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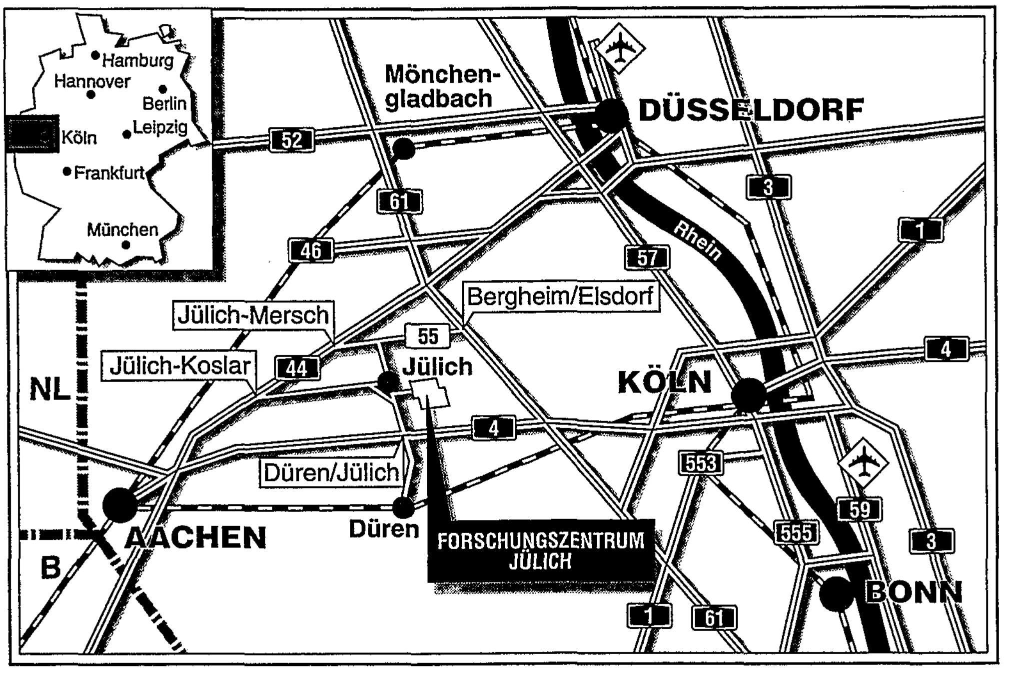

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ABSTRACT

Sequential statistical decision tests for detecting clandestine removal of nuclear material from a running processing plant using material accountancy are investigated. Part of the process material is assumed to be unmeasurable and the moments (mean and variance) of the unmeasured inventory are estimated with the same set of sequential material balance data as is used for the decision test. For uncorrelated, normally distributed process fluctuations detection times comparable to those obtained with a priori known mean and variance of the unmeasured inventory are obtained.

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1. Introduction

A well-known problem in the application of Near Real Time Accountancy (NRTA), especially to plutonium reprocessing, is that a part of the process inventory is inaccessible to direct measurement during plant operations. Typically, this is the material in the contactors and evaporators of the various plutonium separation and purification cycles. It was pointed out some time ago /1,2,3/ and also more recently /4/, that in such cases it may be necessary to make use of estimates of unmeasured inventories as well as of their variances. Numerical calculations /1,2,3/ indicate that, even if the expected value and variance of the unmeasured process inventory are not negligible, the loss in NRTA detection capability is rather small.

In /4/ it has also been proposed that the unmeasured process inventory and/or its variance might be estimated from the same set of sequential material balance data which is used for testing for material diversion. It is the purpose of the present study to investigate quantitatively how this procedure might work and to compare it with the conventional approach of using *a priori* estimates.

We begin, in Section 2, with a quantitative formulation of the problem and the methods of solution. In Section 3 the solution concept is illustrated for the case that the effectiveness of NRTA is measured by its *probability* of detecting material diversions. It is pointed out in Section 4.1 that in this instance, provided there are regular (e.g. annual) washouts of the process area, the unmeasured inventory is irrelevant.

Unmeasured inventory can be important, however, if the criterion of effectiveness is the *timeliness* of diversion detection. Section 4.2 examines in some detail the definition of an appropriate measure of timeliness and presents an approximate analytical solution of the problem which again indicates the relative insensitivity to unmeasured inventory. Finally, in Section 5 we examine with the aid of simulation the impact on the timeliness of NRTA of estimating unmeasured inventory mean and variance and testing for diversion simultaneously.

2. Quantitative Formulation

We focus attention on the process material balance area of a commercial reprocessing plant, and consider a ***reference time period*** $[t_0, t_n]$ (typically an operating year, or the time interval between consecutive clean-out physical inventories), which we divide into n ***inventory periods*** $[t_{i-1}, t_i]$, $i = 1, ..., n$. At time t_0 the inventory I_0 is accurately determined (for instance by wash-out, in which case the expected value of the inventory can be taken to be nil) and in each subsequent inventory period $[t_{i-1}, t_i]$ the cumulative receipts R_i and shipments S_i (including waste) are determined, giving the net transfers $D_i = R_i - S_i$. At time t_i the ***main part*** P_i of the in-process inventory is also measured, but there remains some portion inaccessible to measurement.

Now let us define

$$Y_i := I_0 + \sum_{j=1}^{i}(R_i - S_i) - P_i = I_0 + \sum_{j=1}^{i} D_i - P_i, \ i = 1, ..., n. \tag{2.1}$$

The physical meaning of this quantity will be explained shortly in connection with the definition of the null hypothesis.

All quantities introduced so far have measurement errors associated with them. Hence, in the usual terminology, we define

$$I_0 = X_0, \ E(X_0) = 0, \ var(X_0) = \sigma_0^2 \tag{2.2}$$

$$D_i = E(D_i) + X_i^D, \ E(X_i^D) = 0, \ var(X_i^D) = \sigma_D^2, \ i = 1, ..., n \tag{2.3}$$

$$P_i = E(P_i) + X_i^P, \ E(X_i^P) = 0, \ var(X_i^P) = \sigma_P^2, \ i = 1, ..., n. \tag{2.4}$$

The X are thus random variables which characterize the measurement errors. Their expectation values are taken to be zero, that is we assume there are no systematic errors,[1] and their variances are presumed known. Furthermore, all errors are taken to be uncorrelated:

$$cov(X_0, X_j^P) = cov(X_0, X_j^D) = cov(X_i^P, X_j^D) = cov(D_i, D_j) = 0 \ \text{ for } \ i \neq j. \tag{2.5}$$

[1]This simplifies the algebra in the following discussion considerably, without detracting from the essence of the problem. In the simulation experiments to be discussed in Section 5 systematic error components were in fact included.

Under the *null hypothesis* there is no material loss or diversion, so that from (2.1)

$$E_0^X(Y_i) = 0 + \sum_{j=1}^{i} E(D_j) - E(P_i) = M_i, \ i = 1, ..., n, \tag{2.6}$$

whereby M_i is the ***unmeasured process inventory*** at time t_i. This is then the physical interpretation of the Y_i: under the null hypothesis Y_i is an estimate of the unmeasured process inventory at time t_i. We now write, with $F \equiv$ fluctuation,

$$M_i = \theta + F_i, \ E_0^F(F_i) = 0, \ var_0^F(I_i) = \sigma_F^2, \tag{2.7}$$

that is, we treat M_i as a random variable with expectation value θ and variance σ_F^2.

Substituting (2.2-2.7) into (2.1) we obtain

$$Y_i = X_0 + \sum_{j=1}^{i} X_i^D - X_i^P + \theta + F_i, \ i = 1, ..., n. \tag{2.8}$$

From (2.8) the null hypothesis may now be expressed formally as

$$E_0(Y_i) = \theta \tag{2.9}$$

$$var_0(Y_i) = \sigma_0^2 + i \cdot \sigma_D^2 + \sigma_P^2 + \sigma_F^2 =: \sigma_i^2, \ i = 1, ..., n. \tag{2.10}$$

It should be remarked that we have formed an expected value and variance for a linear combination of two different types of random variable. It is methodologically clearer if we make use of the *conditional expected value* of (2.6): We obtain with the help of well-known formulae (see e.g. /5/)

$$E_0(Y_i) = E_0^F(E_0^X(Y_i \mid M_i)) = \theta$$

$$\begin{aligned} var_0(Y_i) &= var_0^F(E_0^X(Y_i \mid M_i)) + E_0^F(var_0^X(Y_i \mid M_i)) \\ &= \sigma_F^2 + \sigma_0^2 + i \cdot \sigma_D^2 + \sigma_P^2, \ i = 1, ..., n, \end{aligned}$$

which corresponds to (2.9) and (2.10).

