

Field Theory for Nonlinear Stochastic Rate Neurons

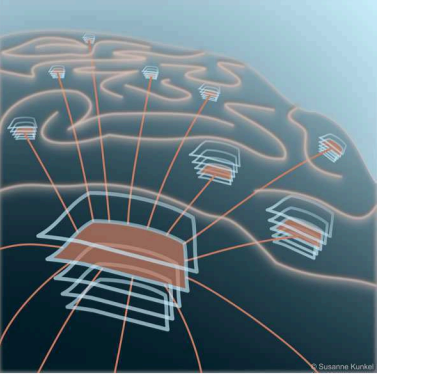
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Motivation

- neuronal dynamics is noisy (see Fig. 1) and nonlinear
- interplay between nonlinearity and stochasticity leads to interesting effects like memory
- goal: find effective description of the moments, considering the fluctuations self-consistently

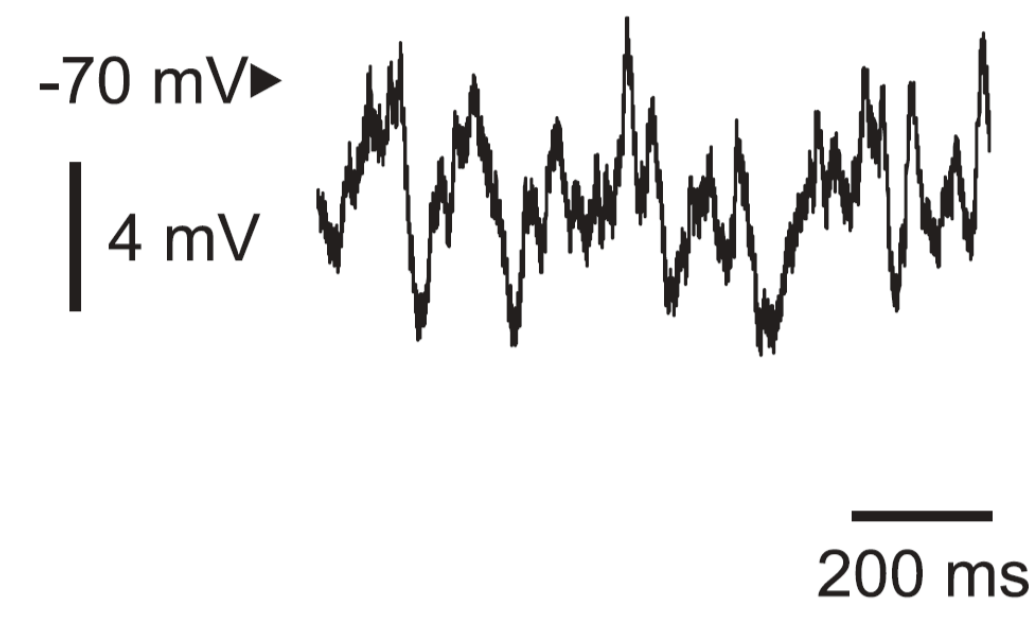


Fig. 1: Voltage trace from an intracellular recording from a cortical neuron of an anesthetized cat. Adapted from Rudolph et al. 2005 [1].

procedure: consider approximations with increasing accuracy

- tree-level
- one-loop
- functional Renormalization Group

$$\partial_t x(t) = f(x(t)) + \xi(t)$$

$$x \Rightarrow \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} =: X \Downarrow \text{MSRDJ-formalism}$$

$$S[X] \Rightarrow W[J] \sim \ln \int \mathcal{D}X e^{S[X] + JX} \xrightarrow{\text{Legendre transform}} \Gamma[X]$$

tree level approximation

$$\frac{d}{dX} \Gamma = 0 \Downarrow \text{loop-expansion, fRG, ...}$$

$$\partial_t \langle x \rangle(t) = F[\langle x \rangle(t'), t' \leq t]$$

Neuron Model in Field Theory

neuron dynamics:

$$dx = f(x(t)) dt + dW(t),$$

where $f(x) = -lx + \beta x^2$

From Langevin equation to field theory using the Martin-Siggia-Rose-De Dominicis-Janssen (MSRDJ) formalism [2, 3, 4]

$$S[x, \tilde{x}] = \tilde{x}^T (\partial_t + l)x + \tilde{x}^T \frac{D}{2} \tilde{x} + S_{\text{int}}[x, \tilde{x}]$$

