Field Theory for Nonlinear Stochastic Rate Neurons

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Motivation

- neuronal dynamics is noisy (see Fig. 1) and nonlinear
- interplay between nonlinearity and stochasticity leads to interesting effects like memory
- goal: find effective description of the moments, considering the fluctuations self-consistently

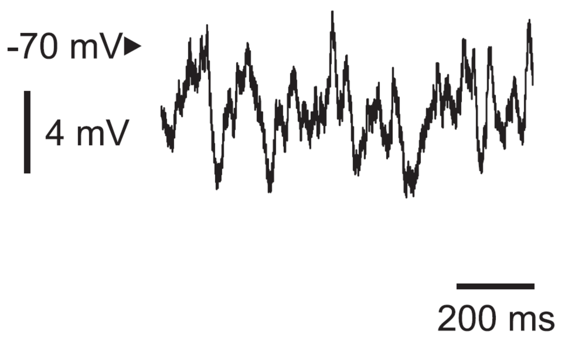


Fig. 1: Voltage trace from an intracellular recording from a cortical neuron of an anesthetized cat. Adapted from Rudolph et al. 2005 [1].

procedure: consider approximations with increasing accuracy

- tree-level
- one-loop
- functional Renormalization Group

$$\boxed{\partial_t x(t) = f(x(t)) + \xi(t)}$$

$$x \Rightarrow \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} =: X \Downarrow \text{MSRDJ-formalism}$$

$$S[X] \Longrightarrow W[J] \sim \ln \int \mathcal{D}X\, e^{S[X]+JX} \xrightarrow[\text{transform}]{\text{Legendre}} \Gamma[X]$$

tree level approximation

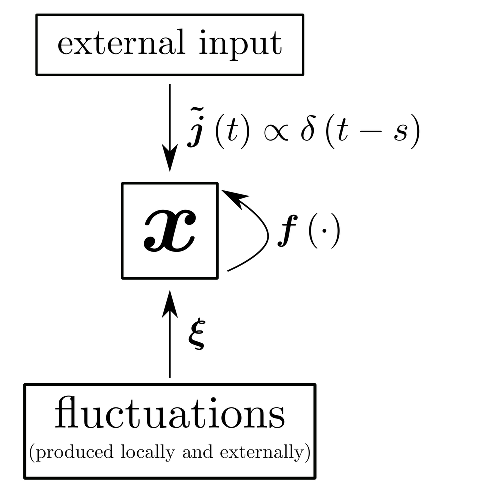
$$\frac{d}{dX}\Gamma = 0 \Downarrow \text{loop-expansion, fRG,...}$$

$$\boxed{\partial_t \langle x\rangle(t) = F[\langle x\rangle(t'), t' \le t]}$$

Neuron Model in Field Theory

neuron dynamics:

$$dx = f(x(t))\,dt + dW(t), \quad \text{where} \quad f(x) = -lx + \beta x^2$$



Single neuron with nonlinear gain function f as well as input $\tilde{j}$ and noise ξ.

From Langevin equation to field theory

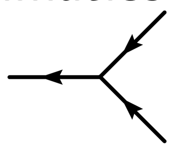
using the Martin-Siggia-Rose-De Dominicis-Janssen (MSRDJ) formalism [2, 3, 4]

$$S[x,\tilde{x}] = \tilde{x}^{\mathrm{T}}(\partial_t + l)x + \tilde{x}^{\mathrm{T}}\frac{D}{2}\tilde{x} + S_{\text{int}}[x,\tilde{x}]$$
$$S_{\text{int}}[x,\tilde{x}] = -\beta\,\tilde{x}^{\mathrm{T}}x^2, \quad \text{where}$$

- $x^{\mathrm{T}}y = \int_{-\infty}^{\infty} x(t)\,y(t)\,dt$ and
- $\tilde{x}$: "response field", all moments identical to zero
- $\mathcal{Z}[j,\tilde{j}] = \int \mathcal{D}x \int \mathcal{D}\tilde{x}\, e^{S[x,\tilde{x}]+j^T x+\tilde{j}^T\tilde{x}}$
- $W[j,\tilde{j}] = \ln \mathcal{Z}[j,\tilde{j}]$
- $\Gamma[x,\tilde{x}] = \sup_{j,\tilde{j}}\, j^T x + \tilde{j}^T\tilde{x} - W[j,\tilde{j}] = -S + \Gamma_{\text{fl.}}$
- equation of state with $\langle\tilde{x}\rangle = 0$, $\langle x\rangle(t) = x^*(t)$:

$$0 \overset{!}{=} \tilde{j} = \Gamma^{(1)}_{\tilde{x}^*}[x^*,0]$$
$$= -\partial_t x^*(t) \underbrace{-l x^*(t) + \beta(x^*(t))^2 + \Gamma^{(1)}_{\tilde{x},fl.}[x^*,0]}_{=:F[\langle x\rangle(t'),t'\le t]}$$