Turning now to the *alternative hypothesis* H_1, assume that within the ith inventory period $[t_{i-1}, t_i]$ the amount of material μ_i, $i = 1, ..., n$ is diverted. Then from (2.1) under H_1

$$E_1(Y_i) = \theta + \sum_{j=1}^{i} \mu_j \tag{2.11}$$

$$var_1(Y_i) = var_0(Y_i) = \sigma_i^2, \; i = 1, ..., n. \tag{2.12}$$

Now assuming, as mentioned above, that all moments (in particular θ and σ_F^2) are known, we can carry out a simple significance test to decide between H_0 and H_1 after the ith inventory period: We conclude

$$\begin{aligned} Y_i - \theta \leq s &: H_0 \text{ correct} \\ Y_i - \theta > s &: H_1 \text{ correct.} \end{aligned} \tag{2.13}$$

The test threshold is determined by the desired *false alarm probability* α_i according to

$$1 - \alpha_i = Pr_0(Y_i - \theta \leq s) = \Phi(\frac{s}{\sigma_i}), \; i = 1, ..., n,$$

where $\Phi(\cdot)$ is the standard normal distribution. The *detection probability* $1 - \beta_i$ is given by

$$\beta_i = Pr_1(Y_i - \theta \leq s) = \Phi(\frac{s - \sum_{j=1}^{i} \mu_j}{\sigma_i}). \tag{2.14}$$

Eliminating s in favour of α_i from (2.14), we obtain

$$1 - \beta_i = \Phi(\frac{1}{\sigma_i} \cdot \sum_{j=1}^{i} \mu_j - U(1 - \alpha_i)), \tag{2.15}$$

where $U(\cdot)$ is the inverse function of $\Phi(\cdot)$.

Nothing will be said at this point about the fact that the test statistics $Y_i - \theta, \; i = 1, 2, \ldots$ are not independent and that consequently, for a series of n tests, the resulting non-false alarm probability can't be obtained simply as the product of the single-period non-false alarm probabilities. Consequently, we leave open for the time being the question of the appropriate choice of the individual false alarm probabilities $\alpha_i, \; i = 1, 2, \ldots$.

3. Estimating the Variance of the Unmeasured Inventory

Let us now assume that it is not possible (or perhaps not desirable) to obtain an 'advance estimate' of the moments of the distribution of process fluctuations in the unmeasured inventory. One reason could be that actual process variations may differ considerably from one reprocessing campaign to the next, or that historical data are unreliable (or, given the mandate of the IAEA, *a priori* suspect). The moments must then be estimated from the same set of data as is used for the statistical test. We consider here an estimator for the variance of the process fluctuation distribution. Estimation of the mean will be included in section 5.

Corresponding to a suggestion in /4/, let us define

$$\bar{Y}_n^2 := \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} (Y_{i+2} - Y_i)^2. \tag{3.1}$$

The choice of averaging over the differences of every *second* Y_i-value in (3.1) was proposed in /4/ to avoid the effect of (presumably weak) autocorrelations in the time sequence of process fluctuations F_i. Under the null hypothesis H_0, from the linearity of the expected value,

$$\begin{aligned} E_0(\bar{Y}_n^2) &= \frac{1}{2}\frac{1}{n-2} \sum_{i=1}^{n-2} E_0((Y_{i+2} - Y_i)^2) \\ &= \frac{1}{2}\frac{1}{n-2} \sum_{i=1}^{n-2} E_0((Y_{i+2} - \theta - (Y_i - \theta))^2) \\ &= \frac{1}{2}\frac{1}{n-2} \sum_{i=1}^{n-2} E_0((Y_{i+2} - \theta)^2 + (Y_i - \theta)^2 - 2 \cdot (Y_{i+2} - \theta) \cdot (Y_i - \theta)). \end{aligned} \tag{3.2}$$

Now, according to (2.8),

$$\begin{aligned} E_0((Y_{i+2} - \theta)^2) &= \sigma_0^2 + (i+2) \cdot \sigma_D^2 + \sigma_P^2 + \sigma_F^2 \\ E_0((Y_i - \theta)^2) &= \sigma_0^2 + i \cdot \sigma_D^2 + \sigma_P^2 + \sigma_F^2 \\ E_0((Y_{i+2} - \theta)(Y_i + \theta)) &= \sigma_0^2 + i \cdot \sigma_D^2, \end{aligned} \tag{3.3}$$

from which

$$\begin{aligned} E_0(\bar{Y}_n^2) &= \frac{1}{2}\frac{1}{n-2} \sum_{i=1}^{n-2} [\sigma_0^2 + (i+2)\sigma_D^2 + \sigma_P^2 + \sigma_F^2 + \sigma_0^2 + i\sigma_D^2 + \sigma_P^2 + \\ &\quad \sigma_F^2 - 2 \cdot (\sigma_0^2 + i\sigma_D^2)] \\ &= \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} [2 \cdot \sigma_D^2 + 2 \cdot \sigma_P^2 + 2 \cdot \sigma_F^2] \\ &= \sigma_D^2 + \sigma_P^2 + \sigma_F^2, \end{aligned} \tag{3.4}$$

and we see that $\bar{Y}_n^2$ is an unbiased estimate for $\sigma_D^2 + \sigma_P^2 + \sigma_F^2$. Given that the measurement error variances σ_D^2 and σ_P^2 ar known, then the estimator $\hat{\sigma}_F^2$ for σ_F^2 from (3.4) is[2]

$$\hat{\sigma}_F^2 = \bar{Y}_n^2 - \sigma_D^2 - \sigma_P^2. \tag{3.5}$$

Under the alternative hypothesis H_1, from (2.11),

$$\begin{aligned}
E_1(\bar{Y}_n^2) &= \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} E_1((Y_{i+2} - Y_i)^2) \\
&= \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} E_1((Y_{i+2} - \theta - \sum_{j=1}^{i+2} \mu_j - (Y_i - \theta - \sum_{j=1}^{i+2} \mu_j) - \mu_{i+1} - \mu_{i+2})^2) \\
&= \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} [var_1(Y_{i+2}) + var_1(Y_i) - 2 \cdot cov(Y_{i+2}, Y_i) + (\mu_{i+1} + \mu_{i+2})^2] \\
&= \sigma_D^2 + \sigma_P^2 + \sigma_F^2 + \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} (\mu_{i+1} + \mu_{i+2})^2,
\end{aligned} \tag{3.6}$$

or, defining

$$\mu_{(n)}^2 := \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} (\mu_{i+2} + \mu_{i+1})^2 \tag{3.7}$$

then

$$E_1(\bar{Y}_n^2) = \sigma_D^2 + \sigma_P^2 + \sigma_F^2 + \mu_{(n)}^2. \tag{3.8}$$

We consider two special cases: For

$$\mu_i = \frac{1}{n} \cdot \mu \text{ for } i = 1, ..., n$$

we get with (3.7)

$$\mu_{(n)}^2 = \frac{1}{2} \cdot \frac{1}{n-2} \cdot \sum_{i=1}^{n-2} (\mu_{i+2} + \mu_{i+1})^2 = \frac{1}{2} \cdot \frac{1}{n-2} \cdot (n-2) \cdot \frac{4\mu^2}{n^2} = 2 \cdot \frac{\mu^2}{n^2}. \tag{3.9}$$

On the other hand, for

$$\mu_i = \begin{cases} 0 & \text{for } i = 1, \ldots, n-1 \\ \mu & \text{for } i = n \end{cases}$$

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[2]It should be mentioned that this expression can become negative, see the comment in /4/, p. 3-28.

again with (3.7)

$$\mu_{(n)}^2 = \frac{1}{2}\cdot\frac{1}{n-2}\cdot\sum_{i=1}^{n-2}(\mu_{i+2}+\mu_{i+1})^2 = \frac{\mu^2}{2\cdot(n-2)}. \tag{3.10}$$

