ANOMALOUS DIFFUSION, DILATION, AND EROSION IN IMAGE PROCESSING

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Part I: Introduction & motivation

INTRODUCTION & MOTIVATION

General idea

- Time-dependent partial differential equations (PDEs) arise naturally in image processing.
- For example: convolution of image with Gaussian kernel which is equivalent to solving a linear diffusion equation.
- Other PDEs: dilation/erosion (evolution equations).
- Can serve as building blocks for higher morphological operations (opening, closing, gradients) or deblurring filters.

INTRODUCTION & MOTIVATION

What is new?

- Different type of generalization of an evolution equation.
- Temporal derivative of fractional order α: $\frac{\partial^\alpha}{\partial t^\alpha}$ with $\alpha \in (0, 2)$.
- Definition of the fractional derivative as an extension of integration concatenated with regular differentiation (Caputo).
- Global information are considered.
- Also interesting for other applications.
- Up to now this approach was only considered for specific fractional orders as $\alpha = 1/2$ and not for morphological operations.

Part II: Anomalous diffusion

ANOMALOUS DIFFUSION

Mathematical model

- Diffusion equation:

$$\frac{{}^{c}\partial^{\alpha}}{\partial t^{\alpha}} u = \mathrm{div}(\kappa \,\mathrm{grad}\, u)\,,$$

  where κ is a constant.
- Caputo fractional derivative:

$$\frac{{}^{c}\partial^{\alpha}}{\partial t^{\alpha}} u = \frac{1}{\Gamma(m+1-\alpha)} \int_0^t \frac{u^{(m+1)}(\tau)}{(t-\tau)^{\alpha-m}} d\tau\,,$$

  where $m = \lfloor \alpha \rfloor$ and $0 < \alpha < 1$ or $1 < \alpha < 2$.
- Initial condition(s): given gray-value image and in case of super-diffusion we need a second initial condition.
- Boundary condition: homogeneous Neumann.

ANOMALOUS DIFFUSION

Space discretization

- 2D-grid with $h = 1$ and $M \times N$ grid points.
- Approximation of Laplace operator with centered differences for interior nodes: $\kappa(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})$.
- Homogeneous Neumann boundaries for exterior nodes.

ANOMALOUS DIFFUSION

Time discretization

- Grid of the form $t_k = k\Delta t, k = 0, \ldots, P$ with grid size $\Delta t = T/P$.
- Approximation of Caputo derivative by Grünwald-Letnikov formula:

$$\frac{{}^C\partial^\alpha u}{\partial t^\alpha}\bigg|_{\mathbf{x}=(x_i,y_j)}^{t=t_{k+1}} \approx \sum_{\ell=0}^{k+1} c_\ell^{(\alpha)} u_{i,j}^{k+1-\ell} - \sum_{n=0}^{m} \frac{(t_{k+1})^{n-\alpha}}{\Gamma(n-\alpha+1)} u^{(m)}(x_i, y_j)\,,$$

where

$$c_0^{(\alpha)} = (\Delta t)^{-\alpha}\,, \qquad c_k^{(\alpha)} = \left(1 - \frac{1+\alpha}{k}\right) c_{k-1}^{(\alpha)}\,.$$

ANOMALOUS DIFFUSION

Numerical schemes

Explicit:

$$\sum_{\ell=0}^{k+1} c_\ell^{(\alpha)} u_{i,j}^{k+1-\ell} - \sum_{n=0}^{m} \frac{(t_{k+1})^{n-\alpha}}{\Gamma(n-\alpha+1)} u^{(m)}(x_i, y_j) = \kappa \left(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k\right) .$$

Implicit:

$$\sum_{\ell=0}^{k+1} c_\ell^{(\alpha)} u_{i,j}^{k+1-\ell} - \sum_{n=0}^{m} \frac{(t_{k+1})^{n-\alpha}}{\Gamma(n-\alpha+1)} u^{(m)}(x_i, y_j) = \kappa \left(u_{i+1,j}^{k+1} + u_{i-1,j}^{k+1} + u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1} - 4u_{i,j}^{k+1}\right) .$$

