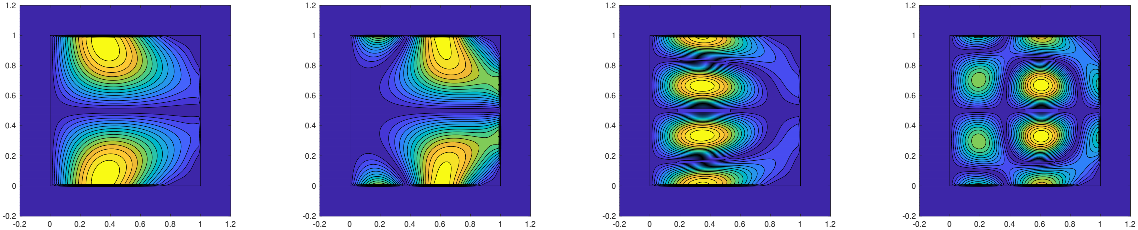

MIXED INTERIOR TRANSMISSION EIGENVALUES

joint work with Jijun Liu

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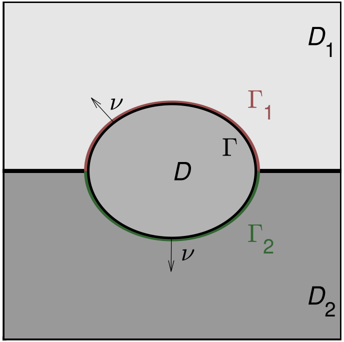
Part I: Introduction & motivation

INTRODUCTION & MOTIVATION

A general physical configuration

- Obstacle D is located in perfect conducting substrate D_2 with boundary $\Gamma_2 \subset \Gamma$.
- Remaining part of boundary $\Gamma_1 = \Gamma \setminus \Gamma_2$ contacts with surface of background dielectric medium D_1.
- We assume $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \neq \emptyset$, and $\Gamma_2 \neq \emptyset$.



- Scattering problem for isotropic inhomogeneous media (TE mode electromagnetic scattering) leads to ...

INTRODUCTION & MOTIVATION

A general physical configuration

- ... (acoustic) interior transmission problem with mixed boundary condition:

$$\begin{cases} \Delta u + k^2 u = 0\,, & x \in D\,, \\ \Delta v + k^2 n v = 0\,, & x \in D\,, \\ u = v\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}\,, & x \in \Gamma_1\,, \text{ (transmission condition)} \\ u = v = 0\,, & x \in \Gamma_2\,. \text{ (hom. Dirichlet condition)} \end{cases} \tag{1}$$

- Here, $n \neq 1$ is the real-valued index of refraction (constant).
- Find $k \neq 0$ and non-trivial (u, v) such that (1) is satisfied.
- Such k will be called mixed interior transmission eigenvalues (MITEs).

INTRODUCTION & MOTIVATION

Goal

- This is a non-standard eigenvalue problem.
- It is neither elliptic nor self-adjoint.
- How to solve this problem numerically?
- No results for the computation of MITEs have yet been reported.

Part II: Some theory

SOME THEORY

A short review

- The set of MITEs is at most discrete.
- Does not accumulate at zero.
- There exists an infinite number of real MITEs.
- Only accumulation point is ∞ .

- Nothing is known for complex-valued MITEs.

Part III: Boundary integral equations

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Fundamental solution: $\Phi_k(x,y) = \mathrm{i}H_0^{(1)}(k|x-y|)/4, x \neq y$.
- Single- and double-layer potentials over Γ given for $x \notin \Gamma$ by

$$\begin{aligned} \mathrm{SL}_k^\Gamma[\psi](x) &= \int_\Gamma \Phi_k(x,y)\,\psi(y)\,\mathrm{d}s(y)\,,\\ \mathrm{DL}_k^\Gamma[\psi](x) &= \int_\Gamma \partial_{\nu(y)}\Phi_k(x,y)\,\psi(y)\,\mathrm{d}s(y)\,. \end{aligned}$$

- Green's representation theorem:

$$u(x) = \mathrm{SL}_k^\Gamma[\partial_\nu u|_\Gamma](x) - \mathrm{DL}_k^\Gamma[u|_\Gamma](x)\,, \qquad x \in D\,.$$