One can show that $\mu_{(n)}^2$, for given $\mu = \sum_i \mu_i$, is maximized by the distribution

$$\mu_i^* = \begin{cases} 0 & \text{for } i = 1, 2 \text{ and } n \\ \mu & \text{for } i = 3, 4, \dots \text{ or } n-1 \end{cases}$$

with

$$\max \mu_{(n)}^2 = \frac{\mu^2}{n-2}.$$

We shall now illustrate the difficulties involved in making use of the estimator (3.5) for σ_F^2 in a test for diversion, i.e. in a procedure for deciding between H_0 and H_1. In the following section it is explained that the optimal test procedure in the sense of maximum detection probability is to ignore intermediate inventories altogether and to perform a single significance test at the end of the reference period. Accordingly, let us consider time t_n and the quantity Y_n which involves only the beginning and ending inventories I_0 and I_n. Using (3.5) as an estimator for σ_F^2, the estimator for $var(Y_n)$ is, from (2.10),

$$v\hat{a}r(Y_n) = \sigma_0^2 + n\cdot\sigma_D^2 + \sigma_P^2 + \hat{\sigma}_F^2 = \sigma_0^2 + (n-1)\cdot\sigma_D^2 + \hat{\hat{Y}}{}_n^2. \tag{3.11}$$

Therefore we choose as a convenient test statistic the random variable

$$\frac{Y_n - \theta}{\sqrt{v\hat{a}r(Y_n)}} = \frac{Y_n - \theta}{\sqrt{\sigma_0^2 + (n-1)\cdot\sigma_D^2 + \hat{Y}_n^2}}. \tag{3.12}$$

Now, we certainly know the expected values of Y_n and $v\hat{a}r(Y_n)$, since from (2.9-12) and (3.5),

$$\begin{aligned}
&E_0(Y_n - \theta) = 0 \\
&E_0(v\hat{a}r(Y_n)) = \sigma_0^2 + (n-1)\cdot\sigma_D^2 + \sigma_D^2 + \sigma_P^2 + \sigma_F^2 = var(Y_n) = \sigma_n^2 \\
&E_1(Y_n - \theta) = \sum_{i=1}^{n}\mu_i = \mu \\
&E_1(v\hat{a}r(Y_n)) = \sigma_0^2 + (n-1)\cdot\sigma_D^2 + \sigma_D^2 + \sigma_P^2 + \sigma_F^2 + \mu_{(n)}^2 = \sigma_n^2 + \mu_{(n)}^2.
\end{aligned} \tag{3.13}$$

What we do *not* know is the *distribution* of expression (3.12).

The significance test that we would like to carry out is

$$\frac{Y_n - \theta}{\sqrt{v\hat{a}r(Y_n)}} < s : H_0 \text{ is rejected}. \tag{3.14}$$

To determine the relationship between significance threshold s and false alarm probability α, we shall simply assume (3.12) to be approximately normally distributed. We will examine this assumption in more detail in Section 5. We then have

$$1-\alpha = Pr_0\left(\frac{Y_n-\theta}{\sqrt{\hat{var}(Y_n)}} \leq s\right) = \Phi(s) \tag{3.15}$$

and, further, with $(\mu_1, \ldots, \mu_n) = \underline{\mu}$,

$$\beta(\underline{\mu}) = Pr_1\left(\frac{Y_n-\theta}{\sqrt{\hat{var}(Y_n)}} \leq s\right). \tag{3.16}$$

Replacing now $\hat{var}(Y_n)$ by its expected value $\sigma_n^2 + \mu_{(n)}^2$ we obtain from (3.16)

$$\begin{aligned}\beta(\underline{\mu}) &\approx Pr_1\left(\frac{Y_n-\theta}{\sqrt{\sigma_n^2+\mu_{(n)}^2}} \leq s\right) = Pr_1\left(\frac{Y_n-\theta-\mu}{\sqrt{\sigma_n^2+\mu_{(n)}^2}} \leq s - \frac{\mu}{\sqrt{\sigma_n^2+\mu_{(n)}^2}}\right)\\ &= \Phi\left(s - \frac{\mu}{\sqrt{\sigma_n^2+\mu_{(n)}^2}}\right)\end{aligned}$$

or, eliminating the significance threshold s,

$$1-\beta(\underline{\mu}) = \Phi\left(\frac{\mu}{\sqrt{\sigma_n^2+\mu_{(n)}^2}} - U(1-\alpha)\right). \tag{3.17}$$

For the special case (3.9), the detection probability becomes

$$\Phi\left(\frac{\mu}{\sqrt{\sigma_n^2+2\cdot\frac{\mu^2}{n^2}}} - U(1-\alpha)\right), \tag{3.18}$$

whereas for (3.10) it is

$$\Phi\left(\frac{\mu}{\sqrt{\sigma_n^2+\frac{1}{2}\frac{\mu^2}{n-2}}} - U(1-\alpha)\right). \tag{3.19}$$

Quite generally we can write

$$\mu_{(n)}^2 = k\cdot\mu^2. \tag{3.20}$$

Then we have from (3.17)

$$1-\beta(\underline{\mu}) = \Phi\left(\frac{\mu}{\sqrt{\sigma_n^2+k\cdot\mu^2}} - U(1-\alpha)\right). \tag{3.21}$$

Now (3.21) is a monotonically increasing function of the function

$$f(\mu) = \frac{\mu}{\sqrt{\sigma_n^2 + k \cdot \mu^2}} \tag{3.22}$$

which has the properties (see Figure 1),

$$f(0) = 0, \; f(\infty) = \frac{1}{\sqrt{k}}$$

$$f'(\mu) = \frac{\sigma_n^2}{(\sigma_n^2 + k \cdot \mu^2)^{\frac{3}{2}}} > 0, \; f'(0) = \frac{1}{\sigma_n},$$

and therefore,

$$0 < f'(\mu) < 1/\sigma_n \text{ for } 0 \leq \mu$$

which implies

$$f(\mu) \leq \tilde{f}(\mu) := \mu/\sigma_n \text{ for } 0 \leq \mu.$$

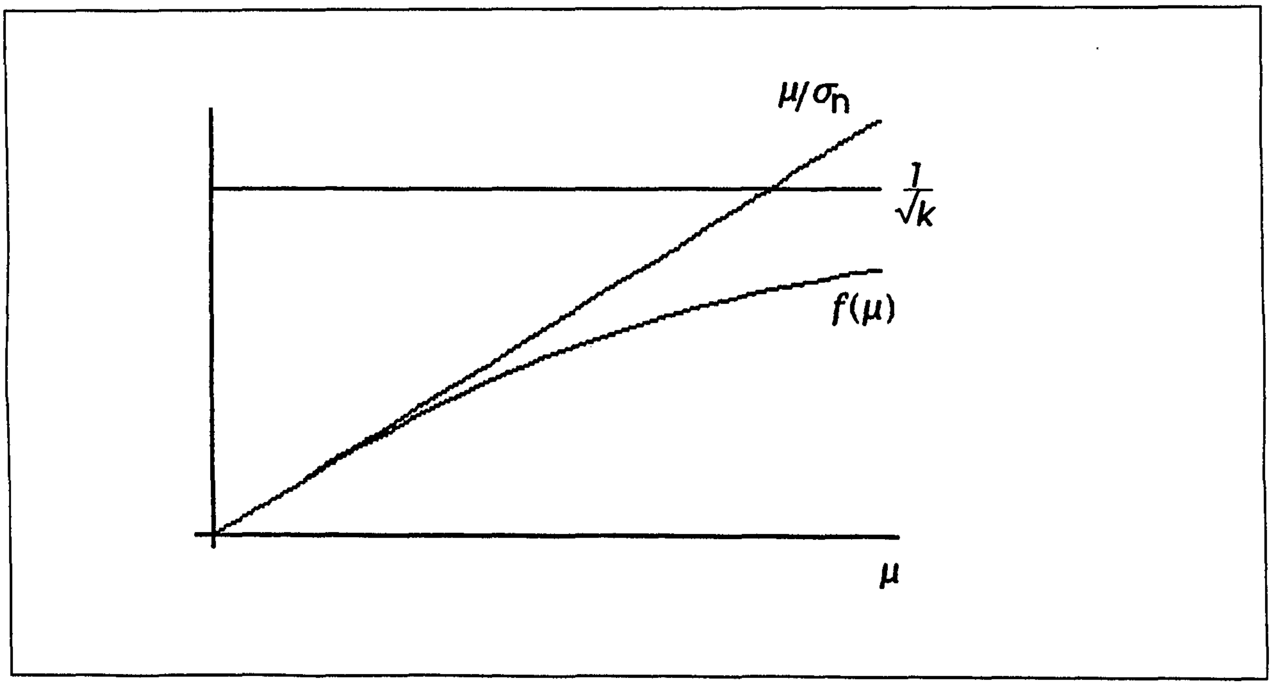

Fig. 1: Plot of $f(\mu)$ according to (3.22).