ANOMALOUS DIFFUSION

Numerical schemes

Explicit:

$$\mathbf{u}^{k+1} = A\,\mathbf{u}^k - \mathbf{b}_{ex} \text{ with } A = \alpha I_{MN} + (\Delta t)^\alpha \kappa \cdot D_2\,.$$

Implicit:

$$B\,\mathbf{u}^{k+1} = \mathbf{b}_{im} \text{ with } B = -(\Delta t)^{-\alpha} I_{MN} + \kappa \cdot D_2\,.$$

D_2 is the 2D-Laplacian and $\mathbf{b}_{ex}$ and $\mathbf{b}_{im}$ are given by

$$\mathbf{b}_{ex} = (\Delta t)^\alpha \left(\sum_{\ell=2}^{k+1} c_\ell^{(\alpha)} \mathbf{u}^{k+1-\ell} - \sum_{n=0}^{m} \frac{(t_{k+1})^{n-\alpha}}{\Gamma(n-\alpha+1)} u^{(m)}(x_i, y_j) \right) ,$$

$$\mathbf{b}_{im} = -\alpha(\Delta t)^{-\alpha}\mathbf{u}^k + \left(\sum_{\ell=2}^{k+1} c_\ell^{(\alpha)} \mathbf{u}^{k+1-\ell} - \sum_{n=0}^{m} \frac{(t_{k+1})^{n-\alpha}}{\Gamma(n-\alpha+1)} u^{(m)}(x_i, y_j) \right) .$$

Part III: Modified dilation & erosion

MODIFIED DILATION & EROSION

Mathematical model & discretization

- Dilation & erosion equation:

$$\frac{{}^{c}\partial^{\alpha}}{\partial t^{\alpha}} u = \pm\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} .$$

- Approximation of Caputo fractional derivative as before.
- Approximation in space by first-order finite difference scheme of Rouy-Tourin:

$$\left[\max(-u_{i,j} + u_{i-1,j}, u_{i+1,j} - u_{i,j}, 0)^2 + \max(-u_{i,j} + u_{i,j-1}, u_{i,j+1} - u_{i,j}, 0)^2\right]^{1/2} .$$

MODIFIED DILATION & EROSION

Numerical schemes

As before, we obtain an iterative scheme of the form

$$\mathbf{u}^{k+1} = \alpha \mathbf{u}^k + (\Delta t)^\alpha \mathbf{b}_{dt} \pm (\Delta t)^\alpha \sqrt{\mathbf{b}_{dx}^2 + \mathbf{b}_{dy}^2}\,,$$

where

$$\mathbf{b}_{dt} = -\sum_{l=2}^{k+1} c_l^{(\alpha)} \mathbf{u}^{k+1-l} + \frac{t_{k+1}^{-\alpha}}{\Gamma(1-\alpha)} \mathbf{u}^0$$

and the i-th, j-th entry of $\mathbf{b}_{dx}$ is given by

$$\max\left(-u_{i,j}^k + u_{i-1,j}^k, u_{i+1,j}^k - u_{i,j}^k, 0\right)$$

and $\mathbf{b}_{dy}$ analogously.

Part IV: Numerical results

NUMERICAL RESULTS

Stability

- Linear test problem:

$$\frac{{}^{c}\partial^{\alpha} u(t)}{\partial t^{\alpha}} = \lambda u(t)\,, \quad \lambda \in \mathbb{C}\,,$$
$$u(0) = u_0 \text{ for } 0 < \alpha \leq 1\,,$$
$$\text{and additionally } u'(0) = u_1 \text{ for } 1 < \alpha < 2\,.$$

- Explicit method: $\mathbb{C} \backslash \{(1-z)^{\alpha}/z : |z| \leq 1\}$.
- Implicit method: $\mathbb{C} \backslash \{(1-z)^{\alpha} : |z| \leq 1\}$.

