Theoretical Foundation of the Weighted Laplace Inpainting Problem

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Abstract. Laplace interpolation is a popular approach in image inpainting using partial differential equations. The classic approach considers the Laplace equation with mixed boundary conditions. Recently a more general formulation has been proposed where the differential operator consists of a point-wise convex combination of the Laplacian and the known image data. We provide the first detailed analysis on existence and uniqueness of solutions for the arising mixed boundary value problem. Our approach considers the corresponding weak formulation and aims at using the Theorem of Lax-Milgram to assert the existence of a solution. To this end we have to resort to weighted Sobolev spaces. Our analysis shows that solutions do not exist unconditionally. The weights need some regularity and fulfil certain growth conditions. The results from this work complement findings which were previously only available for a discrete setup.

Key words. Image Inpainting, Image Reconstruction, Laplace Equation, Laplace Interpolation, Mixed Boundary Conditions, Partial Differential Equations, Weighted Sobolev Space

AMS subject classifications. 35J15, 35J70, 46E35, 94A08

1. Introduction.

Image inpainting deals with recovering lost image regions or structures by means of interpolation. It is an ill-posed process; as soon as a part of the image is lost it cannot be recovered correctly with absolute certainty unless the original image is completely known. The inpainting problem goes back to the works of Masnou and Morel as well as Bertalmío and colleagues [4, 35], although similar problems have been considered in other fields already before. There exist many inpainting techniques, often based on interpolation algorithms, but partial differential equation (PDE)-based approaches are among the most successful ones, see e.g. [16]. Among these, strategies based on the Laplacian stand out [5, 32, 41, 46]. In that context, the elliptic mixed boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \Omega_K \\ u = f & \text{in} \quad \partial\Omega_K \\ \partial_n u = 0 & \text{in} \quad \partial\Omega \setminus \partial\Omega_K \end{cases} \tag{1.1}$$

is very popular. Here, f represents known image data in a region $\Omega_K \subset \Omega$ (resp. on the boundary $\partial\Omega_K$) of the whole image domain Ω. Further, $\partial_n u$ denotes the derivative in outer normal direction. An exemplary sketch of the layout of the problem is given in Figure 1. Equations like (1.1), that involve different kinds of boundary conditions, are commonly referred to as mixed boundary value problems and in rare cases also as Zaremba's problem [50]. Image inpainting based on (1.1) appears under various names in the literature: Laplace interpolation [42],

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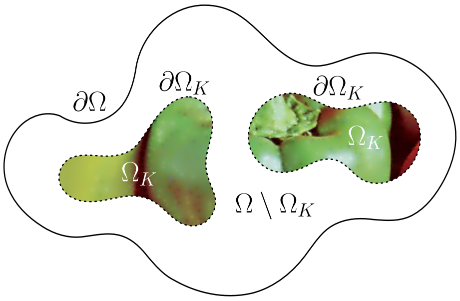
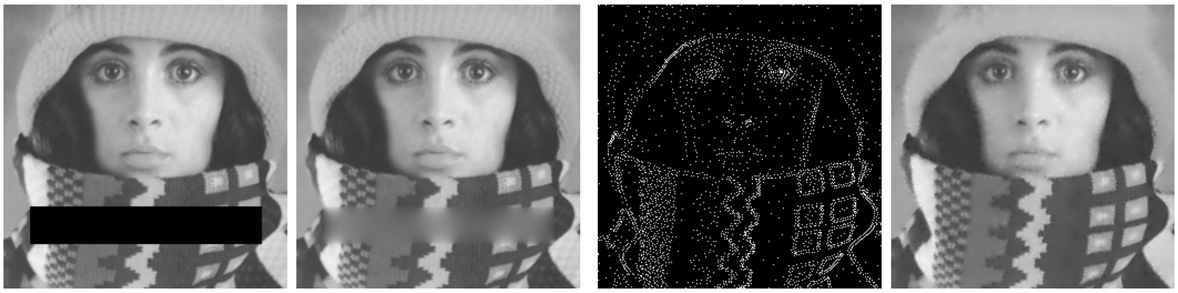


Figure 1. *Generic inpainting model as given in (1.1) with known data image f in Ω_K. The task consists in recovering a reasonable reconstruction of the image f in $\Omega \setminus \Omega_K$ by solving the PDE in (1.1).*

harmonic interpolation [45], or homogeneous diffusion inpainting [32]. The latter name is often used in combination with the steady state solution of the parabolic counterpart of (1.1).

Applications of image inpainting are manifold and range from art restoration to image compression. The earliest uses of (1.1) go back to Noma and Misulia (1959) [37] and Crain (1970) [10] for generating topographic maps. Further applications include the works of Bloor and Wilson (1989) [5], who studied partial differential equations for generating blend surfaces. Finally, refer to [23, 45] for a broad overview on PDE-based inpainting and the closely related problem of PDE-based image compression.