Since the detection probability corresponding to (3.21) for *known* variance σ_F^2 is given by

$$\Phi(\frac{\mu}{\sigma_n} - U(1-\alpha)),$$

we see that the detection probability will always be *diminished* by the estimation of σ_F^2.

4. Safeguards Objectives and Appropriate Test Procedures

The tests involving the quantity Y_i discussed in the preceding sections permit a statement to be made regarding diversion in the interval $[t_0, t_i]$, $i = 1, ..., n$. In this section we turn to the question of choice of an appropriate test procedure for the reference period $[t_0, t_n]$, making the important distinction between the safeguards goals of maximizing the ***sensitivity*** of diversion detection and maximizing *timeliness*. This distinction is essential, as it is well-known that these two optimization criteria are mutually exclusive.

4.1. Sensitive Detection

An obvious goal of any safeguards measure, such as NRTA, is to detect material diversion with the largest possible probability within a given reference period for some acceptable false alarm probability. Given 'known' or 'nominal' values for the moments of the unmeasured process inventory θ and σ_F^2 the optimal test procedure in the sense of maximum overall detection probability is to test the material balance once over the entire reference period /6/. The test statistic is thus simply Y_n and the maximum guaranteed detection probability is given by

$$1 - \beta = \Phi\left(\frac{\mu}{\sqrt{var(Y_n)}} - U(1-\alpha)\right). \tag{4.1}$$

The intermediate inventories I_i, $i = 1 \ldots n-1$ thus don't enter into the test at all. Moreover, if t_n, like t_0, corresponds to a clean out inventory, the 'unmeasured inventory' becomes measurable: process fluctuations and 'adjusted running book inventories' /4/ are wholly irrelevant to the decision procedure.

4.2. Timely Detection

Particularly in the case of direct use material such as plutonium, timely detection of diversion is an important criterion for safeguards effectiveness. Obviously timeliness is best achievable when decision tests for diversion are performed as frequently as possible, and this is the basic reason for NRTA and the associated determination of in-process inventories. Necessarily, NRTA test procedures deviate from the single period test discussed above and thus incur a loss of detection sensitivity.

In the following, we will consider timeliness to be the exclusive effectiveness criterion, present a measure to characterize it and discuss a simple sequential test for which a closed expression for timeliness can be obtained.

In order to choose an appropriate test procedure for timely detection it is necessary to quantify the concept of timeliness as a scalar quantity. Let $t_0, t_1, t_2, \ldots$, be the times at which the plant inventory is, at least partially, determined. Suppose diversion of material begins at t_0. Then $T = t_i$ is the realized time to detection if the sequential test alarms at $T = t_i$. Since this does not occur with certainty, T is a (discrete) random variable with realizations $t_1, t_2, \ldots$. Let its distribution be given by the function $Pr(T = t_i)$.

In a recent paper /7/ it is shown that, under reasonable assumptions, the expectation value of the time to detection, given by

$$E(T) = \sum_i t_i \cdot Pr(T = t_i), \tag{4.2}$$

is an appropriate measure of timeliness. For simplicity, replace t_i by i, $i = 1, 2, \ldots$, so that

$$E(T) = \sum_i i \cdot Pr(T = i). \tag{4.3}$$

The quantity $E(T)$ is then referred to as the ***average run length to detection*** and is expressed in units of inventory periods. Here we must be careful to specify that detection occurs with certainty within a given reference time period, or that the time horizon is infinite, since otherwise this expectation value is undefined.

The boundary condition for optimizing detection probability was the desired false alarm probability. When optimizing timeliness it is natural to replace false alarm probability with the expected time to detection when no diversion takes place. Thus if we define

$$L_0 = E_0(T) = \sum_i i \cdot Pr_0(T = i) \text{ under } H_0, \tag{4.4}$$

$$L_1 = E_1(T) = \sum_i i \cdot Pr_1(T = i) \text{ under } H_1, \tag{4.5}$$

our task is to find, for a given diversion pattern, that sequential test procedure which minimizes L_1 for a given value of L_0.

Unfortunately a general solution of this optimization problem, in contrast to the sensitive detection case, is not possible as there is no counterpart to the Neyman-Pearson Lemma for sequential test procedures. One has to resort to inventing 'reasonable' tests and examining their performance (i.e. expected detection times) against a variety of diversion strategies (protracted, abrupt, etc.). Over the years a great deal of effort has been invested in studying the performance of such sequential tests; the results are ambiguous.

A further problem arises through the fact that the detection probabilities $Pr(T = i)$ are in general very complicated expressions, so that analytical comparison of different NRTA procedures is not possible. To illustrate, suppose we perform a simple significance test on the Y_i, assuming θ and σ_F^2 to be known, repeating this after each period (sequential CUMUF test):

$$Y_i - \theta \leq U(1-\alpha) \cdot \sqrt{var(Y_i)} \; =: \; s_i : H_0 \text{ true}, \tag{4.6}$$

where α is the single period false alarm probability, chosen here to be period - independent. Then

$$Pr(T = i) = Pr(Y_1 - \theta \leq s_1, ..., Y_{i-1} - \theta \leq s_{i-1}, Y_i - \theta > s_i). \tag{4.7}$$

But since the Y_i, $i = 1, 2, \ldots$ are not independent, (4.7) can't be factored, i.e. written as a closed expression.

We can obtain a closed expression for expected detection time by applying the so-called *independence transformation.* It is more convenient to start from the series of single period MUF-values Z_i, which can be obtained from the Y_i according to

$$\begin{aligned} Z_1 &= Y_1 \\ Z_i &= Y_i - Y_{i-1}, \quad i = 2, 3, \ldots \end{aligned} \tag{4.8}$$

The simplest representation of the independence transformation is then by induction:

$$\begin{aligned} \tilde{Z}_1 &= Z_1 \\ \tilde{Z}_i &= a_i \cdot \tilde{Z}_{i-1}, \; i = 2, 3, ... \end{aligned} \tag{4.9}$$

where the transformation coefficients a_i are given inductively by

$$\begin{aligned} a_0 &= 0 \\ a_{i+1} &= \frac{var(P_i) + var(M_i)}{var(Z_i) - a_i \cdot (var(P_{i-1}) + var(M_{i-1}))}, \; i = 1, 2, \ldots \; . \end{aligned} \tag{4.10}$$

The variances of the transformed variables $\tilde{Z}_i$ are given by

$$\begin{aligned} var(\tilde{Z}_1) &= var(Z_1) \\ var(\tilde{Z}_{i+1}) &= var(Z_i) - \frac{(var(P_i) + var(M_i))^2}{var(\tilde{Z}_i)}, \; i = 2, 3, \ldots \; . \end{aligned} \tag{4.11}$$

Under the null hypothesis H_0 we have

$$E_0(\tilde{Z}_1) = \theta, \; E_0(\tilde{Z}_i) = 0, \; i = 2, \ldots, n,$$

whereas for H_1 we have

$$E_1(\tilde{Z}_1) = \theta + \mu_1, \ E_1(\tilde{Z}_i) = \mu_i, \ i = 2, \ldots, n,$$

which leads, with (4.9), to the following recursive relation for the expected values:

$$\begin{aligned} E_1(\tilde{Z}_1) &= \mu_1 + \theta \\ E_1(\tilde{Z}_i) &= a_i \cdot E_1(\tilde{Z}_{i-1}) + \mu_i, \ i = 2, 3, \ldots \ . \end{aligned} \tag{4.12}$$

Analogously to (4.6) we then define the sequential test

$$\begin{aligned} \tilde{Z}_1 - \theta \le U(1-\alpha) \cdot \sqrt{var(\tilde{Z}_1)} &=: \tilde{s}_1 : H_0 \text{ correct} \\ \tilde{Z}_i \le U(1-\alpha) \cdot \sqrt{var(\tilde{Z}_i)} &=: \tilde{s}_i : H_0 \text{ correct}, i = 2, 3, \ldots . \end{aligned} \tag{4.13}$$