Diagrammatics

$S_{\text{int}}[x,\tilde{x}] =$ 

$$\Delta(t-s) = \left[\Gamma^{(2)}\right]^{-1} = \begin{pmatrix} x(t) \text{—} x(s) & x(t) \text{←} \tilde{x}(s) \\ \tilde{x}(t) \text{→} x(s) & 0 \end{pmatrix}$$

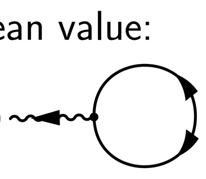
Loop-wise expansion

Tree-level/mean-field theory

- discard the noise → use $\Gamma \approx -S$ → deterministic ODE
- two stable fixed points $x_0 = 0$, $x_\infty = \infty$ and one unstable fixed point $x_1 = \frac{l}{\beta}$
- $\langle x\rangle = 0$ and $\langle\langle x^2\rangle\rangle = -\frac{D}{2l}$

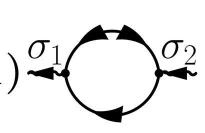
One-loop

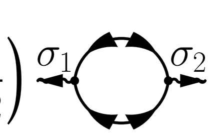
first correction to the mean value:

$$0 \overset{!}{=} \tilde{j} = \Gamma^{(1)}_{\tilde{X},\text{fl.}}(\sigma) = (-1)$$ 

$$= \frac{2\pi\beta}{(2\pi)^2}\int d\omega \frac{D}{\omega^2+m^2}\delta(\sigma) = \frac{\beta D}{2|m|}\delta(\sigma)$$

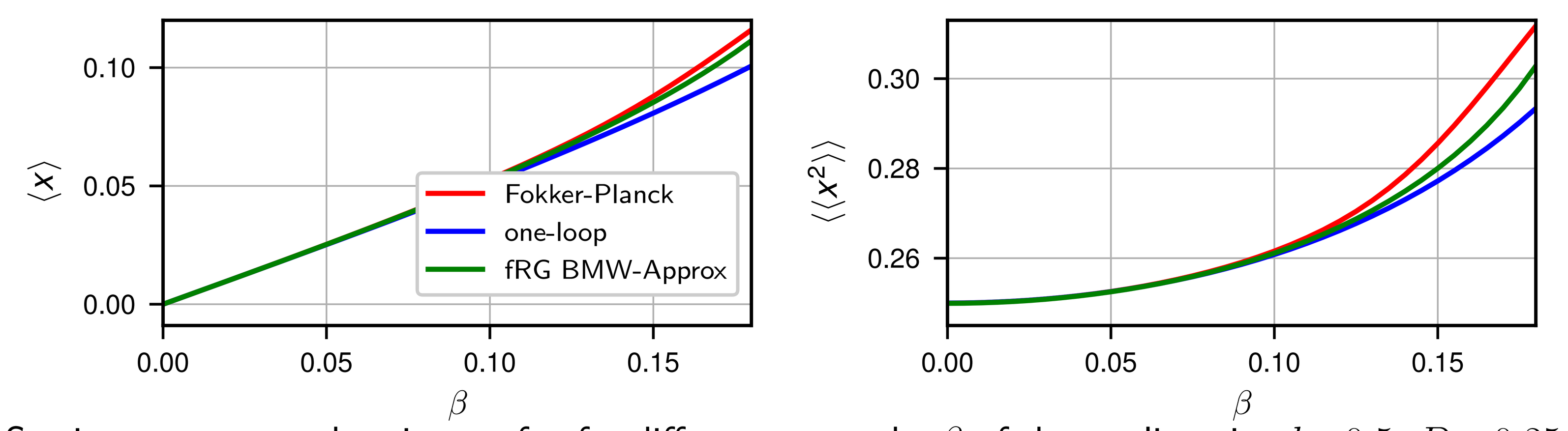
first correction to propagator:

$$\Gamma^{(2)}_{\tilde{X}X,\text{fl.}}(\sigma_1,\sigma_2) = (-1)$$ 

$$\Gamma^{(2)}_{\tilde{X}\tilde{X},\text{fl.}}(\sigma_1,\sigma_2) = \left(-\frac{1}{2}\right)$$ 