In the context of image reconstructions, (1.1) is often favoured over other more complex models due to its mathematically sound theory, even though the strong smoothing properties may yield undesirable blurry reconstructions. Models based on anisotropic diffusion [15, 44] or total variation [46] may be more powerful, but are much harder to grasp from a mathematical point of view. In the context of image compression, the data Ω_K used for the reconstruction can be freely chosen, since the complete image is known beforehand. The difficulty in compressing an image with a PDE lies in the fact that one has to optimise two contradicting constraints. On the one hand, the size of the data Ω_K should be small to allow an efficient coding, but on the other hand one wishes to have an accurate reconstruction from this sparse amount of information, too. The optimal data depends on the choice of the differential operator and the simplicity of the Laplacian offers many design choices for optimisation strategies to find the best Ω_K. Some of these approaches belong to the state-of-the-art methods in PDE-based image compression. We refer to [40] for a comparison of different PDE-based models and to [14, 24, 33] for data optimisation strategies in the compression context. Figure 2 demonstrates the potential of such a careful data optimisation. Figure 2a and Figure 2b show the reconstruction of an arbitrary rectangle. The reconstruction is severely blurred and the texture of the scarf is almost completely lost. On the other hand, Figure 2c represents an optimised set of 5% of the data points with the corresponding colour values. Figure 2d depicts the corresponding reconstruction. Although the reconstruction has a few artefacts, its overall quality is very convincing. As already mentioned, finding the best pixel-data is a very challenging task. Mainberger et al. [33] consider the combinatorial point of view of this task while Belhachmi and colleagues [3] approach the topic from the analytic side. Recently [24], the "hard" boundary conditions in (1.1) have been



(a) Arbitrary data (b) Reconstruction (c) Optimal data (d) Reconstruction

Figure 2. *(a) Image data with an arbitrary missing rectangular region (marked in black). (b) Corresponding reconstruction with (1.1). The reconstruction suffers from blurring effects. (c) Remaining data (5% of all pixels) with optimal reconstruction property. Missing data is black. (d) Corresponding reconstruction with (1.1). The reconstruction is sharp although the Laplacian causes strong smoothing.*

replaced by softer weighting schemes that lead to models such as

$$\begin{cases} c\,(u - f) + (1 - c)\,(-\Delta)\,u = 0 & \text{in } \Omega \setminus \Omega_K \\ \partial_n u = 0 & \text{in } \partial\Omega \end{cases} \tag{1.2}$$

with a weighting function c. Optimising such a weighting function is notably simpler, at least in discrete setups.

Equation (1.1) is well understood and there exist many results on existence, uniqueness, and regularity of solutions, see [7, 12] for a generic analysis and [9, 46] for a more specific analysis in the inpainting context with Dirichlet boundary conditions only. Finite difference discretisations of (1.1) and (1.2) have also been subject of several investigations in the past. One can show that the discrete counterpart of (1.1) admits a unique solution as soon as the Dirichlet boundary set is non-empty [32]. Similarly, the discrete finite difference formulation of (1.2) admits a unique solution if c is positive in at least one position [20].

An important question that arises in this context is what these discrete requirements relate to in the continuous setting. If we consider for example the following model problem that one may extract from the formulation of (1.1),

$$\begin{cases} -\Delta u = 0 & \text{in} \quad B_1 \setminus B_\varepsilon \\ u = 0 & \text{in} \quad \partial B_1 \\ u = 1 & \text{in} \quad \partial B_\varepsilon \end{cases} \tag{1.3}$$

where B_r is a ball or radius r with centre at the origin, then one can show that a smooth solution exists for every $\varepsilon > 0$, but that no solution exists in the limiting case $\varepsilon \to 0$. Indeed, the solution is given by

$$u(x, y) = \frac{\ln\left(x^2 + y^2\right)}{2\ln(\varepsilon)} \tag{1.4}$$

Yet, the discrete formulation will admit a unique solution independently of the choice of ε. It suffices that the corresponding matrix is block irreducible. We refer to [20, 32] for a detailed discussion on the existence of solutions. To remedy the situation for the continuous formulation in (1.1), the authors of [3] have required that the set Ω_K should have positive α-capacity. The α-capacity ($\alpha > 0$) of a subset $E \subset D$ of a smooth, bounded, and open set D is given by

$$\inf \left\{ \int_D |\nabla u|^2 + \alpha |u|^2 \, \mathrm{d}x \,\middle|\, u \in U_E \right\} \tag{1.5}$$

where U_E is the set of all functions u of the Sobolev space $H_0^1(D)$ such that $u \geqslant 1$ almost everywhere in a neighbourhood of E. If Ω_K has positive α-capacity, then a solution of (1.1) exists in the Sobolev space $H^1(\Omega)$ [3]. This requirement, that Ω_K must have positive capacity, can be understood as requiring that image pixels are "fat enough" to allow a reconstruction. It reconciles the continuous and discrete worlds and leads to a consistent theory on both sides. A higher regularity than $H^1(\Omega)$ can be achieved for specific constellations of the boundary data. A rather general theory is given in [2, 13, 34]. The author of [36] shows that a Hölder continuous solution exists if the data is regular enough. Finally, [6] discusses the regularity of solutions on Lipschitz domains. Let us mention that the authors of [8] have also discussed this inability of the Laplacian to recover images from isolated points and that they suggested absolutely minimising Lipschitz extensions as an alternative.

The authors of this manuscript are not aware of any similar theory that would remedy the apparent discrepancy between (1.2) and its discrete counterpart. The discrete setup is almost always solvable. On the other hand, solutions for the continuous model are only known for some special cases such as c being bounded between two positive constants in the interval $(0, 1)$, or c being itself a constant [7, 12]. For inpainting purposes it is important that c may map to the whole unit interval and even beyond. Regions with $c \equiv 1$ keep the data fixed and if c exceeds the value 1, then contrast enhancing in the reconstruction is possible, see [21, 25].