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Γ is disjoint union of Γ_1 and Γ_2. Hence,

$$\begin{aligned} u(x) &= \mathrm{SL}_k^{\Gamma_1}\left[\partial_\nu u|_{\Gamma_1}\right](x) + \mathrm{SL}_k^{\Gamma_2}\left[\partial_\nu u|_{\Gamma_2}\right](x) \\ &- \mathrm{DL}_k^{\Gamma_1}\left[u|_{\Gamma_1}\right](x) - \mathrm{DL}_k^{\Gamma_2}\left[u|_{\Gamma_2}\right](x)\,, \qquad x \in D\,, \end{aligned} \tag{2}$$

$$\begin{aligned} v(x) &= \mathrm{SL}_{k\sqrt{n}}^{\Gamma_1}\left[\partial_\nu v|_{\Gamma_1}\right](x) + \mathrm{SL}_{k\sqrt{n}}^{\Gamma_2}\left[\partial_\nu v|_{\Gamma_2}\right](x) \\ &- \mathrm{DL}_{k\sqrt{n}}^{\Gamma_1}\left[v|_{\Gamma_1}\right](x) - \mathrm{DL}_{k\sqrt{n}}^{\Gamma_2}\left[v|_{\Gamma_2}\right](x)\,, \qquad x \in D\,. \end{aligned} \tag{3}$$

- Using $u|_{\Gamma_2} = v|_{\Gamma_2} = 0$, equations (2) and (3) can be simplified to

$$u(x) = \mathrm{SL}_k^{\Gamma_1}\left[\partial_\nu u|_{\Gamma_1}\right](x) + \mathrm{SL}_k^{\Gamma_2}\left[\partial_\nu u|_{\Gamma_2}\right](x) - \mathrm{DL}_k^{\Gamma_1}\left[u|_{\Gamma_1}\right](x)\,, \quad x \in D\,, \tag{4}$$

$$v(x) = \mathrm{SL}_{k\sqrt{n}}^{\Gamma_1}\left[\partial_\nu v|_{\Gamma_1}\right](x) + \mathrm{SL}_{k\sqrt{n}}^{\Gamma_2}\left[\partial_\nu v|_{\Gamma_2}\right](x) - \mathrm{DL}_{k\sqrt{n}}^{\Gamma_1}\left[v|_{\Gamma_1}\right](x)\,, \quad x \in D\,. \tag{5}$$

BOUNDARY INTEGRAL EQUATIONS

Preliminaries

- Boundary integral operators:

$$
\begin{aligned}
\mathrm{S}_k^{\Gamma_i\to\Gamma_j}\left[\psi|_{\Gamma_i}\right](x) &= \int_{\Gamma_i} \Phi_k(x,y)\psi(y)\,\mathrm{d}s(y)\,, \quad x\in\Gamma_j\,,\\
\mathrm{K}_k^{\Gamma_i\to\Gamma_j}\left[\psi|_{\Gamma_i}\right](x) &= \int_{\Gamma_i} \partial_{\nu_i(y)}\Phi_k(x,y)\psi(y)\,\mathrm{d}s(y)\,, \quad x\in\Gamma_j\,,\\
\mathrm{K}_k^{\top\,\Gamma_i\to\Gamma_j}\left[\psi|_{\Gamma_i}\right](x) &= \int_{\Gamma_i} \partial_{\nu_j(x)}\Phi_k(x,y)\psi(y)\,\mathrm{d}s(y)\,, \quad x\in\Gamma_j\,,\\
\mathrm{T}_k^{\Gamma_i\to\Gamma_j}\left[\psi|_{\Gamma_i}\right](x) &= \partial_{\nu_j(x)}\int_{\Gamma_i} \partial_{\nu_i(y)}\Phi_k(x,y)\psi(y)\,\mathrm{d}s(y)\,, \quad x\in\Gamma_j\,,
\end{aligned}
$$

where $i,j\in\{1,2\}$.

