.Fxx + A.x^2.Fxx + A.z^2.Fxx + x^3.Fxx + \\ & + x.z^2.Fxx - A.Fzz - x.Fzz + A.x^2.Fzz + \\ & + A.z^2.Fzz + x^3.Fzz + x.z^2.Fzz \end{aligned}\right\} \tag{A.9}$$

(Fx $= \partial F/\partial x$, Fz $= \partial F/\partial z$, Fxx $= \partial^2 F/\partial x^2$, Fzz $= \partial^2 F/\partial z^2$).

Eq. (A.9) shows that the left side of Eq. (A.8) only depends on the aspect ratio A.

(b) The two dimensional array C(k-2i,2i) and the vector CS(l)

In order to determine the unknown values C(k-2i,2i) we compare the coefficients of equal terms of $x^m.z^n$ of both sides of Eq. (A.8). Thus a system of linear equations will be obtained. The number of the equations NEQ or the dimension of the corresponding matrix depends on KE in the following way:

$$\text{NEQ} = (\text{KE}+2)^2 / 4 \qquad \text{if KE is an even number}$$

$$= (\text{KE}+1).(\text{KE}+3)/4 \qquad \text{if KE is an odd number}$$

For example KE = 5 gives NEQ = 12, KE = 10 gives NEQ = 36, KE = 20 gives NEQ = 121.

The two dimensional array of the unknown C(k-2i,2i) must be written as an one dimensional array or vector, here denoted by CS(l). This vector is stored and used in the program. The index l depends on k and i ≤ [k/2] in the following manner:

$$l = k.(k+2)/4 + 1 + i \qquad \text{if k is an even number}$$

$$= (k+1)^2 /4 + 1 + i \qquad \text{if k is an odd number.}$$

It follows for example

k	i	l
0	0	1
1	0	2
2	0	3
2	1	4
3	0	5
3	1	6
4	0	7
4	1	8
4	2	9

and so on.

The program may be modified so that the magnetic axis is adjusted at the centre of the plasma in all cases. That means that C(1,0) is omitted. Then NEQ and l are to be modified in a suitable manner. Here we will not treat this case.

(c) The calculation of the C(k-2i,2i) or CS(l)

For a given current distribution the right side of Eq. (A.8) is a polynomial in $x^m.z^n$ (m,n ≥ 0) of a given degree with well known coefficients. The left side of this equation is a polynomial in $x^m.z^n$ too, containing the C(k-2i,2i). Its degree depends on KE and is greater than the corresponding number NEQ. Of course, the degree of this polynomial must be greater than that of the right side; otherwise the accuracy test Eq. (8) would fail.

We start with a NEQ x NEQ matrix and a column vector of dimension NEQ. All the elements of the matrix and of the vector are zero. Then successively we compare the coefficients of $x^m.z^n$ in the order m = 0,1,2... and n = 0,1,..≤ m, which gives the successive number j of the rows of the matrix and of the column vector. In the column vector the zero at the j th position is replaced by the corresponding coefficient of the right side. The corresponding coefficient at the left side containes a sum of expressions of the form (number x C(k-2i,2i)). The "number" is written at the position of the matrix indicated by j (the row) and l (the column), where l is given by k and i as described above. In the next section we will explain how these terms can be determined from the various terms of the left side of Eq. (8), that is from D F(x,z) in Eq. (A.9). The structure of the matrix is shown in Tab. A.1.

	j	l=	C(0,0) CS(1) 1	C(1,0) CS(2) 2	C(2,0) CS(3) 3	C(0,2) CS(4) 4	C(3,0) CS(5) 5	C(1,2)... CS(6) ... 6 ...
$x^0.z^0$	1		14	1	-7	-7	0	0
x	2		2	28	14	-2	-21	-7
x^2	3							
z^2	4							
x^3	5							
$x\ .z^2$	6							
x^4	7							
$x^2.z^2$	8							
z^4	9							
.....								

Tab. A.1: The structure of the matrix of the system of equations to calculate the unknown C(k-2i,2i). The matrix elements of the first two rows are calculated using A = 3,5.

(d) The coefficient of $x^m .z^n$ of D F(x,z) according Eq. (A.9)

Of course, the simplest way to construct the matrix elements is to use a program language as FORMAC or REDUCE, which is able to handle algebraic formulas with non numeric quantities in a general way. This method was used in an earlier paper [7], but for numerical calculations and for practical purposes it seems to be not appropriate. It is only applicable to check parts of the program, and this has been really done.

Here the coefficients of $x^m .z^n$ of D F(x,z) in Eq. (A.9) are determined in the following way. Let us consider the term

$$x^3.Fxx = x^3.\sum_{k=2}^{KE}\sum_{i=0}^{[k/2]} (k-2i)(k-2i-1).C(k-2i,2i).x^{k-2i-2}.z^{2i}$$

Then the following relations are worth

$$2i = n$$
$$k - 2i - 2 + 3 = m$$

It follows: $k = m - 1 + n$.
For example, the coefficient of $x^5.z^4$ in this case is 4.3.C(4,4) with i = 2 and k = 8.

In this way all the terms of D F(x,z) in Eq. (A.9) are investigated with respect to the various $x^m . z^n$.