Here, we attempt to bridge that gap between the discrete setup and the continuous model for the case when c maps to $[0, 1]$. We show that a weak solution exists if certain assumptions on the weight functions are met. Special interest will be paid to occurring requirements on c and whether they correspond to discrete counterparts. We aim at applying the theorem of Lax-Milgram in purpose-built weighted Sobolev spaces. As a such, the contributed novelties of this manuscript are twofold. First we complement the well-posedness study of (1.1) and $c > 1$, which has recently been discussed in [22] with the missing case where c maps to $[0, 1]$ and secondly, we introduce weighted Sobolev spaces to the image processing community. These spaces bear a certain number of interesting properties that can also be useful for other image analysis tasks.

In the next section we first derive the weak formulation corresponding to (1.1) and introduce the weighted Sobolev spaces where the solution is sought. Then we will state the necessary conditions on the weight function c that must be fulfilled to assert the existence of a solution. Finally we show that a unique solution exists.

2. Inpainting with the weighted Laplacian.

As already mentioned in the previous section, the classic formulation for PDE-based inpainting with the Laplacian reads

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \Omega_K \\ u = f & \text{in} \quad \partial\Omega_K \\ \partial_n u = 0 & \text{in} \quad \partial\Omega \setminus \partial\Omega_K \end{cases} \tag{2.1}$$

Using the findings from [12, 22], it is easy to show that (2.1) is well-posed and that a unique weak solution exists in a subspace of $H^1(\Omega)$. With a weight function c that maps from Ω to $\{0, 1\}$, (2.1) can also be rewritten as

$$\begin{cases} c\,(u - f) + (1 - c)\,(-\Delta)\,u = 0 & \text{in} \quad \Omega \\ \partial_n u = 0 & \text{in} \quad \partial\Omega \end{cases} \tag{2.2}$$

Interestingly, the latter formulation also makes sense if $c : \Omega \to \mathbb{R}$, a fact which was first exploited in [24]. If c has binary values in the set $\{0, 1\}$, then (2.2) is equivalent to (2.1) with the Dirichlet boundary conditions specified by f at those regions where c equals 1. Equation (2.2) can also be interpreted from a physical or chemical point of view. We are in the presence of a stationary reaction-diffusion equation. The diffusive term $(c-1)\Delta u$ is responsible for spreading the information generated by the reactive term $c\,(u - f)$. The weight c is responsible for the speed at which information is generated and spread.

If c is bounded between two non-negative numbers strictly smaller than one, then it follows from [7, 22] that a solution exists in $C^{2,\alpha}\left(\overline{\Omega}\right)$. For inpainting purposes it is however important to allow $c(x) = 1$ or even $c(x) > 1$. In order to derive the weak formulation of (2.2) we follow the presentation in [22], where the setup in (2.2) with $c > 1$ was discussed by outlining its relationship to the Helmholtz equation.

Let us now rewrite (2.2) in a more suitable form. In a first step we explicitly set the regions where $c \equiv 1$ apart.

$$\begin{cases} c\,(u - f) + (1 - c)\,(-\Delta)\,u = 0 & \text{in } \Omega \setminus \Omega_K \\ u = f & \text{in } \partial\Omega_K \\ \partial_n u = 0 & \text{in } \partial\Omega \setminus \partial\Omega_K \end{cases} \tag{2.3}$$

The previous reformulation implies that $c < 1$ almost everywhere in $\Omega \setminus \Omega_K$. A small detail that will become important in the forthcoming discussions. If we further assume that $c \in H^1(\Omega, [0, 1])$, then we can apply the product rule and rewrite (2.3) as

$$\begin{cases} -\operatorname{div}\left((1 - c)\nabla u\right) - \nabla c \cdot \nabla u + c(u - f) = 0 & \text{in} \quad \Omega \setminus \Omega_K \\ u = f & \text{in} \quad \partial\Omega_K \\ \partial_n u = 0 & \text{in} \quad \partial\Omega \setminus \partial\Omega_K \end{cases} \tag{2.4}$$

In order to derive the weak formulation of (2.4) within weighed Sobolev spaces let us first remark that if u solves (2.4), then $v := u - f$ solves

$$\begin{cases} -\operatorname{div}\left((1 - c)\nabla v\right) - \nabla c \cdot \nabla v + cv = g & \text{in } \Omega \setminus \Omega_K \\ v = 0 & \text{in } \partial\Omega_K \\ \partial_n v = h & \text{in } \partial\Omega \setminus \partial\Omega_K \end{cases} \tag{2.5}$$

with $g := (1 - c)\Delta f$ and $h := -\partial_n f$. For convenience of writing, we will continue calling the sought solution of (2.5) u and not v. Being able to solve (2.5) is equivalent to being able to solve (2.4). Yet, this change lets us reduce the problem to the case with homogeneous

Dirichlet boundary conditions. Deriving the associated weak formulation is now straightforward. Multiplying with a test function φ and integrating (2.5) by parts implies that we must seek a function $u \in V$, which solves

$$\underbrace{\int_{\Omega\setminus\Omega_K} (1-c)\nabla u \cdot \nabla\varphi - (\nabla c \cdot \nabla u)\varphi + cu\varphi \,\mathrm{d}x}_{=:B^c(u,\varphi)} = \underbrace{\int_{\Omega\setminus\Omega_K} g\varphi \,\mathrm{d}x + \int_{\partial\Omega\setminus\partial\Omega_K} h\varphi \,\mathrm{d}x}_{=:F(\varphi)} \quad \forall \varphi \in V \tag{2.6}$$

Since c maps to the unit interval, we are in the presence of a so called degenerate elliptic equation [43, 48] or sometimes also referred to as a PDE with non-negative characteristic form [38]. Such PDEs are characterised by the fact, that their highest order term is allowed to vanish. This fact that the second order differential operator may vanish locally requires a more sophisticated analysis. The key issue to approach this kind of problems is to select the correct function space V and to place certain necessary restrictions onto c.

