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Institut für Plasmaphysik
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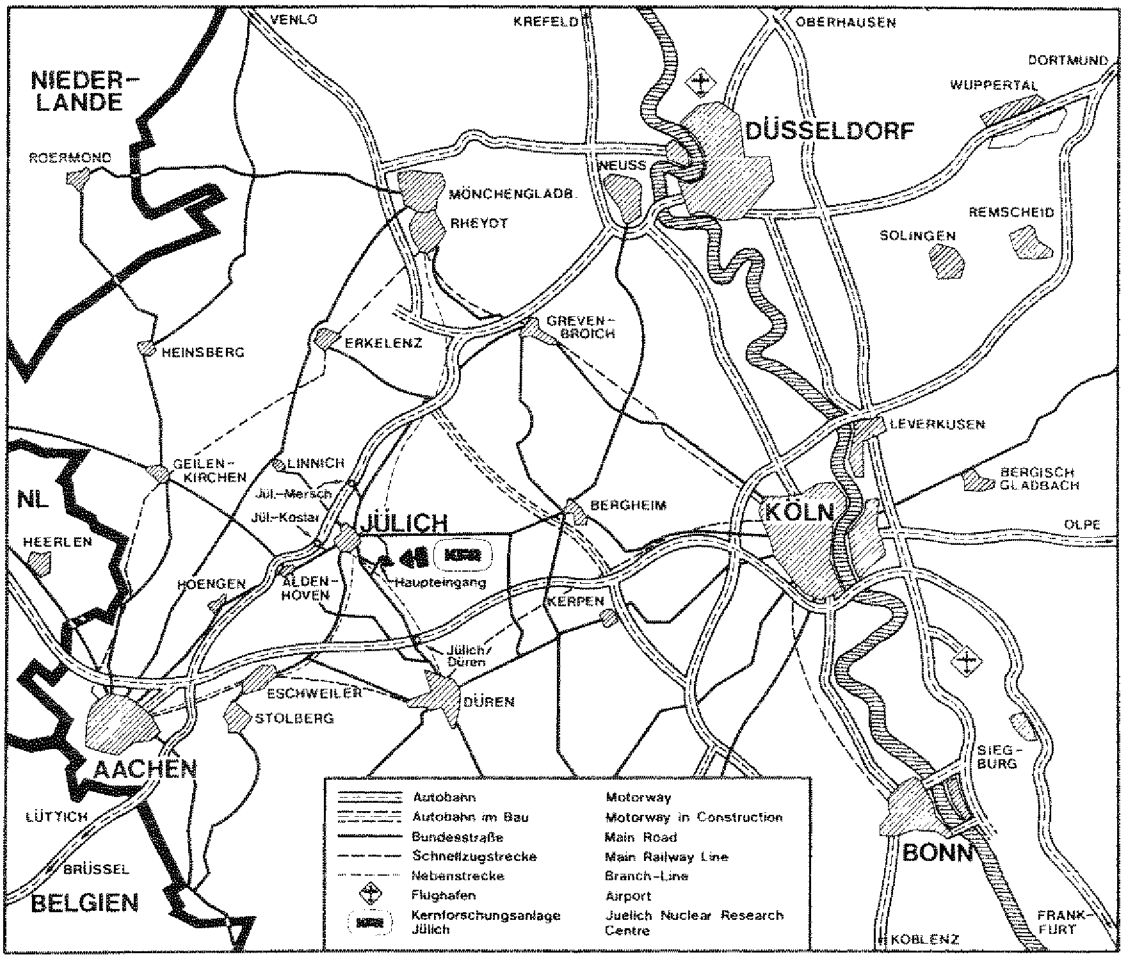
Jül-1992
April 1985
ISSN 0366-0885



Als Manuskript gedruckt

Berichte der Kernforschungsanlage Jülich – Nr. 1992
Institut für Plasmaphysik, Association EURATOM-KFA Jül-1992

Zu beziehen durch: ZENTRALBIBLIOTHEK der Kernforschungsanlage Jülich GmbH
Postfach 1913 · D-5170 Jülich (Bundesrepublik Deutschland)
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TABLE OF CONTENTS

THEORY OF THE RIPPLING INSTABILITY IN TOROIDAL DEVICES

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Abstract

The theory of the rippling instability is developed for axisymmetric toroidal plasmas including ion viscosity and parallel electron heat conduction, but assuming that the growth rate is small compared to the wave angular frequency $\omega_o = (1 + 1.71\ \eta_e)\ \omega_e^*$. ω_e^* is the electron diamagnetic frequency and $\eta_e \equiv (d\ln T_e/dr)/(d\ln N/dr)$. Three (in)stability regions must be considered, viz. $(c_i/R\omega_o)^2 < 0.12$ where cylindrical effects dominate and the plasma is stable; $0.12 < (c_i/R\omega_o)^2 < 0.26\ q^2$ where the magnetic drift destabilizes the mode; and $0.26\ q^2 < (c_i/R\omega_o)^2$ where the plasma is stable again. (The numerical values are given for $\eta_e = \eta_i = 2$ and $T_e = T_i$). Parallel electron heat conduction is stabilizing but ion viscosity broadens the instability domain. Under certain conditions, an important top-bottom asymmetry of the density fluctuation spectrum may arise.

if $T_e = T_i$ and $\eta_e = \eta_i = \eta >> 1$; this simple result does, however, not account for the stabilizing role of electron heat conduction and ion viscosity; these are considered in the text.

The paper is organized as follows. The two-fluid equations are simplified in sections 2 and 3 via the introduction of a scaling appropriate to the conditions prevailing in the plasma edge on the one hand, and to the rippling instability on the other hand. These reduced equations are solved order by order in section 4. Given the complexity of the complete eigenmode problem - which retains ion parallel viscosity and electron and ion parallel heat conduction -, the various asymptotic limits which can be considered are discussed and solved in section 5. Section 6 concludes with a brief summary and a table showing the instability and stability regions in terms of the relevant dimensionless parameters $\omega_o \tau_i$ and $(c_i/R\omega_o)^2$.

II. SCALING CONSIDERATIONS AND REDUCTION OF THE TWO FLUIDS EQUATIONS

In order to reduce the two fluids equations (Braginskii, 1965) to a tractable and yet consistent form, we order the variable characterizing the plasma on the one hand and the rippling mode on the other hand in powers of an expansion parameter. It is convenient to assume here that

$$\frac{\nu_{ei}}{\Omega_i} \sim \frac{m_e}{m_i} \sim \frac{a_i}{a} \sim \mu^2 \quad ; \quad \frac{a}{R} \sim \mu \tag{5}$$

where Ω_i is the ion gyrofrequency, $a_i = c_i/\Omega_i$ the ion Larmor radius, and a the minor radius of the plasma. Since the rippling mode is localized and its radial mode number is much larger than its poloidal mode number, we also scale

$$a_i \; \hat{p}\cdot\vec{\nabla} \ln f \simeq ik_r \, a_i \sim 1 \quad ; \quad a_i \; \hat{b}\cdot\vec{\nabla} \ln f \simeq ik_\theta a_i \sim \mu \tag{6}$$

where r is the radial variable, θ the poloidal angle, f a fluctuating variable (e.g. density, electrostatic potential, temperature, velocity component) and $\hat{p}$, respectively $\hat{b} = \hat{n}x\hat{p}$, is the unit vector normal to the flux surface, respectively is the unit vector binormal to the field line. Under these conditions it can be shown that

$$a_i \; \hat{n}\cdot\vec{\nabla} \ln \tilde{f} \sim \frac{B_\theta}{B} \frac{a_i}{r} \frac{\partial}{\partial\theta} \ln \tilde{f} + O(\mu^4) \sim O(\mu^3) \qquad (7)$$

Here $\partial/\partial\theta \sim 1$ accounts for those poloidal variations induced by the toroidal geometry. The correction $O(\mu^4)$ will be explicited later on. We note that there results from the scaling considerations (5-7) that

$$(c_i \; \hat{n}\cdot\vec{\nabla}\ln \tilde{f})/\nu_{ii} \sim (c_e \; \hat{n}\cdot\vec{\nabla} \ln f)/\nu_{ei} \sim 1, \qquad (8)$$

$$\omega_e^* \; \tau_i \sim \omega_i^* \; \tau_i \sim \omega_o \; \tau_i \sim 1. \qquad (9)$$

These relations imply that the parallel length scale of the instability (i.e. in the direction of the magnetic field) is of the order of the parallel mean free paths and insures that ion viscosity will not be deleted by the scaling procedure. To start with the reduction of the fluid equations, it is also necessary to choose a normalization of the amplitudes of the density, temperature, potential (Φ), and velocity fluctuations. We assume

$$e\Phi/T \sim n/N \sim t_J/T_J \sim \vec{u}_J/c_S \sim \lambda \qquad (10)$$

where $\lambda \to 0$ is the linearization parameter which will drop out from the (linear) analysis of the instability. We note also that the theory of the equilibrium shows that

$$\hat{p}\cdot\vec{V}_J \sim \mu^4 \; c_s, \qquad (11a)$$

$$\hat{b}\cdot\vec{V}_J = \frac{c}{B} \left(\frac{\hat{p}\cdot\vec{\nabla}P_J}{e_J N} + \hat{p}\cdot\vec{\nabla}V\right) \sim \mu^2 c_S \qquad (11b)$$

where $-\hat{p}\cdot\vec{\nabla}V$ is the equilibrium radial electric field that we shall omit in the following for it has no impact on the mode structure and growth rate (it merely yields a frequency shift). The equilibrium pressure P_J and density N depends only on the radial variable r, or more generally on the flux surface coordinate ψ (see later) up to order μ inclusive. Experimental evidence and theory further show that

$$\hat{n}\cdot\vec{V}_e = \frac{e_e \tau_e}{0.51\, m_e} E_\phi + O(\mu) \sim c_S \tag{12a}$$

$$\hat{n}\cdot\vec{V}_i \sim \mu c_S \tag{12b}$$

where E_ϕ is the driving toroidal electric field. In view of the scaling $a/R \sim \mu$, E_ϕ is independent from both ψ and χ (the generalized poloidal coordinate) in lowest order. Eq. (12a) permits also to scale this driving field.

