

MIXED INTERIOR TRANSMISSION EIGENVALUES

joint work with Jijun Liu

CMMSE 2019 (MS 23) | July 2, 2019 | Andreas Kleefeld | Jülich Supercomputing Centre, Germany

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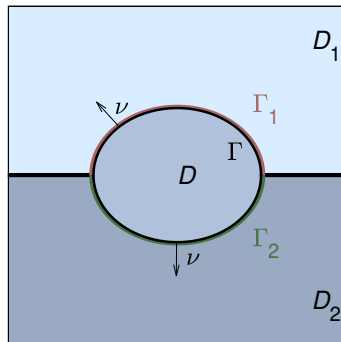
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Part I: Introduction & motivation

INTRODUCTION & MOTIVATION

A general physical configuration

- Obstacle D is located in perfect conducting substrate D_2 with boundary $\Gamma_2 \subset \Gamma$.
- Remaining part of boundary $\Gamma_1 = \Gamma \setminus \Gamma_2$ contacts with surface of background dielectric medium D_1 .
- We assume $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \neq \emptyset$, and $\Gamma_2 \neq \emptyset$.
- Scattering problem for isotropic inhomogeneous media (TE mode electromagnetic scattering) leads to ...



INTRODUCTION & MOTIVATION

A general physical configuration

- ... (acoustic) interior transmission problem with mixed boundary condition:

$$\begin{cases} \Delta u + k^2 u = 0, & x \in D, \\ \Delta v + k^2 n v = 0, & x \in D, \\ u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, & x \in \Gamma_1, \text{ (transmission condition)} \\ u = v = 0, & x \in \Gamma_2. \text{ (hom. Dirichlet condition)} \end{cases} \quad (1)$$

- Here, $n \neq 1$ is the real-valued index of refraction (constant).
- Find $k \neq 0$ and non-trivial (u, v) such that (1) is satisfied.
- Such k will be called mixed interior transmission eigenvalues (**MITEs**).

INTRODUCTION & MOTIVATION

Goal

- This is a non-standard eigenvalue problem.
- It is neither elliptic nor self-adjoint.
- How to solve this problem numerically?
- No results for the computation of MITEs have yet been reported.

Part II: Some theory

SOME THEORY

A short review

- The set of MITEs is at most discrete.
- Does not accumulate at zero.
- There exists an infinite number of real MITEs.
- Only accumulation point is ∞ .

- Nothing is known for complex-valued MITEs.

Part III: Boundary integral equations

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Fundamental solution: $\Phi_k(x, y) = iH_0^{(1)}(k|x - y|)/4, x \neq y$.
- Single- and double-layer potentials over Γ given for $x \notin \Gamma$ by

$$\begin{aligned}\mathrm{SL}_k^\Gamma[\psi](x) &= \int_\Gamma \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y), \\ \mathrm{DL}_k^\Gamma[\psi](x) &= \int_\Gamma \partial_{\nu(y)} \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y).\end{aligned}$$

- Green's representation theorem:

$$u(x) = \mathrm{SL}_k^\Gamma[\partial_\nu u|_\Gamma](x) - \mathrm{DL}_k^\Gamma[u|_\Gamma](x), \quad x \in D.$$

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Γ is disjoint union of Γ_1 and Γ_2 . Hence,

$$\begin{aligned} u(x) &= \text{SL}_k^{\Gamma_1} [\partial_\nu u|_{\Gamma_1}](x) + \text{SL}_k^{\Gamma_2} [\partial_\nu u|_{\Gamma_2}](x) \\ &\quad - \text{DL}_k^{\Gamma_1} [u|_{\Gamma_1}](x) - \text{DL}_k^{\Gamma_2} [u|_{\Gamma_2}](x), \quad x \in D, \end{aligned} \quad (2)$$

$$\begin{aligned} v(x) &= \text{SL}_{k\sqrt{n}}^{\Gamma_1} [\partial_\nu v|_{\Gamma_1}](x) + \text{SL}_{k\sqrt{n}}^{\Gamma_2} [\partial_\nu v|_{\Gamma_2}](x) \\ &\quad - \text{DL}_{k\sqrt{n}}^{\Gamma_1} [v|_{\Gamma_1}](x) - \text{DL}_{k\sqrt{n}}^{\Gamma_2} [v|_{\Gamma_2}](x), \quad x \in D. \end{aligned} \quad (3)$$