It may be remarked that in the case of Eq. (27) the expression D F(x,z) at first is to multiply by h(x,z). The evaluation of the matrix elements than becomes more complicate. This may indicate the necessity to check such parts of the program by other independent methods.

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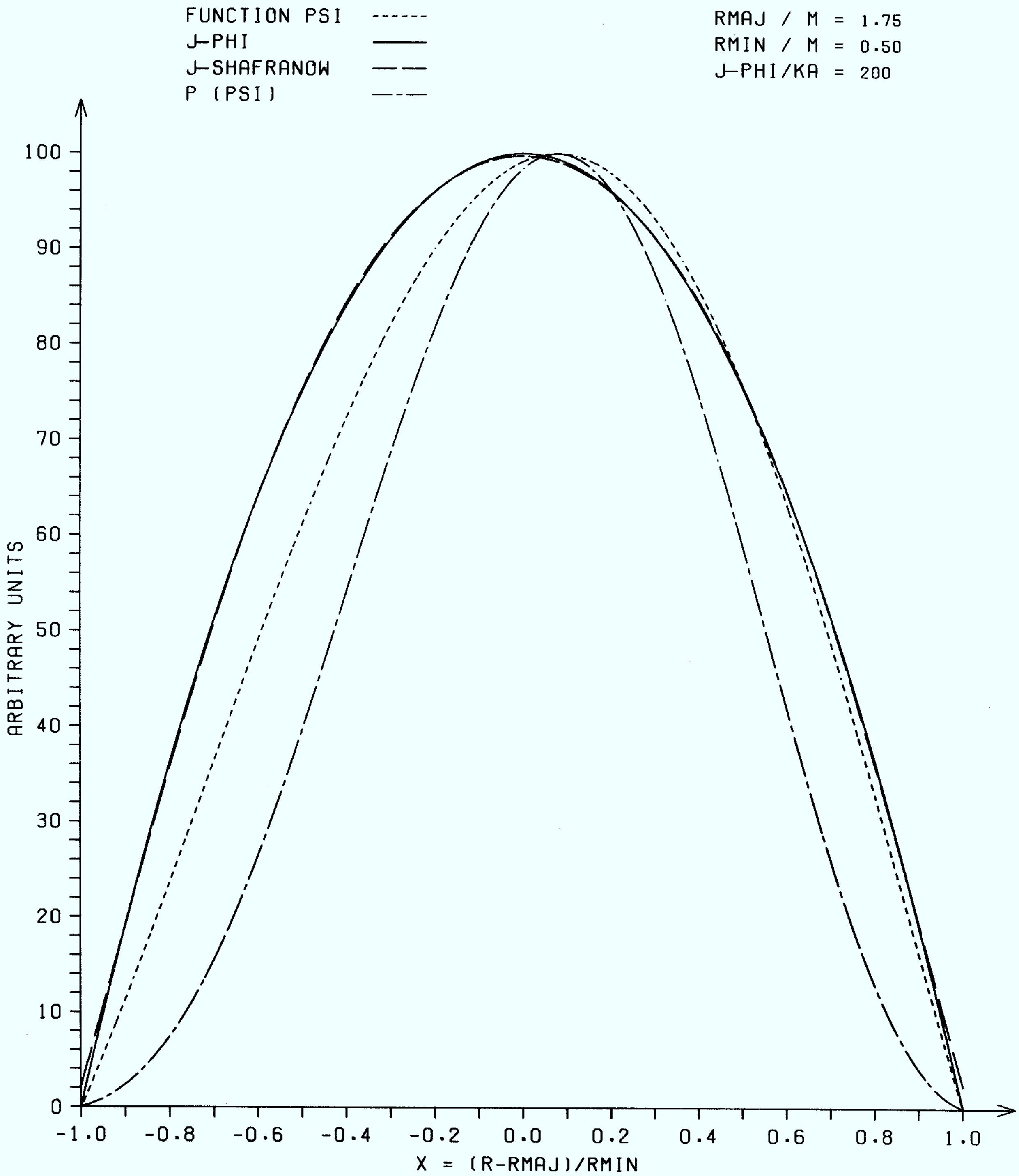
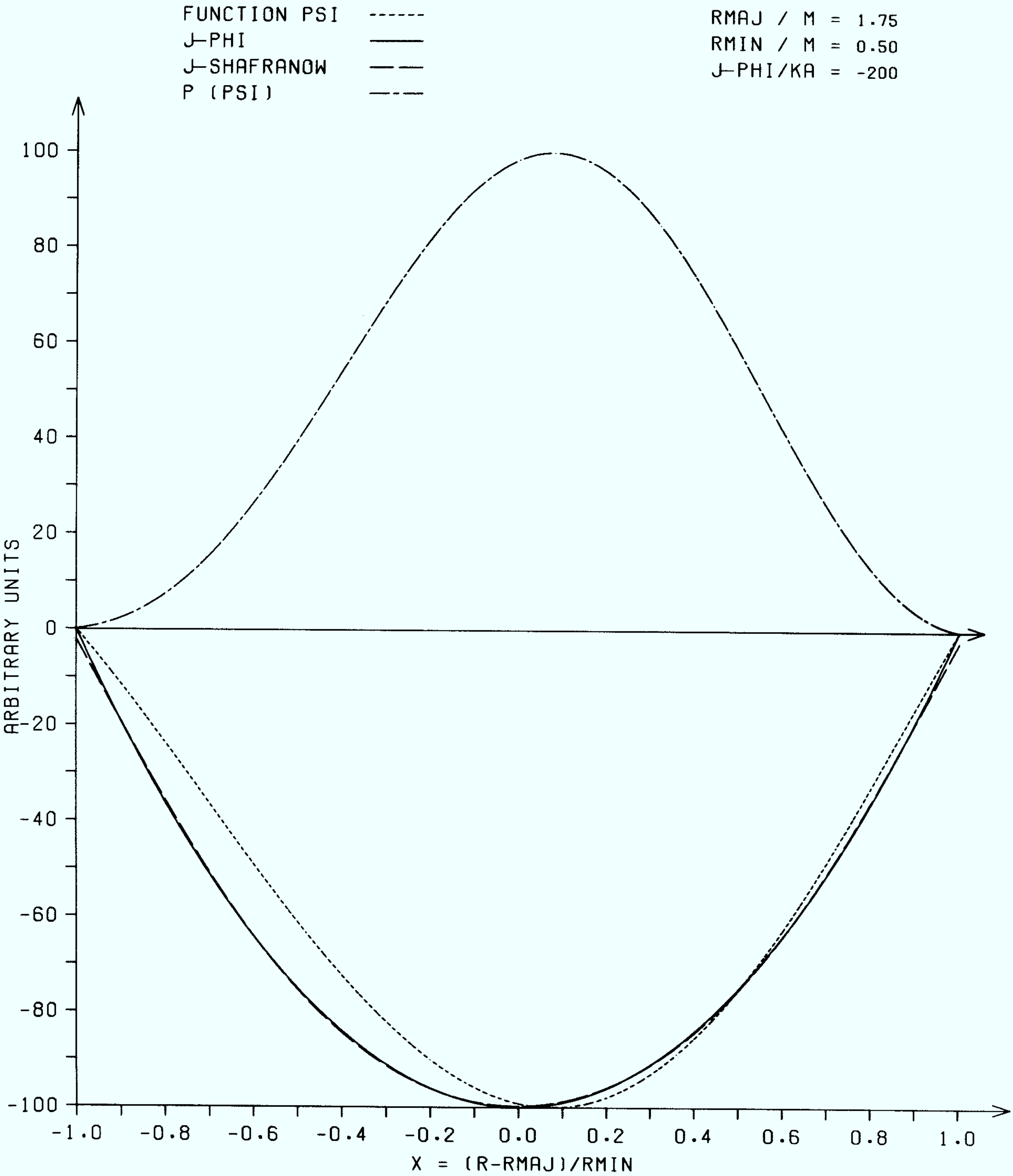