BOUNDARY INTEGRAL EQUATIONS

Derivation of first boundary integral equation

- $D \ni x \to x \in \Gamma_1$ in (4) and (5) and jump relations:

$$u|_{\Gamma_1} = \mathrm{S}_k^{\Gamma_1\to\Gamma_1}[\partial_\nu u|_{\Gamma_1}] + \mathrm{S}_k^{\Gamma_2\to\Gamma_1}[\partial_\nu u|_{\Gamma_2}] - \left(\mathrm{K}_k^{\Gamma_1\to\Gamma_1}[u|_{\Gamma_1}] - \frac{1}{2}u|_{\Gamma_1}\right), \tag{6}$$

$$v|_{\Gamma_1} = \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}[\partial_\nu v|_{\Gamma_1}] + \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_1}[\partial_\nu v|_{\Gamma_2}] - \left(\mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}[v|_{\Gamma_1}] - \frac{1}{2}v|_{\Gamma_1}\right). \tag{7}$$

- Difference of (6) and (7), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(\mathrm{S}_k^{\Gamma_1\to\Gamma_1} - \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}\right)[\partial_\nu u|_{\Gamma_1}] + \mathrm{S}_k^{\Gamma_2\to\Gamma_1}[\partial_\nu u|_{\Gamma_2}] \\ &- \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_1}[\partial_\nu v|_{\Gamma_2}] - \left(\mathrm{K}_k^{\Gamma_1\to\Gamma_1} - \mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}\right)[u|_{\Gamma_1}]. \end{aligned} \tag{8}$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of second boundary integral equation

- $D \ni x \to x \in \Gamma_2$ in (4) and (5):

$$u|_{\Gamma_2} = \mathrm{S}_k^{\Gamma_1\to\Gamma_2}[\partial_\nu u|_{\Gamma_1}] + \mathrm{S}_k^{\Gamma_2\to\Gamma_2}[\partial_\nu u|_{\Gamma_2}] - \mathrm{K}_k^{\Gamma_1\to\Gamma_2}[u|_{\Gamma_1}]\,, \tag{9}$$

$$v|_{\Gamma_2} = \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}[\partial_\nu v|_{\Gamma_1}] + \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_2}[\partial_\nu v|_{\Gamma_2}] - \mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}[v|_{\Gamma_1}]\,. \tag{10}$$

- Difference of (9) and (10), $u|_{\Gamma_2} = v|_{\Gamma_2} = 0$, $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 \;=\; & \left(\mathrm{S}_k^{\Gamma_1\to\Gamma_2} - \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}\right)[\partial_\nu u|_{\Gamma_1}] + \mathrm{S}_k^{\Gamma_2\to\Gamma_2}[\partial_\nu u|_{\Gamma_2}] \\ - \; & \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_2}[\partial_\nu v|_{\Gamma_2}] - \left(\mathrm{K}_k^{\Gamma_1\to\Gamma_2} - \mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}\right)[u|_{\Gamma_1}]\,. \end{aligned} \tag{11}$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of third boundary integral equation

- Normal derivative of (4) and (5), $D \ni x \to x \in \Gamma_1$, and jump relations:

$$\partial_\nu u|_{\Gamma_1} = \mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_1}\left[\partial_\nu u|_{\Gamma_1}\right] + \frac{1}{2}\partial_\nu u|_{\Gamma_1} + \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_1}\left[\partial_\nu u|_{\Gamma_2}\right] - \mathrm{T}_k^{\Gamma_1\to\Gamma_1}\left[u|_{\Gamma_1}\right], \quad (12)$$

$$\partial_\nu v|_{\Gamma_1} = \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_1}\left[\partial_\nu v|_{\Gamma_1}\right] + \frac{1}{2}\partial_\nu v|_{\Gamma_1} + \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_1}\left[\partial_\nu v|_{\Gamma_2}\right] - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}\left[v|_{\Gamma_1}\right]. \quad (13)$$

- Difference of (12) and (13), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$\begin{aligned} 0 &= \left(\mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_1} - \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_1}\right)\left[\partial_\nu u|_{\Gamma_1}\right] + \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_1}\left[\partial_\nu u|_{\Gamma_2}\right] \\ &- \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_1}\left[\partial_\nu v|_{\Gamma_2}\right] - \left(\mathrm{T}_k^{\Gamma_1\to\Gamma_1} - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1}\right)\left[u|_{\Gamma_1}\right]. \end{aligned} \quad (14)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of fourth boundary integral equation