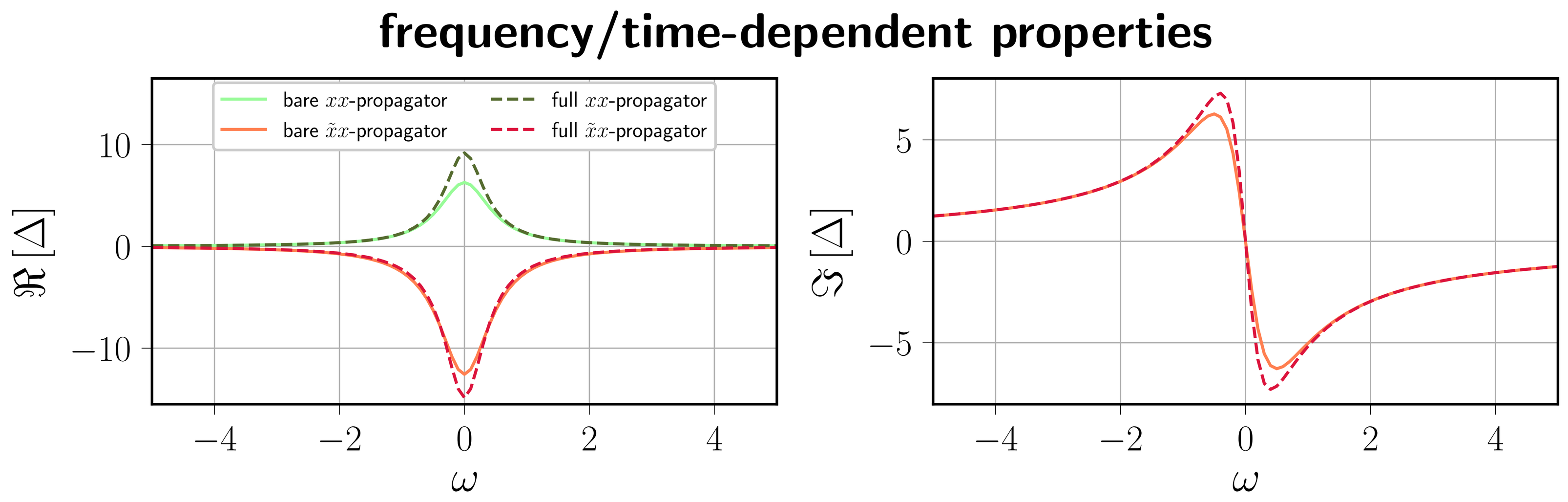
Results

stationary properties

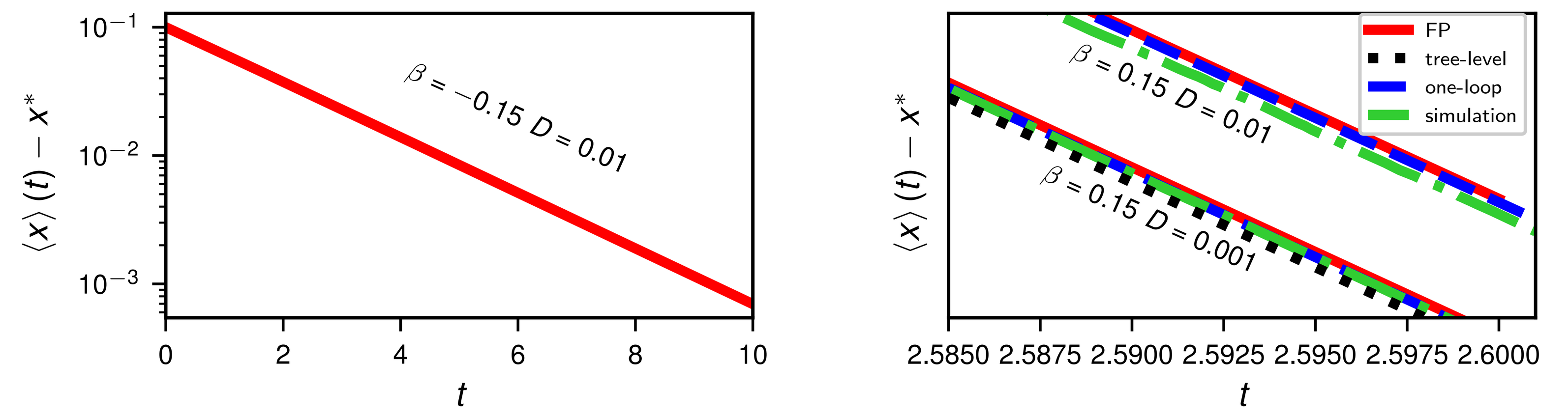


Stationary mean and variance of x for different strengths β of the nonlinearity, $l = 0.5$, $D = 0.25$.

frequency/time-dependent properties



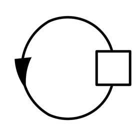
Bare and full propagator (autocorrelation and response function) as a function of the frequency ω, $l = 0.5$, $\beta = 0.18$, $D = 0.25$.