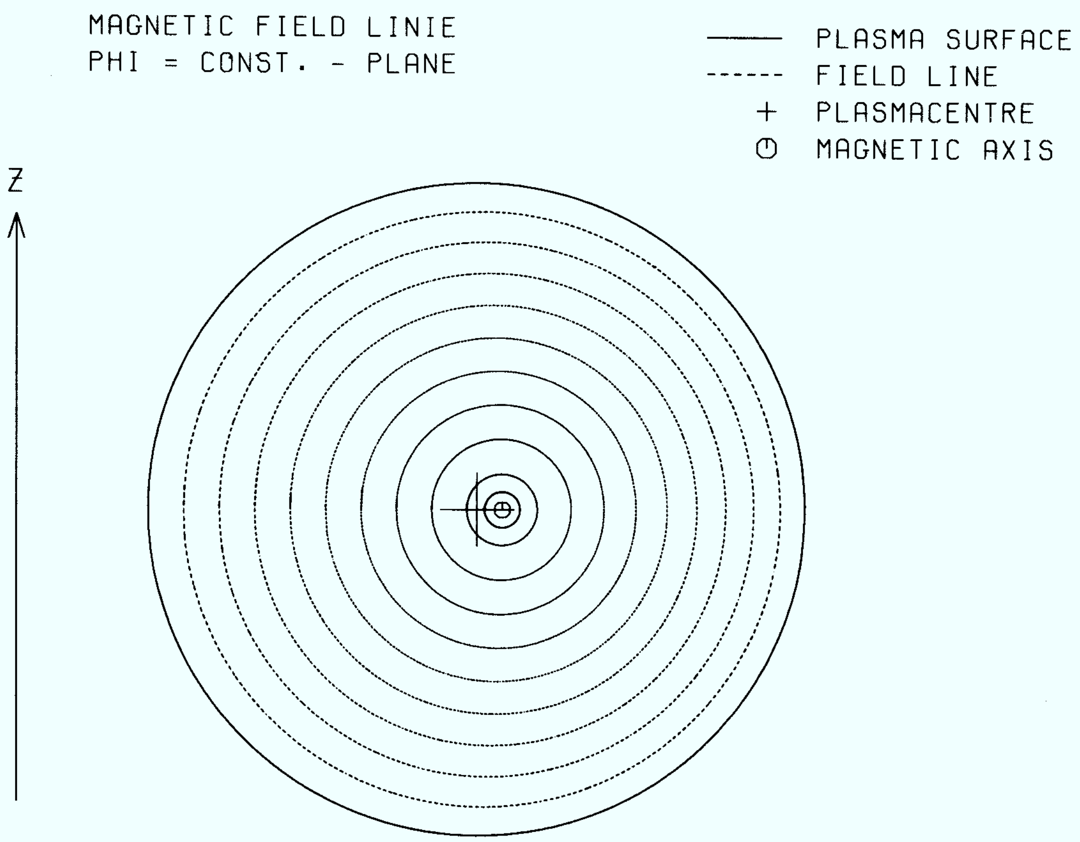
The symmetric current distribution j_φ according to Eqs. (26) and (28) in the plane z = 0, and the calculated curves for the magnetic flux ψ and the plasma pressure $p(\psi)$. J-SHAFRANOW is the approximation of the given j_φ according to Eq. (6) or (11); both curves coincide.

