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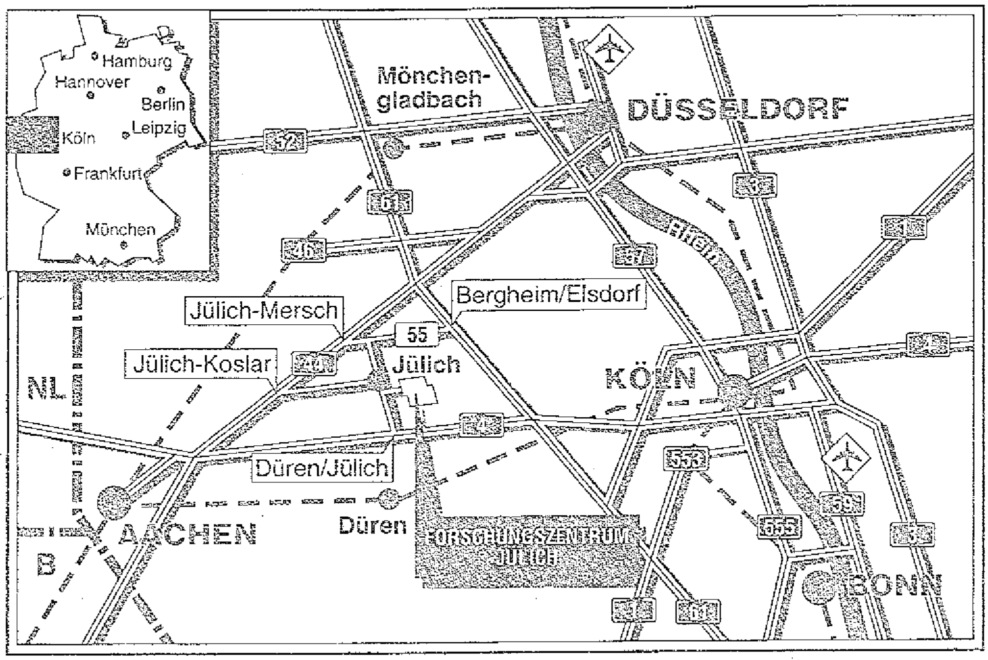

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ABSTRACT

Game-theoretical models of randomized inspections for timely verification of compliance are investigated. Guaranteed average detection times comparable to or shorter than those obtainable without randomization are calculated.

1. Introduction

The verification of compliance to a particular undertaking by use of on-site inspection is often influenced by considerations of timeliness. Brief but frequent visits may need to be made at regular or irregular intervals between more exhaustive inspections in order to obtain, with some degree of assurance, short notice indication of a possible violation. Since interim inspections of this kind generally involve excessive demands on travel expenses and manpower resources for the inspecting party, it is of interest to look for inspection strategies which optimize timeliness under fixed constraints.

In this report we examine the following idealized model: A reference time period is subdivided into n equal intervals, at the end of each of which an interim inspection can take place. An inspection is assumed *always* to take place after the nth interval, i.e. at the end of the reference period, and the total number of inspections, including the last, is defined to be $k \leq n$. It is assumed, furthermore, that *at most* one violation can occur, and if it occurs we assume that it will be detected at the next inspection.

For the case $k < n$, we can formulate a two-person zero-sum game between inspectee and inspector with the payoff to the former, in case of illegal behavior, being the expected time to detection as measured from the time of violation. In case of legal behavior, since no violation occurs and since we exclude the possibility of false alarms, the time to detection is undefined. Formally, therefore, the inspectee's optimal strategy may well be simply to behave legally. Rather than introducing subjective utilities into the payoff function to deal explicitly with this option (and thereby complicating the analysis unnecessarily) we can give the following technical justification for considering the illegal game: we seek a *subgame perfect equilibrium* of the overall two-person game (see e.g. /1/), i.e. a strategy for the inspector which is also optimal in the case in which the inspectee, perhaps against his better judgement, chooses to act illegally within the reference period.

Thus we consider only the illegal subgame. In order to stress this we shall, in the sequel, refer to the inspectee as the 'violator'.

Three versions of the game are considered. In the first, k is interpreted as the precise number of inspections to be carried out, so that an inspector's strategy is to choose a subset of $k-1$ interim inspections out of $n-1$ possibilities. We also require that the violation strategy be chosen prior to the reference period, and that it be irrevocable. The resulting *matrix game*, by virtue of von Neumann's *minimax theorem* /2/, possesses a saddlepoint in the domain of mixed strategies, i.e. for randomized inspections. Numerical solutions are calculated using the simplex algorithm and tabulated for $3 \leq n \leq 12$ and $2 \leq k < n$.

In the second version, k is treated as an expectation value for the number of inspections, when the interim inspection times are chosen according to a probability distribution over the n periods. That is, if p_i is the probability that an inspection takes place after interval i, then $k = \sum_1^n p_i$. It is demonstated that, if both players are ignorant of their opponent's strategies, the game has no saddlepoint. The related *inspector leadership game*, in which the inspector announces his strategy in advance, does however have an equilibrium, which is derived.

Finally, in the third version k is again taken to be the precise number of inspections, but the restriction that the violator choose his strategy in advance without taking into account the information gained during the game is relaxed. He can use *behavioral strategies*, also selected prior to the reference period, which indeed depend on the inspector's actions. The resulting *sequential game* is also solved.

All three results represent improvements over the 'naive' solution in which the interim inspections are simply performed independently with equal probability over $n-1$ periods.

After some preliminary definitions and relations concerning average run lengths in Section 2, a detailed presentation of the game theoretical results is given in Section 3. The solutions are discussed in relation to applicability to interim inspection problems in Section 4.

2. Average Run Lengths

The situation that we shall be considering here is depicted schematically in Figure 1. Beginning at some arbitrary time $i = 0$, there exists a series of detection opportunities for some prior violation, these opportunities occurring at times $i = 1, 2, \ldots$ and distributed at equal intervals to infinity.

We distinguish a reference period of length n intervals. Assume for now that the inspection activity is such that, had a violation occurred at time $t < i$, it would be detected with probablity $1 - \beta_i$ by an inspection at time i. Assuming further that the inspection activities performed at each time are independent of one another, the probability of detecting *for the first time*, at time i, a violation which occurred at time 0 is given by

$$(1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j. \qquad (2.1)$$

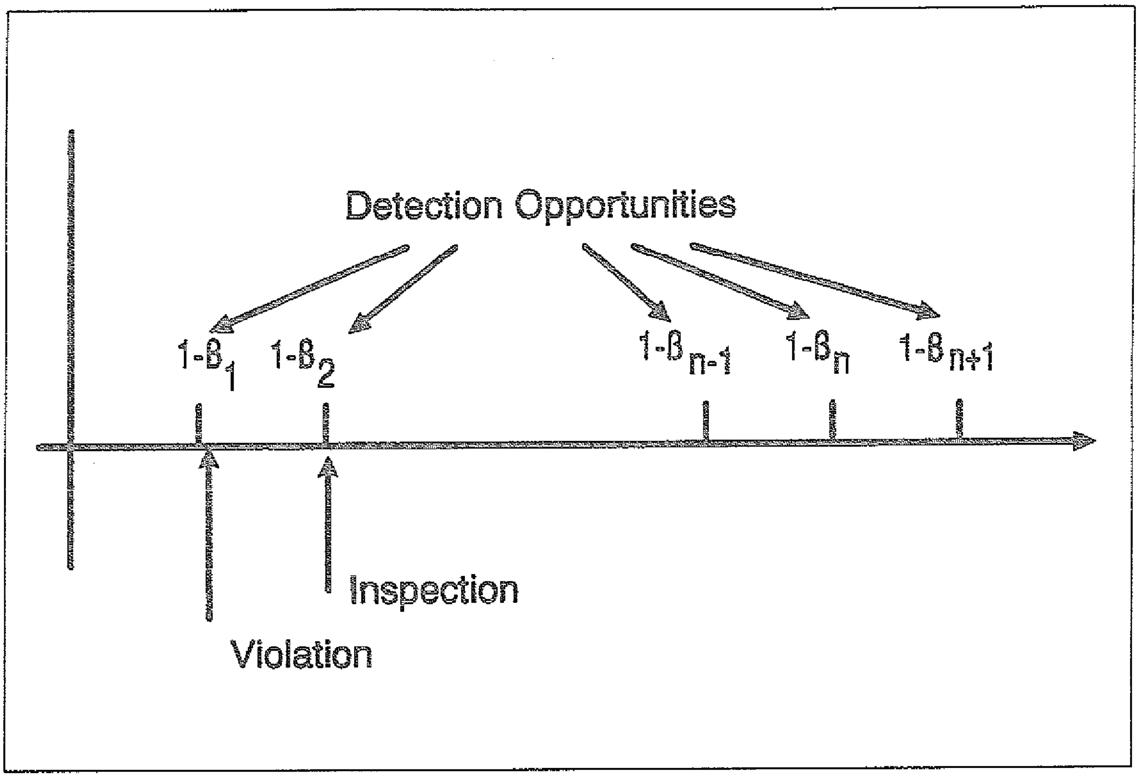

Fig. 1: Violation and detection opportunities and associated non-detection probabilities.

Hence, since

$$\sum_{i=1}^{\infty}(1-\beta_i)\cdot\prod_{j=1}^{i-1}\beta_j = 1 + \prod_{i=1}^{\infty}\beta_i = 1,$$

the expectation value of the number of periods to detection of the violation, i.e. the average run length, is

$$L_0 = \sum_{i=1}^{\infty} i\cdot(1-\beta_i)\cdot\prod_{j=1}^{i-1}\beta_j. \tag{2.2}$$

With the notational conventions $\prod_{j=1}^{0} = 1$ and $\prod_{j=1}^{-1} = 0$ we can simplify this last equation as follows

$$
\begin{aligned}
L_0 &= \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i-1} \beta_j - \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i} \beta_j \\
&= \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i-1} \beta_j - \sum_{i=0}^{\infty} (i-1) \cdot \prod_{j=1}^{i-1} \beta_j \\
&= \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i-1} \beta_j - \sum_{i=1}^{\infty} i \cdot \prod_{j=1}^{i-1} \beta_j + \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \beta_j \\
&= \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} \beta_j
\end{aligned}
\tag{2.3}
$$

We now restrict ourselves to the special case in which the non-detection probability for the inspection which takes place at reference time n is zero, i.e. $1-\beta_n = 1$. The average run length reduces to the finite sum

$$
L_0 = \sum_{i=1}^{n} \prod_{j=1}^{i-1} \beta_j = 1 + \beta_1 + \beta_1 \cdot \beta_2 + \ldots + \beta_1 \cdot \beta_2 \cdot \ldots \cdot \beta_{n-1}. \tag{2.4}
$$

Now suppose that the violation occurred immediately after the first inspection, i.e. just after $t = 1$ rather than $t = 0$. The same considerations lead us to the average run length

$$
L_1 = 1 + \beta_2 + \beta_2 \cdot \beta_3 + \ldots + \beta_2 \cdot \beta_3 \cdot \ldots \cdot \beta_{n-1} \tag{2.5}
$$

and hence

$$
L_0 = 1 + \beta_1 \cdot L_1. \tag{2.6}
$$

Generalizing, we can write the average run length for detection of a violation occurring at time i recursively:

$$
\begin{aligned}
L_i &= 1 + \beta_{i+1} \cdot L_{i+1}, \qquad 0 \le i < n-1 \\
L_{n-1} &= 1.
\end{aligned}
\tag{2.7}
$$

This result is the basis for the discussion in Section 3.2.