The expected detection times under the two hypotheses now read

$$\begin{aligned} L_0 &= Pr_0(\tilde{Z}_1 - \theta > \tilde{s}_1 + \sum_{i>1} i \cdot Pr_0(\tilde{Z}_i > \tilde{s}_i) \cdot \prod_{j=1}^{i-1} Pr_0(\tilde{Z}_j \le \tilde{s}_j) \\ L_1 &= Pr_1(\tilde{Z}_1 - \theta > \tilde{s}_1 + \sum_{i>1} i \cdot Pr_1(\tilde{Z}_i > \tilde{s}_i) \cdot \prod_{j=1}^{i-1} Pr_1(\tilde{Z}_j \le \tilde{s}_j). \end{aligned} \tag{4.14}$$

Even given that the Z_i, and hence the $\tilde{Z}_i$, are normally distributed, the expected detection times are still very complicated expressions. However, assuming that the variances quickly converge to their asymptotic limits and $\mu_i = \mu$, $i = 1, 2, \ldots$, then, for a stationary process $var(D_i) = var(D) =: \sigma_D^2$, $i = 1, 2, \ldots$,

$$\lim_{n \to \infty} \frac{E_1(\tilde{Z}_i)}{\sqrt{var(\tilde{Z}_i)}} = \frac{\mu}{\sigma_D}, \tag{4.15}$$

and we have immediately

$$\begin{aligned} L_0 &= \sum_i i \cdot (1-\alpha)^{i-1} \cdot \alpha = \frac{1}{\alpha} \\ L_1 &= \sum_i i \cdot \beta^{i-1} \cdot (1-\beta) = \frac{1}{1-\beta} = \frac{1}{\Phi(\frac{\mu}{\sigma_D} - U(1-\alpha))}. \end{aligned} \tag{4.16}$$

The relationship between L_0 and L_1 can thus be represented in closed form as

$$U(\frac{1}{L_1}) = \frac{\mu}{\sigma_D} + U(\frac{1}{L_0}). \tag{4.17}$$

In this approximation we see that neither the variance of the measured inventory σ_P^2 nor that of the unmeasured inventory σ_F^2 influences the timeliness of detection. Only if they are known can transformation (4.9) be carried out, but they play no role in determining the effectiveness of NRTA 'in the long run'. Of course we have only examined a very simple sequential test, presumably not a particularly efficient one, and this only in the asymptotic limit. Nevertheless the fact that the fluctuations in the unmeasured inventory don't affect timeliness is a good argument for applying the approximate estimation methods discussed in Section 3 to sequential test procedures. This is the subject of the next Section.

5. Near Real Time Accountancy with Selected Test Procedures

In this Section, we focus attention on two sequential test procedures, the CUMUF and CUSUM tests, and establish with them the influence of simultaneous estimation of the moments θ and/or σ_F^2 on the average run lengths, i.e. on the timeliness of NRTA. We distinguish the following cases:

— both θ and σ_F^2 known,

and the two interesting variants

— θ known, σ_F^2 unknown,

— θ unknown, σ_F^2 unknown.

The appropriate test statistics are $Y_i \equiv \text{CUMUF}_i,\ i = 1, 2, \ldots$, given by (2.1), and $Z_i \equiv \text{MUF}_i,\ i = 1, 2, \ldots$ according to (4.8).

5.1. CUMUF Procedures

<u>θ and σ_F^2 known</u>

The test procedure is

$$\frac{Y_i - \theta}{\sqrt{var(Y_i)}} \leq s_i^{(1)} \ : \ \text{continue}, \tag{5.1}$$

otherwise stop with acceptance of H_1, whereby

$$var(Y_i) = \sigma_i^2 = \sigma_0^2 + i \cdot \sigma_D^2 + \sigma_P^2 + \sigma_F^2. \tag{5.2}$$

The single-period decision thresholds are chosen such that the average run length L_0 under H_0 achieves some desired value. This cannot be calculated analytically, since the Y_i are highly correlated. Numerical methods are required.

<u>θ known, σ_F^2 unknown</u>

Now the test procedure is, for $i > 2$,

$$\frac{Y_i - \theta}{\sqrt{v\hat{a}r(Y_i)}} \leq s_i^{(2)} \ : \ \text{continue}, \qquad (5.3))$$

otherwise stop with acceptance of H_1. Making use of the estimator for σ_F^2 given by (3.5) as in (3.11),

$$v\hat{a}r(Y_i) = \sigma_0^2 + (i-1) \cdot \sigma_D^2 + \bar{Y}_{i-1}^2, \tag{5.4}$$

the test procedure is

$$\frac{Y_i - \theta}{\sqrt{\sigma_0^2 + (i-1)\cdot\sigma_D^2 + \bar{Y}_{i-1}^2}} \leq s_i^{(2)} \quad : \quad \text{continue,} \tag{5.5}$$

otherwise stop with acceptance of H_1. (We have chosen $\bar{Y}_{i-1}$ rather than $\bar{Y}_i$ so as to increase the sensitivity to abrupt diversion.)

As indicated earlier, the significance thresholds $s_i^{(2)}$, $i > 2$, may be determined for the boundary condition of desired average run length L_0 by computer simulation. In order to perform such simulations, the distribution of the Y_i must of course be known, and this in turn presupposes knowledge of σ_F^2. Values may be assumed in a parametric study which span the extremes of expected process fluctuations, in the hope that the significance thresholds, and hence L_0, are insensitive to their choice. This will be illustrated in Section 5.3.

<u>θ unknown, σ_F^2 unknown</u>

Here we introduce two different test procedures. The first is, for $i > 2$,

$$\frac{Y_i - Z_1}{\sqrt{\hat{var}(Y_i - Z_1)}} \leq s_i^{(3)} \quad : \quad \text{continue,}$$

otherwise stop. From (2.10),

$$\begin{aligned} var(Y_i - Z_1) &= var(Y_i) + var(Z_1) - 2cov(Y_i, Z_1) = \\ &= \sigma_i^2 + \sigma_0^2 + \sigma_D^2 + \sigma_P^2 + \sigma_F^2 - 2\cdot(\sigma_0^2 + \sigma_D^2) = \\ &= (i-1)\cdot\sigma_D^2 + 2\sigma_P^2 + 2\sigma_F^2. \end{aligned} \tag{5.6}$$

Thus, with (3.5), we have the estimator

$$\begin{aligned} \hat{var}(Y_i - Z_1) &= (i-1)\cdot\sigma_D^2 + 2\sigma_P^2 + 2\cdot(\bar{Y}_{i-1}^2 - \sigma_D^2 - \sigma_P^2) = \\ &= (i-3)\cdot\sigma_D^2 + 2\bar{Y}_{i-1}^2, \end{aligned} \tag{5.7}$$

so the test procedure is, explicitly,

$$\frac{Y_i - Z_1}{\sqrt{(i-3)\cdot\sigma_D^2 + 2\bar{Y}_{i-1}^2}} \leq s_i^{(3)} \quad : \quad \text{continue} \tag{5.8}$$

otherwise stop.

If a systematic error variance in the throughput is included, the test statistic must be modified to

$$\frac{Y_i - Z_1}{\sqrt{(i-3)\cdot\sigma_{D_r}^2 + (i^2 - 2i - 3)\cdot\sigma_{D_s}^2 + 2\bar{Y}_{i-1}^2}}$$

The second test procedure is, for $i > 2$,

$$\frac{Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j}{\sqrt{\hat{var}(Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j)}} \leq s_i^{(4)} : \quad \text{continue,} \tag{5.9}$$

otherwise stop. It is shown in the appendix that

$$\hat{var}(Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j) = \frac{i}{i-1} \cdot (\frac{2i-7}{6} \cdot \sigma_D^2 + \bar{Y}_{i-1}^2) \tag{5.10}$$

so that the test procedure is

$$\frac{Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j}{\sqrt{\frac{i}{i-1} \cdot (\frac{2i-7}{6} \cdot \sigma_D^2 + \bar{Y}_{i-1}^2)}} \leq s_i^{(4)} : \quad \text{continue,} \tag{5.11}$$

otherwise stop. The corresponding expression including systematic error is given in the appendix.

5.2. CUSUM Procedures

As for the CUMUF procedures above, the effectiveness criterion is the average run length L_1 under H_1. The global boundary condition for choice of the CUSUM test parameters h and k is the desired average run length L_0 under H_0.