With this information in hand one can show that the projection of the curl of the momentum equations along the direction $\hat{n}$ of the magnetic field are

$$m_i N \left(\frac{\partial}{\partial t} + \vec{V}_i\cdot\vec{\nabla}\right) \hat{p}\cdot\vec{\nabla}\, \hat{b}\cdot\vec{u}_i$$

$$= -\frac{\hat{b}\cdot\vec{\nabla}B}{B}\, \hat{p}\cdot\vec{\nabla}A_{0,i} - e_i\, \hat{p}\cdot\vec{\nabla}N\, \hat{b}\cdot\vec{\nabla}\phi - \frac{e_i B}{c}\Big[2N\, \hat{b}\cdot\vec{u}_i\, \frac{\hat{b}\cdot\vec{\nabla}B}{B} + \vec{\nabla}\cdot(N\vec{u}_i)$$

$$- \hat{n}\cdot\vec{\nabla}\, (N\hat{n}\cdot\vec{u}_i) + \vec{\nabla}\cdot(n\vec{V}_i)\Big] + 2(\hat{p}\cdot\vec{\nabla}\eta_{3,i})\, \hat{b}\cdot\vec{\nabla}\, \hat{p}\cdot\vec{\nabla}\, \hat{b}\cdot\vec{u}_i$$

$$+ \eta_{3,i}(\hat{p}\cdot\vec{\nabla})^2 \left(\frac{\partial n}{\partial t} + \frac{1}{N}\, \hat{p}\cdot\vec{u}_i\, \hat{p}\cdot\vec{\nabla}N + \frac{1}{N}\, \hat{b}\cdot\vec{V}_i\, \hat{b}\cdot\vec{\nabla}n - \hat{n}\cdot\vec{\nabla}\, \hat{n}\cdot\vec{u}_i\right)$$

$$+ \eta_{1,i}(p\cdot\vec{\nabla})^3\, \hat{b}\cdot\vec{u}_i \tag{13a}$$

The parallel momentum equations are

$$m_i N \left(\frac{\partial}{\partial t} + \vec{V}_i \cdot \vec{\nabla}\right) (\hat{n} \cdot \vec{u}_i)$$

$$= - \hat{n} \cdot \vec{\nabla} \left(p_i + e_i N \Phi - 0.71\ N t_e + \frac{2}{3} A_{0,i}\right)$$

$$+ 2\ (\hat{p} \cdot \vec{\nabla}\ \eta_{i,3})\ \hat{b} \cdot \vec{\nabla} (\hat{n} \cdot \vec{u}_i) + 4 \eta_{i,3} \frac{\hat{b} \cdot \vec{\nabla} B}{B}\ \hat{p} \cdot \vec{\nabla}\ (\hat{n} \cdot \vec{u}_i) + 2 \eta_{i,3}\ \hat{p} \cdot \vec{\nabla}\ \hat{n} \cdot \vec{\nabla}\ (\hat{b} \cdot \vec{u}_i)$$

$$+ 4\ \eta_{i,1}\ (\hat{p} \cdot \vec{\nabla})^2\ (\hat{n} \cdot \vec{u}_i) \qquad (19a)$$

and

$$0 = - \hat{n} \cdot \vec{\nabla}\ (p_e + e_e N \Phi + 0.71\ N t_e)$$

$$- 0.51 \frac{m_e}{\tau_e} \left[N\ \hat{n} \cdot (\vec{u}_e - \vec{u}_i) + n\ \hat{n} \cdot \vec{V}_e - \frac{3}{2} \frac{t_e}{T_e} N\ \hat{n} \cdot \vec{V}_e \right] - \frac{2}{3}\ \hat{n} \cdot \vec{\nabla} A_{0,e} \qquad (19b)$$

We have included terms of order μ in the last equation because the solution of the lowest order equation (first term = 0) is unknown up to a function independent from the poloidal variables θ or χ.

Finally the ion and electron heat equations are (the latter up to order μ):

$$\frac{3}{2} N \left[\left(\frac{\partial}{\partial t} + \vec{V}_i \cdot \vec{\nabla}\right) t_i + \vec{u}_i \cdot \vec{\nabla} T_i \right] - T_i \left[\left(\frac{\partial}{\partial t} + \vec{V}_i \cdot \vec{\nabla}\right) n + \vec{u}_i \cdot \vec{\nabla} N \right]$$

$$+ \frac{5}{2} \frac{c T_i}{e_i B}\ \hat{n} \cdot \left(\vec{\nabla} T_i \times \vec{\nabla} n - \vec{\nabla} N \times \vec{\nabla} t_i - 2 \vec{\nabla} t_i \times \frac{\vec{\nabla} B}{B} N\right)$$

$$= \hat{n} \cdot \vec{\nabla}\ (\chi_{\parallel,i} \hat{n} \cdot \vec{\nabla} t_i) + \chi_{\perp,i} (\hat{p} \cdot \vec{\nabla})^2 t_i \qquad (20a)$$

and

$$0 = -\frac{c}{B}\,\hat{p}\cdot\vec{\nabla}N\;\hat{b}\cdot\vec{\nabla}\phi - \Big[2N\;\hat{b}\cdot\vec{u}_e\,\frac{\hat{b}\cdot\vec{\nabla}B}{B} + \vec{\nabla}\cdot(n\vec{u}_e) - \hat{n}\cdot\vec{\nabla}(N\hat{n}\cdot\vec{u}_e)$$

$$+ \vec{\nabla}\cdot(n\vec{V}_e) - \hat{n}\cdot\vec{\nabla}\;(n\;\hat{n}\cdot\vec{V}_e) \qquad (13b)$$

Here, $A_{0,i}$ is defined by

$$A_{0,i} = -3\eta_{0,i}\;\Big[\hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}_i) - \frac{\hat{b}\cdot\vec{\nabla}B}{B}\;\hat{b}\cdot\vec{u}_i - \frac{1}{3}\,\vec{\nabla}\cdot\vec{u}_i\Big] \qquad (14)$$

and $\eta_{0,i}$, $\eta_{1,i}$ and $\eta_{3,i}$ the three viscosity coefficients defined by Braginskii (1965):

$$\eta_{0,i} = 0.96\;P_i\tau_i \quad ; \quad \eta_{1,i} = \frac{3P_i}{10\Omega_i^2\tau_i} \quad ; \quad \eta_{3,i} = \frac{P_i}{2\;\Omega_i} \qquad (15)$$

The velocity components $\hat{b}\cdot\vec{u}_J$ and $\hat{p}\cdot\vec{u}_J$ are given by

$$1 + \frac{1}{2}\,a_j^2\;(\hat{p}\cdot\vec{\nabla})^2\;\begin{Bmatrix}\hat{p}\cdot\vec{u}_J\\ \\ \hat{b}\cdot\vec{u}_J\end{Bmatrix} = \frac{c}{B\;e_J\;N}\;\begin{Bmatrix}-\hat{b}\cdot\vec{\nabla}\\ \\ \hat{p}\cdot\vec{\nabla}\end{Bmatrix}\;(p_J - \frac{1}{3}\,A_{0,J} + e_J N\Phi) \qquad (16)$$

We also note that in lowest order the terms $m_i N\;\vec{V}_i\cdot\vec{\nabla}\;\hat{p}\cdot\vec{\nabla}\;\hat{b}\cdot\vec{u}_i$ and $2(\hat{p}\cdot\vec{\nabla}\;\eta_{3,i})\;\hat{b}\cdot\vec{\nabla}\;\hat{p}\cdot\vec{\nabla}\;\hat{b}\cdot\vec{u}_i$ cancel in Eq. (13a). Further considerations on the viscosity tensor are given in Appendix. The typical time scale is ω^*:

$$\frac{\partial}{\partial t} = -\,i\omega \sim -i\omega^* \qquad (17)$$

and the divergence of the flow is given by

$$\vec{\nabla}\cdot\vec{u}_J = -\frac{1}{N}\,\vec{u}_J\cdot\vec{\nabla}N - \frac{1}{N}\,\vec{V}_J\cdot\vec{\nabla}n - \frac{\partial n}{\partial t} \sim \mu^3\Omega_i \qquad (18)$$

since, according to (16), $\hat{b}\cdot\vec{u}_J \sim \lambda c_S$, but $\hat{p}\cdot\vec{u}_J \sim \mu\lambda c_S$.

and

$$\frac{3}{2} N \left[(\frac{\partial}{\partial t} + \vec{V}_e \cdot \vec{\nabla}) t_e + \vec{u}_e \cdot \vec{\nabla} T_e \right] - T_e \left[(\frac{\partial}{\partial t} + \vec{V}_e \cdot \vec{\nabla}) n + \vec{u}_e \cdot \vec{\nabla} N) \right]$$

$$+ \frac{5}{2} \frac{cT_e}{e_e B} \hat{n} \cdot (\vec{\nabla} T_e \times \vec{\nabla} n - \vec{\nabla} N \times \vec{\nabla} t_e - 2\vec{\nabla} t_e \times \frac{\vec{\nabla} B}{B} N)$$

$$= B \, \hat{n} \cdot \vec{\nabla} \{ \frac{1}{B} \chi_{\|,e} \, \hat{n} \cdot \vec{\nabla} t_e - 0.71 \frac{1}{B} \left[P_e \, (\vec{u}_e - \vec{u}_i) \cdot \hat{n} + p_e \vec{V}_e \cdot \hat{n} \right] \} \tag{20b}$$

where the parallel and perpendicular heat conduction coefficients are given by

$$\chi_{\|,i} = 3.9 \frac{P_i \tau_i}{m_i} \quad , \quad \chi_{\perp,i} = 2 \frac{P_i}{m_i \Omega_i^2 \tau_i} \quad , \tag{21a}$$

$$\chi_{\|,e} = 3.16 \frac{P_e \tau_e}{m_e} \tag{21b}$$

Equations (13) through (21) represent a considerable simplification over Bragisnkii two-fluid equations. They are still very general however; it has, for example, been shown previously (Rogister, 1984) that they can describe both the rippling mode and the collisional drift mode. A more restricted scaling, and further simplifications, are thus given in the next section along with a specific representation of the mode structure and of the $\vec{\nabla}$ operator.