3. Game Theoretical Results

Let the total number of inspections that are to take place in the reference time n be $k < n$. As indicated in Section 1, we suppose that inspections are such that prior violations will be detected with certainty at the first subsequent inspection and that an inspection *always* takes place at $i = n$. Suppose further that the violator *must* carry out his violation during the reference time and that he is aware of the number k of inspections that will be performed.

Since detection is inevitable, the violator can at best choose his violation time in such a way as to *maximize* the time to detection. He has precisely $n - 1$ pure strategies at his disposal, namely to violate just after time $i, i = 0 \ldots n-2$. (Time $n - 1$ can be excluded, since this will always lead to the minimum detection time of one period.)

3.1. The Matrix Game

We treat the problem first as a finite game. The violator chooses his strategy irrevocably prior to the start of play, i.e. at time $t < 0$. He cannot change his mind during the course of the game. (We shall return to this point later.) The value k is taken to be exact, that is *precisely* k inspections must take place within the reference time n. Thus the inspector has $\binom{n-1}{k-1}$ pure strategies, recalling that one inspection is always reserved for time n, and he will choose his strategy such as to *minimize* the time to detection. Each strategy combination of violator and inspector leads to a unique detection time which is an integral number of inspection opportunity intervals. This suffices to define a *two-person, zero-sum matrix game* between violator (player I) and inspector (player II) with the payoff to the violator being the time to detection after violation. As an example, Table 1 shows the payoff matrix for the case $n = 5, k = 3$.

None of the matrices constructed in this way possess saddlepoints, that is, there is no pure strategy combination (i^*, j^*) for players I and II, respectively, which satisfies

$$L_{i,j^*} \leq L_{i^*,j^*} \leq L_{i^*,j} \qquad \forall i, j$$

TABLE 1 *The violator-inspector matrix game for* $n = 5, k = 3$. *The violator's pure strategies are given in the first column, and the inspector's pure strategies in the first row.*

$I \setminus II$	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
0	1	1	1	2	2	3
1	1	2	3	1	1	2
2	3	1	2	1	2	1
3	2	2	1	2	1	1

where $L_{i,j}$ is an element of the payoff matrix. However the *minimax* theorem of von Neumann /1/ guarantees the existence of a saddlepoint solution in the domain of mixed strategies, where a mixed strategy for a player is a probability distribution over his set of pure strategies. A mixed strategy for player II, the inspector, is thus an *inspection randomization* strategy. For small matrices, the optimal mixed strategies (solutions of the game) can generally be calculated by hand. A graphical solution for the simplest non-trivial case $n = 3, k = 2$ is shown in Figure 2. For many cases of interest, however, the matrix can be quite large and only numerical solutions are possible. For example for $n = 12, k = 6$ the game matrix is of dimension 11×462.

The optimal inspection randomization strategies and the associated guaranteed detection times for all values $3 \leq n \leq 12$ with $2 \leq k \leq n - 1$ were calculated numerically. The payoff matrices were determined as decribed in Appendix A, and a standard simplex algorithm /3/ was then applied to solve the games. Results are tabulated in Appendix B.

The matrix game solutions are interesting from the point of view of inspection efficiency since they satisfy

$$
\begin{aligned}
L &= \frac{n}{k} \qquad & k = n - 1 \\
L &< \frac{n}{k} \qquad & 1 < k < n - 1
\end{aligned}
\tag{3.1}
$$

where L is the inspector's guaranteed detection time. This means that, by performing k random inspections over the n opportunities, the inspector can obtain *shorter* detection times (at least for $k < n-1$) than by splitting the reference period into k equal intervals and inspecting *with certainty* after each interval. (This latter strategy gives of course a detection time of exactly $\frac{n}{k}$.) For fixed inspection effort, a randomization strategy is thus superior to a deterministic one. We shall discuss a practical application of this in Section 4.

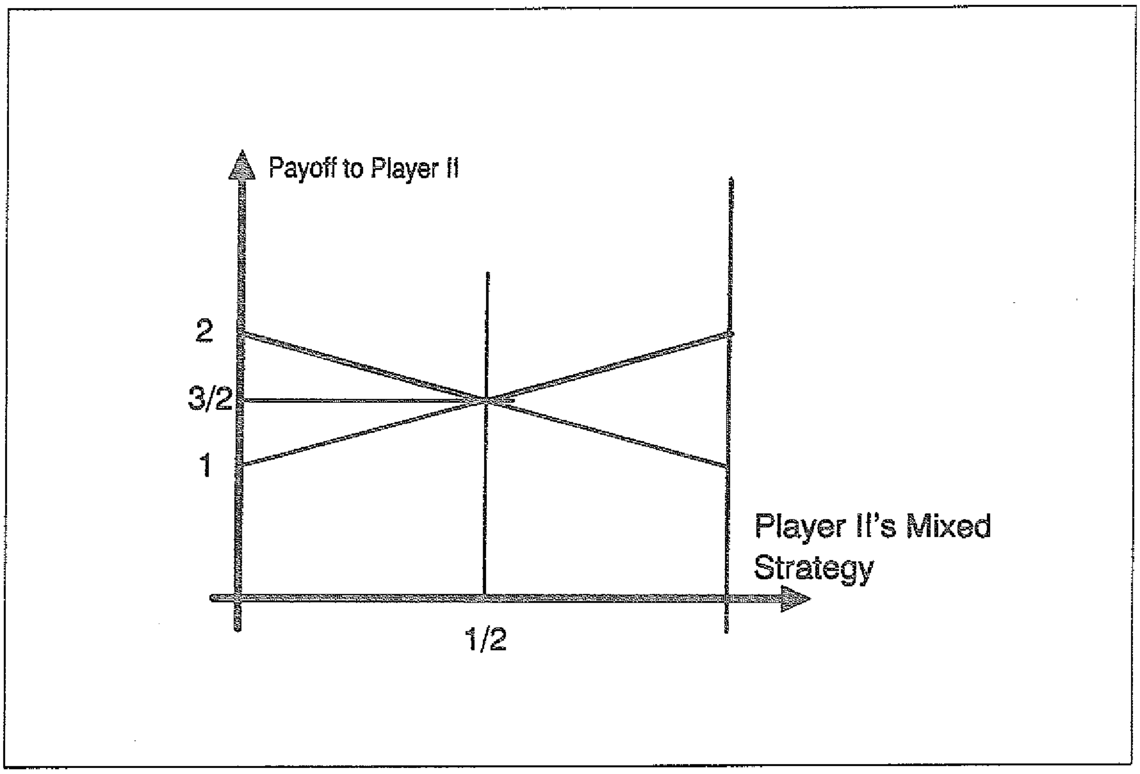

Fig. 2: Graphical solution of the matrix game $n = 3, k = 2$.

The disadvantage of the matrix game approach is that it requires, as we have already indicated above, that player I choose his strategy prior to the reference period without taking into account any information about the inspector's strategy gained in the course of the game. One could imagine that, knowing both n and k, the violator would wait for a few periods in order to observe how many inspections had been 'used up' and then act. Using such a *behavioral strategy* (which is of course also selected prior to the reference period) the violator might, with some justification, hope to push the detection time slightly beyond the saddlepoint value of the game considered here. This is not to say that the solution has no practical application. Depending on external circumstances the violator may have no choice in timing his action (e.g. he may have to plan the precise time of action well in advance).

Alternatively, the violator may be ignorant of the number k of inspections. This would be the case if, for instance, not the precise value but only the expectation value of k is known to the operator. This situation is considered in the following section.

3.2. The Inspector Leadership Game

Suppose the inspector chooses to inspect at the ith opportunity with probability p_i, whereby the p_i are completely independent. Then the expected value of the number of interim inspections carried out (i.e. excluding the last one) is

$$\sum_{i=1}^{n-1} p_i = k - 1. \tag{3.2}$$

As usual, we have required $p_n = 1$. Thus a fixed k is to be interpreted now as an *expectation value* and need no longer be an integer. Again assuming that an inspection, if it takes place, will always detect a prior violation with certainty, we can interpret the non-detection probabilities β_i in equation (2.7) as non-inspection probabilities $1 - p_i$, obtaining

$$\begin{aligned} L_i &= 1 + (1 - p_{i+1}) \cdot L_{i+1}, \qquad 0 \leq i < n - 1 \\ L_{n-1} &= 1. \end{aligned} \tag{3.3}$$

It is convenient to consider first the special case $n = 3$. We will generalize to arbitrary n later. For this case, equation (3.2) is

$$p_1 + p_2 = k - 1, \qquad 1 \leq k \leq 3 \tag{3.4}$$

with the restriction

$$\max(0, k - 2) \leq p_{1(2)} \leq \min(1, k - 1). \tag{3.5}$$

From equation (3.3) we obtain

$$\begin{aligned} L_0 &= 1 + (1 - p_1)L_1 \\ L_1 &= 1 + (1 - p_2)L_2 \\ L_2 &= 1 \end{aligned}$$

Defining $p_1 = p$, and with (3.4), $p_2 = k - p - 1$,

$$\begin{aligned} L_0 &= 1 + (1-p) \cdot (3 - k + p) \\ L_1 &= 3 - k + p. \end{aligned} \tag{3.6}$$

Let $q = q_0$ be the probability that the violation occurs at time 0. Then the probability of violation at time 1 is $q_1 = 1 - q$, since, as already noted, it would not be sensible for player I to time his violation at $i = n - 1 = 2$, this leading to the minimum detection time of one period. The average time to detection can therefore be written

$$\begin{aligned} L(q,p;k) =& q \cdot L_0 + (1-q) \cdot L_1 \\ =& q \cdot (1 + (1-p) \cdot (3 - k + p)) + (1-q) \cdot (3 - k + p). \end{aligned} \tag{3.7}$$

We can now formulate the problem of determining optimal inspection strategy as solving a two-person, zero-sum game $< L, q, p >$ where the payoff function L is given by equation (3.7) and the strategy sets of players I and II satisfy, respectively, $0 \le q \le 1$ and $\max(0, k-2) \le p \le \min(1, k-1)$. We seek q^* and p^* such that the *equilibrium* or *saddlepoint conditions*

$$L(q, p^*; k) \le L(q^*, p^*; k) \le L(q^*, p; k) \tag{3.8}$$

are satisfied.