Let us briefly explain why additional restrictions on c, resp. the solution space V, are vital to solve (2.6). The standard approach to show existence and uniqueness of a weak solution consists in applying the Lax-Milgram Theorem [12]. The crucial part will be the coercivity of the bilinear form B^c and the boundedness of B^c and F. Obviously the boundedness of B^c and F depends a lot on the choice of the space V and c. To show coercivity of the bilinear form, we must study the behaviour of

$$\int_{\Omega\setminus\Omega_K} (1-c)|\nabla u|^2 - (\nabla c \cdot \nabla u)u + cu^2 \,\mathrm{d}x \tag{2.7}$$

If c is, for example, piecewise constant in (2.7), then the middle term vanishes almost everywhere. Any function u which is equal to 0 whenever c is positive and equal to an arbitrary constant when c is 0 will force the bilinear form to be 0. Yet the norm of u can be arbitrarily large, which hinders us from showing coercivity. In order to prevent this situation, the following assumptions seem reasonable:

1. The function c should have a certain regularity, e.g. being continuous. Then arbitrary switching between regions where c takes different constants is not possible anymore.
2. The space V in which we seek solutions should fix its elements at certain boundaries. This would prevent solutions u "slipping" away by adding constants that are invisible to the bilinear form.

From these observations it becomes apparent that the function c must go, at least partially, into the definition of the space V. We consider such an approach in the following section by using weighted Sobolev spaces and provide precise requirements that assert the well-posedness of (2.4).

2.1. Weighted Sobolev Spaces. Weighted Sobolev spaces have been studied intensively in the past. Their uses are manifold, but they are most often found in the analysis of PDEs with vanishing or singular diffusive term. The works [26, 31, 38, 43, 48] give an almost complete overview of their usefulness. For the sake of completeness, we shortly summarise how these spaces are set up.

In the following we denote by W_Ω the set of weight functions ω, i.e. ω is a measurable and almost everywhere positive function in some domain Ω. For $1 \leqslant p < \infty$ and $\omega \in W_\Omega$ we define the corresponding weighted L^p space as

$$L^p(\Omega;\omega) := \left\{ u\colon \Omega \to \mathbb{R} \;\middle|\; \|u\|_{L^p(\Omega;\omega)} := \left(\int_\Omega |u(x)|^p \omega(x) \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\} \tag{2.8}$$

In a similar way as Sobolev spaces refine the Lebesgue spaces we can also refine our weighted L^p spaces by including the weak derivatives (defined in the usual sense) into the norm. In such cases, different weights for different derivatives are also possible. For a given collection $S_k := \{\omega_\alpha \in W_\Omega \mid \; |\alpha| \leqslant k\}$ of weight functions, we denote by $W^{k,p}(\Omega; S_k)$ the set of all functions u defined on Ω whose (weak) derivatives $D^\alpha u$ of order $|\alpha| \leqslant k$ (α being a multi-index) belong to $L^p(\Omega; \omega_\alpha)$. We can equip this vector space with the norm

$$\|u\|_{W^{k,p}(\Omega;S_k)} := \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha u(x)|^p \omega_\alpha(x) \, \mathrm{d}x \right)^{\frac{1}{p}} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(\Omega;\omega_\alpha)} \right)^{\frac{1}{p}} \tag{2.9}$$

One can show that the space $W^{k,p}(\Omega; S_k)$ is a Banach space if $\omega_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ and $\omega_\alpha^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega)$ for all $|\alpha| \leqslant k$, see [28, 30]. Note that this requires that all derivatives up to the order k must be attributed such a weight ω_α. However, one can also show that $W^{k,p}(\Omega; \tilde{S}_k)$ is still complete if $\tilde{S}_k \subsetneqq S_k$ contains at least one weight ω_α with $|\alpha| = k$ and a weight for $|\alpha| = 0$, see [27, 29].

We remark that for $p = 2$ there is a canonical choice for a scalar product:

$$\langle u\,, v\rangle_{W^{k,2}(\Omega;S_k)} := \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha u(x) D^\alpha v(x) \omega_\alpha(x) \, \mathrm{d}x \tag{2.10}$$

Thus, with a suitable choice of weights we obtain a Hilbert space. If all the weight functions are constant and equal to one, then our weighted spaces coincide with the usual definition of Sobolev spaces. We refer to [26, 31] for a more complete listing of possible weighted Sobolev space constructions. Finally, we remark that an alternative description of reasonable weight functions can be given in terms of so called Muckenhoupt A_p weights. We refer to [47] for more information.

By looking at (2.4) it becomes apparent why these weighted Sobolev spaces are useful. The function c (resp. $1-c$) can be considered as a weight function and simply be integrated into the space definition. This simplifies the proofs to show existence and uniqueness, since boundedness and coercivity are easier to show and theorems such as Lax-Milgram can by applied in any Hilbert space.