NUMERICAL RESULTS

Stability

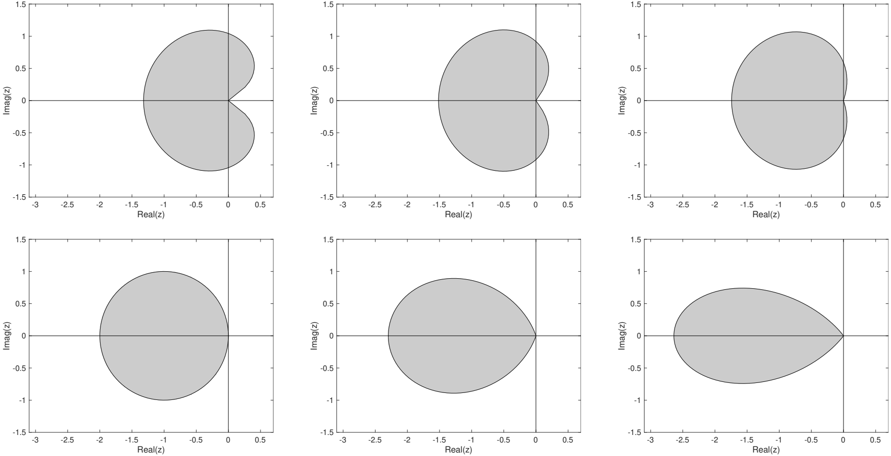


Figure: Stability regions for explicit Euler method using parameters $\alpha = 0.4$, $\alpha = 0.6$, and $\alpha = 0.8$ (first row) and $\alpha = 1.0$, $\alpha = 1.2$, and $\alpha = 1.4$ (second row).

NUMERICAL RESULTS

Stability

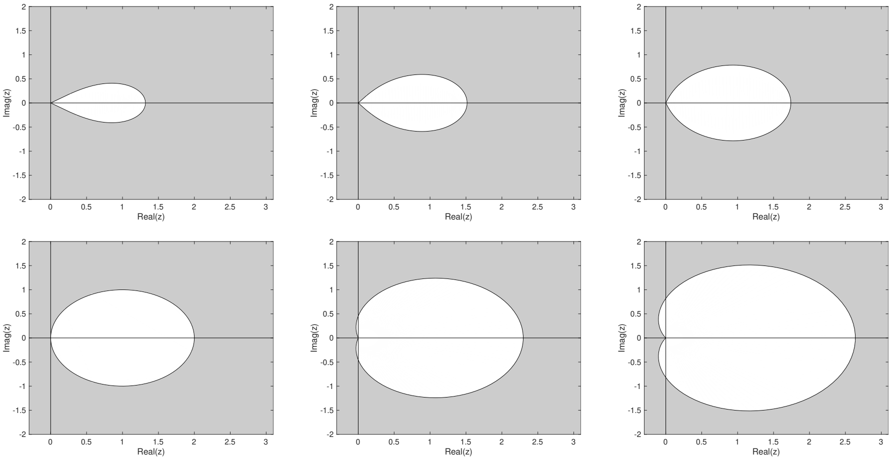


Figure: Stability regions for implicit Euler method using parameters $\alpha = 0.4$, $\alpha = 0.6$, and $\alpha = 0.8$ (first row) and $\alpha = 1.0$, $\alpha = 1.2$, and $\alpha = 1.4$ (second row).

NUMERICAL RESULTS

Stability

- Interval of stability is $(-2^\alpha, 0)$.
- Implicit Euler method is *A*-stable for $0 < \alpha \leq 1$ whereas we loose this property for $1 < \alpha < 2$.
- Could investigate $A(\theta)$ stability, where $\theta \leq \pi/2$ will depend on α.
- We obtain the θ angles (in degrees $^\circ$) 90, 81, 72, 63, 54, 45, 36, 27, 18, and 9 for the parameters $\alpha =$ 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, and 1.9, respectively.
- Hence, it appears to be that θ is given by $(2-\alpha)\cdot 90^\circ$ for $1 \leq \alpha < 2$ (the proof remains open).

NUMERICAL RESULTS
Convergence

- Homogeneous initial conditions: convergence order 1
- Non-homogenous initial conditions: convergence order depends on α
- Calculation of error:

$$\frac{{}^{c}\partial^{\alpha} u(t)}{\partial t^{\alpha}} = t^2\,, \quad u(0) = 0\,, \quad 0 \le t \le 1\,, \quad 1 < \alpha \le 2$$

with exact solution

$$u(t) = \frac{\Gamma(3+\alpha)}{\Gamma(3)} t^{2+\alpha}\,.$$

Estimated convergence order (EOC):

$$\mathrm{EOC} = \frac{log(E_{\Delta t}/E_{\Delta t/2})}{log(2)}, \text{ where } E_{\Delta t} = |u(1) - \tilde{u}_{\Delta t}(1)|\,.$$