Unlike the matrix game considered before, there is no guarantee that a saddlepoint exists. In fact a saddlepoint exists if and only if

$$\min_p \max_q L(q,p;k) = \max_q \min_p L(q,p;k)$$

under the above restrictions on p and q, and we shall show that this is in general not the case. We begin by proving

Theorem 1: *Define $L(q^*, p^*; k) = \min_p \max_q L(q,p;k)$, where $L(q,p;k)$ is given by (3.7). Then for $3/2 < k \le 3$*

$$\begin{aligned} L(q^*, p^*; k) &= \frac{2}{k - 3 + \sqrt{(k-3)^2 + 4}} \\ q^* &= 0 \text{ or } 1 \\ p^* &= \frac{1}{L} \end{aligned} \tag{3.9}$$

and for $1 \leq k \leq 3/2$

$$\begin{aligned} L(q^*, p^*; k) &= 5 - 2k \\ q^* &= 1 \\ p^* &= k - 1. \end{aligned} \tag{3.10}$$

Proof: From (3.7) we can write L(q,p;k) as

$$L(q, p; k) = q \cdot (1 - p \cdot (3 - k) - p^2) + 3 - k + p. \tag{3.11}$$

Since L is linear in q, it will be maximized for $q = 1$ or $q = 0$ depending upon whether the coefficient of q in (3.11) is positive or negative, respectively. This in turn depends upon the value of p. The coefficient of q in (3.11) changes sign for $p = p^*$ satisfying

$$1 - p \cdot (3 - k) - p^2 = 0$$

or

$$p^* = \frac{k - 3 + \sqrt{(k-3)^2 + 4}}{2} \tag{3.12}$$

being positive for $p < p^*$ and negative for $p > p^*$.

Thus we have

$$\begin{aligned} &L(q^* = 0, p; k) = 3 - k + p, \qquad p \geq p^* \\ &L(q^* = 1, p; k) = 2 + (2 - k) \cdot (1 - p) - p^2, \qquad p \leq p^*. \end{aligned} \tag{3.13}$$

For $3/2 \leq k \leq 3$ this function takes its minimum wrt p at $p = p^*$. (See Fig. 3.) Substituting for p^* in either of the expressions (3.13), we get

$$L(q^*, p^*; k) = \frac{2}{k - 3 + \sqrt{(k-3)^2 + 4}} = \frac{1}{p^*}$$

which is (3.9) as required.

For $k \leq 3/2$ the situation is as shown in Fig. 4 where, by virtue of (3.5), $0 \leq p \leq k - 1 < 1/2$. Now the violator will always choose $q^* = 1$ (upper curve) and the minimum wrt p occurs with p set to its maximum allowed value of $p^* = k - 1$. Substituting this value into $L(q^* = 1, p; k)$ in equation (3.13), we get

$$L(q^* = 1, p^* = k - 1; k) = 5 - 2k$$

which is (3.10), and the proof is complete.

We note the following properties of Theorem 1:

i. The condition for validity of solution (3.9) can be written

$$n/2 < k \leq n/1$$

where $n = 3$. For k satisfying this condition, the violator is *indifferent* to strategies $q = 1$ and $q = 0$, i.e. to violation at times 0 or $1 = n - 2$.

ii. Similarly, the condition for solution (3.10) can be written

$$n/n \leq k \leq n/2$$

and now the violator chooses $q = 1$, that is he does not consider violation at time $n - 2$, knowing that k is so small that he can do better than achieving a detection time of 2 periods. We shall see shortly that these results can be generalized.

That a saddlepoint need not exist can be easily seen by considering the special case $n = 3, k = 2$. From (3.7)

$$L(q, p; 2) = q \cdot (2 - p^2) + (1 - q) \cdot (1 + p).$$

Since this equation is quadratic in p with nonnegative slope at $p = 0$,

$$\left[\frac{dL}{dp}\right]_{p=0} = 1 - q \geq 0,$$

it can take its minimum wrt p only at $p = 0$ or $p = 1$. At these points we have

$$L(q, 0; 2) = 1 + q$$
$$L(q, 1; 2) = 2 - q.$$

and it follows that

$$\min_p L(q, p; 2) = \begin{cases} 1 + q, & \text{for } q \leq 1/2 \\ 2 - q, & \text{for } q \geq 1/2 \end{cases}.$$

The maximum of this function wrt q occurs for q satisfying $1 + q = 2 - q$. Thus $q = 1/2$ and $\max_q \min_p L(q, p; 2) = 3/2$, whereas from (3.9) we obtain

$$\min_p \max_q L(q, p; 2) = \frac{2}{\sqrt{5} - 1} > 3/2 = \max_q \min_p L(q, p; 2).$$

and there is no saddlepoint.

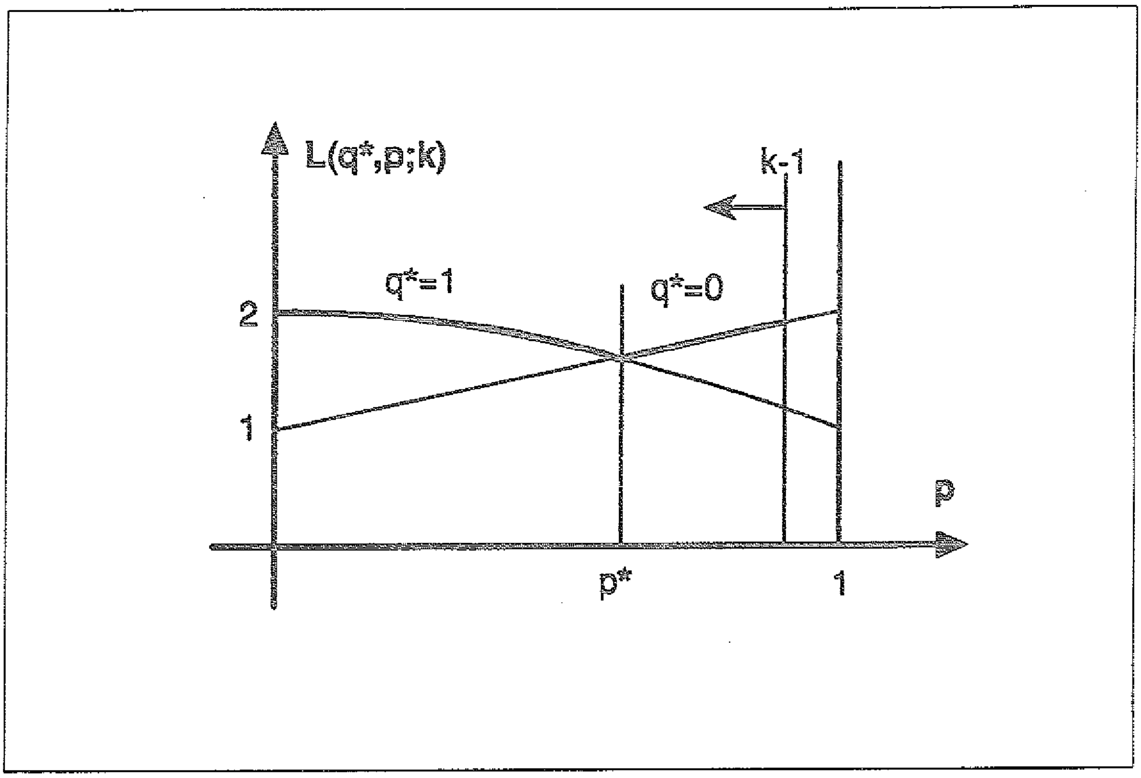

Fig. 3: Illustrating the situation of Theorem 1 for $3/2 < k \leq 3$.

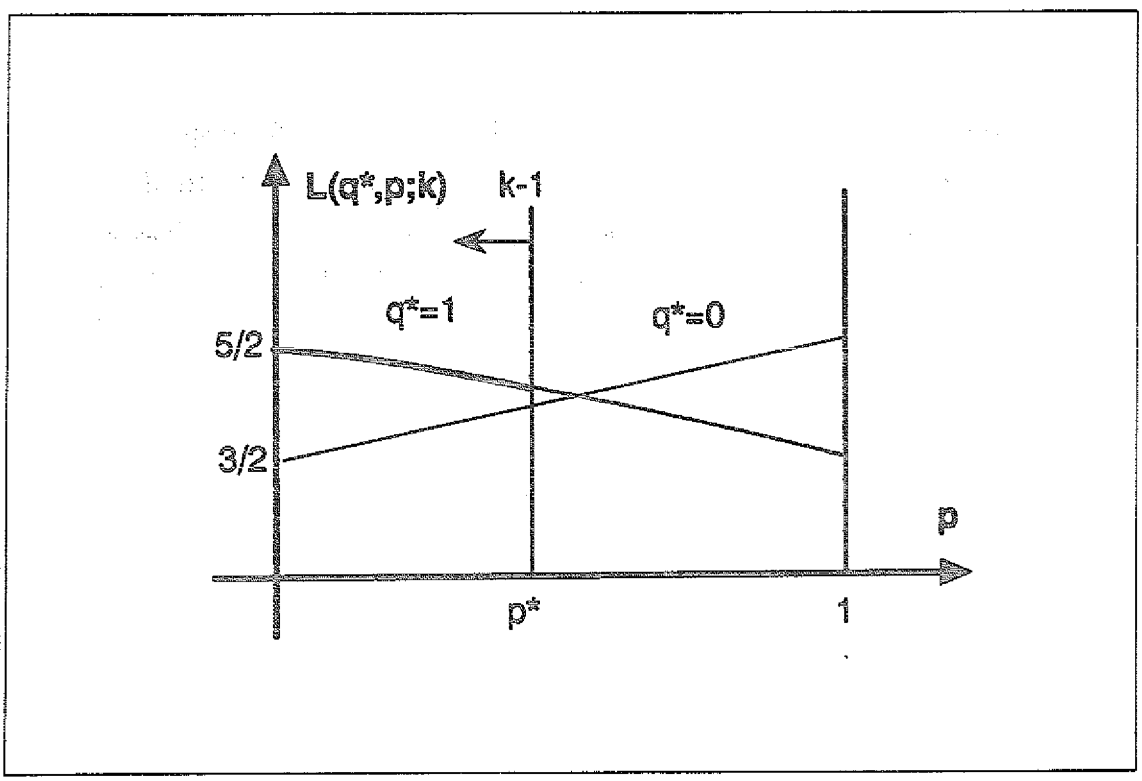

Fig. 4: Illustrating the situation of Theorem 1 for $1 \leq k \leq 3/2$.

In non-cooperative games, equilibrium (or for our 2-person zero sum game the saddlepoint) is the only available solution concept. In order to calculate the inspector's randomization strategy and his associated guaranteed detection time, we therefore need a new game theoretical model. This is provided by a change of rules. Rather than the players not informing one another of their strategies, we assume that the inspector *announces his strategy in advance*, that is his inspection probabilities $p_i, i = 1 \dots n$. We call this the *inspector leadership game* and it is obviously realizable in an actual inspection regime. In this model, the equilibrium solution for both players is the *minmax* solution given by Theorem 1. This can be seen by means of backward induction as follows: Player II (the inspector) announces his strategy, p^0 say, in advance. Player I (the violator) will then choose his strategy q^0 so as to maximize the detection time, $q^0 = \arg\max_q L(q, p^0; k)$. Knowing this, player II will choose his p^0 so as to minimize $\max_q L(q, p; k)$, i.e. he announces $p^0 = \arg\min_p \max_q L(q, p; k)$.

It is important to realize that, although we have defined a new game, the *solution concept* is unchanged, namely that of equilibrium /1,4/. The point is illustrated in Figure 5 where the simultaneous matrix game of Section 3.1 and the inspector leadership game are shown in *extensive form* for the case $n = 3$ and $k = 2$.