- Normal derivative of (4) and (5), $D \ni x \to x \in \Gamma_2$, and jump relations:

$$\partial_\nu u|_{\Gamma_2} = \mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_2}\left[\partial_\nu u|_{\Gamma_1}\right] + \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_2}\left[\partial_\nu u|_{\Gamma_2}\right] + \frac{1}{2}\partial_\nu u|_{\Gamma_2} - \mathrm{T}_k^{\Gamma_1\to\Gamma_2}\left[u|_{\Gamma_1}\right], \quad (15)$$

$$\partial_\nu v|_{\Gamma_2} = \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_2}\left[\partial_\nu v|_{\Gamma_1}\right] + \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_2}\left[\partial_\nu v|_{\Gamma_2}\right] + \frac{1}{2}\partial_\nu v|_{\Gamma_2} - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}\left[v|_{\Gamma_1}\right]. \quad (16)$$

Equations (15) and (16) can be rewritten as

$$0 = \mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_2}\left[\partial_\nu u|_{\Gamma_1}\right] + \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_2}\left[\partial_\nu u|_{\Gamma_2}\right] - \mathrm{T}_k^{\Gamma_1\to\Gamma_2}\left[u|_{\Gamma_1}\right] - \frac{1}{2}\partial_\nu u|_{\Gamma_2}, \quad (17)$$

$$0 = \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_2}\left[\partial_\nu v|_{\Gamma_1}\right] + \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_2}\left[\partial_\nu v|_{\Gamma_2}\right] - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}\left[v|_{\Gamma_1}\right] - \frac{1}{2}\partial_\nu v|_{\Gamma_2}. \quad (18)$$

BOUNDARY INTEGRAL EQUATIONS

Derivation of fourth boundary integral equation

- Difference of (17) and (18), $u|_{\Gamma_1} = v|_{\Gamma_1}$ and $\partial_\nu u|_{\Gamma_1} = \partial_\nu v|_{\Gamma_1}$:

$$
\begin{aligned}
0 \;=\;& \left(\mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_2} - \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_2}\right)[\partial_\nu u|_{\Gamma_1}] + \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_2}[\partial_\nu u|_{\Gamma_2}] \\
-\;& \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_2}[\partial_\nu v|_{\Gamma_2}] - \left(\mathrm{T}_k^{\Gamma_1\to\Gamma_2} - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2}\right)[u|_{\Gamma_1}] - \frac{1}{2}\partial_\nu u|_{\Gamma_2} \\
+\;& \frac{1}{2}\partial_\nu v|_{\Gamma_2}\,. \qquad (19)
\end{aligned}
$$

BOUNDARY INTEGRAL EQUATIONS

4×4 system of boundary integral equations

- Four equations (8), (11), (14), and (19) can be written as

$$Z(k)g = 0$$

with

$$Z(k) = \begin{pmatrix} \mathrm{S}_k^{\Gamma_1\to\Gamma_1} - \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1} & \mathrm{K}_k^{\Gamma_1\to\Gamma_1} - \mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1} & \mathrm{S}_k^{\Gamma_2\to\Gamma_1} & \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_1} \\ \mathrm{S}_k^{\Gamma_1\to\Gamma_2} - \mathrm{S}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2} & \mathrm{K}_k^{\Gamma_1\to\Gamma_2} - \mathrm{K}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2} & \mathrm{S}_k^{\Gamma_2\to\Gamma_2} & \mathrm{S}_{k\sqrt{n}}^{\Gamma_2\to\Gamma_2} \\ \mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_1} - \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_1} & \mathrm{T}_k^{\Gamma_1\to\Gamma_1} - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_1} & \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_1} & \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_1} \\ \mathrm{K}_k^{\top\,\Gamma_1\to\Gamma_2} - \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_1\to\Gamma_2} & \mathrm{T}_k^{\Gamma_1\to\Gamma_2} - \mathrm{T}_{k\sqrt{n}}^{\Gamma_1\to\Gamma_2} & \mathrm{K}_k^{\top\,\Gamma_2\to\Gamma_2} - \frac{1}{2}\mathrm{I} & \mathrm{K}_{k\sqrt{n}}^{\top\,\Gamma_2\to\Gamma_2} - \frac{1}{2}\mathrm{I} \end{pmatrix} \tag{20}$$

$$g = \begin{pmatrix} \alpha & -\beta & \gamma & -\delta \end{pmatrix}^\top ,$$

where we used the notation

$$\alpha = \partial_\nu u|_{\Gamma_1}\,,\; \beta = u|_{\Gamma_1}\,,\; \gamma = \partial_\nu u|_{\Gamma_2}\,,\; \text{and } \delta = \partial_\nu v|_{\Gamma_2}\,. \tag{21}$$