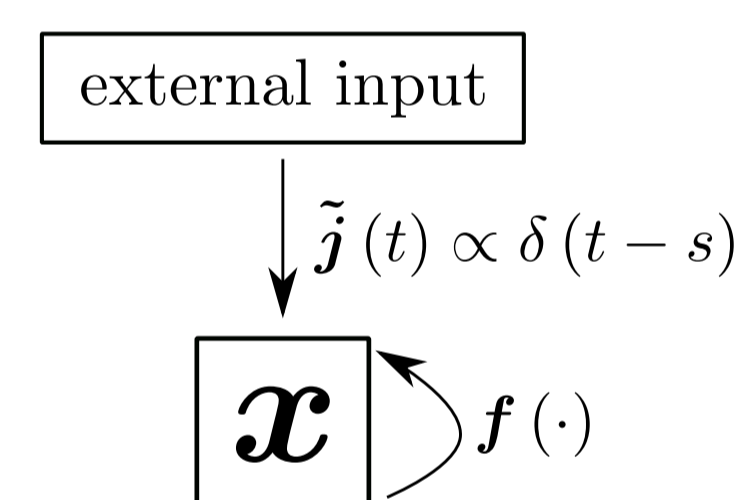
$$S_{\text{int}}[x, \tilde{x}] = -\beta \tilde{x}^T x^2, \quad \text{where}$$

- $x^T y = \int_{-\infty}^{\infty} x(t) y(t) dt$ and
- \tilde{x} : "response field", all moments identical to zero
- $\mathcal{Z}[j, \tilde{j}] = \int \mathcal{D}x \int \mathcal{D}\tilde{x} e^{S[x, \tilde{x}] + j^T x + \tilde{j}^T \tilde{x}}$
- $W[j, \tilde{j}] = \ln \mathcal{Z}[j, \tilde{j}]$
- $\Gamma[x, \tilde{x}] = \sup_{j, \tilde{j}} j^T x + \tilde{j}^T \tilde{x} - W[j, \tilde{j}] = -S + \Gamma_{\text{fl}}$
- equation of state with $\langle \tilde{x} \rangle = 0$, $\langle x \rangle(t) = x^*(t)$:

$$0 \stackrel{!}{=} \tilde{j} = \Gamma_{\tilde{x}, \text{fl}}^{(1)}[x^*, 0]$$

$$= -\partial_t x^*(t) - l x^*(t) + \beta (x^*(t))^2 + \Gamma_{\tilde{x}, \text{fl}}^{(1)}[x^*, 0]$$

$=: F[\langle x \rangle(t'), t' \leq t]$



Single neuron with nonlinear gain function f as well as input \tilde{j} and noise ξ .

Diagrammatics

$$S_{\text{int}}[x, \tilde{x}] = \text{diagram of a vertex with two incoming lines (x and x-tilde) and one outgoing line (x-tilde)}$$

$$\Delta(t-s) = [\Gamma^{(2)}]^{-1} = \begin{pmatrix} x(t) & x(s) & x(t) & \tilde{x}(s) \\ \tilde{x}(t) & x(s) & 0 & \end{pmatrix}$$

Loop-wise expansion

Tree-level/mean-field theory

- discard the noise \rightarrow use $\Gamma \approx -S \rightarrow$ deterministic ODE
- two stable fixed points $x_0 = 0$, $x_\infty = \infty$ and one unstable fixed point $x_1 = \frac{l}{\beta}$
- $\langle x \rangle = 0$ and $\langle x^2 \rangle = -\frac{D}{2l}$

One-loop

first correction to the mean value:

$$0 \stackrel{!}{=} \tilde{j} = \Gamma_{\tilde{x}, \text{fl}}^{(1)}(\sigma) = (-1) \text{diagram of a loop with a vertex}$$

$$= \frac{2\pi\beta}{(2\pi)^2} \int d\omega \frac{D}{\omega^2 + m^2} \delta(\sigma) = \frac{\beta D}{2|m|} \delta(\sigma)$$

first correction to propagator:

$$\Gamma_{\tilde{x}, \text{fl}}^{(2)}(\sigma_1, \sigma_2) = (-1) \text{diagram of a loop with two vertices}$$

$$\Gamma_{\tilde{x}, \text{fl}}^{(2)}(\sigma_1, \sigma_2) = \left(-\frac{1}{2}\right) \text{diagram of a loop with two vertices}$$