- Using $u|_{\Gamma_2} = v|_{\Gamma_2} = 0$, equations (2) and (3) can be simplified to

$$u(x) = \text{SL}_k^{\Gamma_1} [\partial_\nu u|_{\Gamma_1}](x) + \text{SL}_k^{\Gamma_2} [\partial_\nu u|_{\Gamma_2}](x) - \text{DL}_k^{\Gamma_1} [u|_{\Gamma_1}](x), \quad x \in D, \quad (4)$$

$$v(x) = \text{SL}_{k\sqrt{n}}^{\Gamma_1} [\partial_\nu v|_{\Gamma_1}](x) + \text{SL}_{k\sqrt{n}}^{\Gamma_2} [\partial_\nu v|_{\Gamma_2}](x) - \text{DL}_{k\sqrt{n}}^{\Gamma_1} [v|_{\Gamma_1}](x), \quad x \in D. \quad (5)$$

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Boundary integral operators:

$$S_k^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_i} \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y), \quad x \in \Gamma_j,$$

$$K_k^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_i} \partial_{\nu_i(y)} \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y), \quad x \in \Gamma_j,$$

$$K_k^{\top \Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \int_{\Gamma_i} \partial_{\nu_j(x)} \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y), \quad x \in \Gamma_j,$$

$$T_k^{\Gamma_i \rightarrow \Gamma_j} [\psi|_{\Gamma_i}] (x) = \partial_{\nu_j(x)} \int_{\Gamma_i} \partial_{\nu_i(y)} \Phi_k(x, y) \psi(y) \, d\mathbf{s}(y), \quad x \in \Gamma_j,$$

where $i, j \in \{1, 2\}$.

BOUNDARY INTEGRAL EQUATIONS

Derivation of first boundary integral equation

- $D \ni x \rightarrow x \in \Gamma_1$ in (4) and (5) and jump relations:

$$u|_{\Gamma_1} = S_k^{\Gamma_1 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_1}] + S_k^{\Gamma_2 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_2}] - \left(K_k^{\Gamma_1 \rightarrow \Gamma_1} [u|_{\Gamma_1}] - \frac{1}{2} u|_{\Gamma_1} \right), \quad (6)$$

$$v|_{\Gamma_1} = S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_1}] + S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_2}] - \left(K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} [v|_{\Gamma_1}] - \frac{1}{2} v|_{\Gamma_1} \right). \quad (7)$$

- Difference of (6) and (7), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(S_k^{\Gamma_1 \rightarrow \Gamma_1} - S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} \right) [\partial_\nu u|_{\Gamma_1}] + S_k^{\Gamma_2 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_2}] \\ &\quad - S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_2}] - \left(K_k^{\Gamma_1 \rightarrow \Gamma_1} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} \right) [u|_{\Gamma_1}]. \end{aligned} \quad (8)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of second boundary integral equation

- $D \ni x \rightarrow x \in \Gamma_2$ in (4) and (5):

$$u|_{\Gamma_2} = S_k^{\Gamma_1 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_1}] + S_k^{\Gamma_2 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_2}] - K_k^{\Gamma_1 \rightarrow \Gamma_2} [u|_{\Gamma_1}] , \quad (9)$$

$$v|_{\Gamma_2} = S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_1}] + S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_2}] - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} [v|_{\Gamma_1}] . \quad (10)$$