θ and σ_F^2 known

The first step in the test procedure is, for $i = 1$,

$$\frac{Z_1 - \theta}{\sqrt{var(Z_1)}} \leq h : \quad \text{continue,} \tag{5.12}$$

otherwise stop, accepting H_1. For $i = 2, 3, \ldots$

$$S_i^{(1)} \leq h : \quad \text{continue,} \tag{5.13}$$

otherwise stop, where the $S_i^{(1)}$ are defined recursively as

$$S_i^{(1)} = \max(0, S_{i-1}^{(1)} - k + \frac{Z_i}{\sqrt{var(Z_i)}}). \tag{5.14}$$

σ_F^2 unknown

The test procedure, for $i > 2$, is now

$$S_i^{(2)} \leq h: \quad \text{continue}, \tag{5.15}$$

otherwise stop, with the recursive definition

$$S_i^{(2)} = \max(0, S_{i-1}^{(2)} - k + \frac{Z_i}{\sqrt{\hat{var}(Z_i)}}). \tag{5.16}$$

But we have

$$var(Z_i) = 2 \cdot (\sigma_P^2 + \sigma_F^2) + \sigma_D^2,$$

and therefore, with (3.5),

$$\hat{var}(Z_i) = 2 \cdot \hat{\bar{Y}}_{i-1}^2 - \sigma_D^2, \tag{5.17}$$

that is, $S_i^{(2)}$ is defined recursively by

$$S_i^{(2)} = \max(0, S_{i-1}^{(2)} - k + \frac{Z_i}{\sqrt{2 \cdot \bar{Y}_{i-1}^2 - \sigma_D^2}}), \quad i = 3, 4, \ldots . \tag{5.18}$$

Again, if systematic errors are included, this becomes

$$S_i^{(2)} = \max(0, S_{i-1}^{(2)} - k + \frac{Z_i}{\sqrt{2 \cdot \bar{Y}_{i-1}^2 - \sigma_{D_r}^2 - 3\sigma_{D_s}^2}}), \quad i = 3, 4, \ldots . \tag{5.19}$$

Since for $i > 2$ the expectation value θ of the unmeasured inventory doesn't enter the distribution of the $S_i^{(2)}$, the third case, θ and σ_F^2 unknown, is irrelevant for the CUSUM procedures.

5.3. Numerical Calculations and Discussion

A Monte Carlo simulation of a simple reference process based on the reprocessing throughput, inventory and measurement data of Ref. /4/ was employed in order to test the effectiveness (timeliness) of the various estimate-and-test procedures described above. The process data and the numerical simulation parameters are summarized in Table 1.

TABLE 1 *Data set for numerical simulations*

Parameter	Symbol	Value
Operating year		50 periods
Time horizon		1000 periods
Test begin		5th period
Diversion	μ	37.4 kg/a
Plant throughput		200 kg/period
Random throughput error	σ_{D_r}	1.0 kg
Systematic throughput error	σ_{D_s}	0.1 kg
Mean measurable inventory	$E(P)$	173 kg
Random inventory error	σ_P	5.2 kg
Mean unmeasurable inventory	θ	25 kg
Process fluctuation	σ_F	0 or 5 kg

Figure 2 illustrates the validity of assuming that the basic test statistic (3.12) is normally distributed. It shows two histograms of 3000 random samples of the quantity on the left hand side of (3.12) for $n = 50$ periods and $\sigma_F = 0$ and 5 kg. A threshold is set at 1.645 corresponding to a false alarm rate of 5% for the standard normal distribution. The number of simulated events above this threshold is indicated in the figure. If (3.12) is really normally distributed, then this number should be close to 5% of the total number of samples. Both distributions appear to a good approximation to be standard normal and are thus roughly independent of σ_F. This may be fortuitous, although there exists a similarity between (3.12) and the Student's t-statistic

$$t = \frac{\bar{X} - \mu}{S\sqrt{n}}, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

S^2 is an unbiased estimate for the variance σ_X^2 of the X_i, $i = 1 \ldots n$. Given that the X_i are independently identically (μ, σ_X^2)- normally distributed then the distribution of t is known to be independent of σ_X.

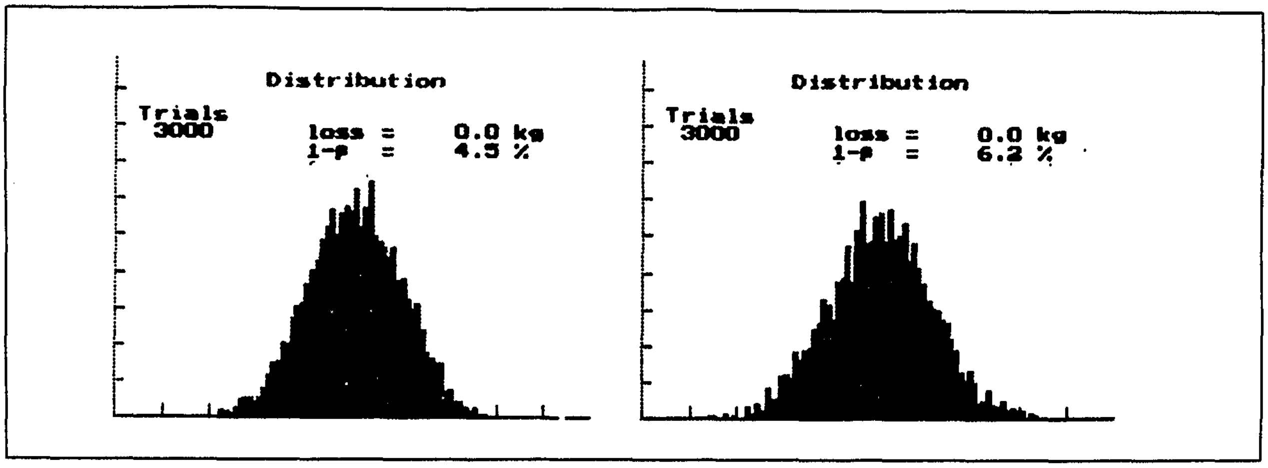

Fig. 2: Simulated distribution of (3.12) for $\sigma_F = 0$ kg (left) and $\sigma_F = 5$ kg (right).

The CUMUF sequential tests were all simulated with constant single period decision thresholds parameterized by

$$s_i^{(k)} = U(1-\alpha),\ k = 1 \ldots 3.$$

A typical simulation over 100 inventory periods of the sequential CUMUF test statistic for constant diversion and known mean and variance of process fluctuations (model A) is shown in Figure 3. Figure 4 gives the same plot for the test in which the mean and variance are simultaneously estimated. Note that in the latter case the test threshold, being estimated as the test proceeds, is not a smooth function. Its form also varies considerably from run to run.

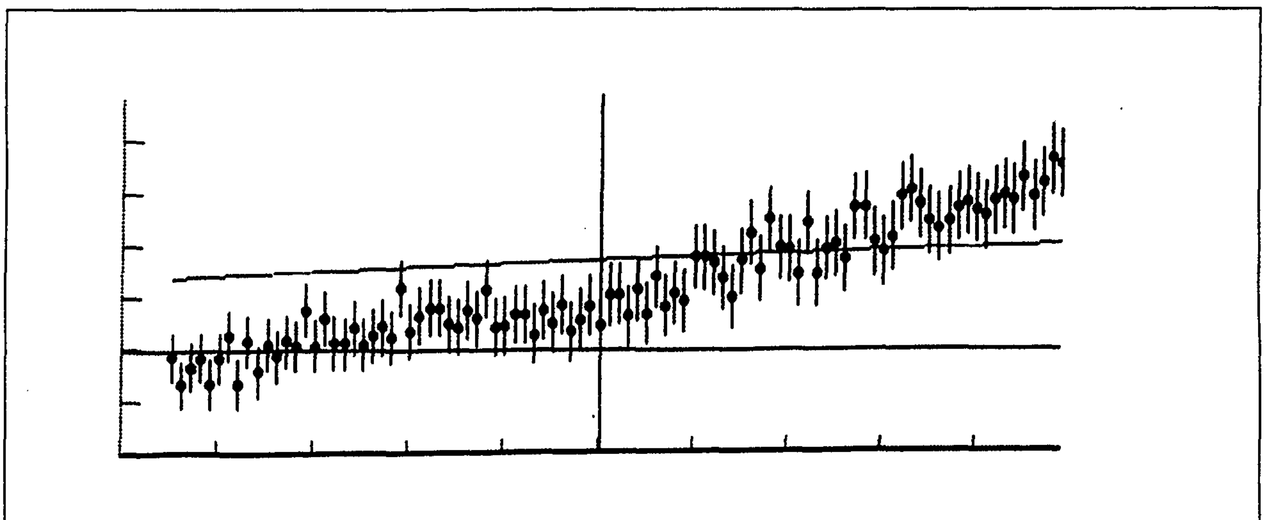

Fig. 3: Typical sequence of CUMUF statistics and thresholds for model A.