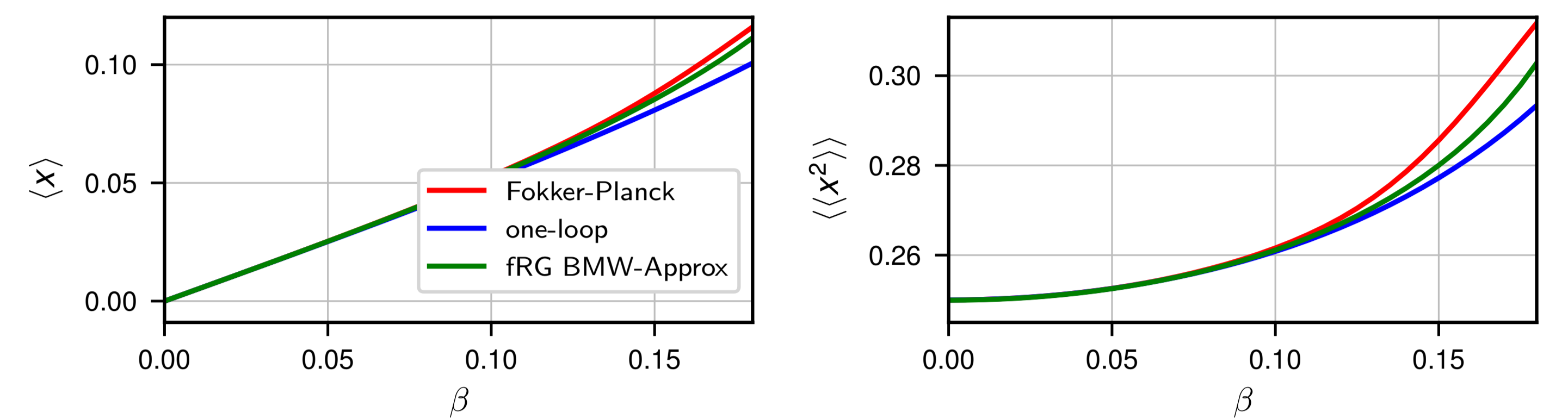
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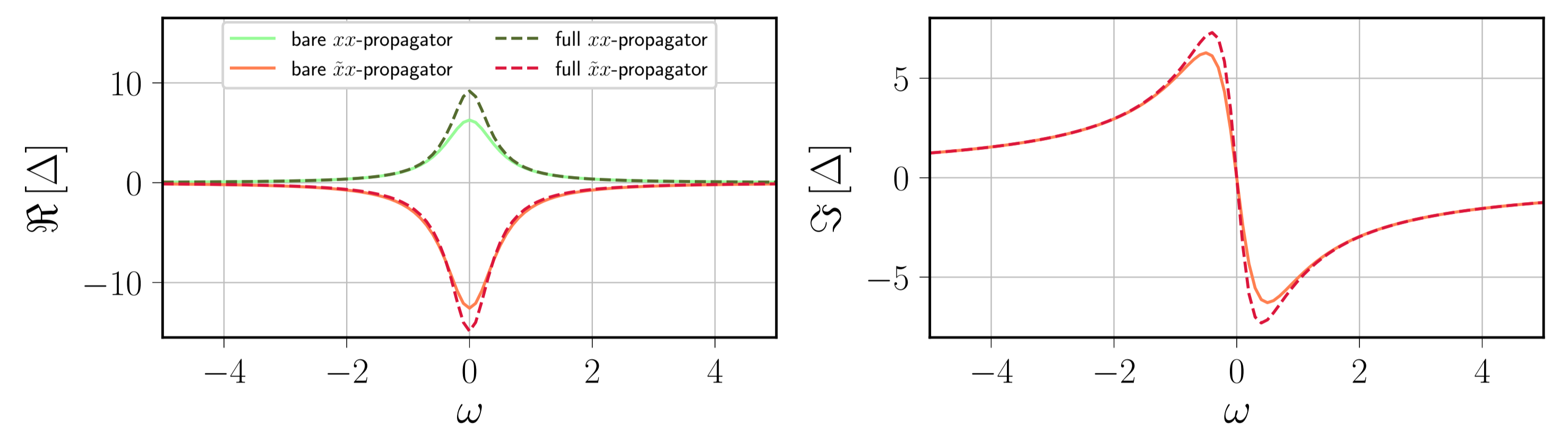
Results