NUMERICAL RESULTS

Convergence

	$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1.0$		$\alpha = 1.2$	
Δt	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC
1/10	0.1220		0.0685		0.0483		0.0324	
1/20	0.0627	0.96	0.0350	0.97	0.0246	0.98	0.0164	0.99
1/40	0.0318	0.98	0.0177	0.98	0.0124	0.99	0.0082	1.00
1/80	0.0160	0.99	0.0089	0.99	0.0062	0.99	0.0041	1.00
1/160	0.0080	1.00	0.0045	1.00	0.0031	1.00	0.0021	1.00
1/320	0.0040	1.00	0.0022	1.00	0.0016	1.00	0.0010	1.00
1/640	0.0020	1.00	0.0011	1.00	0.0008	1.00	0.0005	1.00

Table: Estimated order of convergence for the explicit Euler method using the parameters $\alpha = 0.4, 0.8, 1.0$, and 1.2.

NUMERICAL RESULTS

Convergence

	$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1.0$		$\alpha = 1.2$	
Δt	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC	$E_{\Delta t}$	EOC
1/10	0.0323		0.0487		0.0517		0.0519	
1/20	0.0161	1.00	0.0241	1.01	0.0254	1.02	0.0253	1.03
1/40	0.0081	1.00	0.0120	1.01	0.0126	1.01	0.0125	1.02
1/80	0.0040	1.00	0.0060	1.00	0.0063	1.00	0.0062	1.01
1/160	0.0020	1.00	0.0030	1.00	0.0031	1.00	0.0031	1.00
1/320	0.0010	1.00	0.0015	1.00	0.0016	1.00	0.0015	1.00
1/640	0.0005	1.00	0.0007	1.00	0.0008	1.00	0.0008	1.00

Table: Estimated order of convergence for the implicit Euler method using the parameters $\alpha = 0.4, 0.8, 1.0$, and 1.2.

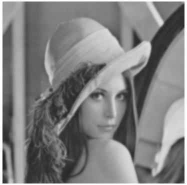
NUMERICAL RESULTS

Anomalous diffusion

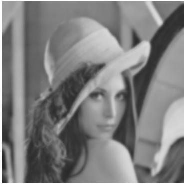
(a) $T = 1$, $\alpha = 1/2$

(b) $T = 1$, $\alpha = 3/4$

(c) $T = 1$, $\alpha = 1$



(d) $T = 10$, $\alpha = 1/2$



(e) $T = 10$, $\alpha = 3/4$

(f) $T = 10$, $\alpha = 1$

Figure: Anomalous sub-diffusion with $T = 1, 10$ and $\alpha = 1/2, 3/4, 1$ for the Lena image.

NUMERICAL RESULTS

Modified dilation

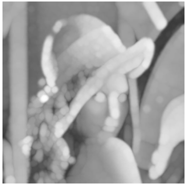
(a) $T = 1, \alpha = 1/2$

(b) $T = 1, \alpha = 3/4$

(c) $T = 1, \alpha = 1$

(d) $T = 10, \alpha = 1/2$

(e) $T = 10, \alpha = 3/4$



(f) $T = 10, \alpha = 1$

Figure: Modified dilation with $T = 1, 10$ and $\alpha = 1/2, 3/4, 1$ for the Lena image.

Part V: Summary & outlook

SUMMARY & OUTLOOK

- Modified standard diffusion as well as dilation & erosion for gray-valued images.
- Treated numerically by explicit and implicit Euler method.
- Showed convergence and stability.

- Consider second-order approximation of the Caputo fractional derivative.
- Multistep methods (BDF, Adams-Moulton, and Adams-Bashforth methods).
- Consider corresponding inverse problems (denoising).
- Extension for higher morphological operations.
- Extending the approach to color images.

REFERENCE

A. Kleefeld, S. Vorderwülbecke, & B. Burgeth, *Anomalous diffusion, dilation, and erosion in image processing*, International Journal of Computer Mathematics 95 (6–7), 1375–1393 (2018), special issue: "Advances on Computational Fractional Partial Differential Equations".