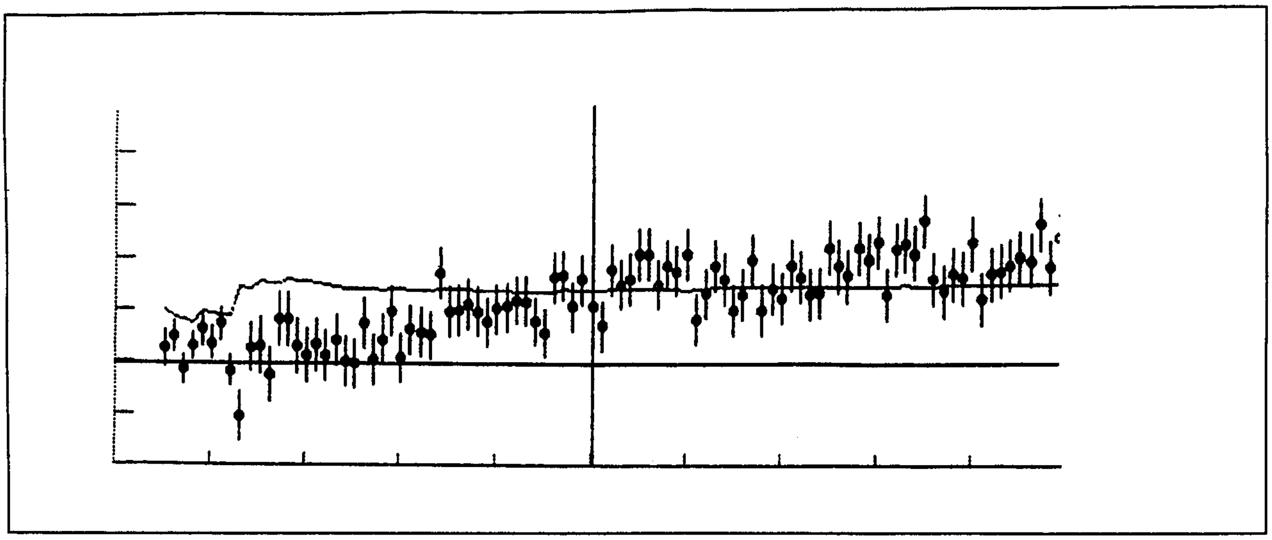

Fig. 4: Typical sequence of CUMUF statistics and thresholds for model D.

Table 2 summarizes the results of the simulation experiments carried out for the sequential CUMUF and CUSUM tests. The assumed diversion per reference period was taken to be the so-called accountancy verification goal quantity: that amount of material whose diversion is detectable with 95% probability for a false alarm probability of 5% using conventional accountancy (i.e. applying the Neyman-Pearson test at the end of the reference period). For the reference process, this was 37.4 kg Pu (see Table 1). Each simulation was repeated for two values of the standard deviation σ_F of the unobserved process inventory, expressed in the table as a percentage of the mean. These values straddle the assumed variation of 10% used in /4/. The 1000 period time series of material balance statistics was sampled 1000 times for zero loss and for protracted or abrupt diversion in determining the average run lengths shown. All four CUMUF tests use the common single period false alarm probability of $\alpha = 0.01$ to establish the single period decision thresholds. Similarly, both CUSUM tests use the same parameter set $h = 5.0$, $k = 0$. The abrupt diversions were simulated in inventory periods 5 and 30. For the CUSUM tests (E and F) and for zero loss or protracted diversion, the Bagshaw - Johnson asymptotic approximation to the average run lengths /8/ is included in parentheses. The approximation is not particularly good, partly because of the small systematic error included in the simulation which the Bagshaw - Johnson formulae do not take into account.

TABLE 2 *Numerical results for sequential tests*

Test[1]	$\sigma_F(\%)$	L_0	L_1(protr)	L_1(abr, 5)	L_1(abr, 30)
A	0	> 846	37	1	> 6
	20	> 800	43	1	> 32
B	0	> 828	36	> 2	> 23
	20	> 753	42	> 12	> 56
C	0	> 730	20	> 6	> 8
	20	> 681	23	> 25	> 18
D	0	> 466	16	> 3	-7
	20	> 380	18	> 12	-7
E	0	$> 445(1377)$	40(49)	> 3	1
	20	$> 558(2627)$	51(68)	> 86	> 6
F	0	$> 406(1377)$	33(49)	> 60	0
	20	$> 492(2627)$	40(68)	> 139	> 18

[1] Test A: Sequential CUMUF, known θ and σ_F^2

Test B: Sequential CUMUF, known θ, unknown σ_F^2

Test C: Sequential CUMUF, unknown θ estimated with Z_1, unknown σ_F^2

Test D: Sequential CUMUF, unknown θ estimated by averaging Y_i, unknown σ_F^2

Test E: CUSUM, known θ and σ_F^2

Test F: CUSUM, known θ, unknown σ_F^2

A technical problem arises in the determination of average run lengths for some simulated diversion strategies, stemming from the fact that the computer cannot provide an infinite time horizon. All average run lengths for which at least one sampled test exceeded the time horizon of 1000 periods are identified in the table by a '>' indicating that the tabulated value is a lower limit. In the case of abrupt diversion, the run length distributions are extremely skew but in all such cases an alarm occurred with $> 90\%$ probability within a few periods of the onset of loss. Negative run lengths occur in the table and are due to the finite probability of a false alarm prior to an abrupt diversion.

Because of the uncertainty in numerical determination of L_0, intercomparisons are difficult, but all L_0 values seem to be acceptably long and are calculationally reproducible to within about 10 periods. The lower limits allow at least relative comparison of the effectiveness of the models. Table 3 compares models A and D, whereby in the latter the single period false alarm parameter α was chosen so as to give a comparable lower limit on L_0 ($\alpha = 0.001$). The model D test procedure is superior to the reference procedure (model A) for both protracted and abrupt diversion, unless the latter occurs near the beginning of the reference period.

TABLE 3 *Comparison of Two CUMUF Tests*

Test	$\sigma_F(\%)$	L_0	L_1(protr)	L_1(abr, 5)	L_1(abr, 30)
A	0	> 846	37	1	> 6
	20	> 800	43	1	> 32
D	0	> 790	28	> 11	-2
	20	> 754	30	> 95	> -1

Examining Tables 2 and 3 it is difficult to see any particular disadvantage in making use of the estimate/test models (B,C,D and F) rather than the conventional procedures (A and E). The former perform generally worse for early abrupt diversions and better for the other diversion strategies.

From the calculational point of view, a better measure of timeliness might be a percentile, i.e. the run length at which detection occurs to a given probability. This measure is in fact proposed in /4/. On the other hand such a measure, unlike the average run length, lacks a decision theoretical justification /7/. Moreover for the time horizon of 1000 periods, chosen so as to give an acceptable computing time on a fast 486 PC, even the 50% percentile was not calculable for zero-loss.

It is important to note that the numerical simulations reproduce uncorrelated Gaussian fluctuations in the amount of unmeasured inventory in the process, reflecting the assumptions of Section 3. A significant autocorrelation in this random variable would certainly lead to poorer performance of the estimate/test models. This would also be the case if the operator can *manipulate* the unmeasured inventory in some systematic way, although the fact that it is unmeasurable would seem to preclude this possibility. The estimate/test models are also less sensitive to abrupt diversions near the beginning of the reference period, as the process variance estimate is poor for small i-values. An abrupt diversion before the test begins (in the calculations at period 5) will usually not be detected at all.

A final disadvantage of the estimate/test procedures is that they cannot be applied to the (analytically less intransigent) independently transformed MUF time series, since the independence transformation requires prior knowledge of the covariance matrix. As it is by no means clear that tests involving the independently transformed MUFs are in any way more effective than the tests considered here[3], this objection is rather formal.