Fig. 1



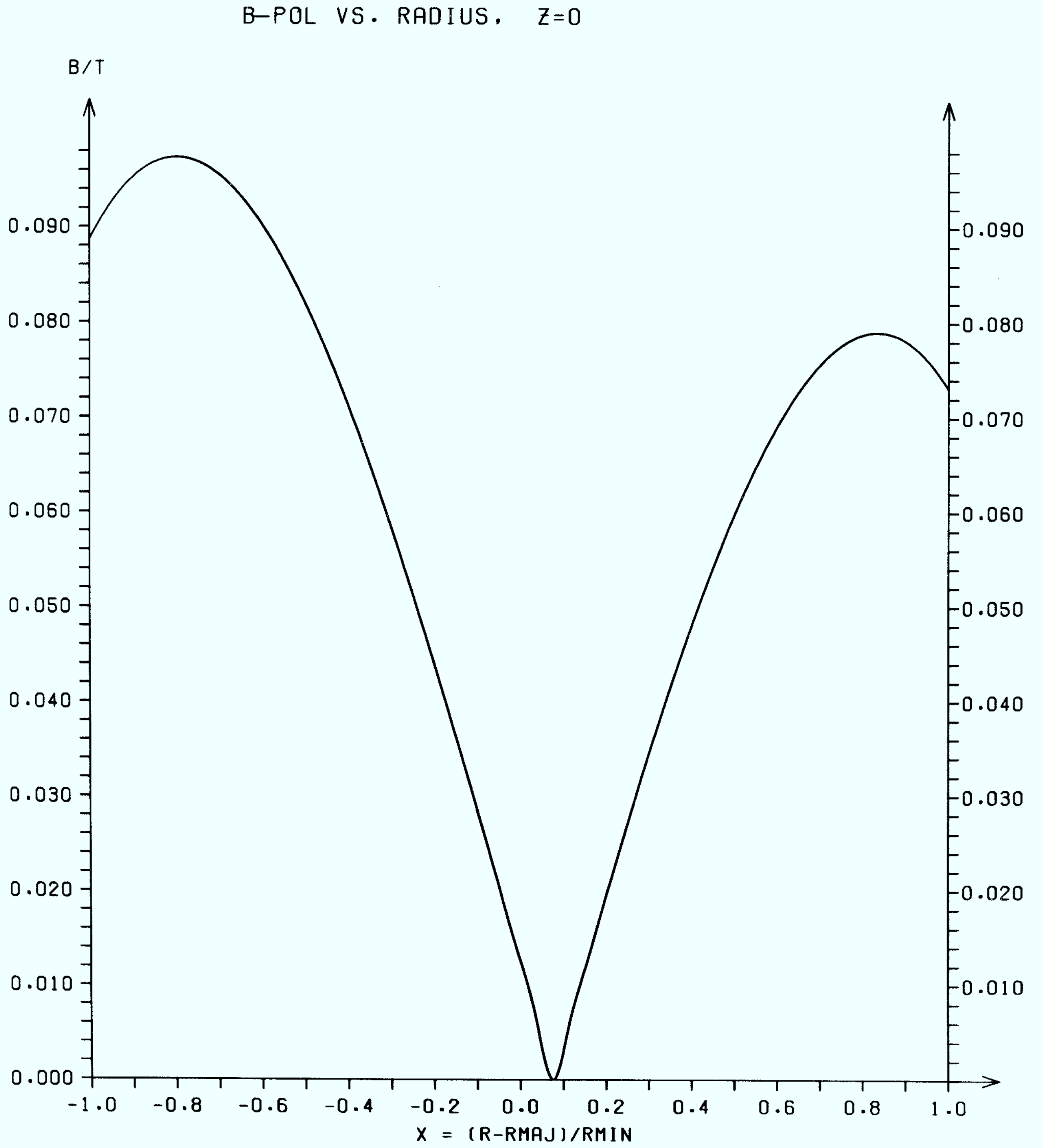
The plasma pressure is independend of the direction of the plasma current.