III. MODE REPRESENTATION AND REDUCED SCALING

In general axisymmetric geometry, the modes can be represented by series of the form (Rogister and Hasselberg, 1985)

$$f(\psi,\chi,\phi) = \sum_{\ell} \exp(i\ell\phi) \; f_{\ell}(\psi,\chi) \quad , \tag{22a}$$

$$f_{\ell}(\psi,\chi) = \sum_{m} \exp \left[-i\ell \int_{\chi_o}^{\chi} \nu(\chi',\psi_{\ell,m}) d\chi' \right] f_{\ell,m}(\psi-\psi_{\ell,m},\psi_1,\chi) \tag{22b}$$

if the "quasi-modes" $f_{\ell,m}$ are localized in the vicinity of the rational surfaces $\Psi_{\ell,m}$ defined by

$$q(\psi_{\ell,m}) = \oint \nu(\chi,\psi_{\ell,m})\, d\chi/2\pi = -m/\ell, \tag{23}$$

where the m's are integers akin to poloidal mode numbers; ℓ is the toroidal mode number. Hence the introduction of two "length" scales $\psi-\psi_{\ell,m}$ and ψ_1. (Connor et al. (1979) have shown that ψ_1 enters the theory only in higher orders; the ψ_1 dependence will be of no interest here.). ψ, χ, and ϕ are respectively the flux surface coordinate, the generalized poloidal variable (also suited for arbitrary cross sections), and the toroidal angle. The corresponding gradient operator can take the different forms.

$$\begin{aligned}
\vec{\nabla} &\equiv \hat{e}_\psi h_\psi^{-1} \frac{\partial}{\partial\psi} + \hat{e}_\chi h_\chi^{-1} \frac{\partial}{\partial\chi} + \hat{e}_\chi h_\phi^{-1} \frac{\partial}{\partial\phi} \\
&= \hat{e}_\psi\, RB_\chi \frac{\partial}{\partial\psi} + \hat{e}_\chi \frac{1}{JB_\chi} \frac{\partial}{\partial\chi} + \frac{1}{R}\frac{\partial}{\partial\phi} \\
&= \hat{p}\, h_\psi^{-1} \frac{\partial}{\partial\psi} + \hat{b}\left(\frac{B_\phi}{B} h_\chi^{-1} \frac{\partial}{\partial\chi} - \frac{B_\chi}{B} h_\phi^{-1} \frac{\partial}{\partial\phi}\right) + \hat{n}\left(\frac{B_\chi}{B} h_\chi^{-1} \frac{\partial}{\partial\chi} + \frac{B_\phi}{B} h_\phi^{-1} \frac{\partial}{\partial\phi}\right)
\end{aligned} \tag{24}$$

where J is the Jacobian of the transformation. We recall that

$$\nu \equiv B_\phi h_\chi \;/\; B_\chi h_\phi \tag{25}$$

We introduce the refined scaling

$$a_i\; \hat{p}\cdot\vec{\nabla}\, \ell n\, f_{\ell,m} \sim \mu^{1/3} \tag{26}$$

and obtain

$$\hat{n}\cdot\vec{\nabla}\left[\exp\ (i\ell\phi)\ f_\ell(\psi,\chi)\right] = \sum_m \exp\ \left\{ i\ell\left[\phi-\int_{\chi_o}^{\chi} \nu(\chi',\psi_{\ell,m})d\chi'\right]\right\} \frac{B_\chi}{B}\ h_\chi^{-1}$$

$$\times\ \left\{ \frac{\partial}{\partial\chi} + i\ell\ \left[\nu(\chi,\psi) - \nu(\chi,\psi_{\ell,m})\right]\right\}\ f_{\ell,m}(\psi-\psi_{\ell,m},\chi) \tag{27a}$$

$$1; \qquad\qquad \mu^{2/3};$$

$$\hat{b}\cdot\vec{\nabla}\left[(\exp(i\ell\phi)\ f_\ell(\psi,\chi)\right] = -\sum_m \exp\ \left\{i\ell\ \left[\phi-\int_{\chi_o}^{\chi} \nu(\chi',\psi_{\ell,m})d\chi\right]\right\} \frac{B_\phi}{B}\ h_\chi^{-1}$$

$$\times\ \left\{ i\ell\ \frac{B^2}{B_\phi^2}\ \nu(\psi,\chi) - \frac{\partial}{\partial\chi} - i\ell\left[\nu(\chi,\psi) - \nu(\chi,\psi_{\ell,m})\right]\right\}\ f_{\ell,m}(\psi-\psi_{\ell,m},\chi)$$

$$1; \qquad\qquad \mu; \qquad\qquad \mu^{5/3};$$

where we have indicated the relative order of the various contributions considering that $h_\psi(\psi-\psi_{\ell,m}) \sim \mu^{-1/3}\ a_i$, as implied by (26). We shall introduce later the notations

$$k_\beta \equiv -\ell\nu(\chi,\psi)\ \frac{B}{B_\phi}\ h_\chi^{-1} \tag{28a}$$

$$k_{||} \equiv \ell(\psi-\psi_{\ell,m})\ \frac{\partial\nu(\chi,\psi)}{\partial\psi}\ \frac{B_\chi}{B}\ h_\chi^{-1} \tag{28b}$$

Noting that the distance between neighbouring rational surfaces is

$$\Delta_\ell \equiv h_\psi(\psi_{\ell,m+1} - \psi_{\ell,m}) = -2\pi/\ell h_\psi^{-1} \oint d\chi\ \partial\nu(\chi,\psi)/\partial\psi \sim \mu^{-1}\ a_i$$

$$\gg (\hat{p}\cdot\vec{\nabla}\ \ell n\ f_{\ell,m})^{-1} \tag{29}$$

we conclude that the toroidal coupling between the quasi-modes ..., $(\ell,m+1)$, (ℓ,m), ... is negligible (no overlap): we are therefore entitled to consider the various terms in the series (22b) independently.

We now rewrite the two-fluid equations, simplified according to (26), up to the order requested by the solution procedure, and indicate the relative order of magnitude of the different terms. We obtain

$$m_i N \frac{\partial}{\partial t}\, \hat{p}\cdot\vec{\nabla}\, \hat{b}\cdot\vec{u}_i = \frac{e_i B}{c}\left[\frac{\partial n}{\partial t} + N\, \hat{n}\cdot\vec{\nabla}\,(\hat{n}\cdot\vec{u}_i) - 2N\hat{b}\cdot\vec{u}_i\, \frac{\hat{b}\cdot\vec{\nabla}B}{B}\right]$$

$$\mu^{2/3}; \qquad 1; \qquad 1; \qquad \mu^{1/3};$$

$$-e_i \hat{p}\cdot\vec{\nabla}N\, \hat{b}\cdot\vec{\nabla}\phi - \frac{\hat{b}\cdot\vec{\nabla}B}{B}\, \hat{p}\cdot\vec{\nabla}A_{0,i} - \eta_3\, (\hat{p}\cdot\vec{\nabla})^2 \left[\vec{\nabla}\cdot\vec{u}_i + \hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}_i)\right], \tag{30}$$

$$1; \qquad \mu^{1/3}; \qquad \mu^{2/3}$$

$$0 = \frac{\partial n}{\partial t} + N\, \hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}_e) + \hat{n}\cdot\vec{V}_e\, \hat{n}\cdot\vec{\nabla}n - 2N\hat{b}\cdot\vec{u}_e\, \frac{\hat{b}\cdot\vec{\nabla}B}{B} - \frac{c}{B}\, \hat{p}\cdot\vec{\nabla}N\hat{b}\cdot\vec{\nabla}\Phi\,, \tag{31}$$

$$1; \qquad 1; \qquad 1; \qquad \mu^{1/3}; \qquad 1;$$

$$m_i N \frac{\partial}{\partial t}\, \hat{n}\cdot\vec{u}_i - 2\,\frac{P_i}{\Omega_i}\,\frac{\hat{b}\cdot\vec{\nabla}B}{B}\, \hat{p}\cdot\vec{\nabla}\, \hat{n}\cdot\vec{u}_i = -\hat{n}\cdot\vec{\nabla}\,\left(p_i + \frac{2}{3}A_{0,i} + e_i N\Phi - 0.71\, Nt_e\right), \tag{32}$$

$$1; \qquad \mu^{1/3}; \qquad 1; \qquad 1; \qquad 1; \qquad 1;$$

$$0 = \hat{n}\cdot\vec{\nabla}\,\left(p_e + \frac{2}{3}A_{0,e} + e_e N\Phi + 0.71\, Nt_e\right) + 0.51\,\frac{m_e}{\tau_e}\left[N\,\hat{n}\cdot(\vec{u}_e - \vec{u}_i) + n\,\hat{n}\cdot\vec{V}_e\right.$$

$$1; \qquad \mu; \qquad 1; \qquad 1; \qquad \mu; \qquad \mu;$$

$$\left. - \frac{3}{2}\,\frac{t_e}{T_e}\, N\, \hat{n}\cdot\vec{V}_e\right] \tag{33}$$

$$\mu;$$

$$A_{0,i} = -3\eta_{0,i}\left[\hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}_i) - \frac{\hat{b}\cdot\vec{\nabla}B}{B}\, \hat{b}\cdot\vec{u}_i - \frac{1}{3}\,\vec{\nabla}\cdot\vec{u}_i\right]$$

$$1; \qquad \mu^{1/3}; \qquad 1;$$

$$\begin{Bmatrix} \hat{p}\cdot\vec{u}_J \\ \hat{b}\cdot\vec{u}_J \end{Bmatrix} = \frac{c}{B\, e_J\, N} \begin{Bmatrix} -\hat{b}\cdot\vec{\nabla} \\ \hat{p}\cdot\vec{\nabla} \end{Bmatrix} \left(p_J - \frac{1}{3}A_{0,J}\,\delta_{iJ} + e_J N\Phi\right)$$

$$1; \qquad 1; \qquad 1;$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}\frac{t_i}{T_i}-\frac{n}{N}\right)-\frac{cT_i}{e_iB}\frac{\hat{p}\cdot\vec{\nabla}N}{N}\left(\frac{3}{2}\eta_i-1\right)\hat{b}\cdot\vec{\nabla}\frac{e_i\phi}{T_i}+\frac{1}{3}\frac{cT_i}{e_iB}\frac{\hat{p}\cdot\vec{\nabla}N}{N}\left(\frac{3}{2}\eta_i-1\right)b\cdot\vec{\nabla}\frac{A_{0,i}}{NT_i}$$