Our goal now will be to consider the corresponding weak formulation of (2.6) in a suitable weighted Sobolev space V. By applying the Theorem of Lax-Milgram in these spaces we will show the existence and uniqueness of a weak solution of (2.4).

We make the following assumptions on our setup. These assumptions will hold throughout the whole paper, unless mentioned otherwise. We remark that some of these assumptions can probably be weakened, nevertheless they are not uncommon for image processing purposes and ease the discussion on a few occasions.

1. Ω is an open, connected and bounded subset of $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary $\partial\Omega$.

2. $\Omega_K \subsetneqq \Omega$ is a closed subset of Ω with positive Lebesgue measure. It represents the known data locations used to recover the missing information on $\Omega \setminus \Omega_K$. The interpolation data is given by $f(\Omega_K)$. The boundary $\partial\Omega_K$, is assumed to be $\mathcal{C}^\infty$, too. This set Ω_K is characterised by $c(x) \equiv 1$.
3. $f\colon \Omega \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function representing the given image data to be interpolated by the underlying PDE.
4. The boundaries $\partial\Omega$ and $\partial\Omega_K$ do not intersect and neither of the boundaries $\partial\Omega$ or $\partial\Omega_K$ are empty.
5. The function c maps from Ω to the interval $[0, 1]$, admits weak first order derivatives, and is an element of $L^1_{\text{loc}}(\Omega \setminus \Omega_K)$.

Let us briefly comment on these requirements. The first part of Item 1 is trivially fulfilled by images. Its second part is more restrictive. Assuming the boundary of Ω to be piecewise $\mathcal{C}^\infty$ would be more realistic, but this would also reduce the regularity of the solution. Item 2 and Item 3 do not impose any severe restrictions for image processing tasks. Images can always be rendered $\mathcal{C}^\infty$ by convolving them with a Gaussian. Item 4 is necessary for technical reasons. If the Neumann and Dirichlet boundary conditions meet each other, it is possible to generate setups that lead to contradicting requirements. Finally, Item 5 is necessary to assert the existence of our weighted Sobolev spaces.

The weights for our space definition should be chosen such that the bilinear form is equivalent to the norm of our space. Often, the multiplicative factors of the individual derivatives in the bilinear form offer themselves as viable choices for this task. In our case however, the function c may vanish locally. This prevents us from using $1 - c$ and c as weights to define a norm. They only give us a seminorm structure. Such a situation is briefly described in [27]. We mostly follow that presentation and we propose the following correspondence between multi-indices $\alpha \in \mathbb{N}_0^2$ and weights ω_α

$$\omega_{\binom{0}{0}} \coloneqq 1, \quad \omega_{\binom{1}{0}} \coloneqq 1 - c, \quad \omega_{\binom{0}{1}} \coloneqq 1 - c \tag{2.11}$$

This yields the scalar product and norm

$$\langle u\,, v\rangle_V \coloneqq \int_{\Omega\setminus\Omega_K} (1 - c)\nabla u \cdot \nabla v + uv \,\mathrm{d}x \tag{2.12a}$$

$$\|u\|_V \coloneqq \left(\int_{\Omega\setminus\Omega_K} (1 - c)|\nabla u|^2 + u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{2.12b}$$

as well as the following definition for our space V:

$$V \coloneqq \{\phi \in W^{1,2}(\Omega \setminus \Omega_K; S_c) \mid \phi \equiv 0 \text{ on } \partial\Omega_K\} \tag{2.13}$$

In addition, we define the following seminorm

$$|||u|||_V \coloneqq \left(\int_{\Omega\setminus\Omega_K} (1 - c)|\nabla u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{2.14}$$

Finally, following the presentation in [31], we note that the bilinear form B^c in (2.6) can be written compactly as a ternary quadratic form

$$B^c(u,\varphi) = \sum_{|\alpha|,|\beta|\leqslant 1} \int_{\Omega\setminus\Omega_K} a_{\alpha,\beta} D^\beta u D^\alpha \varphi \, \mathrm{d}x \tag{2.15}$$

where α, β are multi-indices in $\mathbb{N}_0^2$. The weights $a_{\alpha,\beta}$ must be set as follows to yield our model:

$$a_{\binom{1}{0},\binom{1}{0}} = a_{\binom{0}{1},\binom{0}{1}} = 1 - c(x), \quad a_{\binom{0}{0},\binom{0}{0}} = c(x) \tag{2.16a}$$

$$a_{\binom{0}{0},\binom{1}{0}} = -\partial_x c(x), \quad a_{\binom{0}{0},\binom{0}{1}} = \partial_y c(x) \tag{2.16b}$$

and $a_{\alpha,\beta} = 0$ for any other combination of multi-indices. In addition to the previous assumptions, we assume further:

6. There exists a constant $\kappa > 0$, such that for all $|\alpha|$, $|\beta| \leqslant 1$, $\alpha \neq \beta$,

$$|a_{\alpha,\beta}| \leqslant \kappa\sqrt{a_{\alpha,\alpha} a_{\beta,\beta}} \tag{2.17}$$

   almost everywhere in $\Omega \setminus \Omega_K$. For our choice in (2.16), this reduces to

$$|\partial_x c| \leqslant \kappa\sqrt{c(1-c)}, \quad |\partial_y c| \leqslant \kappa\sqrt{c(1-c)} \tag{2.18}$$

   almost everywhere in $\Omega \setminus \Omega_K$.