Part IV: Numerical results

NUMERICAL RESULTS

A short description

- Discretize the resulting boundary integral operator via boundary element collocation method.
- Curved boundary is approximated by lines.
- Collocation nodes are the midpoints having m collocation points in total.
- Unknown function is approximated by constant interpolation at each midpoint.
- Hence, we can regard (20) as non-linear eigenvalue problem of the form $\mathbf{Z}(k)\tilde{g} = 0$ with $\mathbf{Z}(k) \in \mathbb{C}^{m\times m}$ and $\tilde{g}$ the discretized version of g given by (21).
- Solved with Beyn's algorithm (based on complex-valued contour integration of the resolvent).

NUMERICAL RESULTS

Unit circle, $n = 4$

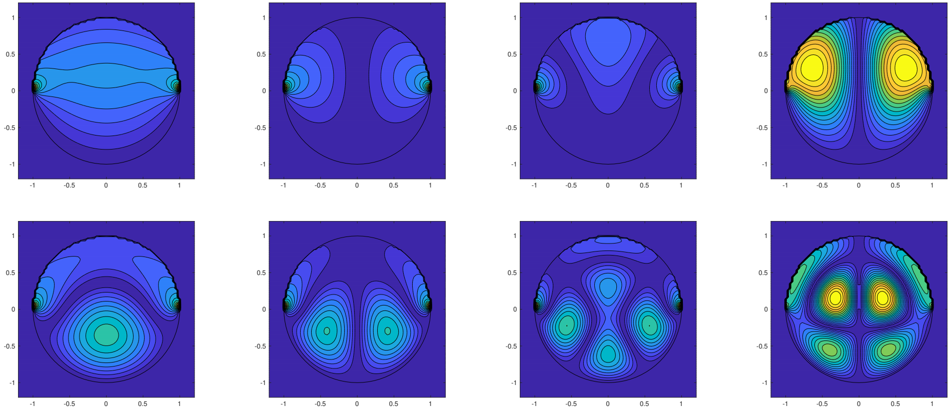


Figure: Absolute value of u (first row) and v (second row) for the first four real-valued MITEs. The MITEs are 1.6818, 2.3185, 2.9533, 3.0791.

NUMERICAL RESULTS

Ellipse, major semi-axis 1, minor semi-axis $4/5$, $n = 4$

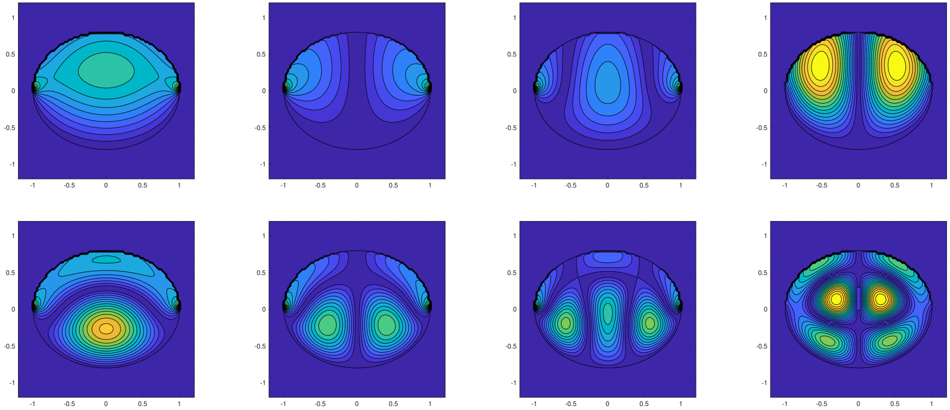


Figure: Absolute value of u (first row) and v (second row) for the first four real-valued MITEs. The MITEs are 1.9111, 2.4973, 3.1282, 3.4609.

NUMERICAL RESULTS

More on ellipses, major semi-axis 1

Table: MITEs for ellipses with various minor semi-axis using $n = 4$.