$1;\quad 1;\qquad\qquad 1;\qquad\qquad 1;$

$$-5\frac{cT_i}{e_iB}\frac{\hat{b}\cdot\vec{\nabla}B}{B}\hat{p}\cdot\vec{\nabla}\frac{t_i}{T_i}=\hat{n}\cdot\vec{\nabla}\left(\frac{\chi_{\|,i}}{N}\hat{n}\cdot\vec{\nabla}\frac{t_i}{T_i}\right) \qquad (35)$$

$\mu^{1/3};\qquad\qquad 1;$

$$\left(\frac{\partial}{\partial t}+\hat{n}\cdot\vec{V}_e\,\hat{n}\cdot\vec{\nabla}\right)\left(\frac{3}{2}\frac{t_e}{T_e}-\frac{n}{N}\right)-\frac{cT_e}{e_eB}\frac{\hat{p}\cdot\vec{\nabla}N}{N}\left(\frac{3}{2}\eta_e-1\right)\hat{b}\cdot\vec{\nabla}\frac{e_e\phi}{T_e}-5\frac{cT_e}{e_eB}\frac{b\cdot\vec{\nabla}B}{B}\hat{p}\cdot\vec{\nabla}\frac{t_e}{T_e}$$

$\mu;\quad \mu;\qquad\qquad \mu;\qquad\qquad \mu^{4/3}$

$$=B\,\hat{n}\cdot\vec{\nabla}\left\{\frac{1}{B}\frac{\chi_{e,\|}}{N}\hat{n}\cdot\vec{\nabla}\frac{t_e}{T_e}-0.71\frac{1}{B}\left[\hat{n}\cdot(\vec{u}_e-\vec{u}_i)+\frac{p_e}{P_e}\hat{n}\cdot\vec{V}_e\right]\right\} \qquad (37)$$

$1;\qquad\qquad \mu;\qquad \mu;$

where $\eta_J=\hat{p}\cdot\vec{\nabla}\ell nT_J\,/\,\hat{p}\cdot\vec{\nabla}\ell nN$.

The solution of these equations order by order is tedious but straightforward. We shall thus proceed, in the next section, as briefly as possible by omitting detailed comments.

IV. SOLUTION OF THE TWO-FLUID EQUATIONS

We first recall that $\hat{n}\cdot\vec{\nabla}\ell nf=(JB)^{-1}\partial/\partial\chi+ik_{\|}$ where the second contribution is of order $\mu^{2/3}$ with respect to the first one. The parallel electron momentum equation, Eq. (33), thus reads at order μ^{o}:

$$\frac{1}{JB}\frac{\partial}{\partial\chi}\left(p_e^{(o)}+e_eN\Phi^{(o)}+0.71\,Nt_e^{(o)}\right)=0$$

where we now define $f = f^{(o)} + \mu^{1/3} f^{(1)} + \ldots$ The constant of integration which will appear here is obtained by considering the χ average (defined by $\oint JB\, d\chi$) of Eq. (33) at order $\mu^{2/3}$. Hence

$$p_e^{(o)} + e_e N\Phi^{(o)} + 0.71\, Nt_e^{(o)} = 0 \tag{38}$$

Applying the same procedure to the electron heat equation, we obtain

$$\frac{1}{JB}\frac{\partial}{\partial\chi} t_e^{(o)} = 0 \tag{39}$$

(the constant of integration which should appear on the right-hand side must vanish because of periodicity requirements) at order μ^o, and

$$\oint JB\, d\chi \left[\frac{\partial}{\partial t_o}\left(\frac{3}{2} Nt_e^{(o)} - n^{(o)}\, T_e\right) - i\omega_e^* \left(\frac{3}{2}\eta_e - 1\right) e_e N\Phi^{(o)}\right] = 0$$

at order μ; we have made use of the identity $(JB)^{-1}\, \partial V_{\|,e}/\partial\chi = 0$ which holds (up to order μ) because of the $a/R \sim \mu$ expansion (Eq. 5). Here

$$\omega_e^* = k_\beta \frac{cT_e}{e_e B}\frac{\hat{p}\cdot\vec{\nabla}N}{N} = -\ell\, \frac{cT_e}{e_e}\frac{\partial N/\partial\psi}{N} \tag{40}$$

is the electron diamagnetic frequency which is rigorously independent from χ. From Eq. (31) we obtain at order μ^o:

$$\oint JB\, d\chi \left(\frac{\partial}{\partial t_o} n^{(o)} - i\omega_e^* N \frac{e_e\Phi^{(o)}}{T_e}\right) = 0$$

which closes the system of lowest order equations. Combining these results, we obtain

$$\left[\frac{\partial}{\partial t_o} + i\omega_e^* (1 + 1.71\ \eta_e)\right] \oint JB\, d\chi\ n^{(o)} = 0 \tag{41}$$

and

$$\oint JB\, d\chi\ (Nt_e^{(o)} - \eta_e n^{(o)} T_e) = 0 \tag{42a}$$

or, alternatively

$$\oint JB\, d\chi\ \left[(1 + 1.71\ \eta_e)\ n^{(o)} T_e + e_e N\Phi^{(o)}\right] = 0. \tag{42b}$$

Equations (38), (39), (41) and (42a) or (42b) yield the lowest order solution. Equation (41), together with the definition (40), is the corresponding dispersion relation.

Consider now Eq. (33) at orders $\mu^{1/3}$ and μ. It yields:

$$\frac{1}{JB}\frac{\partial}{\partial\chi}\ (p_e^{(1)} + e_e N\Phi^{(1)} + 0.71\ Nt_e^{(1)}) = 0$$

and

$$(p_e^{(1)} + e_e N\Phi^{(1)} + 0.71\ Nt_e^{(1)}) + i\ \frac{3\text{x}0.51}{2}\ \frac{m_e V_{\|,e}}{k_{\|} T_e \tau_e}\ Nt_e^{(o)}$$

$$= i\ 0.51\ \frac{m_e}{k_{\|}\tau_e} \oint JB\, d\chi\ \left[N(u_{\|,e}^{(o)} - u_{\|,i}^{(o)}) + n^{(o)} V_{\|,e}\right] \Big/ \oint JB\, d\chi \tag{43}$$

where $u_{\|} = \hat{n}\cdot\vec{u}$. The electron heat equation in turn yields

$$\frac{1}{JB}\frac{\partial}{\partial\chi}\ t_e^{(1)} = 0 \tag{44}$$

and

$$\oint JB\, d\chi\ \left[\frac{\partial}{\partial t_o}\ (\tfrac{3}{2}\ Nt_e^{(1)} - n^{(1)} T_e) - i\omega_e^*\ (\tfrac{3}{2}\ \eta_e - 1)\ e_e N\Phi^{(1)} + \frac{\partial}{\partial t_1}\ (\tfrac{3}{2}\ Nt_e^{(o)} - n^{(o)} T_e)\right.$$

$$\left. + k_{\|}^2\ \chi_{\|,e} t_e^{(o)}\right] = 0$$

From Eq. (31) at order $\mu^{1/3}$ we obtain:

$$\oint JB d\chi\ \left(\frac{\partial}{\partial t_o}\ n^{(1)} + \frac{\partial}{\partial t_1}\ n^{(o)} - i\omega_e^*\ N\ \frac{e\phi^{(1)}}{T_e}\right) = 0, \tag{45}$$

which, combined with the previous equation, yields

$$\oint JB\, d\chi \left[\frac{\partial}{\partial t_o} (Nt_e^{(1)} - \eta_e n^{(1)} T_e) + \frac{2}{3} k_\|^2 \chi_{\|,e} t_e^{(o)}\right] = 0 \tag{46}$$

Eliminating further $\Phi^{(1)}$ and $t_e^{(1)}$ from Eq. (43) and removing the secularity from the result (see, e.g., Frieman and Book, 1963) leads to

$$\left[\frac{\partial}{\partial t_o} + i\omega_e^* (1+1.71\eta_e)\right] \oint JB\, d\chi\, n^{(1)} = 0$$

and

$$\left(\frac{\partial}{\partial t_1} + \frac{2}{3} \frac{1.71\eta_e}{1+1.71\eta_e} k_\|^2 \chi_{\|,e}\right) \oint JB\, d\chi\, n^{(o)}$$

$$+ 0.51\, \omega_e^* \frac{m_e}{k_\| \tau_e T_e} \oint JB\, d\chi \left[N\, (u_{\|,e}^{(o)} - u_{,i}^{(o)}) + (1 - \frac{3}{2}\eta_e)\, n^{(o)}\, V_{\|,e}\right] = 0 \tag{47}$$