stationary properties

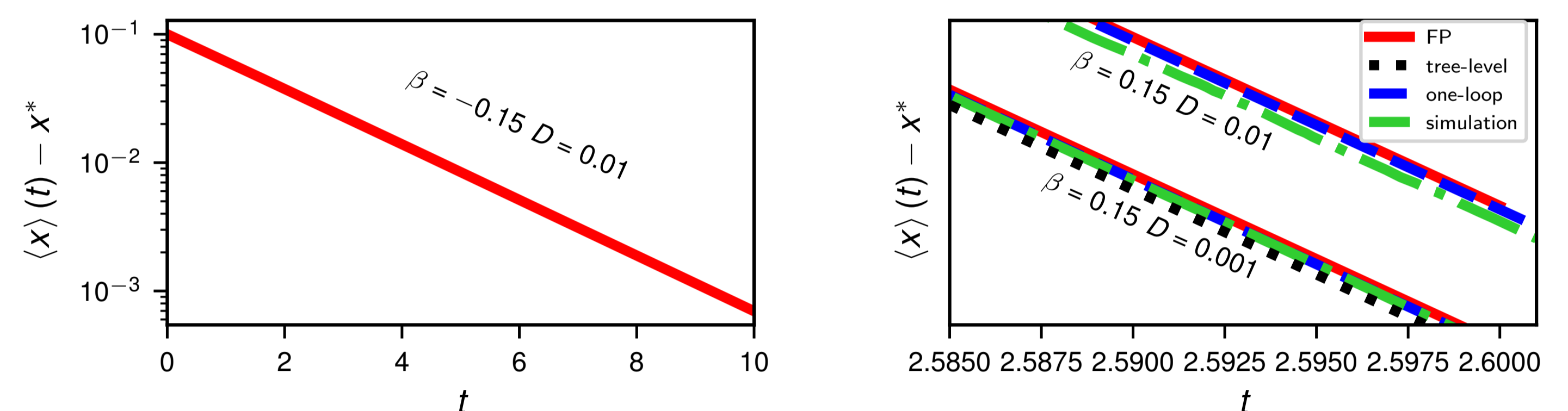


Stationary mean and variance of x for different strengths β of the nonlinearity, $l = 0.5$, $D = 0.25$.

frequency/time-dependent properties



Bare and full propagator (autocorrelation and response function) as a function of the frequency ω , $l = 0.5$, $\beta = 0.18$, $D = 0.25$.



Relaxation of the mean value back to its stationary value after a δ -input at time $t = 0$, $l = 0.5$.

Functional Renormalization Group

Wetterich equation [5, 6] describes the flow of the effective action Γ_λ from $\Gamma_\lambda = -S$ to $\Gamma_0 = \Gamma$.

$$\frac{\partial \Gamma_\lambda[x^*, \tilde{x}^*]}{\partial \lambda} = \frac{1}{2} \text{Tr} \left\{ \Delta_{\tilde{x}, \lambda} \frac{\partial R_\lambda}{\partial \lambda} \right\}$$

$$= \frac{1}{2} \text{Tr} \left\{ \left[\Gamma_\lambda^{(2)}[x^*, \tilde{x}^*] + R_\lambda \right]^{-1} \frac{\partial R_\lambda}{\partial \lambda} \right\} = \frac{1}{2} \text{diagram of a loop with a vertex}$$

- $\frac{\partial R_\lambda}{\partial \lambda}$ denotes the derivative of the regulator.
- Δ_λ is the second cumulant of the theory and is known as the "propagator".