In (a) player II, the inspector, chooses the probability $p_1 = p$ for inspection at time 1, thus fixing the probability $p_2 = 1 - p$ for inspection at time 2. He does so without informing player I, the violator, as to his decision. Player I decides with probability q to violate at time 0 and $1 - q$ at time 1. The shaded area denotes his information set. The game can be transformed immediately into the following 2×2 matrix game:

I/II	1	2
0	1	2
1	2	1

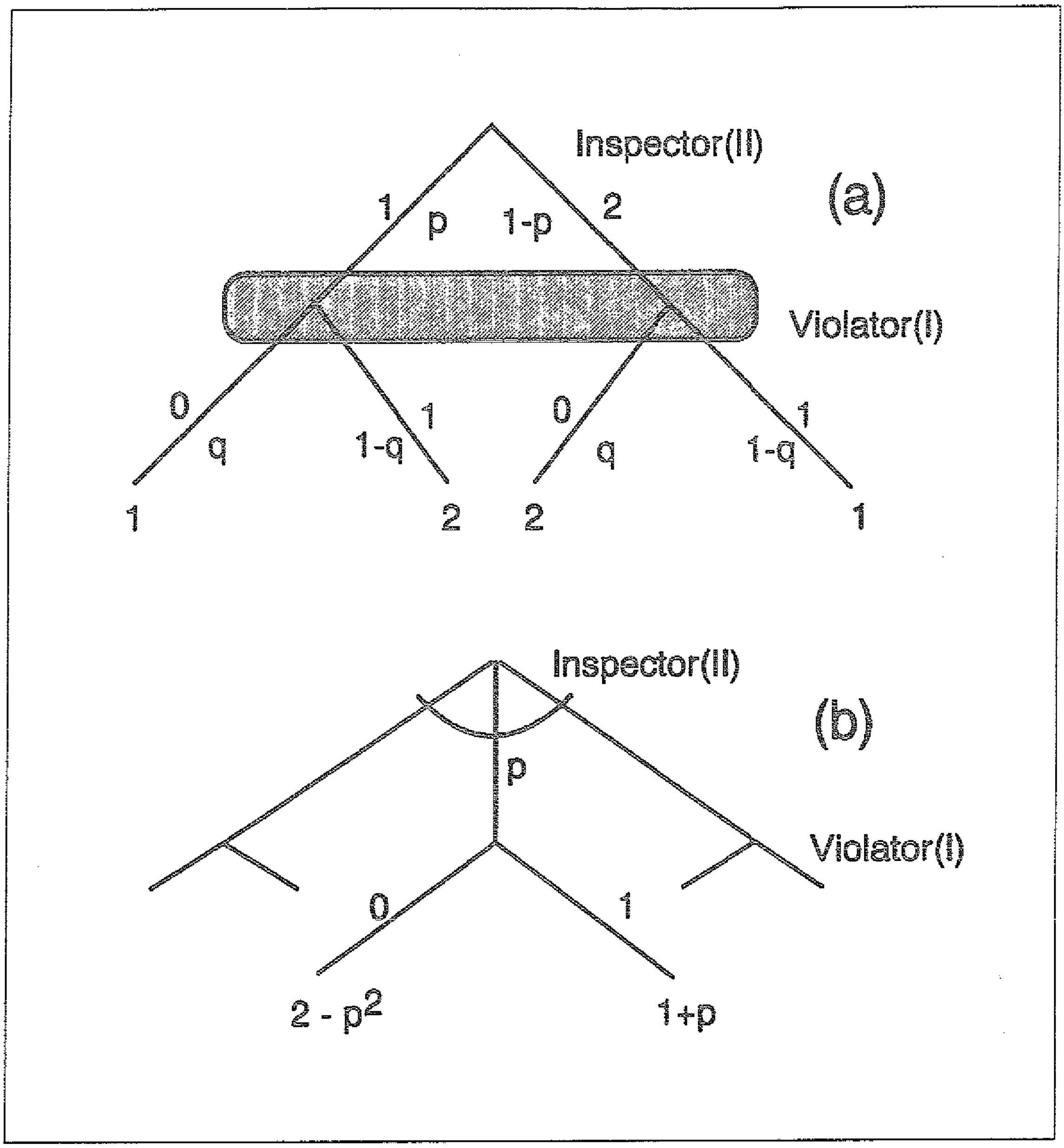

Fig. 5: Extensive forms of the simultaneous (a) and inspector leadership (b) games for $n = 3$, $k = 2$. For further explanations see text.

The expected losses to player II, therefore, are

$$\begin{aligned} L^{(a)}(0,p) &= 1 \cdot p + 2 \cdot (1-p) = 2 - p \\ L^{(a)}(1,p) &= 2 \cdot p + 1 \cdot (1-p) = 1 + p \end{aligned} \tag{3.14}$$

which leads immediately to the equilibrium solution (see Fig. 2)

$$p^* = 1/2, \qquad L^{(a)^*} = 3/2.$$

Now consider the situation depicted in (b). Player II's strategy is to choose, according to (3.2), p_1 and p_2 such that $p_1 + p_2 = 1$. Let us therfore again define $p_1 = p$ and $p_2 = 1 - p$. The probability p is now announced to player I, who is then in a position to calculate his expected payoff $L^{(b)}(\tau, p)$, where τ is the violator's strategy. With $\tau = 0(1)$ denoting violation at time $0(1)$ irrespective of the announced values of p_1, p_2, we get

$$\begin{aligned} L^{(b)}(0,p) &= 3 \cdot (1-p_1) \cdot (1-p_2) + 2 \cdot (1-p_1) \cdot p_2 + 1 \cdot p_1 \\ &= 2 - p^2, \\ L^{(b)}(1,p) &= 2 \cdot (1-p_2) + 1 \cdot p_2 \\ &= 1 + p. \end{aligned} \tag{3.15}$$

The difference between (3.14) and (3.15) is that the former represents *expectation values of payoffs*, expressed in terms of true run lengths, whereas (3.15) represents average run lengths.

Expressions (3.15) are just the two functions illustrated in Figure 3 when $k = 2$. They intersect at the point

$$p^* = \frac{\sqrt{5}-1}{2}.$$

Let us now define the following strategy τ^* for player I in the leadership game:

If the announced strategy $p \leq p^*$ then violate at time 0 otherwise violate at time 1.

It follows that the strategy pair (τ^*, p^*) is an equilibrium for the inspector leadership game: from the definition of τ^*

$$L^{(b)}(\tau, p^*) = L^{(b)}(\tau^*, p^*), \qquad \forall \tau.$$

Furthermore

$$L^{(b)}(\tau^*, p) \geq L^{(b)}(\tau^*, p^*), \qquad \forall p \in [0, 1]$$

as is obvious from Figure 3. Thus the equilibrium or saddlepoint condition

$$L^{(b)}(\tau, p^*) \leq L^{(b)}(\tau^*, p^*) \leq L^{(b)}(\tau^*, p)$$

is satisfied.

We see then that the inspector leadership game, which is solved in Theorem 1 for $n = 3$, provides a satisfactory solution to our problem. We clearly wish to generalize Theorem 1 to arbitrary n, leading us to

Theorem 2: *Given the function*

$$L(\underline{q}, \underline{p}; k) = \sum_{i=0}^{n-1} q_i \cdot L_i, \qquad n \geq 3$$

where L_i satisfies

$$\begin{aligned} L_i &= 1 + (1 - p_{i+1}) \cdot L_{i+1}, \qquad i < n-1 \\ L_{n-1} &= 1 \end{aligned} \tag{3.16}$$

and where

$$0 \leq q_i \leq 1, \qquad \sum_{i=0}^{n-1} q_i = 1$$

and

$$0 \leq p_i \leq 1, \qquad \sum_{i=1}^{n} p_i = k, \qquad p_n = 1$$

then for k in the range

$$\frac{n}{r+1} < k \leq \frac{n}{r}, \qquad r = 1, 2, \ldots n-1$$

the quantity

$$L(\underline{q}^*, \underline{p}^*; k) \equiv \min_{\underline{p}} \max_{\underline{q}} L(\underline{q}, \underline{p}; k)$$

is the positive solution of the quadratic equation

$$L^2 + (kr - 2r - 1) \cdot L - r \cdot (n - r - 1) = 0 \tag{3.17}$$

with the vector $\underline{p}^* = (p_1^* \ldots p_n^*)$ given by

$$\begin{aligned} p_i^* &= \frac{1}{L}, \qquad 1 \leq i \leq n - r - 1 \\ p_{n-r}^* &= \frac{(r + 1 - L)}{r} \\ p_i^* &= 0, \qquad n - r + 1 \leq i \leq n - 1 \\ p_n^* &= 1. \end{aligned} \tag{3.18}$$

and the vector $\underline{q}^* = (q_0^* \ldots q_{n-1}^*)$ by

$$(1, 0, 0 \ldots 0) \text{ or } (0, 1, 0 \ldots 0) \text{ or } (0, 0, 1 \ldots 0) \text{ or } \ldots \tag{3.19}$$

the last $n - r$ components being all 0's.

Proof: We begin by noting that L as defined by (3.17) is a strictly decreasing function of k:

$$L(k = \frac{n}{r}) = r, \qquad r = 1, 2 \ldots n - 1 \tag{3.20}$$

as can be seen by substitution into (3.17). Applying (3.16) to the definition of $\underline{p}^*$ in (3.18) we have

$$\begin{aligned} L_{n-1}(\underline{p}^*) &= 1 \\ L_{n-2}(\underline{p}^*) &= 2 \\ &\ldots \\ L_{n-r}(\underline{p}^*) &= r \\ L_i(\underline{p}^*) &= L, \qquad i = 0, 1 \ldots n - r - 1. \end{aligned} \tag{3.21}$$

Thus, with (3.20) it follows for $\frac{n}{r+1} < k \leq \frac{n}{r}$ that $L \geq r$ or

$$L = \max_i L_i(\underline{p}^*). \tag{3.22}$$

From (3.18) and (3.20)

$$0 \leq p_i^* \leq 1, \qquad i = 1 \ldots n$$

as required. Also using (3.17) and (3.18) we see that

$$\begin{aligned} \sum_{i=1}^{n} p_i^* &= (n - r - 1) \cdot \frac{1}{L} + \frac{r + 1 - L}{r} + 1 \\ &= \frac{(n - r - 1) \cdot r + L \cdot (r + 1) - L^2 + L \cdot r}{L \cdot r} \\ &= \frac{L^2 + (kr - 2r - 1) \cdot L + L \cdot (r + 1) - L^2 + L \cdot r}{L \cdot r} \\ &= k \end{aligned}$$

again as required.

We now show that

$$L = \min_{\underline{p}} \max_i L_i(\underline{p}), \tag{3.23}$$

that is, there is no probability vector $\underline{p}^o$ for which

$$\max_i L_i(\underline{p}^o) < L. \tag{3.24}$$

Assume such a $\underline{p}^o$ satisfying (3.24) exists. Then

$$L_i(\underline{p}^o) < L, \qquad i = 0 \ldots n - 1. \tag{3.25}$$

To simplify notation in the following, we shall write L_i for $L_i(\underline{p}^o)$ and p_i for p_i^o. Summing (3.16) from $i = j$ to $n-2$ it is easily seen that

$$L_j = n - j - \sum_{j+1}^{n-1} p_i L_i. \tag{3.26}$$

In particular for $j = 0$,

$$\begin{aligned} L_0 &= n - \sum_{1}^{n-1} p_i L_i \\ &= n - \sum_{1}^{n-r-1} p_i L_i - \sum_{n-r}^{n-1} p_i L_i. \end{aligned}$$

Again, from (3.26) with $j = n - r - 1$,

$$L_{n-r-1} = r + 1 - \sum_{n-r}^{n-1} p_i L_i.$$

Combining these last two expressions and making use of (3.25),

$$\begin{aligned} L_0 &= n - r - 1 - \sum_{1}^{n-r-1} p_i L_i + L_{n-r-1} \\ &> n - r - 1 - L \sum_{1}^{n-r-1} p_i + L_{n-r-1} \\ &= n - r - 1 - L(k - 1 - \sum_{n-r}^{n-1} p_i) + L_{n-r-1} \\ &= n - r - 1 - L(k-1) + p_{n-r}L + L_{n-r-1} + L \sum_{n-r+1}^{n-1} p_i \end{aligned}$$

where we use the convention $\sum_{n}^{n-1} p_i = 0$.