It remains to eliminate $\oint JB\, d\chi\, (u_{\|,e}^{(o)} - u_{\|,i}^{(o)})$. This quantity enters in Eq. (33) which has already been used up and in the difference between (31) and (30). We thus obtain

$$\frac{1}{JB} \frac{\partial}{\partial \chi} \left[N(u_{\|,e}^{(o)} - u_{\|,i}^{(o)}) + n^{(o)}\, V_{\|,e}\right] = 0 \tag{48}$$

in lowest order (u^o) and

$$\frac{1}{\Omega_i} \frac{\partial}{\partial t_o} (\hat{p}\cdot\vec{\nabla})^2 \oint JB\, d\chi\, (p_i^{(o)} + e_i N\Phi^{(o)} - \frac{1}{3} A_{0,i}^{(o)})$$

$$= \frac{e_e B}{c}\, ik_\| \left[N(u_{\|,e}^{(o)} - u_{\|,i}^{(o)}) + n^{(o)}\, V_{\|,e}\right] \oint JB\, d\chi$$

$$- \hat{p}\cdot\vec{\nabla} \oint JB\, d\chi\, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[2\, (p_i^{(1)} + e_i N\Phi^{(1)}) + \frac{1}{3} A_{0,i}^{(1)} \right]$$

$$- \eta_{3,i} (\hat{p}\cdot\vec{\nabla})^2 \oint JB\, d\chi\, (\vec{\nabla}\cdot\vec{u}_i)^{(o)} \quad ;$$

at order $\mu^{2/3}$. We have noted that

$$e_e N \oint JB\, d\chi\, \frac{B}{c} \frac{\hat{b}\cdot\vec{\nabla}B}{B} u_e^{(o,1)}\cdot\hat{b} = - \oint d\chi\, \frac{B-<B>}{B_\chi}\, \hat{p}\cdot\vec{\nabla}\frac{\partial}{\partial\chi}\, (p_e^{(0,1)} + e_e N\Phi^{(0,1)})$$

$$= 0 \tag{50}$$

in view of $a/R \sim \mu$ [Eq. (5)] on the one hand, and of (43) and (44), or (38) and (39) on the other hand. Here $<B> = \oint JB^2 d\chi / \oint JB d\chi$.

Equation (49) shows that one must now consider systematically the ion dynamics. The latter is complicated by the parallel viscosity [The $A_{o,i}$ term in Eq. (35)]. The poloidal variation of $t_i^{(o)}$, $A_{0,i}^{(o)}$, and $n^{(o)}$ results from the following equations: (i) the ion heat equation at order μ^o:

$$\left[\frac{3}{2}\, i + \frac{m_i \chi_{\|,i} \omega_o}{P_i} \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2 \right] Nt_i^{(o)}$$

$$= i \left[n^{(o)}\, T_i + \left(\frac{3}{2}\eta_i - 1\right) \frac{\omega_i^*}{\omega_o} \left(e_e N\Phi^{(o)} + \frac{1}{3} A_{0,i}\right) \right] \; ; \tag{51}$$

(ii) a proper combination of the parallel ion momentum equation and of the ion continuity equation [which yields $(\vec{\nabla}\cdot\vec{u}_i)^{(o)}$]:

$$\left[\left(i\, \frac{P_i}{\omega_o \eta_{o,i}} - \frac{1}{3}\frac{\omega_i^*}{\omega_o}\right) + 2\, \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2 \right] A_{o,i}^{(o)}$$

$$= \frac{\omega_i^*}{\omega_o} \left[(1+1{,}71\eta_e)\, n^{(o)} T_e + e_e N\Phi^{(o)} - (Nt_i^{(o)} - \eta_i n^{(o)} T_i) \right]$$

$$- 3\, \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2 (n^{(o)}T_i + Nt_i^{(o)} + e_i N\Phi^{(o)}) \quad ; \tag{52}$$

and (iii) a straightforward consequence of Eqs. (30) and (32):

$$(n^{(o)} + \frac{\omega_e^*}{\omega_o} N \frac{e_e\Phi^{(o)}}{T_e} + (\frac{c_i}{JB\omega_o} \frac{\partial}{\partial\chi})^2 \left[n^{(o)}(1 + \frac{T_e}{T_i}) + N \frac{t_i^{(o)}}{T_i} + \frac{2}{3}\frac{A_{0,i}^{(o)}}{T_i}\right]. \tag{53}$$

To complete the set of lowest order equations, we note that the relation

$$e_e N\Phi^{(o)} = -n^{(o)}T_e - 1.71\ \eta_e \oint JBd\chi n^{(o)}\ T_e \ / \oint JBd\chi \tag{54}$$

follows from (38, 39) and (42a). These relations can be combined to yield a sixth order ordinary differential equation of the form

$$\sum_{k=1}^{3} \alpha_{2k} \frac{\partial^{2k}}{\partial\chi^{2k}} n^{(o)} = n^{(o)} + \frac{\omega_e^*}{\omega_o} e_e N \frac{\Phi^{(o)}}{T_e}$$

$$= \frac{\omega_e^*}{\omega_o}(n^{(o)} - \langle n^{(o)}\rangle) \quad .$$

Since the solutions of the homogeneous equations do not have the required periodicity, we conclude that

$$n^{(o)} = \langle n^{(o)}\rangle \quad , \tag{55a}$$

$$e_e\Phi^{(o)}/T_e = e_e\langle\Phi^{(o)}\rangle/T_e = -(\omega_o/\omega_e^*)n^{(o)}/N \quad , \tag{55b}$$

$$t_i^{(o)}/T_i = \langle t_i^{(o)}\rangle/T_i = \eta_i n^{(o)}/N, \tag{55c}$$

$$A_{0,i}^{(o)} = 0 \tag{55d}$$

$$t_e^{(o)}/T_e = \langle t_e^{(o)}\rangle/T_e = \eta_e n^{(o)}/N \quad . \tag{55e}$$

We note also that $u_{\|,i}^{(o)} = (\vec{\nabla}\cdot\vec{u}_i)^{(o)} = 0$.

We now proceed to eliminate $t_i^{(1)}$, $A_{0,i}^{(1)}$, $n^{(1)}$, and $\Phi^{(1)}$ from Eq. (49). The ion heat equation yields at order $\mu^{1/3}$.

$$\oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[-\frac{3}{2} i + \frac{m_i \chi_{\|,i}\omega_o}{P_i} \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2\right] Nt_i^{(1)}$$

$$= i \oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[n^{(1)} T_i + \left(\frac{3}{2}\eta_i - 1\right) \frac{\omega_i^*}{\omega_o} \left(e_e N\Phi^{(1)} + \frac{1}{3} A_{0,i}^{(1)}\right)\right]$$

$$- 5 \frac{cT_i}{e_i B} \frac{1}{\omega_o} \hat{p}\cdot\vec{\nabla} \oint JBd\chi \left(\frac{\hat{b}\cdot\vec{\nabla}B}{B}\right)^2 Nt_i^{(o)} . \tag{56}$$

A proper combination of the parallel ion momentum equation and of the ion continuity equation at order $\mu^{1/3}$ shows that

$$\oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[\left(i \frac{P_i}{\omega_o \eta_{o,i}} - \frac{1}{3}\frac{\omega_i^*}{\omega_o}\right) + 2 \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2\right] A_{0,i}^{(1)}$$

$$= \oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left\{\left[\left(- n^{(1)}T_i + \frac{\omega_i^*}{\omega_o} e_e N\Phi^{(1)}\right) - \frac{\omega_i^*}{\omega_o} \left(Nt_i^{(1)} - \eta_i n^{(1)} T_i\right)\right]\right.$$

$$\left. - 3 \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2 \left(n^{(1)}T_i + Nt_i^{(1)} + e_i N\Phi^{(1)}\right)\right\} \tag{57}$$

From Eq. (30) at order $\mu^{1/3}$, one obtains after elimination of $u_{\|,i}^{(1)}$ (the latter is given by the parallel ion momentum equation):

$$\oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[1 + \left(1 + \frac{T_i}{T_e}\right) \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2\right] n^{(1)}$$

$$= N \oint JBd\chi \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[-\frac{\omega_e^*}{\omega_o}\frac{e_e\Phi^{(1)}}{T_e} - \left(\frac{c_i}{JB\omega_o}\frac{\partial}{\partial\chi}\right)^2 \left(\frac{t_i^{(1)}}{T_i} + \frac{2}{3}\frac{A_{0,i}^{(1)}}{NT_i}\right)\right]$$

$$+ 2 \frac{i}{\omega_o} \frac{cT_i}{e_i B} \hat{p}\cdot\vec{\nabla} \oint JBd\chi \left(\frac{\hat{b}\cdot\vec{\nabla}B}{B}\right)^2 \frac{1}{T_i} \left(p_i^{(o)} + e_i N\Phi^{(o)}\right) \tag{58}$$

Instead of Eq. (57), one may prefer

$$\oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left(i \frac{P_i}{\omega_o \eta_{o,i}} - \frac{1}{3}\frac{\omega_i^*}{\omega_o}\right) A_{0,i}^{(1)}$$

$$= \oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[2 \frac{1.71\eta_e}{1+1.71\eta_e} n^{(1)}T_i - \frac{\omega_i^*}{\omega_o} (Nt_i^{(1)} - \eta_i n^{(1)} T_i)\right]$$

$$- 6 \frac{i}{\omega_o} \frac{cT_i}{e_i B} \hat{p}\cdot\vec{\nabla} \oint JBd\chi \left(\frac{\hat{b}\cdot\vec{\nabla}B}{B}\right)^2 (p_i^{(o)} + e_i N\Phi^{(o)}) \qquad (59)$$

which is a linear combination of (57) and (58). As concerns the operator $(c_i/JB\omega_o)\partial/\partial\chi$, we note that for circular cross-sections we have the identity

$$\oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \frac{\partial^2}{\partial\chi^2} f = - \oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} f \qquad (60)$$

if χ is identified with the poloidal angle. We note also that, according to Eqs. (43) and (44):

$$e_e N \oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \Phi^{(1)} = - \oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} n^{(1)} T_e \quad . \qquad (61)$$

We now combine Eqs. (47) and (49) to obtain

$$\left(\frac{\partial}{\partial t_1} + \frac{2}{3} \frac{1.71\eta_e}{1+1.71\eta_e} k_\parallel^2 \chi_{\parallel,e} - \frac{3}{2}\eta_e \, 0.51 \frac{V_{\parallel,e}\omega_e^*}{k_\parallel \tau_e c_e^2}\right) n^{(o)} \; T_i$$

$$+ 0.51 \frac{\omega_e^* \omega_o}{k_\parallel^2 \tau_e c_e^2} \left[(1+\eta_i)T_i + (1 + 1.71\eta_e) \; T_e\right] a_i^2 \, (\hat{p}\cdot\vec{\nabla})^2 \, n^{(o)}$$

$$= -0.51 \; i \frac{\omega_e^* c_i}{k_\parallel^2 \tau_e c_e^2} a_i \; \hat{p}\cdot\vec{\nabla} \oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[2 \, (p_i^{(1)} + e_i N\Phi^{(1)}) + \frac{1}{3} A_{0,i}^{(1)}\right] / \oint JBd\chi \quad (62$$

Equations (56), (58-62) together with (55a-e) form a complete set. They are still cumbursome to solve. We shall thus consider in the next section the various asymptotic limits.