Fig. 2



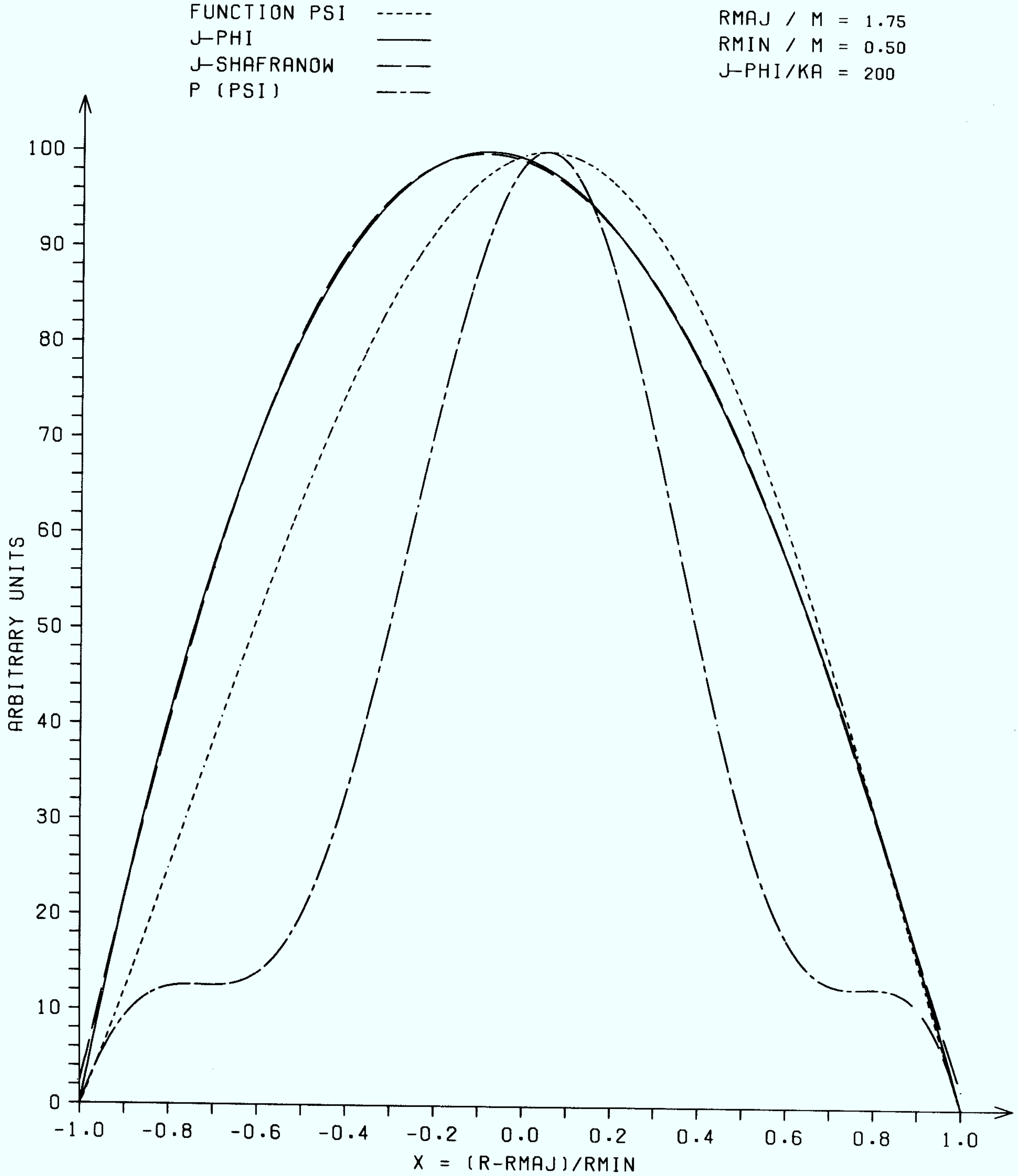
The magnetic field lines.