- Difference of (9) and (10), $u|_{\Gamma_2} = v|_{\Gamma_2} = 0$, $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(S_k^{\Gamma_1 \rightarrow \Gamma_2} - S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \right) [\partial_\nu u|_{\Gamma_1}] + S_k^{\Gamma_2 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_2}] \\ &\quad - S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_2}] - \left(K_k^{\Gamma_1 \rightarrow \Gamma_2} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \right) [u|_{\Gamma_1}] . \end{aligned} \quad (11)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of third boundary integral equation

- Normal derivative of (4) and (5), $D \ni x \rightarrow x \in \Gamma_1$, and jump relations:

$$\partial_\nu u|_{\Gamma_1} = K_k^{\top \Gamma_1 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_1}] + \frac{1}{2} \partial_\nu u|_{\Gamma_1} + K_k^{\top \Gamma_2 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_2}] - T_k^{\Gamma_1 \rightarrow \Gamma_1} [u|_{\Gamma_1}] , \quad (12)$$

$$\partial_\nu v|_{\Gamma_1} = K_{k\sqrt{n}}^{\top \Gamma_1 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_1}] + \frac{1}{2} \partial_\nu v|_{\Gamma_1} + K_{k\sqrt{n}}^{\top \Gamma_2 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_2}] - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} [v|_{\Gamma_1}] . \quad (13)$$

- Difference of (12) and (13), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(K_k^{\top \Gamma_1 \rightarrow \Gamma_1} - K_{k\sqrt{n}}^{\top \Gamma_1 \rightarrow \Gamma_1} \right) [\partial_\nu u|_{\Gamma_1}] + K_k^{\top \Gamma_2 \rightarrow \Gamma_1} [\partial_\nu u|_{\Gamma_2}] \\ &\quad - K_{k\sqrt{n}}^{\top \Gamma_2 \rightarrow \Gamma_1} [\partial_\nu v|_{\Gamma_2}] - \left(T_k^{\Gamma_1 \rightarrow \Gamma_1} - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} \right) [u|_{\Gamma_1}] . \end{aligned} \quad (14)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of fourth boundary integral equation

- Normal derivative of (4) and (5), $D \ni x \rightarrow x \in \Gamma_2$, and jump relations:

$$\partial_\nu u|_{\Gamma_2} = K_k^{\top \Gamma_1 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_1}] + K_k^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_2}] + \frac{1}{2} \partial_\nu u|_{\Gamma_2} - T_k^{\Gamma_1 \rightarrow \Gamma_2} [u|_{\Gamma_1}] , \quad (15)$$

$$\partial_\nu v|_{\Gamma_2} = K_{k\sqrt{n}}^{\top \Gamma_1 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_1}] + K_{k\sqrt{n}}^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_2}] + \frac{1}{2} \partial_\nu v|_{\Gamma_2} - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} [v|_{\Gamma_1}] . \quad (16)$$

Equations (15) and (16) can be rewritten as

$$0 = K_k^{\top \Gamma_1 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_1}] + K_k^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_2}] - T_k^{\Gamma_1 \rightarrow \Gamma_2} [u|_{\Gamma_1}] - \frac{1}{2} \partial_\nu u|_{\Gamma_2} , \quad (17)$$

$$0 = K_{k\sqrt{n}}^{\top \Gamma_1 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_1}] + K_{k\sqrt{n}}^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_2}] - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} [v|_{\Gamma_1}] - \frac{1}{2} \partial_\nu v|_{\Gamma_2} . \quad (18)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of fourth boundary integral equation