V. MODE STRUCTURE AND GROWTH RATE IN THE VARIOUS ASYMPTOTIC LIMITS

The first order response obtained in section IV is characterized by only two independent dimensionaless parameters: $\omega_o \tau_i$ and $(c_i/qR\omega_o)^2$. We shall thus consider the following asymptotic limits.

(A) $\omega_o \tau_i << 1 \quad , \quad (c_i/qR\omega_o)^2 \sim (c_i/R\omega_o)^2 \sim 1$

(B) $\omega_{o\ i} \gtrsim 1 \quad , \quad (c_i/qR\omega_o)^2 << 1 \sim (c_i/R\omega_o)^2$

We justify this choice as follows. Case (A) allows one to treat the transition between the essentially cylindrical solution $[(c_i/R\omega_o)^2<<1]$ and the magnetic drift dominated solution $[(c_i/R\omega_o)^2>>1]$ on the one hand, and the transition between the regimes where the ions are able $[c_i/qR\omega_o)^2>>1]$ or unable $[c_i/qR\omega_o)^2<<1]$ to travel a full connection length qR in one wave period on the other hand. Case (B) allows to study the transition between the non-viscous ($\omega_o\tau_i<<1$) and viscous ($\omega_o\tau_i>>1$) solutions in a regime where the mean free path is still smaller than the connection length ($c_i\tau_i/qR<<1$). Since we know that wave damping prevails in the cylindrical limit (Rogister, 1984), we have allowed here $(c_i/R\omega_o)^2 \sim 1$, implying that q^2 is considered as a large number; this procedure is justified to the extent that $q \gtrsim 2$ in the plasma boundary of a Tokamak. We note finally that the dimensionless parameter $(c_i/R\omega_o)^2$ will enter via the "driving" terms in Eqs. (56, 58, 59).

(A) $\omega_o\tau_i << 1; \quad (c_i/qR\omega_o)^2 \sim (c_i/R\omega_o)^2 \sim 1.$

The solution of Eqs. (56, 58, 59) is:

$$\oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[\frac{A^{(1)}_{o,i}}{P_i} \; ; \; \frac{t^{(1)}_i}{T_i} \; ; \; \frac{n^{(1)}}{N}\right] = i \, \frac{cT_i}{e_i B} \, \frac{1}{\omega_o} \, \frac{\alpha_k}{\beta} \, \hat{p}\cdot\vec{\nabla} \oint JBd\chi \, \left(\frac{\hat{b}\cdot\vec{\nabla}B}{B}\right)^2 \frac{n^{(0)}}{N} \qquad (63)$$

where

$$\alpha_{A_i} = 0 \quad , \qquad (64a)$$

$$\alpha_{t_i} = \frac{4}{3} \left[1 + \eta_i + (1+1.71\eta_e) \frac{T_e}{T_i} \right] (1.71\eta_e + 1.5\eta_i)$$

$$+ \frac{10}{3} \eta_i \left[1.71\ \eta_e - (1+1.71\eta_e)\ (1 + \frac{T_i}{T_e})\ (\frac{c_i}{qR\omega_o})^2 \right] \qquad (64b)$$

$$\alpha_n = 2\ (1+1.71\eta_e) \left[1 + \eta_i + (1+1.71\eta_e) \frac{T_e}{T_i} + \frac{5}{3} \eta_i\ (\frac{c_i}{qR\omega_o})^2 \right] \qquad (64c)$$

$$\beta = 1.71\ \eta_e - \left[(1+1.71\eta_e)(1 + \frac{T_e}{T_i}) + \frac{2}{3}\ (1.71\eta_e + 1.5\eta_i) \right] (\frac{c_i}{qR\omega_o})^2 \qquad (64d)$$

The eigenvalue equation (62) thus reads:

$$\left[A(\partial^2/\partial\xi^2) - B(\gamma/\omega_o)\ \xi^2 + C\xi \right] n^{(o)} = 0 \qquad (65)$$

where we have introduced the definitions

$$A = \frac{\omega_e^*}{\omega_o} \{ \frac{1}{\beta} \left[\alpha_n (1 + \frac{T_e}{T_i}) + \alpha_{t_i} \right] <2\ (\frac{c_i}{\omega_o} \frac{\hat{b}\cdot\vec{\nabla}B}{B})^2>$$

$$- \left[(1+\eta_i) + (1+1.71\eta_e) \frac{T_e}{T_i} \right] \} \qquad (66a)$$

$$B = \frac{k_\parallel'^2 a_i^2 c_e^2 \tau_e}{0.51\omega_o} \qquad (66b)$$

$$C = \frac{3}{2} \frac{\eta_e \omega_e^*}{\omega_o} \frac{k_\parallel' V_{\parallel,e} a_i}{\omega_o} \qquad (66c)$$

Here $\xi = x/a_i$ is a normalized distance measured from the rational surface; $k_\parallel' = \partial k_\parallel/\partial x$ and

$$\gamma \equiv \frac{\partial}{\partial t_1} + \frac{2}{3} \frac{1.71\eta_e}{1+1.71\eta_e} (k_\parallel^2)_r \frac{\chi_{\parallel,e}}{N}\ ; \qquad (66d)$$

the latter definition where $(k_\parallel^2)_r$, is the radial average, implies that we treat the parallel electron heat conductivity by perturbation: proceeding otherwise would indeed be very cumbersome.

The solution of Eq. (65) - which, it is worth noting, is valid for arbitrary cross sections if $(c_i/qR\omega_o)^2 << 1$ - is of the form

$$n^{(o)} \propto \exp \left[- \Theta^{1/2} (\xi+\delta)^2/2 \right] \tag{67}$$

with $(\gamma/\omega_o) = (-C/2B)\ (-C/2A)^{1/3}$, $\delta = (-C/2A)^{-1/3}$, and $\Theta^{1/2} = \delta^{-2}$. It can be shown that there is no instability if A is negative, i.e. if cylindrical effects [the second term in (66a) dominate over toroidal effects (the first term in (66a))]. Assuming e.g. $\eta_e = \eta_i = 2$, $T_e = T_i$, and $(c_i/qR\omega_o)^2 << 1$, this occurs if $(c_i/R\omega_o)^2 > 0.117$. Indeed A is then given by

$$A = \frac{14.39}{1-3.84(c_i/qR\omega_o)^2} \left(\frac{c_i}{R\omega_o}\right)^2 - 1.68 \tag{66c}$$

Neglecting now cylindrical effects, A is also negative when $(c_i/qR\omega_o)^2 > 0.261$, i.e. if the ions are able to travel a full connection length in one wave period. For circular cross-sections the growth rate can be rewritten as

$$\tau_e \frac{\partial}{\partial t_1} = \gamma\tau_e - \frac{2}{3} \frac{1.71\eta_e}{1+1.71\eta_e} (k_\parallel^2)_r \frac{\chi_{\parallel,e}\tau_e}{N}$$

$$= 0.35 \left(\frac{T_i}{AT_e}\right)^{1/3} \frac{m_e}{m_i} \left| \frac{\eta_e^2}{1+1.71\eta_e} \frac{L_S}{L_N} \frac{V_{\parallel,e}^2}{c_i^2} \right|^{2/3}$$

$$- \frac{12.77A}{1+1.71\eta_e} (\omega_o\tau_i)^2 \frac{m_i\tau_e^2}{m_e\tau_i^2} \left| \frac{L_N}{L_S} \frac{c_i}{V_{\parallel,e}} \right| \tag{68}$$

Finally we note that the parallel ion heat conduction disappears from the solution. We stress that Eqs. (63, 64) imply that there is a (first order) top-bottom asymmetry of the density and ion temperature spectra. This question has been treated in more details elsewhere (Rogister, 1985) with a simpler, but less complete, scaling which however emphasized this aspect of the instability.