Continuing to make use of (3.16), (3.25) and also $L_{n-i} \leq i$,

$$\begin{aligned} L_0 - L \sum_{n-r+1}^{n-1} p_i &> n - r - 1 - L(k-1) + p_{n-r}L + 1 + L_{n-r} - p_{n-r}L_{n-r} \\ &\geq n - r - L(k-1) + p_{n-r}L + L_{n-r} - rp_{n-r} \\ &= n - r - L(k-1) + \frac{1 + L_{n-r} - L_{n-r-1}}{L_{n-r}}(L - r) + L_{n-r} \\ &> n - r - L(k-1) + \frac{1 + L_{n-r} - L}{r}(L - r) + L_{n-r} \end{aligned}$$

Multiplying through by r,

$$\begin{aligned} r(L_0 - L \sum_{n-r+1}^{n-1} p_i) &> r(n-r) - rL(k-1) + (1 + L_{n-r} - L)(L - r) + rL_{n-r} \\ &= (kr - 2r - 1)L + r(n - r - 1) - L^2 + L_{n-r}L \\ &= L_{n-r}L \end{aligned}$$

from (3.17). Thus we have

$$L_0 > L\left(\frac{L_{n-r}}{r} + \sum_{n-r+1}^{n-1} p_i\right).$$

Again using (3.26) now with $j = n - r$,

$$L_{n-r} = r - \sum_{n-r+1}^{n-1} p_i L_i$$

and hence

$$L_0 > L(1 + \sum_{n-r+1}^{n-1} p_i(1 - L_i/r)).$$

But since $L_i < r$ for $i > n - r$,

$$L_0(\underline{p}^o) = L_0 > L$$

which contradicts (3.25). We conclude that there is no such $\underline{p}^o$ and (3.23) holds.

To complete the proof we note that $L(\underline{q}, \underline{p}; k)$ is a linear form in $\underline{q}$. Therefore for any $\underline{p}$

$$\max_{\underline{q}} L(\underline{p}, \underline{q}; k) = \max_i L_i(\underline{p}) \tag{3.27}$$

and

$$\underline{q}^* = \arg\max_{\underline{q}} L(\underline{p}, \underline{q}; k) = (0, 0, \ldots, 1, \ldots, 0)$$

where the 1 occurs in the jth position if $\max_i L_i(\underline{p}) = L_j(\underline{p})$, say. In particular $\underline{p} = \underline{p}^*$, minimizes (3.27) and from (3.21) and (3.22) the last $n - r$ components of $\underline{q}^*$ will be 0. Finally, substituting (3.27) into (3.23) gives

$$L = \min_{\underline{p}} \max_{\underline{q}} L(\underline{q}, \underline{p}; k)$$

and we are done.

We note the following properties of the inspector leadership solution:

i. For $k = n/r$, that is when n is an integer multiple of k, equation (3.17) has the solution $L = n/k$. The optimal inspection strategy for the inspector leadership game gives an average detection time equal to the detection time for the deterministic strategy of splitting the reference time n into k equal intervals and inspecting with certainty after each interval (compare with Section 3.1 above). If n is not an integer multiple of k, then L exceeds n/k very slightly (see the numerical example in Section 4 below).

ii. Since according to (3.19), the violator is *indifferent* as to which of the first $n-r$ periods in which to act, he cannot improve his payoff by delaying his decision. This game thus avoids the objection to the matrix game solution.

Apart from giving somewhat longer guaranteed detection times than the matrix game solution, the present solution has an additional disadvantage. Player II, the inspector, cannot control the number of inspections *actually performed* in a given instance of the game. This can lead to practical difficulties in the efficient planning of inspection resources, which brings us to our third (and final) game theoretical model.

3.3. The Sequential Game

As in the matrix game dealt with in Section 3.1 we again assume that the inspector will perform precisely $k-1$ interim inspections during the reference period of n intervals. However now we relax the requirement that the players' strategies do not take into account information obtained over the course of the reference period. On the contrary, the violator's strategy takes into account the inspector's action during the course of the game. Similarly for the inspector.

The method of solution that we shall attempt to employ is that of recursion, first used successfully in a similar inspection problem by Dresher /5/. Let us call the sought after value of the sequential game $\Gamma(n,k)$. Then it is obvious that, as in the preceeding two models,

$$\Gamma(n,1)=n, \qquad \Gamma(n,n)=1. \tag{3.28}$$

Before considering the general case (n,k) we examine first two special cases.

The special case $(n,n-1)$

If the inspector uses his first opportunity, and if the violator behaves legally in the first interval, then play continues with the game $(n-1,n-2)$. If the violator behaves illegaly, the game terminates with payoff 1 to the violator. If an inspection does not occur and the violator behaves legally, play continues with the game $(n-1,n-1)$, with payoff 1 to the violator, whereas if he had chosen to violate in the first interval, detection will occur for certain after the second interval, i.e. his payoff is then 2. We thus obtain, in extensive form, the recursive game shown in Figure 6.

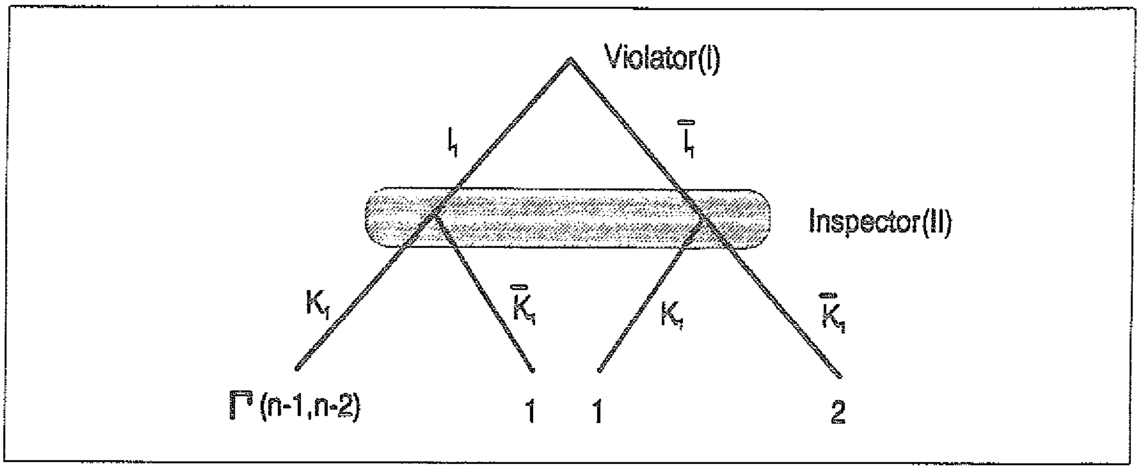

Fig. 6: Recursive extensive form of the $(n, n-1)$ game. The symbols $l_1, \bar{l}_1, K_1$ and $\bar{K}_1$ refer respectively to legal behavior, illegal behavior, inspection and no inspection in the players' initial moves.

The normal form of this game is therefore

I/II	K_1	$\bar{K}_1$
l_1	$\Gamma(n-1, n-2)$	1
$\bar{l}_1$	1	2

and the mixed strategies $(p, 1-p)$ and $(q, 1-q)$ for the violator and inspector, as well as the value $\Gamma(n, n-1)$ of the game, are determined by the following equations

$$\begin{aligned} p \cdot \Gamma(n-1, n-2) + (1-p) \cdot 1 &= \Gamma(n, n-1) \\ p \cdot 1 + (1-p) \cdot 2 &= \Gamma(n, n-1) \\ q \cdot \Gamma(n-1, n-2) + (1-q) \cdot 1 &= \Gamma(n, n-1) \\ q \cdot 1 + (1-q) \cdot 2 &= \Gamma(n, n-1). \end{aligned} \tag{3.29}$$

Note that q is the probability of *legal* behavior. Solving,

$$p = q = 2 - \Gamma(n, n-1) \tag{3.30}$$

and substitution into any of equations (3.29) gives the following recursive expression for the value of the game:

$$\Gamma(n, n-1) = 2 - \frac{1}{\Gamma(n-1, n-2)}. \tag{3.31}$$

The solution of this equation, satisfying the boundary condition $\Gamma(2, 1) = 2$, is

$$\Gamma(n, n-1) = \frac{n}{(n-1)}, \qquad n = 2, 3, \ldots \tag{3.32}$$

as can be seen by induction on n, i.e. by substitution we obtain

$$\Gamma(n, n-1) = 2 - \frac{1}{\frac{n-1}{n-2}} = \frac{n}{n-1}.$$

Substitution into (3.30) gives

$$p = q = 2 - \frac{n}{n-1} = \frac{n-2}{n-1} \qquad n = 2, 3, \ldots. \tag{3.33}$$

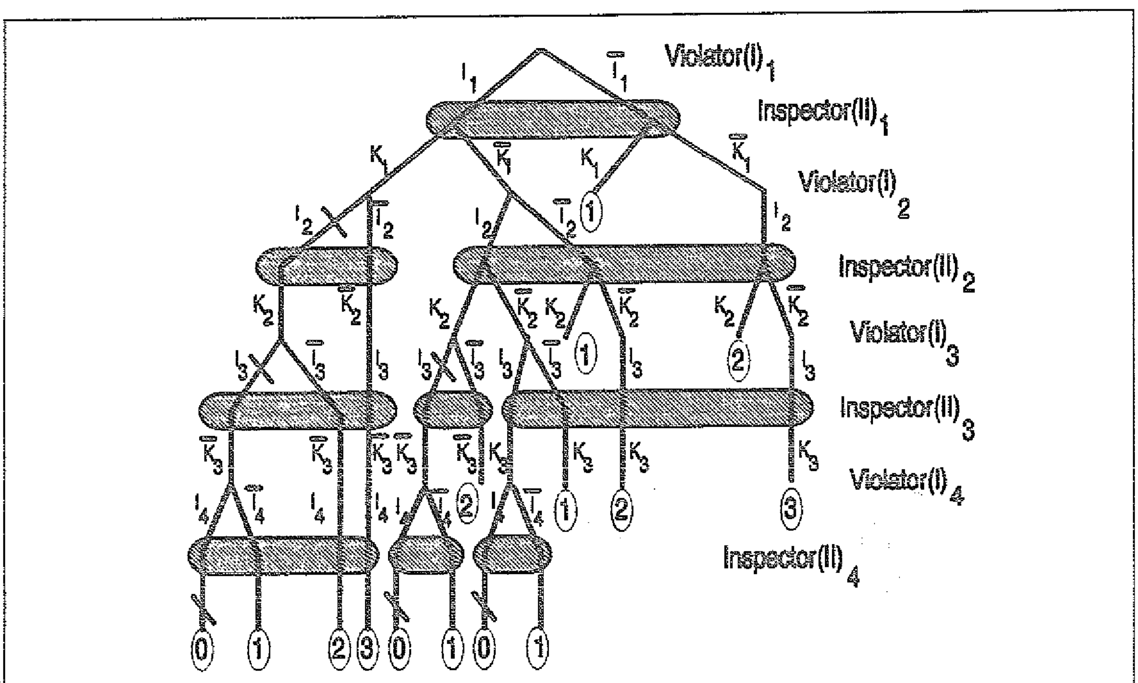

Fig. 7: Extensive form of the (4,2) game. Dominated substrategies can be deleted as indicated.