- Difference of (17) and (18), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(K_k^{\top \Gamma_1 \rightarrow \Gamma_2} - K_{k\sqrt{n}}^{\top \Gamma_1 \rightarrow \Gamma_2} \right) [\partial_\nu u|_{\Gamma_1}] + K_k^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu u|_{\Gamma_2}] \\ &\quad - K_{k\sqrt{n}}^{\top \Gamma_2 \rightarrow \Gamma_2} [\partial_\nu v|_{\Gamma_2}] - \left(T_k^{\Gamma_1 \rightarrow \Gamma_2} - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} \right) [u|_{\Gamma_1}] - \frac{1}{2} \partial_\nu u|_{\Gamma_2} \\ &\quad + \frac{1}{2} \partial_\nu v|_{\Gamma_2}. \end{aligned} \tag{19}$$

BOUNDARY INTEGRAL EQUATIONS

4×4 system of boundary integral equations

- Four equations (8), (11), (14), and (19) can be written as

$$Z(k)g = 0$$

with

$$Z(k) = \begin{pmatrix} S_k^{\Gamma_1 \rightarrow \Gamma_1} - S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} & K_k^{\Gamma_1 \rightarrow \Gamma_1} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} & S_k^{\Gamma_2 \rightarrow \Gamma_1} & S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_1} \\ S_k^{\Gamma_1 \rightarrow \Gamma_2} - S_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} & K_k^{\Gamma_1 \rightarrow \Gamma_2} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} & S_k^{\Gamma_2 \rightarrow \Gamma_2} & S_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} \\ K_k^{\Gamma_1 \rightarrow \Gamma_1} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} & T_k^{\Gamma_1 \rightarrow \Gamma_1} - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_1} & K_k^{\Gamma_2 \rightarrow \Gamma_1} & K_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_1} \\ K_k^{\Gamma_1 \rightarrow \Gamma_2} - K_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} & T_k^{\Gamma_1 \rightarrow \Gamma_2} - T_{k\sqrt{n}}^{\Gamma_1 \rightarrow \Gamma_2} & K_k^{\Gamma_2 \rightarrow \Gamma_2} - \frac{1}{2}I & K_{k\sqrt{n}}^{\Gamma_2 \rightarrow \Gamma_2} - \frac{1}{2}I \end{pmatrix} \quad (20)$$

$$g = \begin{pmatrix} \alpha & -\beta & \gamma & -\delta \end{pmatrix}^T,$$

where we used the notation

$$\alpha = \partial_\nu u|_{\Gamma_1}, \quad \beta = u|_{\Gamma_1}, \quad \gamma = \partial_\nu u|_{\Gamma_2}, \quad \text{and} \quad \delta = \partial_\nu v|_{\Gamma_2}. \quad (21)$$

Part IV: Numerical results

NUMERICAL RESULTS

A short description

- Discretize the resulting boundary integral operator via boundary element collocation method.
- Curved boundary is approximated by lines.
- Collocation nodes are the midpoints having m collocation points in total.
- Unknown function is approximated by constant interpolation at each midpoint.
- Hence, we can regard (20) as non-linear eigenvalue problem of the form $\mathbf{Z}(k)\tilde{g} = 0$ with $\mathbf{Z}(k) \in \mathbb{C}^{m \times m}$ and $\tilde{g}$ the discretized version of g given by (21).
- Solved with Beyn's algorithm (based on complex-valued contour integration of the resolvent).