Flow equation for the mean value x^* :

$$\frac{\partial x^*}{\partial \lambda} = -\Delta_{\tilde{x}, \lambda} \left[\frac{\partial \Gamma_{\tilde{x}, \lambda}^{(1)}}{\partial \lambda} + \frac{1}{4\pi} \frac{\partial R_\lambda}{\partial \lambda} x^* \right]$$

contribution to self-energy:

$$\frac{\partial \Gamma_{\tilde{x}, \lambda}^{(2)}(\sigma_1, \sigma_2)}{\partial \lambda} = \frac{1}{2} \text{diagram of a loop with two vertices}$$

$+ \sigma_1 \leftrightarrow \sigma_2$

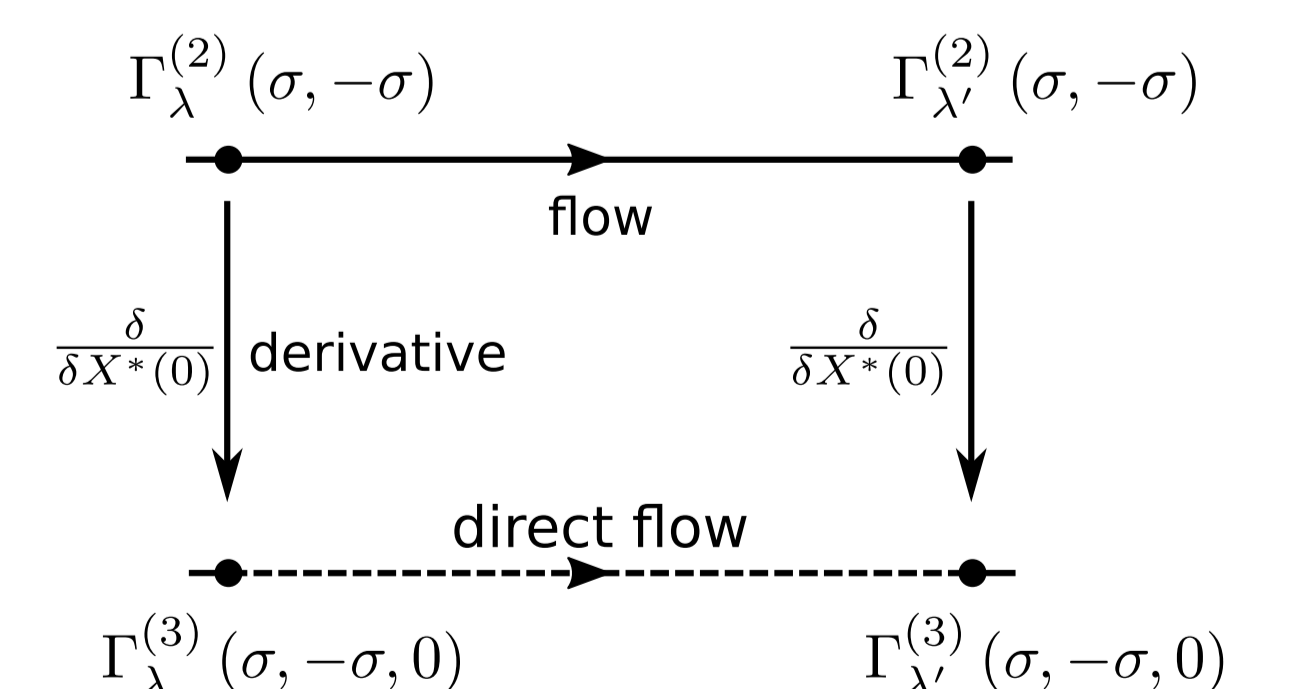
The flow equation for $\Gamma_\lambda^{(n)}$ requires $\Gamma_\lambda^{(n+1)}$ and $\Gamma_\lambda^{(n+2)} \Rightarrow$ infinite hierarchy \Rightarrow approximation needed.

Truncation

- simplest solution: truncate the hierarchy of flow equations $n > m$ and keep all higher vertices fixed at their initial condition $\Gamma^{(n>m)} = -S^{(n)}$.
- include the flow up to $\Gamma_\lambda^{(2)}$ and additionally that for $\Gamma_{\tilde{x}, \lambda}^{(3)}$.

BMW-scheme for vertex expansion

- BMW = Blaizot-Méndez-Wschebor [7, 8]
- express higher order vertices in terms of (ordinary, not functional) derivatives with respect to $X(0)$, $\tilde{X}(0)$ of lower order vertices
- no PDE in fields (here: $X(0)$, $\tilde{X}(0)$) and λ as typical for BMW but just ODE in λ



Outlook

- generalization to spatially extended networks; here renormalization probably intrinsically needed because of diverging bare propagator (compare KPZ-model), possibly emergence of long range phenomena (e.g. cortical waves) from short-range interactions
- investigation of critical phenomena, critical exponents \Rightarrow comparison to experimental data