(B) $\omega_o \tau_i \sim 1 \quad , \quad (c_i/qR\omega_o)^2 << 1 \sim (c_i/R\omega_o)^2$

The solution of Eqs. (56, 58, 59) is

$$\oint JBd\chi \, \frac{\hat{b}\cdot\vec{\nabla}B}{B} \left[\frac{A^{(1)}_{o,i}}{T_i} ; \frac{t^{(1)}_i}{T_i} ; \frac{n^{(1)}}{N}\right] = i \frac{cT_i}{e_i B} \frac{1}{\omega_o} \frac{\gamma_k}{\delta_k} \hat{p}\cdot\vec{\nabla} \oint JBd\chi \, \left(\frac{\hat{b}\cdot\vec{\nabla}B}{B}\right)^2 \frac{n^{(o)}}{N} \qquad (69)$$

where

$$\gamma_{A_i} = 3 \left\{ \gamma_n \left(\frac{2}{3} \frac{1.71\eta_e+1.5\eta_i}{1+1.71\eta_e} + 3.42\,\eta_e \frac{T_e}{T_i} - \eta_i\right) - 6 \frac{T_e}{T_i} (1+1.71\eta_e) \left[1+\eta_i+(1+1.71\eta_e) \frac{T_e}{T_i}\right] + \frac{10}{3} \eta_i \right\} , \qquad (70a)$$

$$\gamma_{t_i} = \frac{2}{3} \left\{ \gamma_n \left[\frac{1.71\eta_e+1.5\eta_i}{1+1.71\eta_e} \left(1-3i \frac{P_i}{\omega_i^{\star}\eta_{o,i}}\right) - \frac{1.5\eta_i-1}{1+1.71\eta_e} \left(3.42\eta_e - \eta_i \frac{T_i}{T_e}\right)\right] + 5\eta_i \left(1-3i \frac{P_i}{\omega_i^{\star}\eta_{o,i}}\right) + 6 \left(\frac{3}{2} \eta_i-1\right) \left[1+\eta_i+(1+1.71\eta_e) \frac{T_e}{T_i}\right]\right\} \qquad (70b)$$

$$\gamma_n = 2 \frac{1+1.71\eta_e}{1.71\eta_e} \left[1+\eta_i+(1+1.71\eta_e) \frac{T_e}{T_i}\right]$$

$$\delta_{A_i} = \delta_{t_i} = 1 - \left(\eta_i - \frac{2}{3}\right) \frac{\omega_i^{\star}}{\omega_o} - 3i \frac{P_i}{\omega_i^{\star}\eta_{o,i}}$$

$$\delta_n = 1$$

The eigenvalue equation is similar to (65) with Eq. (66a) replaced by

$$A = \frac{\omega_e^*}{\omega_o} \left\{ \left[\gamma_n \left(1 + \frac{T_e}{T_i}\right) + \frac{\gamma_{t_i}}{\delta_{t_i}} + \frac{1}{6} \frac{\gamma_{A_i}}{\delta_{A_i}} \right] < 2 \left(\frac{c_i}{\omega_o} \frac{\hat{b}\cdot\vec{\nabla}B}{B} \right)^2 > \right.$$

$$\left. - \left[(1+\eta_i) + (1+1.71\eta_e) \frac{T_e}{T_i} \right] \right\} \quad ; \qquad (71)$$

and (66b–d) remaining unchanged. Let

$$A = |A| e^{i\phi}$$

It is shown in Appendix B that the bounded solution(s) can be expressed in the form

$$\delta^{-1} = - \left| \frac{C}{2A} \right|^{1/3} e^{-i\phi/3} \text{ sign } C. \qquad (73)$$

$$\gamma = \frac{\omega_o}{2B} |C|^{4/3} \, |2A|^{-1/3} \, e^{i\phi/3} \quad , \qquad (74)$$

$$\Theta^{1/2} = \delta^{-2} \qquad (75)$$

Requesting Re $\Theta > 0$ (convergence) implies

$$\text{(i)} \; -\frac{3\pi}{4} < \phi < \frac{3\pi}{4} \qquad \text{or} \qquad \text{(ii)} \; -\frac{15\pi}{4} < \phi < -\frac{9\pi}{4} \qquad (76)$$

which generalizes the condition obtained in the absence of viscosity. The mode is unstable in the first case and stable in the second. The situation is represented in Fig. 1. We note that ion viscosity enters at this order) exclusively via the genuine toroidal contribution in Eq. (71).

In order to assess more concretely the role of this viscosity, we have calculated A for $T_e = T_i$ and $\eta_e = \eta_i = 2$. We obtain

$$A = \frac{14.39 + 0.39\ (\omega_i^* \eta_{o,i}/P_i)^2 - i\ 5.39\ \omega_i^* \eta_{o,i}/P_i}{1 + 0.19\ (\omega_i^* \eta_{o,i}/P_i)^2} \left(\frac{c_i}{R\omega_o}\right)^2 - 1.68 \qquad (77)$$

This result shows that modes which would be stable in the absence of viscosity ($\cos\phi = -1$, $\sin\phi = 0$) can be destabilized by viscous effects when $\cos\phi$ becomes larger than $-\sqrt{2}/2$. Similar results have been obtained with the tearing mode (Schep, 1985). The condition $\cos\phi < -\sqrt{2}/2$ reads presently

$$\left(\frac{c_i}{R\omega_o}\right)^2 > \frac{1.68 + 0.32\ (\omega_i^* \eta_{o,i}/P_i)^2 - 5.39\ (\omega_i^* \eta_{o,i}/P_i)}{14.39 + 0.39\ (\omega_i^* \eta_{o,i}/P_i)^2} \qquad (78)$$

We finally note that ion heat conductivity has again dropped out of the problem, and that Eq. (68) giving the growth rate still applies with A replaced by Eq. (71).

VI. CONCLUSION

We have shown that in the absence of viscosity three different regimes of stability must be considered for the rippling mode:

$$(c_i/R\omega_o)^2 \ll 1 \qquad (i)$$

$$(c_i/qR\omega_o)^2 \ll 1 \ll (c_i/R\omega_o)^2 \qquad (ii)$$

$$1 \ll (c_i/qR\omega_o)^2 \quad . \qquad (iii)$$

The first limit essentially coincides with the cylindrical model and is stable. The mode is unstable in the second regime, the destabilizing effect arising from the radial magnetic drift. Stability is recovered in the third limit implying that the magnetic drift is not destabilizing when averaged over a full connection length.

The effect of viscosity can be treated meaningfully via the two-fluid model only in cases (i) and (ii) where it enters via the toroidicity-induced contribution. Its role is rather subtle and it can modify an otherwise stable situation into an unstable one.

Table I summarizes the situation, giving the precise limits for $T_e = T_i$ and $\eta_e = \eta_i = 2$.

The real wave frequency is in all cases approximately $\omega_o = (1+1.71\eta_e)\ \omega_e^*$.

Finally the top-bottom asymmetry of the density fluctuation spectrum has been considered in another paper (Rogister, 1985).

TABLE I

	$(c_i/R\omega_o)^2 = 0.12$		$(c_i/R\omega_o)^2 = 0.26\ q^2$
$\omega_o\tau_i \ll 1$	Stable Additional damping by parallel electron heat conduction. Eqs. (68, 66a, 66e)	Unstable if damping by parallel electron heat conduction permits. Eqs. (68, 66a, 66e)	Stable Additional damping by parallel electron heat conduction. Eqs. (68, 66a, 66e)
$\omega_o\tau_i \gtrsim 1$	Region of instability extended according to Eq. (78) Additional damping by parallel electron heat conduction.		Outside the scape of fluid theory.

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APPENDIX A.

The stress tensor (Braginskii 1965) can be written in the form

$$\overleftrightarrow{\Pi} = \overleftrightarrow{\Pi}_o + \overleftrightarrow{\Pi}_{1,2} + \overleftrightarrow{\Pi}_{3,4} \qquad \text{(A-1)}$$

where

$$\overleftrightarrow{\Pi}_o = (\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})\ A_o, \qquad \text{(A-2a)}$$

$$A_o = -3\eta_o\ (\hat{n}\cdot\vec{\nabla}\vec{U}\cdot\hat{n} - \frac{1}{3}\ \vec{\nabla}\cdot\vec{U}) \qquad \text{(A-2b)}$$

$$\overleftrightarrow{\Pi}_{3,4} = -\eta_3(\hat{p}\hat{p} - \hat{b}\hat{b})\ \hat{p}\hat{b}:\overleftrightarrow{W} + \frac{1}{2}\eta_3\ (\hat{p}\hat{b} + \hat{b}\hat{p})\ (\hat{p}\hat{p}-\hat{b}\hat{b}):\overleftrightarrow{W}$$

$$- \eta_4(\hat{p}\hat{n} + \hat{n}\hat{p})\ \hat{b}\hat{n}:\overleftrightarrow{W} + \eta_4(\hat{b}\hat{n}+\hat{n}\hat{b})\ \hat{p}\hat{n}:\overleftrightarrow{W} \qquad \text{(A-3)}$$

$$\overleftrightarrow{\Pi}_{1,2} = -\frac{1}{2}\eta_1(\hat{p}\hat{p}-\hat{b}\hat{b})\quad(\hat{p}\hat{p}-\hat{b}\hat{b}):\overleftrightarrow{W} - \eta_1(\hat{p}\hat{b}+\hat{b}\hat{p})\ \hat{p}\hat{b}:\overleftrightarrow{W}$$

$$- \eta_2(\hat{p}\hat{n}+\hat{n}\hat{p})\ \hat{p}\hat{n}:\overleftrightarrow{W} - \eta_2(\hat{b}\hat{n}+\hat{n}\hat{b})\hat{b}\hat{n}:\overleftrightarrow{W} \qquad \text{(A-4)}$$

Here $\overleftrightarrow{U} = \overleftrightarrow{V} + \lambda\vec{u}$ is the total velocity (equilibrium + fluctuating) and $\overleftrightarrow{W}$ is the tensor

$$\overleftrightarrow{W} = \vec{\nabla}\vec{U} - \frac{2}{3}\overleftrightarrow{I}\ \vec{\nabla}\cdot\vec{U} \qquad \text{(A-5)}$$

The coefficients η_o, η_1 and η_3 are given in Eq. (15) and $\eta_2 = 4\ \eta_1$, $\eta_4 = 2\ \eta_3$.