The special case $(4, 2)$

Figure 7 shows the (4,2) game in extensive form. Consider first of all the left-most branch, given by player I's strategy l_1: legal behavior in the first interval. If the inspector plays K_1 then we find ourselves in the (3,1) subgame. The information sets of this subgame are seen to be disjoint from the rest of the game, so we determine its solution immediately as $\Gamma(3, 1) = 3$. (We could have also arrived at this result by deletion of dominated substrategies as indicated in the Figure.)

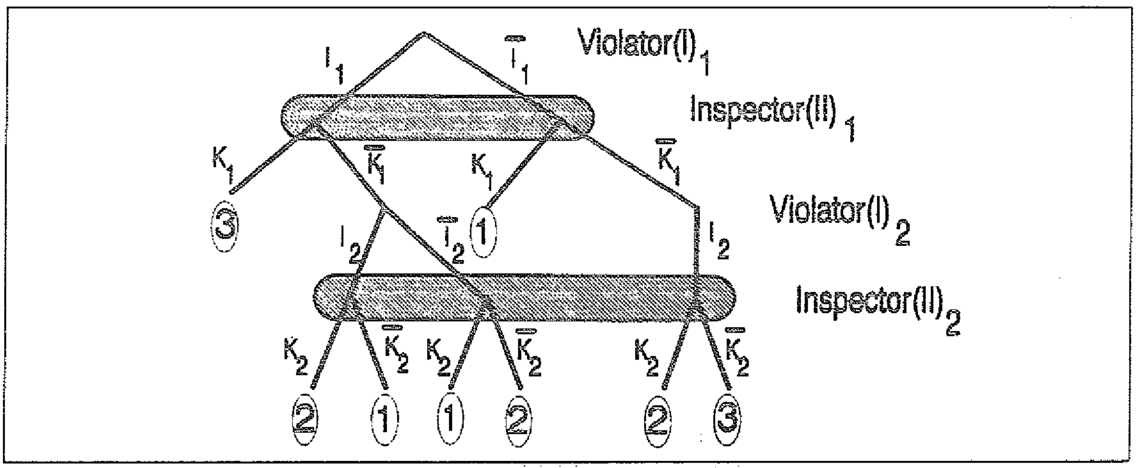

Fig. 8: Reduced extensive form of the (4,2) game.

If we consider the remaining branches and delete dominated substrategies as shown we arrive at the reduced game of Figure 8, the normal form of which can be written as

I/II	K_1	$\bar{K}_1 K_2$	$\bar{K}_1 \bar{K}_2 K_3$
$l_1 l_2$	3	2	1
$l_1 \bar{l}_2$	3	1	2
$\bar{l}_1$	1	2	3

The probabilities $p_1, p_2, p_3 = 1 - p_1 - p_2$ and $q_1, q_2, q_3 = 1 - q_1 - q_2$ with which inspector and violator play their pure strategies at equilibrium in the above matrix game, and the value $\Gamma(4,2)$ of the game, are given by the equations

$$\begin{aligned}
3p_1 + 2p_2 + 1 - p_1 - p_2 &= \Gamma(4,2) \\
3p_1 + p_2 + 2(1 - p_1 - p_2) &= \Gamma(4,2) \\
p_1 + 2p_2 + 3(1 - p_1 - p_2) &= \Gamma(4,2) \\
3q_1 + 3q_2 + (1 - q_1 - q_2) &= \Gamma(4,2) \\
2q_1 + q_2 + 2(1 - q_1 - q_2) &= \Gamma(4,2) \\
q_1 + 2q_2 + 3(1 - q_1 - q_2) &= \Gamma(4,2).
\end{aligned} \tag{3.34}$$

Their solutions are

$$\begin{aligned}
&p_1 = p_2 = 1 - p_1 - p_2 = 1/3 \\
&q_1 = 1 - q_1 - q_2 = 1/2, \;\; q_2 = 0 \\
&\Gamma(4,2) = 2.
\end{aligned} \tag{3.35}$$

We see that, at equilibrium, the inspector will choose each of his pure strategies of inspecting at times 1,2 or 3 with equal probability, whereas the violator will choose to act illegally with 50 % probability at the beginning of the first and third intervals. It is interesting to compare this result with the equivalent *simultaneous* game discussed in Section 3.1 (see Appendix B). There the equilibrium strategies for inspector and violator were calculated to be (1/3,1/2,1/6) and (1/6,1/3,1/2) respectively and the value of the game was $\Gamma = 11/6$. We will discuss these differences in more detail in the Section 4.

Let us return again to Figure 8 and the following heuristic postulate: at play II_2 (the inspector's second move) the inspector will choose between his two possibilities K_2 and $\bar{K}_2$ so as to make the violator *indifferent* to his two strategies l_1 and $\bar{l}_1$. Moreover, in doing so, he will *ignore the rightmost branch* leading from I_1 because there the violator has no further freedom of action – he has *already* behaved illegally, and, according to the rules of the game, he does so only once. To bring about indifference, the inspector will thus choose K_2 with probability p_2 such that

$$2p_2 + (1 - p_2) = p_2 + 2(1 - p_2) \quad \text{or} \quad p_2 = 1/2$$

thus leading to the extensive form of Figure 9.

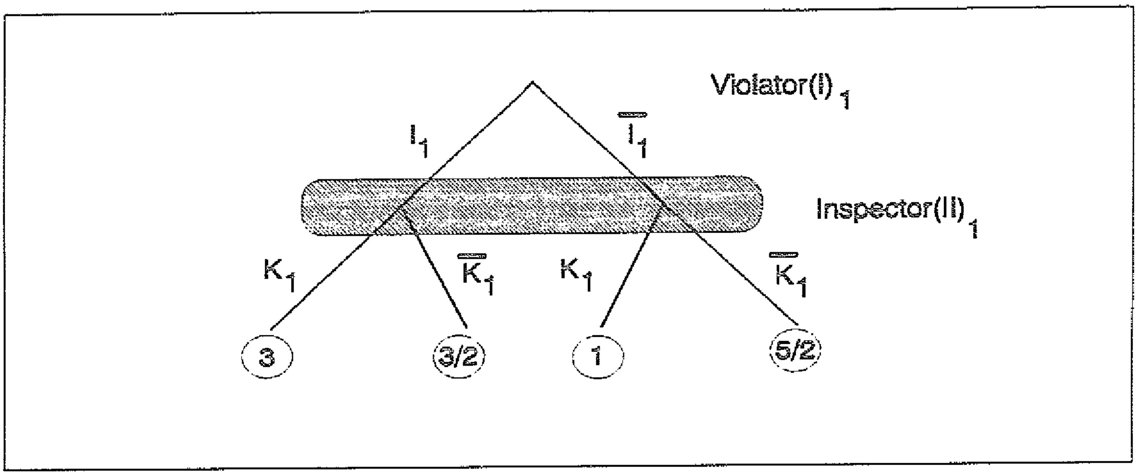

Fig. 9: Further reduction in the extensive form of the (4,2) game.

In fact, if we write this reduced game in normal form and determine its equilibrium, we obtain the correct solution derived previously without our heuristic assumption.

Now we note from Figure 9 that the leftmost branch but one has the payoff for the (3,2) game, $\Gamma(3,2) = 3/2$ (see equation (3.32)), that is we have the impression that we are allowed here to consider the (3,2) subgame in the continuation from this point , while for the rightmost branch the payoff seems to be $1 + \Gamma(3,2)$, as if the game here continues like the (3,2) game, but with the violator 'one step ahead' as it were. The fact that the information sets of the subgames overlap does not seem to matter. These observations, which hold for other special cases of managable size that have been examined, such as the (5,3) game, suggest an easy generalization of the special solutions to arbitrary (n, k).

<u>The general case (n, k)</u>

The extended form of the (n, k) game is shown recursively in Figure 10. We emphasize once again that this representation of the game tree is not strictly justified:

— The inspector, in the left hand branch at II_1 does not know whether the violator has, to that point, behaved legally or not, as indicated by his information set.

— The violator in the subgame beginning at the rightmost branch at I_2 can only behave legally.

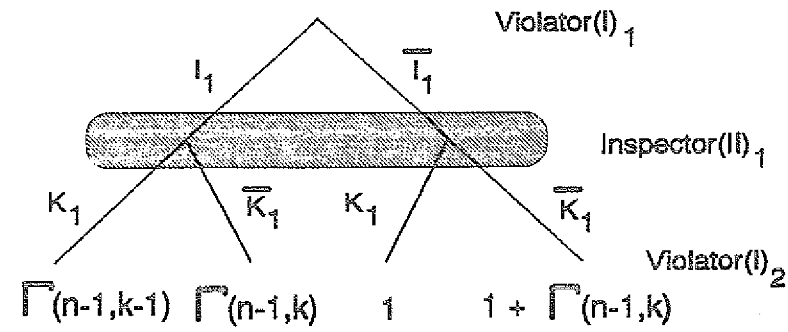

Fig. 10: Recusive extensive form of the (n, k) game.

We shall proceed nevertheless on the assumption, justified by the special cases treated above, that this recursive representation is correct. A mathematical proof is indeed possible /6/.