Relaxation of the mean value back to its stationary value after a δ-input at time $t = 0$, $l = 0.5$.

Functional Renormalization Group

Wetterich equation [5, 6] describes the flow of the effective action Γ_λ from $\Gamma_\Lambda = -S$ to $\Gamma_0 = \Gamma$.

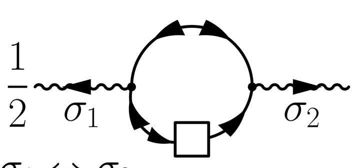
$$\frac{\partial\Gamma_\lambda[x^*,\tilde{x}^*]}{\partial\lambda} = \frac{1}{2}\mathrm{Tr}\left\{\Delta_{\tilde{x}x,\lambda}\frac{\partial R_\lambda}{\partial\lambda}\right\}$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\left[\Gamma^{(2)}_\lambda[x^*,\tilde{x}^*] + R_\lambda\right]^{-1}_{\tilde{x}x}\frac{\partial R_\lambda}{\partial\lambda}\right\} = \frac{1}{2}$$ 

- $\frac{\partial R_\lambda}{\partial\lambda}$ denotes the derivative of the regulator.
- Δ_λ is the second cumulant of the theory and is known as the "propagator".

Flow equation for the mean value x^*:

$$\frac{\partial}{\partial\lambda}x^* = -\Delta_{x\tilde{x},\lambda}\left[\frac{\partial\Gamma^{(1)}_{\tilde{x},\lambda}}{\partial\lambda} + \frac{1}{4\pi}\frac{\partial R_\lambda}{\partial\lambda}x^*\right]$$

contribution to self-energy:

$$\frac{\partial\Gamma^{(2)}_{\tilde{X}\tilde{X},\lambda}(\sigma_1,\sigma_2)}{\partial\lambda} = \frac{1}{2}$$ 

$+\,\sigma_1 \leftrightarrow \sigma_2$

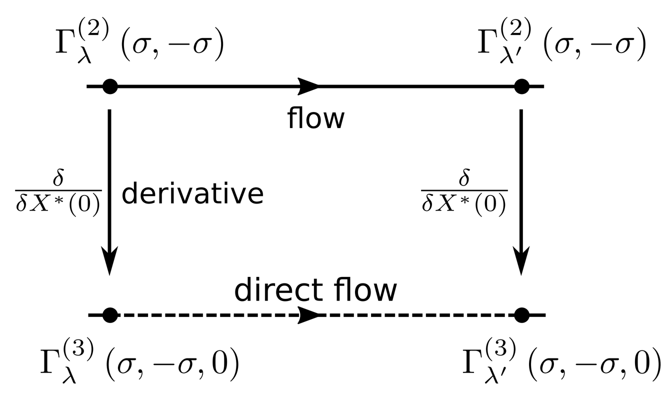
The flow equation for $\Gamma^{(n)}_\lambda$ requires $\Gamma^{(n+1)}_\lambda$ and $\Gamma^{(n+2)}_\lambda$ ⇒ infinite hierarchy ⇒ approximation needed.

Truncation

- simplest solution: truncate the hierarchy of flow equations $n > m$ and keep all higher vertices fixed at their initial condition $\Gamma^{(n>m)} = -S^{(n)}$.
- include the flow up to $\Gamma^{(2)}_\lambda$ and additionally that for $\Gamma^{(3)}_{\tilde{x}xx,\lambda}$.

BMW-scheme for vertex expansion

- BMW = Blaizot-Méndez-Wschebor [7, 8]
- express higher order vertices in terms of (ordinary, not functional) derivatives with respect to $X(0)$, $\tilde{X}(0)$ of lower order vertices
- no PDE in fields (here: $X(0)$, $\tilde{X}(0)$) and λ as typical for BMW but just ODE in λ



Outlook

- generalization to spatially extended networks; here renormalization probably intrinsically needed because of diverging bare propagator (compare KPZ-model), possibly emergence of long range phenomena (e.g. cortical waves) from short-range interactions
- investigation of critical phenomena, critical exponents ⇒ comparison to experimental data

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