Fig. 3



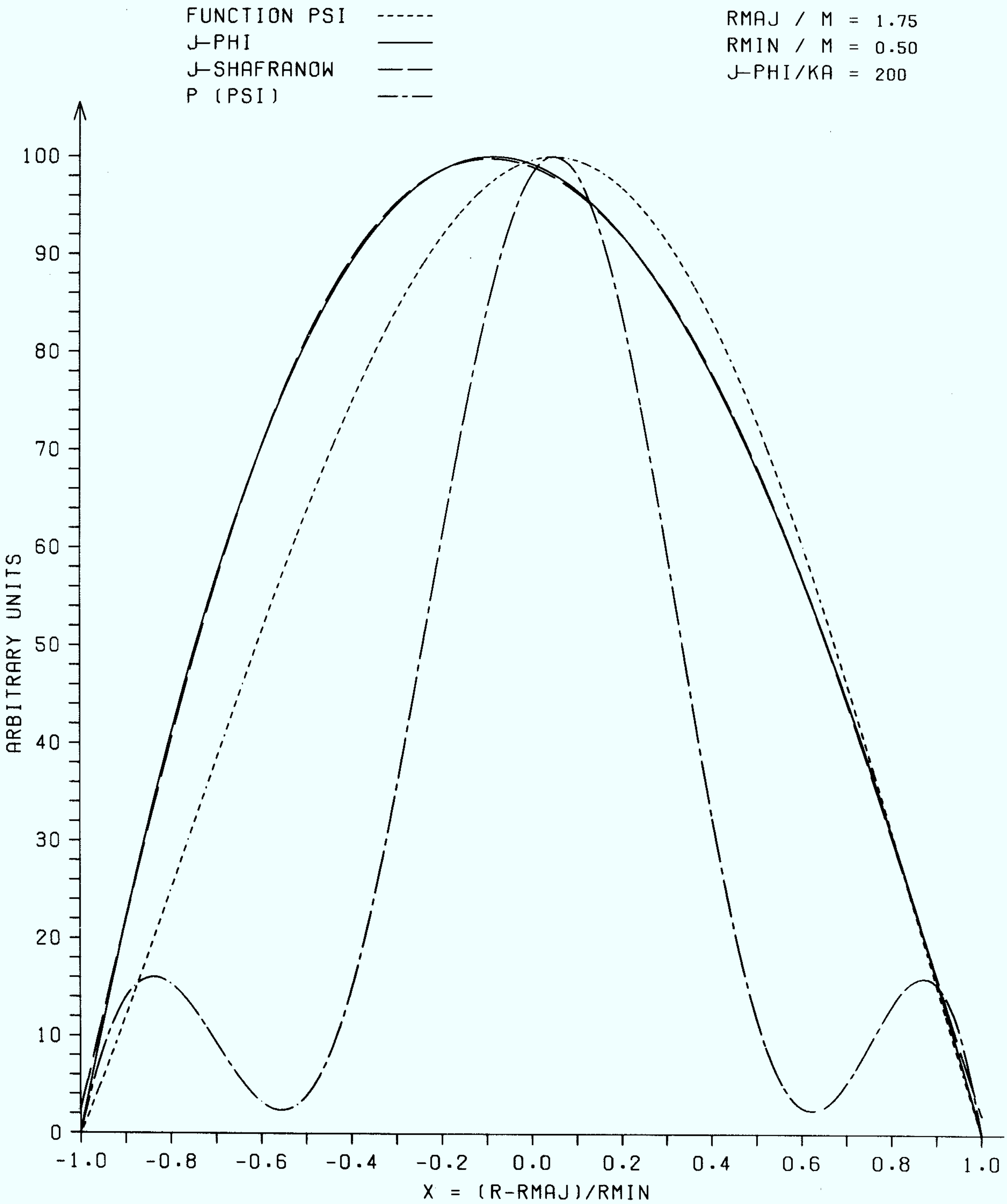
A typical distribution of the poloidal magnetic field B-POL in the plane z = 0.

Fig. 4



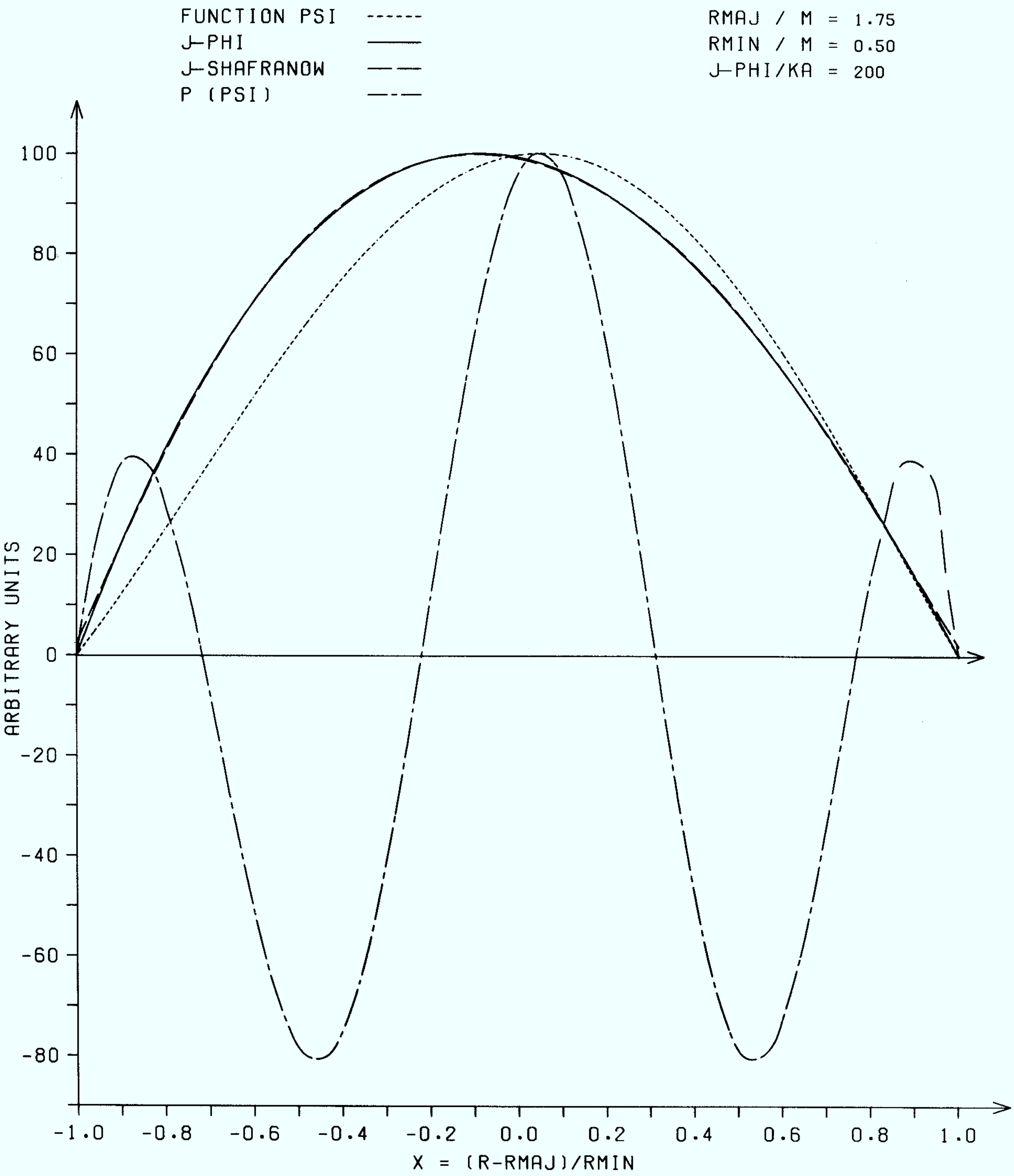
A non symmetrical current distribution according to Eqs. (26) and (29). Position of the maximum of the current density at xc = -4,2 cm. Magnetic axis at xm = 2,435 cm.