We are interested in

$$\hat{n}\cdot\left[\vec{\nabla}\times(\vec{\nabla}\cdot\overleftrightarrow{\Pi})\right] = (\hat{p}\cdot\vec{\nabla} + \hat{b}\cdot\vec{\nabla}\hat{p}\cdot\hat{b})\ \hat{b}\vec{\nabla}:\overleftrightarrow{\Pi}$$

$$-\ (\hat{b}\cdot\vec{\nabla} + \hat{p}\cdot\vec{\nabla}b\cdot\hat{p})\ \hat{p}\vec{\nabla}:\overleftrightarrow{\Pi}$$

$$+\ (b\cdot\vec{\nabla}\hat{p}\cdot\hat{n} - \hat{p}\cdot\vec{\nabla}b\cdot\hat{n})\ \hat{n}\vec{\nabla}:\overleftrightarrow{\Pi} \qquad \text{(A-6)}$$

Up to the order requested in the text:

$$\hat{n}\cdot\left[\vec{\nabla}\times(\vec{\nabla}\cdot\overleftrightarrow{\Pi}_{3,4})\right]$$

$$= \underset{1}{\hat{p}\cdot\vec{\nabla}}\ \{\ \eta_3\underset{\mu^3}{(\hat{p}\cdot\vec{\nabla}\hat{b}\cdot\hat{p} + \vec{\nabla}\cdot\hat{b})}\ \underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}} + \frac{1}{2}\,\eta_3\underset{\mu^2}{(\hat{b}\cdot\vec{\nabla}\hat{p}\cdot\hat{b} + \vec{\nabla}\cdot\hat{p})}\ \underset{\mu}{(\hat{p}\hat{p} - \hat{b}\hat{b}):\overleftrightarrow{W}}$$

$$+\ \underset{\mu}{\hat{b}\cdot\vec{\nabla}}\ (\eta_3\underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}}) + \frac{1}{2}\,\underset{1}{\hat{p}\cdot\vec{\nabla}}\left[\eta_3\underset{\mu}{(\hat{p}\hat{p}-\hat{b}\hat{b}):\overleftrightarrow{W}}\right] + \underset{\mu^3}{\hat{n}\cdot\vec{\nabla}}\ (\eta_4\underset{1}{\hat{p}\hat{n}:\overleftrightarrow{W}})\ \}$$

$$+\ \underset{\mu^2}{\hat{b}\cdot\vec{\nabla}\hat{p}\cdot\hat{b}}\ \{\ \underset{\mu}{\hat{b}\cdot\vec{\nabla}}\ (\eta_3\underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}}) + \frac{1}{2}\,\underset{1}{\hat{p}\cdot\vec{\nabla}}\left[\eta_3\underset{\mu}{(\hat{p}\hat{p}-\hat{b}\hat{b}):\overleftrightarrow{W}}\right]\ \}$$

$$-\ \underset{\mu}{\hat{b}\cdot\vec{\nabla}}\ \{-\eta_3\underset{\mu^2}{(\hat{b}\cdot\vec{\nabla}\hat{p}\cdot\hat{b} + \vec{\nabla}\cdot\hat{p})}\ \underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}} - \underset{1}{\hat{p}\cdot\vec{\nabla}}(\eta_3\underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}}) + \frac{1}{2}\,\underset{\mu}{\hat{b}\cdot\vec{\nabla}}\left[\eta_3\underset{\mu}{(\hat{p}\hat{p}-\hat{b}\hat{b}):\overleftrightarrow{W}}\right]\}$$

$$+\ \underset{\mu^3}{\hat{p}\cdot\vec{\nabla}\hat{b}\cdot\hat{p}}\ \underset{1}{\hat{p}\cdot\vec{\nabla}}(\eta_3\underset{1}{\hat{p}\hat{b}:\overleftrightarrow{W}}) \qquad \text{(A-7)}$$

where we have indicated the order of magnitude of the various factors.

Introducing the commutator

$$\hat{p}\cdot\vec{\nabla}\ \hat{b}\cdot\vec{\nabla} - \hat{b}\cdot\vec{\nabla}\ \hat{p}\cdot\vec{\nabla} = \hat{p}\cdot\vec{\nabla}\ \hat{b}\cdot\hat{p} - \hat{b}\cdot\vec{\nabla}\ \hat{p}\cdot\hat{b} + (\hat{p}\cdot\vec{\nabla}\ \hat{b}\cdot\hat{n} - \hat{b}\cdot\vec{\nabla}\ \hat{p}\cdot\hat{n})\ \hat{n}\cdot\vec{\nabla} \qquad \text{(A-8)}$$

and the identity

$$\hat{p}\cdot\vec{\nabla}\ \hat{p}\cdot\vec{u} + \hat{b}\cdot\vec{\nabla}\ \hat{b}\cdot\vec{u} + \hat{n}\cdot\vec{\nabla}\ \hat{n}\cdot\vec{u} + (\hat{b}\cdot\vec{\nabla}\ \hat{p}\cdot\hat{b} + \hat{n}\cdot\vec{\nabla}\ \hat{p}\cdot\hat{n})\ \hat{p}\cdot\vec{u}$$

$$+ (\hat{p}\cdot\vec{\nabla}\hat{b}\cdot\hat{p} + \hat{n}\cdot\vec{\nabla}b\cdot\hat{n})\ \hat{b}\cdot\vec{u} - \hat{n}\cdot\vec{u}\ \frac{1}{B}\ \hat{n}\cdot\vec{\nabla}B$$

$$= -\frac{1}{N}\ \vec{u}\cdot\vec{\nabla}N - \frac{1}{N}\ \vec{V}\cdot\vec{\nabla}n - \frac{n}{N}\ \vec{\nabla}\cdot\vec{V} - \frac{\partial n}{\partial t}$$

$$\equiv \vec{\nabla}\cdot\vec{u} \quad , \qquad \text{(A-9)}$$

one obtains

$$\hat{n}\cdot \left[\vec{\nabla} \times (\vec{\nabla}\cdot\overleftrightarrow{\Pi}_{3,4})\right]$$

$$= \eta_4\ \hat{p}\cdot\vec{\nabla}\ \hat{n}\cdot\vec{\nabla}\ \hat{p}\cdot\vec{\nabla}\ \hat{n}\cdot\vec{u} + \eta_3(\hat{p}\cdot\vec{\nabla})^2\ (\vec{\nabla}\cdot\vec{u} - \hat{n}\cdot\vec{\nabla}\ \hat{n}\cdot\vec{u})$$

$$- 2(\hat{p}\cdot\vec{\nabla}\eta_3)\ \hat{p}\cdot\vec{\nabla}\ \hat{b}\cdot\vec{\nabla}\ \hat{b}\cdot\vec{u} \qquad \text{(4-10)}$$

where all the terms are of order $\mu^3\eta_3 c_s/a_i^3$.

One can similarily show that up to the same order,

$$\hat{n}\cdot \left[\ \vec{\nabla} \times (\vec{\nabla}\cdot\overleftrightarrow{\Pi}_{1,2})\right] = -\eta_1(\hat{p}\cdot\vec{\nabla})^3\ \hat{b}\cdot\vec{u} \quad . \qquad \text{(A-11)}$$

APPENDIX B

Substitution of the solution (67) into the eigenvalue equation (65) shows that

$$\gamma \equiv \frac{-\omega_o}{2B} C \, \delta^{-1} \quad ,$$

$$\delta^{-1} = (-C/2A)^{1/3} \quad ,$$

$$\Theta^{1/2} = \delta^{-2}$$

If $A = |A| e^{i\phi}$, it is easily shown that δ^{-1} can take one of the three values

$$\delta^{-1} = \text{sign } C \; |C/2A|^{1/3} \; e^{-i\phi/3} \left[e^{i\Pi/3}, \; -1, \; e^{-i\Pi/3} \right] \qquad \text{(B-1)}$$

whatever the sign of C. The solution will be convergent (Re $\Theta^{1/2}$>0) if, respectively

$$-\frac{7\Pi}{6} + 2\,k\Pi < -\frac{2\phi}{3} < -\frac{\Pi}{6} + 2\,k\Pi \; ,$$

$$-\frac{\Pi}{2} + 2\,k\Pi < -\frac{2\phi}{3} < \frac{\Pi}{2} + 2\,k\Pi \qquad \text{(B-2)}$$

$$\frac{\Pi}{6} + 2\,k\Pi < -\frac{2\phi}{3} < \frac{7\Pi}{6} + 2\,k\Pi \; .$$

The argument of the growth rate

$$\gamma = -\frac{\omega_o}{2B} \, |C| \; |C/2A|^{1/3} \, e^{-i\phi/3} \left[+e^{i\Pi/3}, \; -1, \; +e^{-i\Pi/3} \right] \qquad \text{(B-3)}$$

where $\omega_o/2B>0$, is accordingly bounded as follows if k = 0:

$$-\frac{\Pi}{4} < \frac{-\phi}{3} + \frac{\Pi}{3} < \frac{\Pi}{4} \quad , \qquad \text{(stable)}$$

$$\frac{3\Pi}{4} < -\frac{\phi}{3} + \Pi < \frac{5\Pi}{4} \quad , \text{(unstable)} \qquad \text{(B-4)}$$

$$-\frac{\Pi}{4} < -\frac{\phi}{3} - \frac{\Pi}{3} < \frac{\Pi}{4} \quad , \qquad \text{(stable)}$$

where we have indicated the stable and unstable solutions. If k = 1, we have instead

$$\frac{3\Pi}{4} < -\frac{\phi}{3} + \frac{\Pi}{3} < \frac{5\Pi}{4} \qquad \text{(unstable)}$$

$$-\frac{\Pi}{4} < -\frac{\phi}{3} - \Pi < \frac{\Pi}{4} \qquad \text{(stable)}$$

$$\frac{3\Pi}{4} < -\frac{\phi}{3} - \frac{\Pi}{3} < \frac{5\Pi}{4} \qquad \text{(unstable)}$$

Therefore the solution can always be written in the form (73-75) with the stability regions given as in Eq. (76).

FIGURE CAPTION

Fig. 1. Phase angles ϕ for which the mode is stable (discontinuous line) or unstable (continuous line). An unstable and a stable mode can coexist. See Equation (76) for the ranges of definition of ϕ.