NUMERICAL RESULTS

Unit circle, $n = 4$

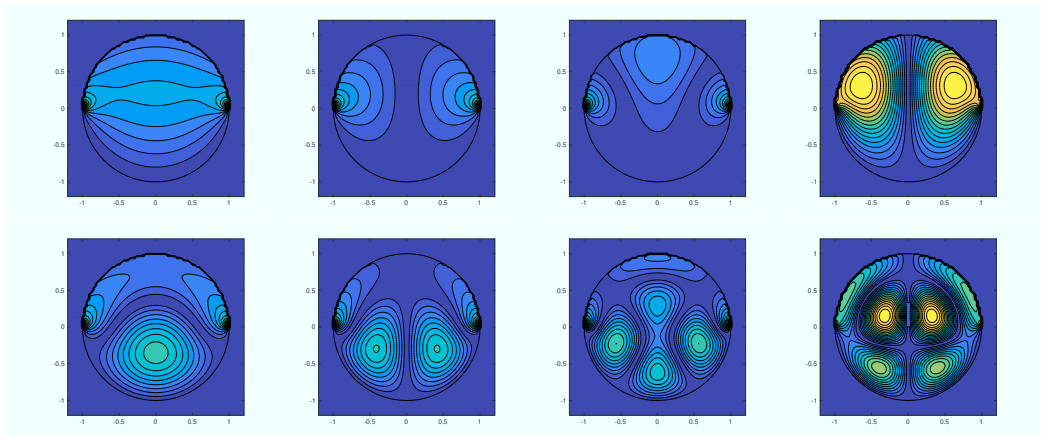


Figure: Absolute value of u (first row) and v (second row) for the first four real-valued MITEs. The MITEs are 1.6818, 2.3185, 2.9533, 3.0791.

NUMERICAL RESULTS

Ellipse, major semi-axis 1, minor semi-axis $4/5$, $n = 4$

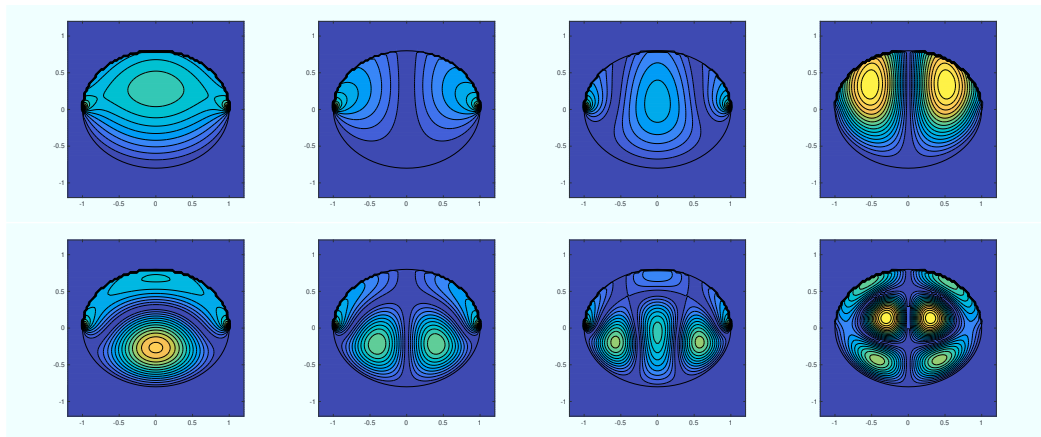


Figure: Absolute value of u (first row) and v (second row) for the first four real-valued MITEs. The MITEs are 1.9111, 2.4973, 3.1282, 3.4609.

NUMERICAL RESULTS

More on ellipses, major semi-axis 1

Table: MITEs for ellipses with various minor semi-axis using $n = 4$.

Minor semi-axis	1 st MITE	2 nd MITE	3 rd MITE	4 th MITE
1	1.6818	2.3185	2.9533	3.0791
4/5	1.9111	2.4973	3.1282	3.4609
1/2	2.7709	3.1764	3.7892	4.3916

Table: MITEs for ellipses with various minor semi-axis using $n = 1/2$.

Minor semi-axis	1 st MITE	2 nd MITE	3 rd MITE	4 th MITE
1	3.1620	4.5193	4.6482	5.8022
4/5	3.5798	4.8518	5.5187	6.2683
1/2	5.1115	6.1186	7.3248	8.4891