Fig. 5



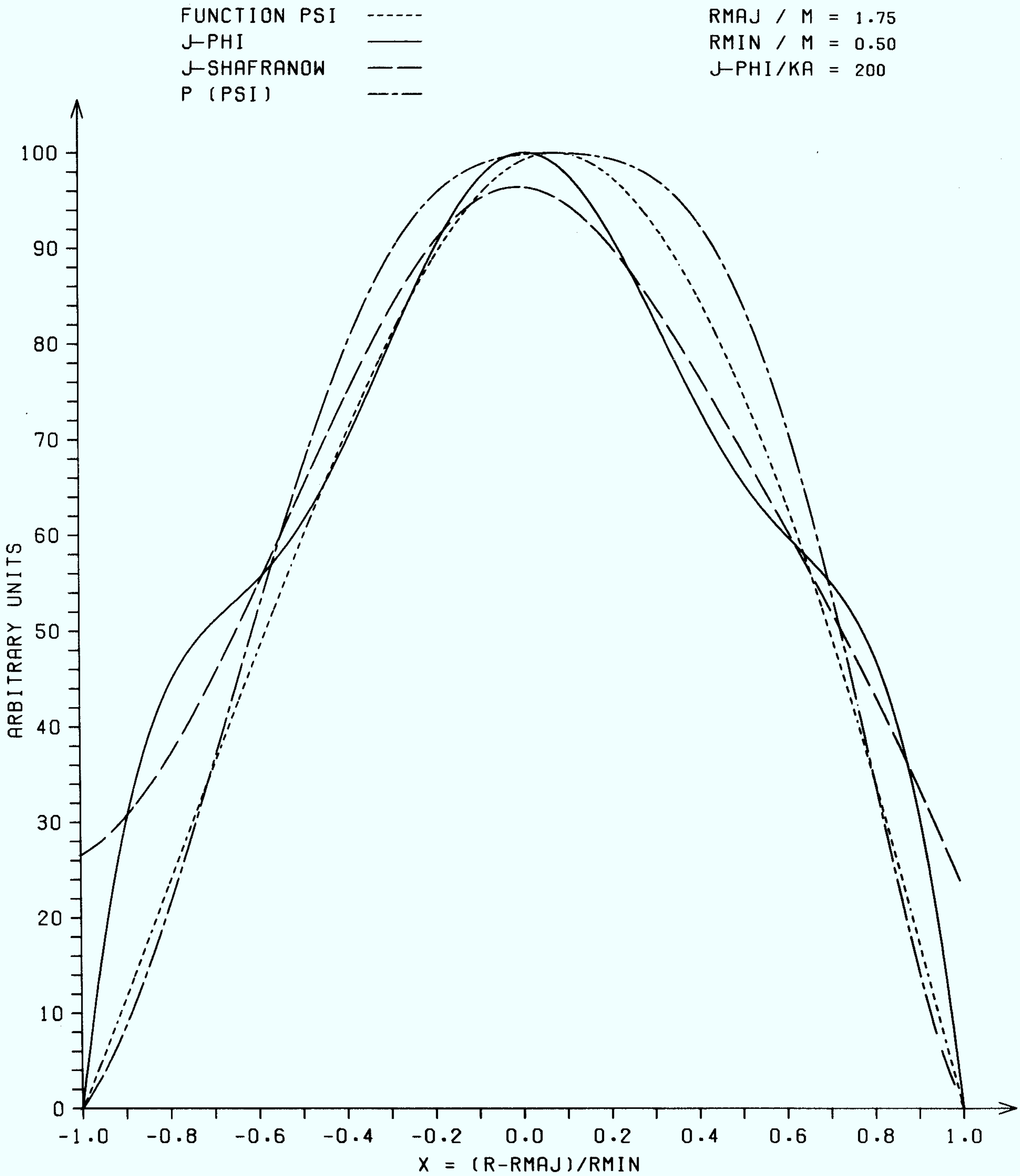
A non symmetrical current distribution according to Eqs. (26) and (29). Position of the maximum of the current density at xc = -4,3 cm. Magnetic axis at xm = 2,400 cm.