The normal form of the game of Figure 10 is then

I/II	K_1	$\bar{K}_1$
I_1	$\Gamma(n-1,k-1)$	$\Gamma(n-1,k)$
$\bar{I}_1$	1	$1+\Gamma(n-1,k)$

The solution of the mixed extension of this game, in which the inspector plays the mixed strategy $(p, 1-p)$ and the violator $(q, 1-q)$, is given by

$$\begin{aligned}
&p \cdot \Gamma(n-1, k-1) + (1-p) \cdot \Gamma(n-1, k) = \Gamma(n, k) \\
&p + (1-p) \cdot (1 + \Gamma(n-1, k)) = \Gamma(n, k) \\
&q \cdot \Gamma(n-1, k-1) + 1 - q = \Gamma(n, k) \\
&q \cdot \Gamma(n-1, k) + (1-q) \cdot (1 + \Gamma(n-1, k)) = \Gamma(n, k).
\end{aligned} \tag{3.36}$$

From the first two equations we obtain immediately

$$p = \frac{1}{\Gamma(n-1, k-1)} \tag{3.37}$$

and

$$\Gamma(n, k) = 1 + (1 - \frac{1}{\Gamma(n-1, k-1)}) \cdot \Gamma(n-1, k). \tag{3.38}$$

The solution of this functional equation, satisfying the boundary conditions (3.28) is given by

$$\Gamma(n,k) = \frac{n}{k}, \qquad 1 \le k \le n \tag{3.39}$$

as can be seen immediately from substitution. We obtain finally from (3.36), (3.37) and (3.38)

$$p = \frac{k-1}{n-1}, \qquad q = \frac{k-1}{k}. \tag{3.40}$$

To conclude this discussion of the sequential game, let us calculate the unconditional probabilities with which the inspector chooses his equilibrium strategies. From (3.40) the probability of inspecting in the first period, $P(K_1)$, and not inspecting, $P(\bar{K}_1)$, are, respectively,

$$P(K_1) = \frac{k-1}{n-1}, \qquad P(\bar{K}_1) = 1 - \frac{k-1}{n-1} = \frac{n-k}{n-1}. \tag{3.41}$$

The conditional probability for an inspection in the second interval, given one in the first interval, is

$$P(K_2 \mid K_1) = \frac{k-2}{n-2} \tag{3.42}$$

so that the combined probability for (K_2, K_1) is

$$P(K_2, K_1) = P(K_2 \mid K_1) \cdot P(K_1) = \frac{k-2}{n-2} \cdot \frac{k-1}{n-1}. \tag{3.43}$$

Similarly, the conditional probability for an inspection in the second interval, given none in the first interval, is

$$P(K_2 \mid \bar{K}_1) = \frac{k-1}{k-2}$$

and we have

$$P(K_2, \bar{K}_1) = P(K_2 \mid \bar{K}_1) \cdot P(\bar{K}_1) = \frac{k-1}{n-2} \cdot \frac{n-k}{n-1}$$

and hence

$$\begin{aligned} P(K_2) &= P(K_2, K_1) + P(K_2, \bar{K}_1) \\ &= \frac{k-2}{n-2} \cdot \frac{k-1}{n-1} + \frac{k-1}{n-2} \cdot \frac{n-k}{n-1} \\ &= \frac{k-1}{n-1}, \end{aligned} \tag{3.44}$$

that is, we obtain the same value as for $P(K_1)$.

We show by induction that, at equilibrium, the inspector chooses his pure strategies, that is all possibilities to distribute $k-1$ inspections among $n-1$ opportunities, with equal probability.

By the induction assumption, let this be the case for the $(n-1, k-1)$ game. Then

$$P(\text{any pure strategy in } (n-1, k-1)) = \frac{1}{\binom{n-2}{k-2}}. \tag{3.45}$$

Consider the leftmost branch of Figure 10. We can write

$$P(\text{any pure strategy in } (n, k) \mid K_1) = \frac{1}{\binom{n-2}{k-2}}, \tag{3.46}$$

and therefore, using (3.40) we have the combined probability

$$\begin{aligned} P(\text{any pure strategy in } (n, k), K_1) &= \frac{1}{\binom{n-2}{k-2}} \cdot \frac{k-1}{n-1} \\ &= \frac{1}{\binom{n-1}{k-1}}. \end{aligned} \tag{3.47}$$

A similar argument gives us

$$P(\text{any pure strategy in } (n, k), \bar{K}_1) = \frac{1}{\binom{n-1}{k-1}} \tag{3.48}$$

so that, irrespective of whether the game (n, k) begins with K_1 or $\bar{K}_1$, the probabilities of all pure inspection strategies are equal. It follows that the probabilities that an inspection occurs ay any given time $i, i = 1 \ldots n-1$, are also equal, as we have already seen in equations (3.41) and (3.44).

4. Discussion

The inspector's guaranteed detection times for the three approaches discussed in Section 3 are compared in Table 2 for the case $n = 10$ and all values of $k = 1, 2 \ldots 10$. Included in the table are the average run lengths that would be obtained if, on the average, $k-1$ interim inspections are distributed with equal and independent probabilities over the $n-1$ opputunities. From equations (2.4) through (2.9) one sees immediately that it is then optimal for the violator to act in the first interval. Accordingly equation (2.4) is used to obtain the figures in the last column, with $\beta_1 = \ldots = \beta_{n-1} = 1 - \frac{k-1}{n-1}$. Except for the trivial cases $k = 1$ and $k = 10$ the matrix game solution always gives the best result. All three game theoretical solutions are better than for equally distributed probabilities.

TABLE 2 *Comparison of solutions for* $n = 10$.

k	Matrix	Leadership	Sequential	Equal Prob.
1	10.00	10.00	10.00	10.00
2	3.987	5.000	5.000	6.228
3	2.558	3.359	3.333	4.135
4	1.946	2.531	2.500	2.948
5	1.645	2.000	2.000	2.244
6	1.444	1.702	1.667	1.799
7	1.333	1.464	1.429	1.500
8	1.222	1.275	1.250	1.286
9	1.111	1.123	1.111	1.125
10	1.000	1.000	1.000	1.000

In addition to the equilibrium solutions in terms of average run lengths, the distributions of the run lengths about the average value are of interest. Monte Carlo simulations of typical distributions are shown in Figure 11. The matrix game solution is seen not only to have the lowest mean detection time, but also the narrowest distribution. Note also that in the inspector leadership game there is always a small but finite probability that detection will occur only after the nth period. In all cases, the probability that a detection occurs *before* the deterministic detection time n/k is greater than 50 % (although the *mean* of the distribution is less than n/k only for the matrix game solution.)

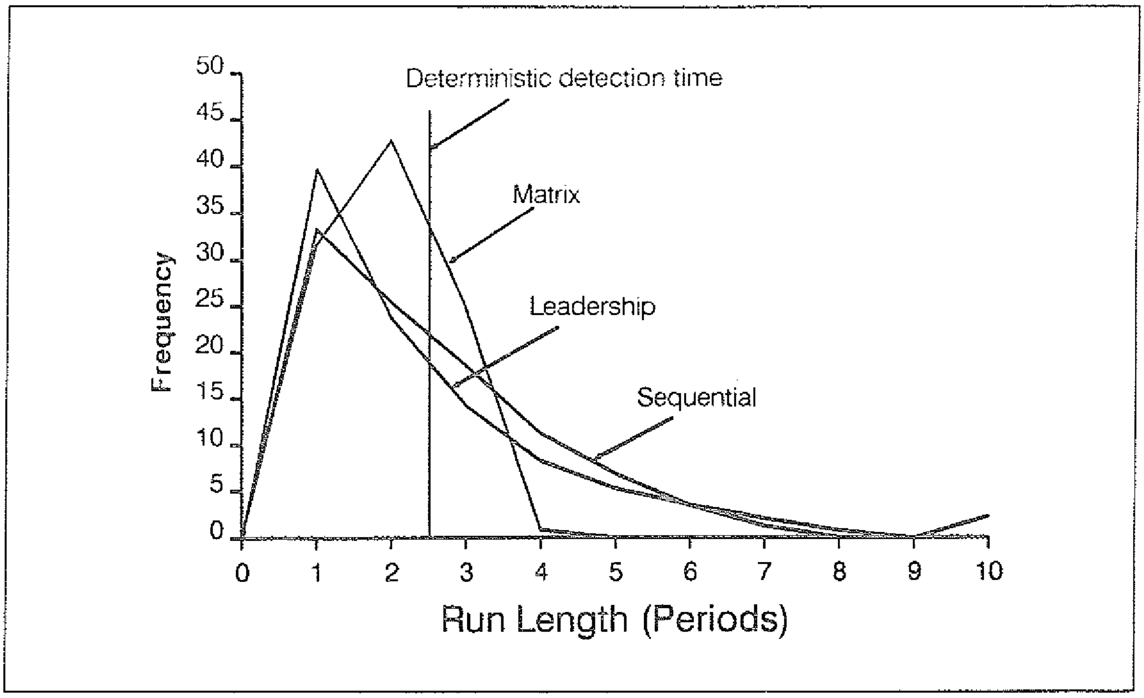

Fig. 11: Simulated probability distribution of run lengths to detection for $n = 10$ and $k = 4$ for the three games. For the inspector leadership game, violation was assumed to be at time $i = 0$.

We conclude with an example of a real application. The detection time goal for international inspections of irradiated fuel elements at light water reactors is currently 3 months /7/. The International Atomic Energy Agency achieves this goal by sending inspectors to the reactor sites at 3 month intervals, with a major inspection (so-called physical inventory verification) occurring once per year during reactor refuelling and maintenance. This corresponds to $n = k = 4$ in our models and a period length of 3 months. The interim inspection activities generally involve the checking of seals and reviewing of surveillance films, with virtually 100 % detection probability for diversion of a complete fuel assembly.

Figure 12 shows a plot of detection time in months as a function of increasing n, as the reference time of one year is subdivided into smaller and smaller intervals, with the number of interim inspections held constant at $k = 4$ and $k = 3$, respectively. The curves correspond to the matrix game saddlepoint solutions of

Section 3.1 (see Appendix B). The guaranteed detection times are seen to decrease substantially with n, so that for $n = 12$ the timeliness goal can be achieved with $k = 3$, that is with only two random interim inspections as opposed to the current practice of three deterministic interim inspections. The prerequisites for realization of such a scheme are that the reactor operator agree to an inspection opportunity once each calendar month, and that the simultaneity assumption (advance commitment to a diversion strategy) is valid.

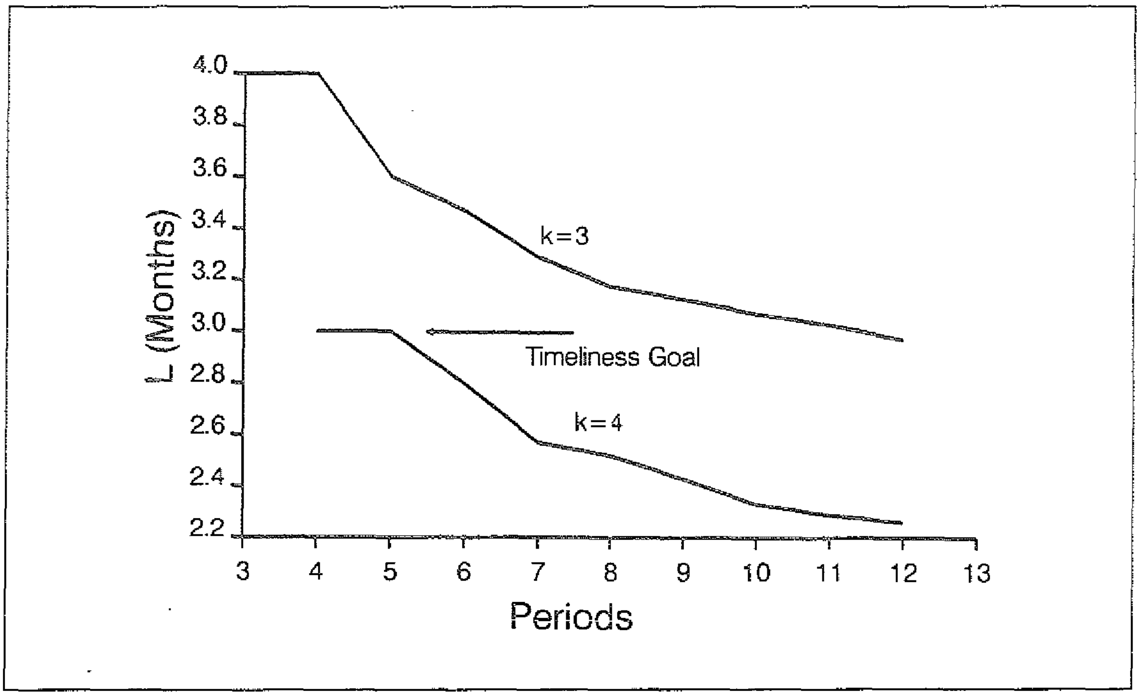

Fig. 12: Guaranteed detection times for a one year reference period, as a function of the number n of inspection intervals, for two values of k.