NUMERICAL RESULTS

Unit square, $n = 4$

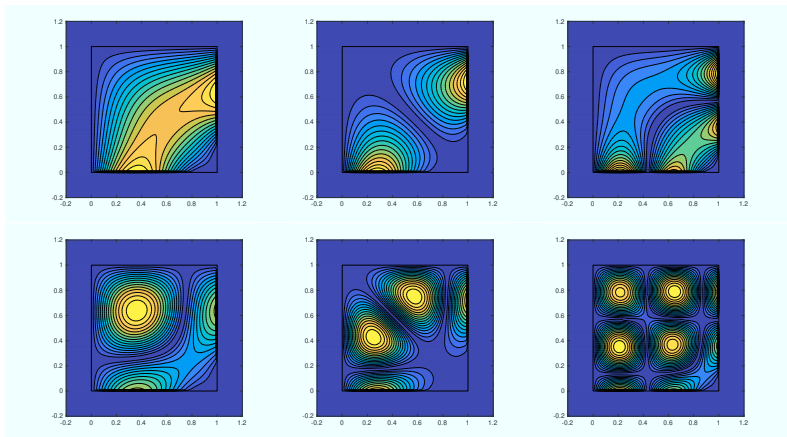


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 3.0503, 4.2622, 5.1805.

NUMERICAL RESULTS

Unit square, $n = 4$

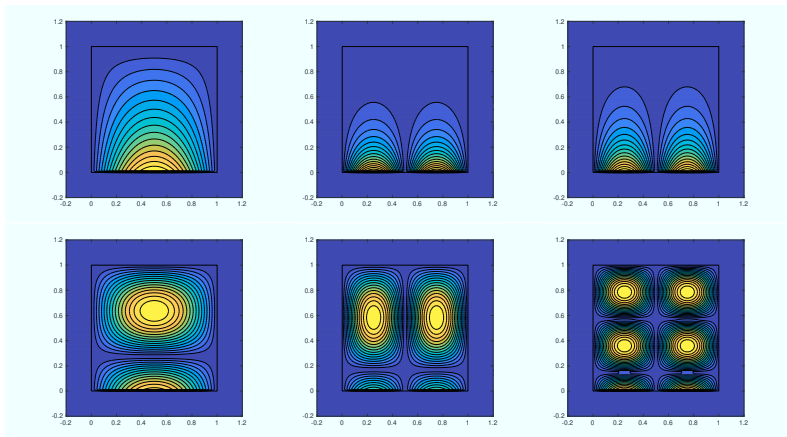


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 2.6717, 3.6662, 4.8367.

NUMERICAL RESULTS

Unit square, $n = 4$

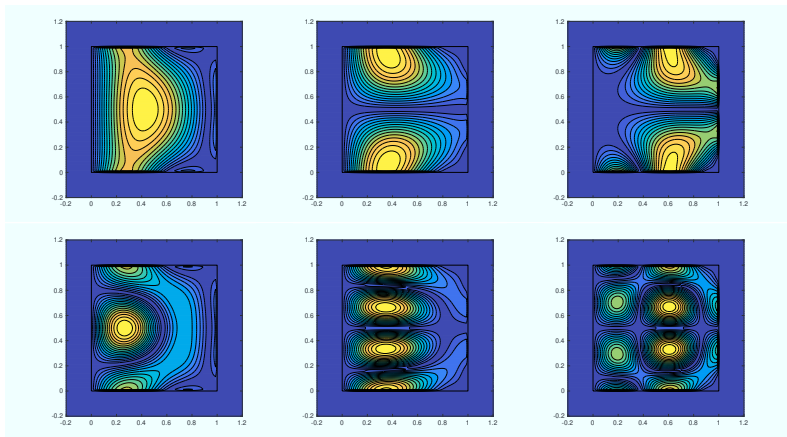


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 4.0802, 5.2285, 5.7030.

Part V: Summary & outlook

SUMMARY & OUTLOOK

- Reviewed existence and discreteness of MITEs for real-valued constant n .
 - Derived a system of boundary integral equations.
 - Showed how to solve it.
 - Provided extensive numerical results for a variety of 2D scatterers.
-
- Study behavior of the MITE eigenfunctions at corners.
 - Investigate inside-outside-duality method both theoretically and practically.

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