Fig. 6



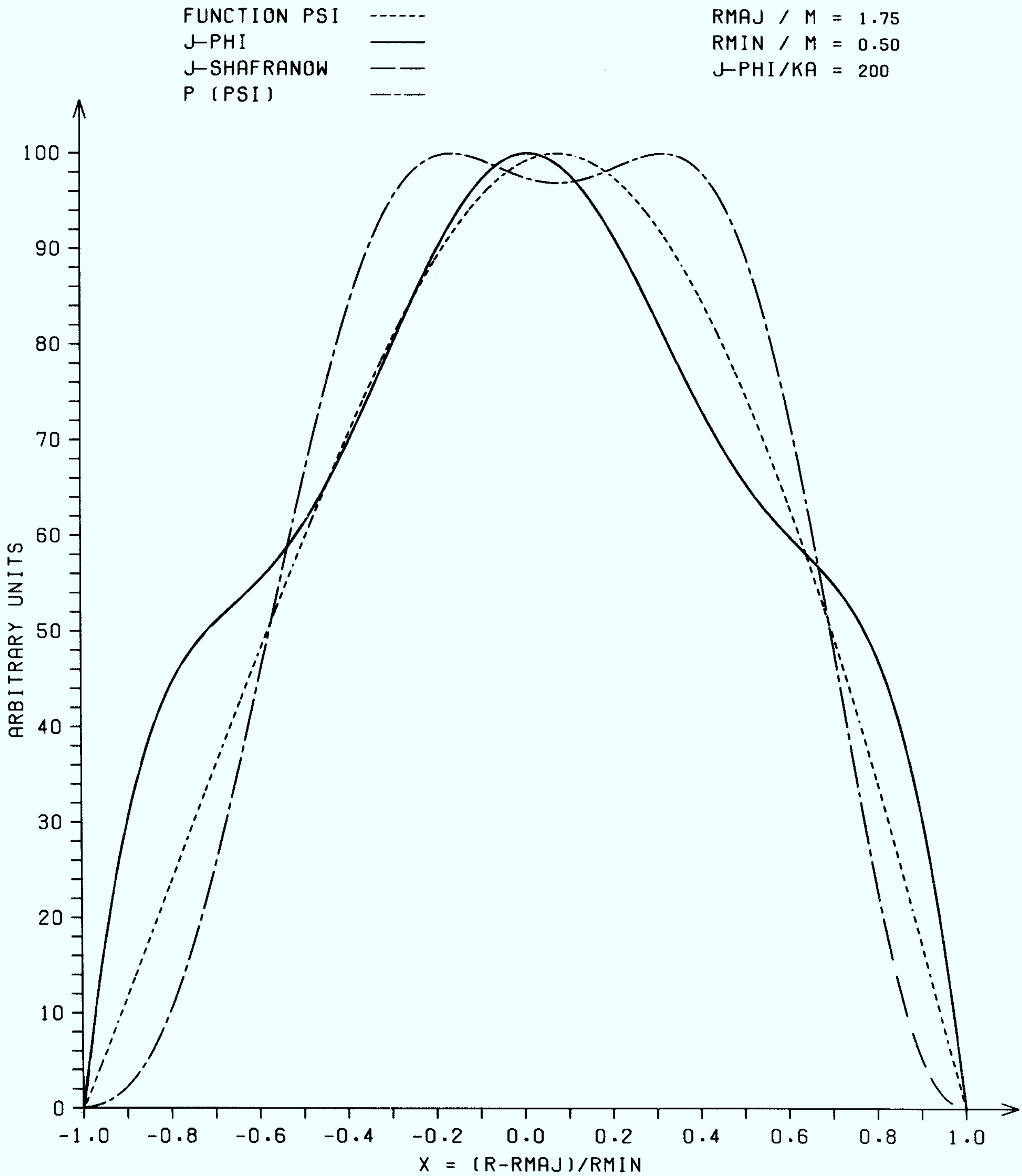
A non symmetrical current distribution according to Eqs. (26) and (29). Position of the maximum of the current density at xc = -4,4 cm. Magnetic axis at xm = 2,365 cm.

Fig. 7



A "shoulder"-shaped distribution of the current density according Eqs. (26) and (30). With JE = 3 J-PHI and J-SHAFRANOW differ especially near the plasma surface at x = ±1.

Fig. 8



A "shoulder"-shaped distribution of the current density according Eqs. (26) and (30). With JE = 5 J-PHI and J-SHAFRANOW agree very well in the whole plasma region.

Fig. 9