7. There exists a constant $\kappa' > 0$, such that for all real vectors $\xi \in \mathbb{R}^3$ with entries ξ_γ (γ being a multi-index in $\mathbb{N}_0^2$ such that $|\gamma| \leqslant 1$) we have

$$\sum_{|\alpha|,|\beta|\leqslant 1} a_{\alpha,\beta}\xi_\alpha\xi_\beta \geqslant \kappa' \sum_{|\gamma|\leqslant 1} a_{\gamma,\gamma}\xi_\gamma^2 \tag{2.19}$$

   almost everywhere in $\Omega \setminus \Omega_K$. For our choice in (2.16), this reduces to

$$\begin{aligned} &c\,\xi_1^2 + (1-c)\xi_2^2 + (1-c)\xi_3^2 - \partial_x c\,\xi_1\xi_3 + \partial_y c\,\xi_1\xi_2 \\ &\qquad\qquad \geqslant \kappa'\left((1-c)\xi_3^2 + (1-c)\xi_2^2 + c\xi_1^2\right) \end{aligned} \tag{2.20a}$$

$$\Leftrightarrow \quad (\partial_y c)\xi_1\xi_2 - (\partial_x c)\xi_1\xi_3 \geqslant (\kappa' - 1)\left((1-c)\xi_3^2 + (1-c)\xi_2^2 + c\xi_1^2\right) \tag{2.20b}$$

   almost everywhere in $\Omega \setminus \Omega_K$.

Item 6 and Item 7 are technical requirements that are necessary for the coercivity and the boundedness of B^c. They cannot be avoided without substantial changes to the forthcoming proofs. Let us remark, that (2.19) can be deduced from (2.17), provided that $\kappa < \frac{1}{2}$ holds. We refer to [31] for a detailed proof. Equations (2.18) and (2.20b) enforce a certain well-behaviour on c, by restricting for example the growth speed.

The following findings are a direct consequence of the foregoing results.

Proposition 2.1. *The bilinear form B^c from (2.15) is continuous.*

Proof. By using (2.17) and the Hölder inequality we obtain.

$$
\begin{aligned}
|B^c(u,\varphi)| &\leqslant \sum_{|\alpha|,|\beta|\leqslant 1} \int_{\Omega\setminus\Omega_K} |a_{\alpha,\beta}||D^\beta u||D^\alpha\varphi| \,\mathrm{d}x \\
&\leqslant \max\{\kappa,1\} \sum_{|\alpha|,|\beta|\leqslant 1} \int_{\Omega\setminus\Omega_K} |D^\beta u|\sqrt{|a_{\beta,\beta}|}|D^\alpha\varphi|\sqrt{|a_{\alpha,\alpha}|}\,\mathrm{d}x \qquad (2.21)\\
&\leqslant K\|D^\beta u\|_V\|D^\alpha\varphi\|_V
\end{aligned}
$$

where K is some positive constant. We emphasise that the last estimate requires $c\leqslant 1$ almost everywhere to be valid. ∎

Proposition 2.2. *There exists a constant $\kappa'>0$ such that the bilinear form B^c from (2.15) satisfies the estimate $B^c(u,u)\geqslant \kappa'\,|||u|||_V^2$.*

Proof. We replace ξ_α by $D^\alpha u$ and ξ_β by $D^\beta u$ in (2.19). Integrating the resulting inequality over $\Omega\setminus\Omega_K$ yields

$$
B^c(u,u) = \sum_{|\alpha|,|\beta|\leqslant 1}\int_{\Omega\setminus\Omega_K} a_{\alpha,\beta}D^\alpha u D^\beta u\,\mathrm{d}x \geqslant \kappa' \sum_{|\gamma|\leqslant 1}\int_{\Omega\setminus\Omega_K} a_{\gamma,\gamma}\left(D^\gamma u\right)^2\mathrm{d}x \geqslant \kappa'\,|||u|||_V^2 \qquad (2.22)
$$ ∎

To complete the proof of the coercivity of the bilinear form B^c we need a Friedrichs-like estimate of the form $\|u\|_V\leqslant K\,|||u|||_V$ with some positive constant K. The particular formulation and preliminaries that we need can be found in [49] as Theorem 2.3. We repeat it here verbatim for the sake of completeness but refer to its source for a detailed proof.

In the following theorem we denote by $W_c(X)$ the subset of weights on the space X which are bounded from above and below by positive constants on each compact subset $Q\subset X$, i.e. we only allow our weights to degenerate at the boundary of the domain. The next theorem also considers a constant A which is defined as follows. For an arbitrary domain X we assume that we can write

$$
X = \bigcup_{k=1}^{\infty} X_k \qquad (2.23)
$$

where $(X_k)_k$ is a sequence of bounded domains whose boundary can be locally described by functions satisfying a Lipschitz condition and where $X_k\subset\overline{X}_k\subset X_{k+1}$ holds for each k. Finally, let $X^k := X\setminus X_k$ and define

$$
A_k = \sup_{\|u\|_{W^{k,p}(X;S_k)}\leqslant 1} \|u\|_{L^p(X^k;w_0)} \qquad (2.24)
$$

where $w_0\in S_k$ is the weight that corresponds to $|\alpha|=0$. We define additionally $A:=\lim_{k\to\infty}A_k$. Obviously $A\in[0,1]$ always holds. This number A is also the ball measure of non-compactness of the embedding $W^{k,p}(X;S_k)\to L^p(X;w_0)$, see [11, 49]. One can interpret the number A as the distance from the embedding operator to the next closest compact operator from $W^{k,p}(X;S_k)$ into $L^p(X;w_0)$. Also, the numbers A_k can be understood as indicators on how much "weight" is put onto the function along the boundary. $A_k < 1$ means that there is at least some weight on the derivatives or inside the domain. Note that in our setup (2.24) simplifies to

$$A_k = \sup_{\|u\|_{W^{1,2}(\Omega\setminus\Omega_K;S_k)}\leqslant 1} \|u\|_{L^2(X^k)} \tag{2.25}$$

where X^k is the complement of a set $X_k \subset \Omega\setminus\Omega_K$ and where S_k is the set of weights from (2.11).