In conclusion, the sequential estimate-and-test procedures seem to provide a satisfactory way of treating *a priori* unknown process fluctuations in inaccessible inventory when applying NRTA to detect clandestine diversions of nuclear material from a running plant.

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[3]Against the operator's Neyman-Pearson optimal diversion strategy, all sequential tests perform miserably /9/.

6. References

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Appendix A. The sequential CUMUF test statistic

The test statistic required for the sequential test (5.9) is

$$\frac{Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j}{\sqrt{v\hat{a}r(Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j)}}, \quad i = 3, 4, ... \tag{A.1}$$

We determine first of all the variance of the random variable in the numerator:

$$var(Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j) = var(Y_i) + \frac{1}{(i-1)^2} \cdot var(\sum_{j=1}^{i-1} Y_j) - \frac{2}{i-1} \cdot cov(Y_i, \sum_{j=1}^{i-1} Y_j). \tag{A.2}$$

We have

(i) from (2.10)

$$var(Y_i) = \sigma_i^2 = \sigma_0^2 + i \cdot \sigma_D^2 + \sigma_P^2 + \sigma_F^2 \tag{A.3}$$

(ii)

$$\begin{aligned} cov(Y_i, \sum_{j=1}^{i-1} Y_j) = \sum_{j=1}^{i-1} cov(Y_i, Y_j) &= \sum_{j=1}^{i-1} (\sigma_0^2 + j \cdot \sigma_D^2) \\ &= (i-1) \cdot \sigma_0^2 + \frac{(i-1) \cdot i}{2} \cdot \sigma_D^2 \end{aligned} \tag{A.4}$$

(iii)

$$var(\sum_{j=1}^{i-1} Y_j) = \underline{1}' \cdot \underline{\underline{\Sigma}}_{i-1} \cdot \underline{1}, \tag{A.5}$$

whereby $\underline{1}$ is the ones vector and $\underline{\underline{\Sigma}}_{i-1}$ is the covariance matrix of $(Y_1, ..., Y_{i-1})$. Setting $i-1 \equiv n$ we have

$$\underline{1}' \cdot \underline{\underline{\Sigma}}_n \cdot \underline{1} = \underline{1}' \cdot \begin{pmatrix} \sigma_1^2 + (n-1) \cdot (\sigma_0^2 + \sigma_D^2) \\ \sigma_0^2 + \sigma_D^2 + \sigma_2^2 + (n-2) \cdot (\sigma_0^2 + 2\sigma_D^2) \\ \sigma_0^2 + \sigma_D^2 + \sigma_0^2 + 2\sigma_D^2 + \sigma_3^2 + (n-3) \cdot (\sigma_0^2 + 3\sigma_D^2) \\ \vdots \\ \sigma_0^2 + \sigma_D^2 + \ldots + \sigma_0^2 + (n-1) \cdot \sigma_D^2 + \sigma_n^2 \end{pmatrix}$$

$$\begin{aligned} &= \sum_{j=1}^{n} \sigma_j^2 + 2 \cdot (n-1) \cdot (\sigma_0^2 + \sigma_D^2) + 2 \cdot (n-2) \cdot (\sigma_0^2 + 2\sigma_D^2) + ... + 2 \cdot (\sigma_0^2 + (n-1) \cdot \sigma_D^2) \\ &= \sum_{j=1}^{n} \sigma_j^2 + 2 \cdot \sum_{j=1}^{n-1} (n-j) \cdot \sigma_0^2 + 2 \cdot \sum_{j=1}^{n-1} (n-j) \cdot j \cdot \sigma_D^2. \end{aligned} \tag{A.6}$$

We obtain with the relations

$$\sum_{j=1}^{n-1}(n-j)=\frac{(n-1)\cdot n}{2}$$

$$\sum_{j=1}^{n-1}(n-j)\cdot j = n\cdot\sum_{j=1}^{n-1} j-\sum_{j=1}^{n-1} j^2 = n\cdot\frac{(n-1)\cdot n}{2}-\frac{(n-1)\cdot n\cdot\overbrace{(2\cdot(n-1)+1)}^{=2n-1}}{6}=$$
$$=\frac{n\cdot(n-1)}{2}\cdot(n-\frac{2n-1}{3})=\frac{n\cdot(n-1)\cdot(n+1)}{6},$$

from (A.6),

$$\begin{aligned}\underline{1}'\cdot\underline{\underline{\Sigma}}_n\cdot\underline{1}&=\sum_{j=1}^{n}(\sigma_0^2+j\cdot\sigma_D^2+\sigma_P^2+\sigma_F^2)+(n-1)\cdot n\cdot\sigma_0^2+\frac{n\cdot(n-1)\cdot(n+1)}{3}\cdot\sigma_D^2\\
&=n\cdot\sigma_0^2+\frac{n\cdot(n-1)}{2}\cdot\sigma_D^2+n\cdot\sigma_P^2+n\cdot\sigma_F^2+(n-1)\cdot n\cdot\sigma_0^2\\
&+\frac{n\cdot(n-1)\cdot(n+1)}{3}\cdot\sigma_D^2\\
&=n^2\cdot\sigma_0^2+n\cdot(n-1)\cdot(\frac{1}{2}+\frac{n-1}{3})\cdot\sigma_D^2+n\cdot\sigma_P^2+n\cdot\sigma_F^2\\
&=n^2\cdot\sigma_0^2+\frac{n\cdot(n+1)\cdot(2n+1)}{6}\cdot\sigma_D^2+n\cdot\sigma_P^2+n\cdot\sigma_F^2.\end{aligned}$$

Returning now with $i-1\equiv n$, we have

$$var(\sum_{j=1}^{i-1}Y_j)=(i-1)^2\cdot\sigma_0^2+\frac{(i-1)\cdot i\cdot(2i-1)}{6}\cdot\sigma_D^2+(i-1)\cdot\sigma_P^2+(i-1)\cdot\sigma_F^2 \qquad (A.7)$$

with the result, from (A.3,4 and 7),

$$\begin{aligned}var(Y_i-\frac{1}{i-1}\cdot\sum_{j=1}^{i-1}Y_j)&=\sigma_0^2+i\cdot\sigma_D^2+\sigma_P^2+\sigma_F^2\\
&+\frac{1}{(i-1)^2}\cdot[(i-1)^2\cdot\sigma_0^2+\frac{(i-1)\cdot i\cdot(2i-1)}{6}\cdot\sigma_D^2\\
&+(i-1)\cdot\sigma_P^2+(i-1)\cdot\sigma_F^2]\\
&-\frac{2}{i-1}\cdot[(i-1)\cdot\sigma_0^2+\frac{(i-1)\cdot i}{2}\cdot\sigma_D^2]\end{aligned}$$

or, finally,

$$var(Y_i-\frac{1}{i-1}\cdot\sum_{j=1}^{i-1}Y_j)=\frac{i}{i-1}\cdot[\frac{2i-1}{6}\cdot\sigma_D^2+\sigma_P^2+\sigma_F^2]. \qquad (A.8)$$

Recalling the estimator for σ_F^2,

$$\hat{\sigma}_F^2 = \bar{Y}_{i-1}^2 - \sigma_D^2 - \sigma_P^2,$$

the estimator for the the variance (A.8) is

$$v\hat{a}r(Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j) = \frac{i}{i-1} \cdot (\frac{2i-7}{6} \cdot \sigma_D^2 + \bar{Y}_{i-1}^2), \qquad (A.9)$$

and the the test statistic (A.1) is therefore

$$\frac{Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j}{\sqrt{\frac{i}{i-1} \cdot (\frac{2i-7}{6} \cdot \sigma_D^2 + \bar{Y}_{i-1}^2)}} \quad i = 3, 4, ..., \qquad (A.10).$$

In the simulations of Section 5.3 a systematic error in the measured throughput, σ_{D_s}, was also included. A somewhat more complicated derivation than the above leads to the test statistic

$$\frac{Y_i - \frac{1}{i-1} \cdot \sum_{j=1}^{i-1} Y_j}{\sqrt{\frac{i}{i-1} \cdot (\frac{2i-7}{6} \cdot \sigma_{D_r}^2 + \frac{3i^2+i-32}{12} \cdot \sigma_{D_s}^2 + \bar{Y}_{i-1}^2)}} \quad i = 3, 4, \dots . \qquad (A.11)$$

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