Minor semi-axis	1st MITE	2nd MITE	3rd MITE	4th MITE
1	1.6818	2.3185	2.9533	3.0791
4/5	1.9111	2.4973	3.1282	3.4609
1/2	2.7709	3.1764	3.7892	4.3916

Table: MITEs for ellipses with various minor semi-axis using $n = 1/2$.

Minor semi-axis	1st MITE	2nd MITE	3rd MITE	4th MITE
1	3.1620	4.5193	4.6482	5.8022
4/5	3.5798	4.8518	5.5187	6.2683
1/2	5.1115	6.1186	7.3248	8.4891

NUMERICAL RESULTS

Unit square, $n = 4$

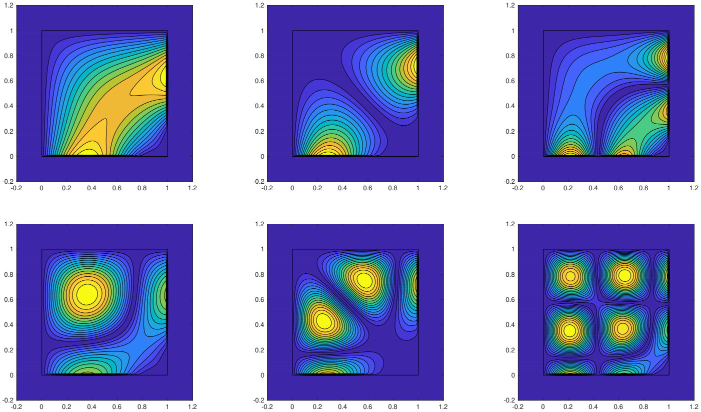


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 3.0503, 4.2622, 5.1805.

NUMERICAL RESULTS

Unit square, $n = 4$

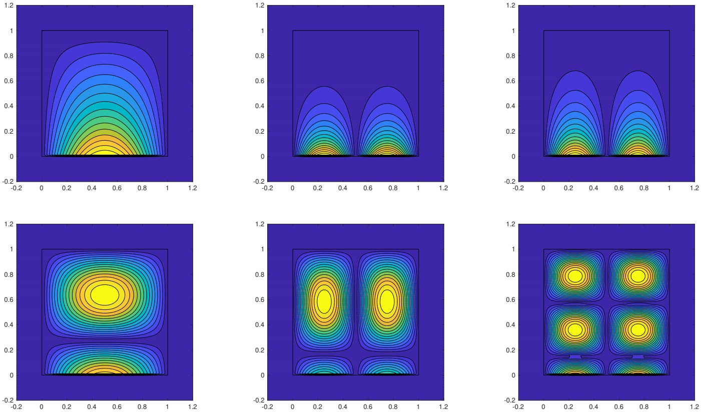


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 2.6717, 3.6662, 4.8367.

NUMERICAL RESULTS

Unit square, $n = 4$

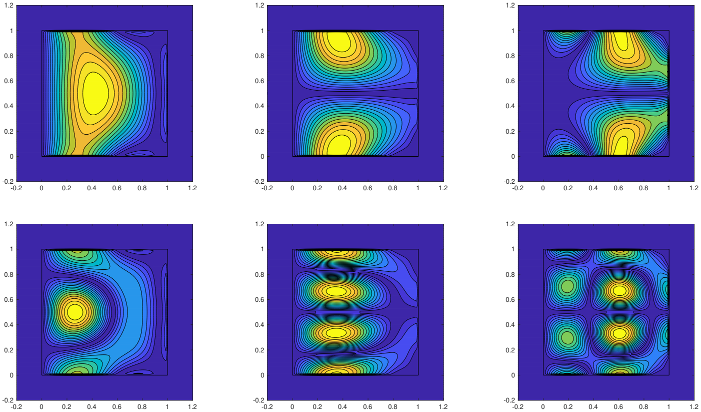


Figure: Absolute value of u (first row) and v (second row) for the first three real-valued MITEs. The MITEs are 4.0802, 5.2285, 5.7030.

Part V: Summary & outlook

SUMMARY & OUTLOOK

- Reviewed existence and discreteness of MITEs for real-valued constant n.
- Derived a system of boundary integral equations.
- Showed how to solve it.
- Provided extensive numerical results for a variety of 2D scatterers.

- Study behavior of the MITE eigenfunctions at corners.
- Investigate inside-outside-duality method both theoretically and practically.

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