For the following theorem it is important that $A < 1$, i.e. the weight is not completely concentrated on the boundary. Let us remark that this requirement is in accordance with the discrete theory established in [20, 32]. In the discrete setting, there should be at least one position with positive weight in the interior of the domain.

Let us emphasise that for our task at hand, such a construction with the requirement that $A < 1$ is an additional regularity assumption on our image data f and the mask function c. Indeed, part of the boundary of the domain that we consider is fixed where $c \equiv 1$. Since the Ω_k need boundaries that can be described locally by functions that fulfil a Lipschitz condition, this requirement carries over to the function c.

As already mentioned, the next theorem is a almost verbatim copy of Theorem 2.3 in [49].

Theorem 2.3. *Suppose $1 \leqslant p < \infty$ and $S_k \subset W_c(X)$. Let ℓ be a functional on $W^{k,p}(X;S_k)$ with the following properties.*

1. *ℓ is continuous on $W^{k,p}(X;S_k)$*
2. *$\ell(\lambda u) = \lambda\ell(u)$ for all $\lambda > 0$ and all $u \in W^{k,p}(X,S_k)$.*
3. *If $u \in P_{k-1} \cap W^{k,p}(X;S_k)$ (P_{k-1} being the set of all polynomials on $\mathbb{R}^n$ of degree less than k) and $\ell(u) = 0$, then $u = 0$.*

Let $A < 1$. Then there is a constant κ_0 such that

$$\int_X |u|^p w_0 \, \mathrm{d}x \leqslant \kappa_0 \left(|\ell(u)|^p + \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(X;w_\alpha)} \right) \tag{2.26}$$

Here, w_0 is the weight that corresponds to $|\alpha| = 0$.

The previous theorem can be seen as a generalisation to weighted spaces of a well-known theorem for constructing equivalent norms out of seminorms in regular Sobolev spaces. See Theorem 7.3.12 in [1]. Equation (2.26) can also be considered as a higher dimensional generalisation of the Hardy inequality. We refer to [39] for an extensive treatise on this inequality.

We now use Theorem 2.3 with $p = 2$, $k = 1$, $n = 2$, $w_0 \equiv 1$, $w_\alpha = 1 - c$ for all α and

$$\ell(u) = \int_{\partial\Omega_K} u \, \mathrm{d}x \tag{2.27}$$

With these choices we obtain the Friedrichs' inequality in our space V:

$$\|u\|^2_{L^2(\Omega\setminus\Omega_K)} \leqslant \kappa_0 \, \|u\|^2_V \tag{2.28}$$

Equation (2.28) is the final key building block in showing the existence and uniqueness of a solution of our PDE. It allows us to show the coercivity of our bilinear form.

Proposition 2.4. *If (2.28) holds, i.e. the requirements of Theorem 2.3 are fulfilled for the choice of ℓ from (2.27) and for our selection of weights for our space V, then the bilinear form B^c from (2.15) is coercive.*

Proof. Equation (2.28) immediately implies that $\|u\|_V^2 \leqslant (1+\kappa_0)\,|||u|||_V^2$. In combination with (2.22) it follows that

$$B^c(u,u) \geqslant \kappa'\,|||u|||_V^2 \geqslant \frac{\kappa'}{1+\kappa_0}\|u\|_V^2 \tag{2.29}$$

■

Proposition 2.4 completes the analysis of our bilinear form B^c. It remains to show that the right-hand side of our weak formulation is continuous if we want to apply the Theorem of Lax-Milgram. This final step is done in the following proposition.

Proposition 2.5. *The linear operator F from (2.6) is continuous, provided that g, Δf, and $\frac{\nabla f}{\sqrt{1-c}}$ are in $L^2(\Omega \setminus \Omega_K)$.*

Proof. We remark that $\varphi \in V$ is 0 along $\partial\Omega_K$, and thus we can extend the boundary integral over that part. Using the Hölder inequality and Green's first identity, we obtain

$$\begin{aligned}
|F(\varphi)| &\leqslant \int_{\Omega\setminus\Omega_K} |g||\varphi|\,\mathrm{d}x + \left|\int_{\partial\Omega\setminus\partial\Omega_K} h\varphi\,\mathrm{d}x\right| \\
&\leqslant \|g\|_{L^2(\Omega\setminus\Omega_K)}\|\varphi\|_{L^2(\Omega\setminus\Omega_K)} + \left|\int_{\Omega\setminus\Omega_K} \Delta f\varphi + \nabla f\cdot\nabla\varphi\,\mathrm{d}x\right| \\
&\leqslant \|g\|_{L^2(\Omega\setminus\Omega_K)}\|\varphi\|_V + \|\Delta f\|_{L^2(\Omega\setminus\Omega_K)}\|\varphi\|_V + \left|\int_{\Omega\setminus\Omega_K} \nabla f\cdot\nabla\varphi\,\mathrm{d}x\right|
\end{aligned} \tag{2.30}$$