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Appendix A. Construction of the Payoff Matrix

In order to solve the finite matrix games of Section 3.1 it is necessary to construct the payoff matrix, that is the matrix of detection times for each strategy combination $(i, j), i = 0 \ldots n-2, j = 1 \ldots \binom{n-1}{k-1}$ for players I and II respectively. Let the matrix elements be denoted $L_{i,j}$. A convenient way of determining the $L_{i,j}$ is with the following

Algorithm:

1. Set the row index of the payoff matrix $i := 0$.
2. Place all pure strategies of player II (the inspector) into a set S. These can be generated lexicographically. (For example for the $n = 5, k = 3$ game the set would consist of the 6 strategies in the first row of Table 1 in Section 3.1, viz. $S = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$). Label the strategies in the set $s = 1 \ldots \binom{n-1}{k-1}$.
3. Set $j := 1$.
4. Remove all strategies from S which contain the index $i+j$. If $i+j = n$ remove all of the rest of the strategies from S as well. Put $L_{i,s} := j$ for each strategy with index s thus removed. If S is empty go to step 6.
5. Set $j := j+1$. If $j \leq \binom{n-1}{k-1}$ go to step 4.
6. Increment the row index, $i := i+1$. If $i = n-1$ then stop, else go to step 2.

Appendix B. Tabulation of Numerical Solutions to the Matrix Game

The following listing presents the saddlepoint solutions to the matrix game for values of n ranging from 3 up to 12. The solutions are given each in three rows. The elements of first row are n, k, and the value of the game (inspector's guaranteed detection time), respectively. The second row lists, in binary code, all of the pure inspection strategies which contribute to the inspector's optimal mixed (randomized) strategy. A '1' in position i indicates that an inspection should take place at time i. Only the first n digits are significant. The third row gives the associated probabilities with which the pure strategies should be chosen.

```
 3  2   1.500
101000000000 011000000000
0.500        0.500

 4  2   1.833
100100000000 010100000000 001100000000
0.333        0.500        0.167

 4  3   1.333
110100000000 101100000000 011100000000
0.333        0.333        0.333

 5  2   2.167
100010000000 010010000000 001010000000
0.250        0.333        0.417

 5  3   1.500
101010000000 010110000000
0.500        0.500

 5  4   1.250
111010000000 110110000000 101110000000 011110000000
0.250        0.250        0.250        0.250

 6  2   2.567
100001000000 010001000000 001001000000 000101000000
0.200        0.250        0.333        0.217

 6  3   1.735
101001000000 100101000000 010101000000 010011000000 001011000000
0.235        0.176        0.324        0.118        0.147

 6  4   1.400
110101000000 101011000000 011101000000 010111000000
0.200        0.400        0.200        0.200

 6  5   1.200
111101000000 111011000000 110111000000 101111000000 011111000000
0.200        0.200        0.200        0.200        0.200

 7  2   2.900
100000100000 010000100000 001000100000 000100100000 000010100000
0.167        0.200        0.250        0.333        0.050

 7  3   1.922
101000100000 100100100000 010100100000 010010100000 001010100000 001001100000
0.088        0.275        0.059        0.294        0.206        0.078

 7  4   1.500
101010100000 010101100000
0.500        0.500

 7  5   1.333
110101100000 101110100000 011011100000
0.333        0.333        0.333

 7  6   1.167
111110100000 111101100000 111011100000 110111100000 101111100000 011111100000
0.167        0.167        0.167        0.167        0.167        0.167

 8  2   3.279
100000010000 010000010000 001000010000 000100010000 000010010000
0.143        0.167        0.200        0.250        0.240
```

8 3 2.120
100100010000 100010010000 010010010000 010001010000 001001010000 000101010000
0.227 0.080 0.253 0.060 0.333 0.047

8 4 1.680
101010010000 101001010000 100101010000 010101010000 010100110000 010010110000 001010110000
0.180 0.135 0.124 0.242 0.090 0.112 0.118

8 5 1.429
110101010000 101101010000 101010110000 011011010000 010110110000 010101110000
0.143 0.143 0.286 0.143 0.143 0.143

8 6 1.286
111010110000 110111010000 110101110000 101111010000 101110110000 011101110000 011011110000
0.143 0.143 0.143 0.143 0.143 0.143 0.143

8 7 1.143
111111010000 111110110000 111101110000 111011110000 110111110000 101111110000 011111110000
0.143 0.143 0.143 0.143 0.143 0.143 0.143

9 2 3.654
100000001000 010000001000 001000001000 000100001000 000010001000 000001001000
0.125 0.143 0.167 0.200 0.250 0.115

9 3 2.358
100100001000 100010001000 010010001000 010001001000 001001001000 001000101000 000100101000
0.096 0.178 0.072 0.196 0.137 0.147 0.174

9 4 1.822
101001001000 100101001000 100100101000 010100101000 010010101000 001010101000 001010011000 001001011000
0.183 0.138 0.073 0.168 0.221 0.038 0.161 0.018

9 5 1.500
101010101000 010101011000
0.500 0.500

9 6 1.375
110110101000 110101011000 101101101000 101101011000 101010111000 011011101000 011010111000 010111011000
0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125

9 7 1.250
111101011000 111011101000 101110111000 010111111000
0.250 0.250 0.250 0.250

9 8 1.125
111111101000 111111011000 111110111000 111101111000 111011111000 110111111000 101111111000 011111111000
0.125 0.125 0.125 0.125 0.125 0.125 0.125 0.125

10 2 3.987
100000000100 010000000100 001000000100 000100000100 000010000100 000001000100 000000100100
0.111 0.125 0.143 0.167 0.200 0.250 0.004

10 3 2.558
100010000100 100001000100 010001000100 010000100100 001000100100 000100100100 000100010100 000010010100
0.195 0.044 0.206 0.035 0.250 0.048 0.214 0.008

10 4 1.946
101001000100 100101000100 100100100100 010101000100 010010010100 001010100100 001010010100 001001010100 001001001100
0.053 0.005 0.293 0.055 0.296 0.040 0.010 0.193 0.054

10 5 1.645
101010100100 101010010100 101001010100 100101010100 010101010100 010101001100 010100101100 010010101100 001010101100
0.145 0.109 0.100 0.098 0.194 0.073 0.091 0.095 0.096

10 6 1.444
110101010100 101101010100 101010101100 011011010100 010110101100 010101110100 010101011100
0.111 0.111 0.333 0.111 0.111 0.111 0.111

10 7 1.333
111010101100 110101011100 101110110100 101011011100 011101101100 010111110100
0.167 0.167 0.167 0.167 0.167 0.167

10 8 1.222
111110101100 111101110100 110111101100 110110111100 101111011100 011011111100
0.111 0.222 0.111 0.111 0.222 0.222

10 9 1.111
111111110100 111111101100 111111011100 111110111100 111101111100 111011111100 110111111100 101111111100 011111111100
0.111 0.111 0.111 0.111 0.111 0.111 0.111 0.111 0.111

11 2 4.383
100000000010 010000000010 001000000010 000100000010 000010000010 000001000010 000000100010
0.100 0.111 0.125 0.143 0.167 0.200 0.154

11 3 2.775
100010000010 100001000010 010001000010 010000100010 001000100010 001000010010 000100010010 000100001010 000010001010
0.104 0.117 0.083 0.133 0.117 0.107 0.227 0.019 0.094

11 4 2.089
100100100010 100100010010 100010010010 100010001010 010010010010 010001001010 001001001010 001000101010 000100101010
0.211 0.060 0.002 0.045 0.272 0.046 0.287 0.035 0.043

11 5 1.765
101010010010 101001010010 100101010010 100100101010 010101001010 010100101010 010010101010 110010100110 001010101010 101001010110
0.165 0.071 0.028 0.147 0.199 0.035 0.046 0.132 0.074 0.103

11 6 1.500
101010101010 110101010110
0.500 0.500

11 7 1.400
110101010110 101101101010 101010101110 111011010110 010110111010
0.200 0.200 0.200 0.200 0.200

11 8 1.300
111101101010 111011010110 111010111010 110101011110 101111011010 101110101110 101011110110 111110110110 111101101110 010111111010
0.057 0.114 0.071 0.157 0.029 0.157 0.114 0.071 0.086 0.143

11 9 1.200
111111010110 111110111010 110111101110 101101111110 111011111110
0.200 0.200 0.200 0.200 0.200

11 10 1.100
111111111010 111111110110 111111101110 111111011110 111110111110 111101111110 111011111110 110111111110 101111111110 111111111110
0.100 0.100 0.100 0.100 0.100 0.100 0.100 0.100 0.100 0.100

12 2 4.746
100000000001 010000000001 001000000001 000100000001 000010000001 000001000001 000000100001 000000010001
0.091 0.100 0.111 0.125 0.143 0.167 0.200 0.063

12 3 2.969
100010000001 100001000001 100000100001 010000100001 010000010001 001000010001 000100010001 000100001001 000010001001 100010000101
0.035 0.167 0.005 0.295 0.004 0.200 0.046 0.163 0.170 0.015

12 4 2.258
100100100001 100100010001 100010010001 010010010001 010010001001 010001001001 110001000101 001001001001 101000100101 100100100101
0.046 0.119 0.127 0.004 0.152 0.019 0.114 0.163 0.128 0.129

12 5 1.868

101001001001	100101001001	100100101001	100100100101	010101010001	110100100101	110010100101	110010010101	101010100101	101010010011	101001010011
0.132	0.102	0.099	0.044	0.034	0.099	0.012	0.233	0.113	0.018	0.115

12 6 1.622

101010101001	101010100101	101010010101	101010010011	101001010101	100101010101	110101010101	110101001011	110100101011	110010101011	101010101011
0.122	0.091	0.023	0.061	0.082	0.081	0.223	0.076	0.080	0.081	0.081

12 7 1.455

110101010101	101101010101	101011010101	101010101011	011010101011	010110110101	010101101011	010101011101	010101010111
0.091	0.091	0.091	0.273	0.091	0.091	0.091	0.091	0.091

12 8 1.364

110101101011	101110110101	101011011101	101011010111	111110101011	111101010111	111010111101	110110111101
0.273	0.091	0.091	0.182	0.091	0.091	0.091	0.091

12 9 1.273

111101101011	111101011011	111010101111	110111010111	110110110111	101110111101	101011111101	111110110111	110111011111
0.227	0.045	0.045	0.091	0.045	0.045	0.227	0.136	0.136

12 10 1.182

111111101011	111111011101	111011110111	110111110111	110110111111	101110111111	101101111111	111101111111	111011111111
0.182	0.182	0.091	0.091	0.091	0.091	0.091	0.091	0.091

12 11 1.091

111111111101	111111111011	111111110111	111111101111	111111011111	111110111111	111101111111	111011111111	110111111111	101111111111	111111111111
0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091