The last integral can be estimated as follows

$$\begin{aligned}
\left|\int_{\Omega\setminus\Omega_K} \nabla f\cdot\nabla\varphi\,\mathrm{d}x\right| &= \left|\int_{\Omega\setminus\Omega_K} \frac{\nabla f}{\sqrt{1-c}}\sqrt{1-c}\,\nabla\varphi\,\mathrm{d}x\right| \\
&\leqslant \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)} \|\nabla\varphi\|_{L^2(\Omega\setminus\Omega_K;1-c)} \\
&\leqslant \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)} \|\varphi\|_V
\end{aligned} \tag{2.31}$$

Therefore, it follows that

$$|F(\varphi)| \leqslant \left(\|g\|_{L^2(\Omega\setminus\Omega_K)} + \|\Delta f\|_{L^2(\Omega\setminus\Omega_K)} + \left\|\frac{\nabla f}{\sqrt{1-c}}\right\|_{L^2(\Omega\setminus\Omega_K)}\right)\|\varphi\|_V \tag{2.32}$$

Thus, F is a bounded linear functional. ■

We can now combine our results to prove our main result.

Theorem 2.6. *The weak formulation (2.6) of the mixed boundary value problem (2.5) has a unique solution in the space V. In addition, we know that*

$$\|u\|_V \leqslant \frac{1+\kappa_0}{\kappa'} \|F\|_{V^*} \tag{2.33}$$

where $\frac{\kappa'}{1+\kappa_0}$ is the constant from (2.29). Here, V^* denotes the dual space of V.

Proof. From Proposition 2.1 and Proposition 2.4 it follows that our bilinear form B^c is bounded and coercive. Proposition 2.5 shows that the corresponding right-hand side F is bounded, too. Therefore, from the Theorem of Lax-Milgram (see [12]) it follows that there exists a unique $u \in V$ such that $B^c(u, \varphi) = F(\varphi)$ holds for all $\varphi \in V$. In addition, this u fulfils $\|u\| \leqslant \frac{1+\kappa_0}{\kappa'} \|F\|_{V^*}$ ■

Our weighted Sobolev space V might be unsuited for other applications as it is very problem specific. Having an embedding from V to some standard Sobolev space $W^{k,2}$ would be very useful in view of the many embedding theorems for these latter spaces which can be used to show a higher regularity of the solution. It would also help in comparing solutions obtained when c maps to the unit interval but does not reach 0 or 1. In that case, the solutions live in $W^{1,2}$. By construction of our space V, we immediately obtain $u \in L^2(\Omega \setminus \Omega_K)$. However, this result does not even acknowledge the existence of the weak derivatives. Any claims beyond that are difficult to do. There exist a certain number of results concerning the embedding of weighted Sobolev spaces into other spaces, however, their assumptions are often very abstract or quite restrictive, e.g. all weights must be identical. We refer the reader to the works [17–19] for a detailed analysis on this topic.

2.2. What happens if $c \geqslant 1$?

Let us shortly discuss the consequences of c exceeding its upper limit 1. Similar conclusions can also be drawn for the case $c \leqslant 0$, however, this latter situation usually does not occur in practice.

There are no restrictions on c when establishing the weak formulation. Applying $c \geqslant 1$, the main difference would be that $1 - c$ and c would have different signs. In order to follow the same strategy as in this paper one would have to find suitable weights for the space definition. In [27] the authors discuss the situation when one of the weights in the weak formulation is negative and they suggest to multiply the negative weight with another negative constant to render it positive. Afterwards, a similar approach as in this paper could be possible.

In our situation there exists a second issue that may be harder to resolve. We required certain restrictions on the growth of the function c, which were of the form

$$|\partial_z c| \leqslant \kappa \sqrt{c(1-c)} \tag{2.34}$$

for z being either x or y. The left-hand side of this inequality will always be a non-negative real number. However, the right-hand side becomes complex-valued once c exceeds 1. These growth restrictions were important to show the coercivity of the bilinear form.

To conclude this section we remark that an alternative approach by means of the Helmholtz equation already exists for the case $c > 1$, see [22]. However, this approach uses different assumptions and yields a well-posedness theory in different spaces.

2.3. Summary: What is needed to assert the existence of a solution?. In this section, we summarise the necessary conditions that we had to impose on our data throughout the paper.

1. The function c must have weak derivatives of first order and map the domain Ω to the interval $[0, 1]$.
2. The function c must fulfil (2.18).
3. The function c must fulfil (2.20b).
4. The sequence $(A_k)_k$ defined by (2.24) must fulfil $\lim_{k\to\infty} A_k < 1$.

The first three requirements can easily be verified if a concrete instance of c is given. However, the last requirement can probably only be checked in particular cases.

3. Conclusion. We have shown that a solution to the inpainting problem with the weighted Laplacian exists if the weight is a function that maps into the interval $[0, 1]$. The effort to assert the existence and uniqueness of such a solution was significant. The well-posedness of the task can be asserted if certain regularity conditions on the weight function c are met. These requirements are similar to what is needed to show existence and uniqueness of a solution in a discrete setting. The results in this manuscript complete the analysis of the inpainting problem with the Laplacian. While the theory for the discrete setup was complete for any choice of $c \geqslant 0$, the continuous theory only covered the setup where $c > 1$. This work complements the setup where c maps to $[0, 1]$.

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